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*Research Article*

# New Class of Close-to-Convex Harmonic Functions Defined by a Fourth-Order Differential Inequality

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In the recent past, various new subclasses of normalized harmonic functions have been defined in open unit disk $U$ which satisfy second-order and third-order differential inequalities. Here, in this study, we define a new class of normalized harmonic functions in open unit disk $U$ which is satisfying a fourth-order differential inequality. We investigate some useful results such as close-to-convexity, coefficient bounds, growth estimates, sufficient coefficient condition, and convolution for the functions belonging to this new class of harmonic functions. In addition, under convex combination and convolution of its members, we prove that this new class is closed, and we also give some lemmas to prove our main results.

## 1. Introduction and Definitions

Let $\mathscr{H}$ represent the class of all harmonic functions $f = s + \overline{v}$ in open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$, and all these harmonic functions are normalized by

$$f(0) = f_z(0) - 1 = 0, \tag{1}$$

and $f$ can be expressed as $f = s + \overline{v}$, where $s$ is the analytic part and $v$ is the coanalytic part of $f$ in $U$, and also, these functions have a series of the form

$$s(z) = z + \sum_{j=2}^{\infty} a_j z^j, \qquad v(z) = \sum_{j=2}^{\infty} b_j z^j. \tag{2}$$

Harmonic functions $f$ is locally univalent and sense-preserving in $U$, if it satisfies a necessary sufficient condition $|s\prime(z)| > |v'(z)|$ [1, 2]. If coanalytic part of $f$ is zero, then class $\mathscr{H}$ of complex valued harmonic functions reduces to class $\mathscr{A}$ of normalized analytic functions.

Let $\mathscr{S}$ denote the family of analytic univalent and normalized functions in $U$ and also $\mathscr{S} \subset \mathscr{S}_{\mathscr{H}}$ which are defined as

$$\mathscr{S} = \{f \in \mathscr{S}_{\mathscr{H}}: v = 0 \text{ in } U\}. \tag{3}$$

Also, let class $\mathscr{H}^0$ define as

$$\mathscr{H}^0 = \{f \in \mathscr{H}: f_{\overline{z}}(0) = 0\}, \qquad \mathscr{S}^0_{\mathscr{H}} = \{f \in \mathscr{S}_{\mathscr{H}}: f_{\overline{z}}(0) = 0\}, \tag{4}$$

where class $\mathcal{S}_{\mathcal{H}}$ represent the class of functions $f$ which are harmonic, univalent, and sense-preserving in open unit disk $U$.

We can see that class $\mathcal{S}_{\mathcal{H}}^{0}$ is compact and normal, but class $\mathcal{S}_{\mathcal{H}}$ is only normal. Let $\mathcal{S}_{\mathcal{H}}^{0,*}$, $\mathcal{K}_{\mathcal{H}}^{0}$, and $\mathcal{C}_{\mathcal{H}}^{0}$ are the subclasses of $\mathcal{S}_{\mathcal{H}}^{0}$ which map open unit disk $U$ onto starlike, convex, and close-to-convex domains for harmonic functions, respectively. We can observe that

$$\begin{aligned} \mathcal{S}^{*} &\subset \mathcal{S}_{\mathcal{H}}^{0,*}, \\ \mathcal{K} &\subset \mathcal{K}_{\mathcal{H}}^{0}, \\ \mathcal{C} &\subset \mathcal{C}_{\mathcal{H}}^{0}. \end{aligned} \tag{5}$$

These subclasses $\mathcal{S}^{*}$, $\mathcal{K}$, and $\mathcal{C}$ map open unit disk $U$ onto their respective domains.

For [2], Ponnusamy et al. defined the class of harmonic functions $f \in \mathcal{H}^{0}$ which satisfy the condition

$$\operatorname{Re}\left(f_{z}(z)\right) > \left|f_{\overline{z}}(z)\right|, \quad \text{for } z \in U. \tag{6}$$

In this class, they studied about close-to-convexity of harmonic functions. After that, Li and Ponnusamy [3, 4] discussed univalency and convexity of the abovementioned class. A class $\mathcal{W}_{\mathcal{H}}^{0}$ of harmonic functions $f = s + \overline{v} \in \mathcal{H}^{0}$ defined by Nagpal and Ravichandran in [5] and the functions in this class satisfy the condition

$$\operatorname{Re}\left(s'(z) + zs''(z)\right) > \left|v'(z) + zs''(z)\right|, \quad \text{for } z \in U. \tag{7}$$

Note that

$$\mathcal{W}_{\mathcal{H}}^{0} \subset \mathcal{S}_{\mathcal{H}}^{0,*}, \tag{8}$$

and members of $\mathcal{W}_{\mathcal{H}}^{0}$ are fully starlike in $U$.

Recently, Ghosh and Vasudevarao [6] for $\alpha \geq 0$ defined a new class $\mathcal{W}_{\mathcal{H}}^{0}(\alpha)$ for harmonic functions $f = s + \overline{v} \in \mathcal{H}^{0}$ satisfying the condition

$$\operatorname{Re}\left(s'(z) + \alpha zs''(z)\right) > \left|v'(z) + \alpha zs''(z)\right|, \quad \text{for } z \in U. \tag{9}$$

Rajbala and Prajapat [7] for $\delta \geq 0, 0 \leq \lambda < 1$, defined a new class $\mathcal{W}_{\mathcal{H}}^{0}(\delta, \lambda)$ of harmonic functions which satisfy the following inequality:

$$\operatorname{Re}\left(s'(z) + \delta zs''(z) - \lambda\right) > \left|s'(z) + \delta zv''(z)\right|. \tag{10}$$

For this class, authors used Gaussian hypergeometric function and created harmonic polynomials for the class $\mathcal{W}_{\mathcal{H}}^{0}(\delta, \lambda)$.

Very recently, for the functions $f = s + \overline{v} \in \mathcal{H}^{0}$, Yaşar and Yalçın [8] introduced the class $\mathcal{R}_{\mathcal{H}}^{0}(\lambda, \delta)$ which satisfy the following inequality:

$$\begin{aligned} &\operatorname{Re}\left(s'(z) + \lambda zs''(z) + \delta z^{2}s'''(z)\right) \\ &> \left|v'(z) + \lambda zv''(z) + \delta z^{2}s'''(z)\right|, \end{aligned} \tag{11}$$

for $\lambda \geq \delta \geq 0$. For further study about harmonic functions, refer [2, 5, 9–11].

By taking the inspiration from the abovementioned work, we define new class of harmonic functions in $U$ which satisfy the fourth-order differential inequality.

*Definition 1.* For $\lambda \geq \delta \geq \gamma \geq 0$, let $\mathcal{R}_{\mathcal{H}}^{0}(\lambda, \delta, \gamma)$ denote the class of functions $f = s + \overline{v} \in \mathcal{H}^{0}$ and satisfy the condition

$$\begin{aligned} &\operatorname{Re}\left(s'(z) + \lambda zs''(z) + \delta z^{2}s'''(z) + \gamma z^{3}s''''(z)\right) \\ &> \left|v'(z) + \lambda zv''(z) + \delta z^{2}v'''(z) + \gamma z^{3}v''''(z)\right|. \end{aligned} \tag{12}$$

*Definition 2.* For $\lambda \geq \delta \geq \gamma \geq 0$, let $\mathcal{R}(\lambda, \delta, \gamma)$ denote a class of functions $f \in \mathcal{A}$ if it satisfies the inequality

$$\operatorname{Re}\left(f'(z) + \lambda zf''(z) + \delta z^{2}f'''(z) + \gamma z^{3}f''''(z)\right) > 0. \tag{13}$$

Note that

$$\mathcal{R}(\lambda, \delta, \gamma) \subset \mathcal{R}_{\mathcal{H}}^{0}(\lambda, \delta, \gamma). \tag{14}$$

Special cases are

(1) $\mathcal{R}_{\mathcal{H}}^{0}(\lambda, \delta, \gamma) = \mathcal{R}_{\mathcal{H}}^{0}(\lambda, \delta)$, defined by Yaşar and Yalçın [8].

(2) $\mathcal{R}_{\mathcal{H}}^{0}(1, 0, 0) = \mathcal{W}_{\mathcal{H}}^{0}$, discussed by Nagpal and Ravichandran [5].

(3) $\mathcal{R}_{\mathcal{H}}^{0}(\alpha, 0, 0) = \mathcal{W}_{\mathcal{H}}^{0}(\alpha)$, defined by Ghosh and Vasudevarao [6].

(4) $\mathcal{R}_{\mathcal{H}}^{0}(\delta, 1 - 2\lambda, 0) = \mathcal{W}_{\mathcal{H}}^{0}(\delta, \lambda)$, defined by Rajbala and Prajapat [7].

In this section, we prove that all the members of the class are close-to-convex. We will derive coefficient bounds, growth estimates, and sufficient coefficient condition for the class $\mathcal{R}_{\mathcal{H}}^{0}(\lambda, \delta, \gamma)$. Furthermore, we investigate example of harmonic polynomial belonging to $\mathcal{R}_{\mathcal{H}}^{0}(\lambda, \delta, \gamma)$.

## 2. Main Results

**Lemma 1** (see [1]). *Let s and v are analytic functions in open unit disk U along with $|s'(0)| < |v'(0)|$ and $\mathscr{L}_\mu = s + \mu v$ is close-to-convex for each $\mu \in \mathbb{C}$ ($|\mu| = 1$). Then,*

$$f = s + \overline{v} \text{ is close to convex in } U. \quad (15)$$

**Theorem 1.** *Let $f = s + \overline{v} \in \mathscr{R}^0_{\mathscr{H}}(\lambda, \delta, \gamma)$ if and only if $\mathscr{L}_\mu = s + \mu v \in \mathscr{R}(\lambda, \delta, \gamma)$ for each $\mu \in \mathbb{C}$ ($|\mu| = 1$).*

*Proof.* Suppose $f = s + \overline{v} \in \mathscr{R}^0_{\mathscr{H}}(\lambda, \delta, \gamma)$. For each $|\mu| = 1$,

$$\begin{aligned}
&\operatorname{Re}\left\{\mathscr{L}'_\mu(z) + \lambda z \mathscr{L}''_\mu(z) + \delta z^2 \mathscr{L}'''_\mu(z) + \gamma z^3 \mathscr{L}''''_\mu(z)\right\}\\
&= \operatorname{Re}\left\{s'(z) + \lambda z s''(z) + \delta z^2 s'''(z) + \gamma z^3 s''''(z)\right.\\
&\quad \left.+ \mu\left(v'(z) + \lambda z v''(z) + \delta z^2 v'''(z) + \gamma z^3 v''''(z)\right)\right\}\\
&> \operatorname{Re}\left\{s'(z) + \lambda z s''(z) + \delta z^2 s'''(z) + \gamma z^3 s''''(z)\right\}\\
&\quad - \left|v'(z) + \lambda z v''(z) + \delta z^2 s'''(z) + \gamma z^3 v''''(z)\right| > 0, \quad z \in U.
\end{aligned} \quad (16)$$

Thus, $\mathscr{L}_\mu \in \mathscr{R}(\lambda, \delta, \gamma)$ for each $\mu$ ($|\mu| = 1$). Conversely, let $\mathscr{L}_\mu = s + \mu v \in \mathscr{R}_{\mathscr{H}}(\lambda, \delta, \gamma)$, then

$$\begin{aligned}
&\operatorname{Re}\left\{s'(z) + \lambda z s''(z) + \delta z^2 s'''(z) + \gamma z^3 s''''(z)\right\}\\
&> \operatorname{Re}\left(-\mu\left(v'(z) + \lambda z v''(z) + \delta z^2 v'''(z) + \gamma z^3 v''''(z)\right)\right), \quad z \in U.
\end{aligned} \quad (17)$$

By choosing $\mu$ ($|\mu| = 1$), we get

$$\begin{aligned}
&\operatorname{Re}\left\{s'(z) + \lambda z s''(z) + \delta z^2 s'''(z) + \gamma z^3 s''''(z)\right\}\\
&> \left|v'(z) + \lambda z v''(z) + \delta z^2 v'''(z) + \gamma z^3 v''''(z)\right|, \quad z \in U,
\end{aligned} \quad (18)$$

Hence, $f \in \mathscr{R}^0_{\mathscr{H}}(\lambda, \delta, \gamma)$.

**Lemma 2** (see [12]). *Let $z_0 \neq 0$, $z_0 \in U$, with $r_0 = |z_0|$. Let $\varphi(z)$ defined by*

$$\varphi(z) = C_j z^j + C_{j+1} z^{j+1} + \cdots, \quad (19)$$

*be analytic in U, such that*

$$\left|\varphi(z_0)\right| = \max_{|z| \leq |z_0|} |\varphi(z)|. \quad (20)$$

Then, for $n \in \mathbb{R}$, $n \geq j \geq 1$, such that

$$\begin{aligned}
&\frac{z_0 \varphi'(z_0)}{\varphi(z_0)} = n,\\
&\operatorname{Re}\left(\frac{z_0''(z_0)}{\varphi'(z_0)}\right) \geq n - 1,\\
&\operatorname{Re}\left(1 + 3\frac{z_0 \varphi''(z_0)}{\varphi'(z_0)} + \frac{z_0^2 \varphi'''(z_0)}{\varphi'(z_0)}\right) \geq n^2,
\end{aligned} \quad (21)$$

or

$$\operatorname{Re}\left(\frac{z_0^2 \varphi'''(z_0)}{\varphi\prime(z_0)}\right) \geq n^2 - 3n + 2. \quad (22)$$

Then,

$$\operatorname{Re}\left(1 + 7\frac{z_0 \varphi''(z_0)}{\varphi\prime(z_0)} + 6\left(\frac{z_0^2 \varphi'''(z_0)}{\varphi\prime(z_0)}\right) + \left(\frac{z_0^3 \varphi''''(z_0)}{\varphi\prime(z_0)}\right)\right) \geq n^3. \quad (23)$$

**Lemma 3.** *If $\mathscr{L} \in \mathscr{R}(\lambda, \delta, \gamma)$ with $\lambda \geq \delta \geq \gamma \geq 0$, then*

$$\operatorname{Re}\left(\mathscr{L}'(z)\right) > 0, \quad (24)$$

*and hence,*

$$\mathscr{L}(z) \text{ is close} - \text{to} - \text{convex in } U. \quad (25)$$

*Proof.* Let $\mathscr{L} \in \mathscr{R}(\lambda, \delta, \gamma)$, and we have

$$\mathscr{L}'(z) + \lambda z \mathscr{L}''(z) + \delta z^2 \mathscr{L}'''(z) + \gamma z^3 \mathscr{L}''''(z) = \vartheta(z). \quad (26)$$

Then,

$$\operatorname{Re}(\vartheta(z)) > 0, \quad \text{for } z \in U. \quad (27)$$

Let $\varphi$ be an analytic function in $U$ with

$$\varphi(0) = 0,$$
$$\mathscr{L}'(z) = \frac{1+\varphi(z)}{1-\varphi(z)}, \tag{28}$$
$$\varphi(z) \neq 1.$$

We have to show that $|\varphi(z)| < 1, \forall z \in U$. Then,

$$\vartheta(z) = \mathscr{L}'(z) + \lambda z\mathscr{L}''(z) + \delta z^2\mathscr{L}'''(z) + \gamma z^3\mathscr{L}''''(z),$$
$$\vartheta(z) = \left\{\left(\frac{1+\varphi(z)}{1-\varphi(z)}\right) + 2\lambda\left(\frac{z\varphi'(z)}{(1-\varphi(z))^2}\right) + 2\delta\frac{z\varphi'(z)}{(1-\varphi(z))^2}\left(\frac{z\varphi''(z)}{\varphi'(z)}\right) + 4\delta\left(\frac{(z\varphi'(z))^2}{(1-\varphi(z))^3}\right) + \frac{2\gamma z\varphi'(z)}{(1-\varphi(z))^2}\left(\frac{z^2\varphi'''(z)}{\varphi'(z)}\right) + \frac{12\gamma z\varphi'(z)}{(1-\varphi(z))^3}\left(\frac{z\varphi''(z)}{\varphi'(z)}\right) + \frac{12\gamma(z\varphi'(z))^3}{(1-\varphi(z))^4}\right\}. \tag{29}$$

Since $\varphi$ is analytic in $U$,

$$\varphi(0) = 0, \quad \text{if } \exists z_0 \in U, \tag{30}$$

such that

$$\max_{|z|\le|z_0|} |\varphi(z)| = |\varphi(z_0)| = 1. \tag{31}$$

Then, by using Lemma 2, we may write

$$\varphi(z_0) = e^{i\theta},$$
$$z_0\varphi'(z_0) = n\varphi(z_0) = ne^{i\theta}, \quad n \ge 1,\ 0 < \theta < 2\pi,$$
$$\operatorname{Re}\left(\frac{z_0\varphi''(z_0)}{\varphi'(z_0)}\right) \ge n-1, \tag{32}$$
$$\operatorname{Re}\left(\frac{z_0^2\varphi''(z_0)}{\varphi'(z_0)}\right) \ge n^2-3n+2.$$

For all $z_0 \in U$, we get

$$\begin{aligned}
\operatorname{Re}\vartheta(z_0) &= \operatorname{Re}\left(\left(\frac{1+\varphi(z)}{1-\varphi(z)}\right) + 2\lambda\left(\frac{z\varphi'(z)}{(1-\varphi(z))^2}\right) + 2\delta\frac{z\varphi'(z)}{(1-\varphi(z))^2}\left(\frac{z\varphi''(z)}{\varphi'(z)}\right) + 4\delta\frac{(z\varphi'(z))^2}{(1-\varphi(z))^3} + \frac{2\gamma z\varphi'(z)}{(1-\varphi(z))^2}\left(\frac{z^2\varphi'''(z)}{\varphi'(z)}\right)\right. \\
&\quad \left. + \frac{12\gamma z\varphi'(z)}{(1-\varphi(z))^3}\left(\frac{z\varphi''(z)}{\varphi'(z)}\right) + \frac{12\gamma(z\varphi'(z))^3}{(1-\varphi(z))^4}\right) \\
&= \operatorname{Re}\left\{\frac{1+e^{i\theta}}{1-e^{i\theta}} + \frac{2\lambda ne^{i\theta}}{(1-e^{i\theta})^2} + \frac{2\delta ne^{i\theta}}{(1-e^{i\theta})^2}\left(\frac{z_0\varphi''(z_0)}{\varphi'(z_0)}\right) + \frac{4\delta(ne^{i\theta})^2}{(1-e^{i\theta})^3} + \frac{2\gamma ne^{i\theta}}{(1-e^{i\theta})^2}\left(\frac{z_0^2\varphi'''(z_0)}{\varphi'(z_0)}\right)\right. \\
&\quad \left. + \frac{12\gamma ne^{i\theta}}{(1-e^{i\theta})^3}\left(\frac{z_0\varphi''(z_0)}{\varphi'(z_0)}\right) + \frac{12\gamma(ne^{i\theta})^3}{(1-e^{i\theta})^4}\right\}, \\
&= \left(\frac{-\lambda n}{1-\cos\theta} - \frac{\delta n}{(1-\cos\theta)}\operatorname{Re}\left(\frac{z_0\varphi''(z_0)}{\varphi'(z_0)}\right) + \frac{\delta n^2}{(1-\cos\theta)} - \frac{\gamma n}{(1-\cos\theta)}\operatorname{Re}\left(\frac{z_0\varphi'''(z_0)}{\varphi'(z_0)}\right)\right. \\
&\quad \left. - \frac{3\gamma n}{(1-\cos\theta)}\operatorname{Re}\left(\frac{z_0\varphi''(z_0)}{\varphi'(z_0)}\right) - \frac{3\gamma n^3}{2(1-\cos\theta)}\right) \\
&\le \frac{-\lambda n}{1-\cos\theta} + \frac{\delta n}{(1-\cos\theta)}(n-1) + \frac{\delta n^2}{(1-\cos\theta)} - \frac{\gamma n}{(1-\cos\theta)}\left(n^2-3n+2\right) - \frac{3\gamma n}{(1-\cos\theta)}(n-1) - \frac{3\gamma n^3}{2(1-\cos\theta)},
\end{aligned} \tag{33}$$
$$\operatorname{Re}(\vartheta(z_0)) \le \frac{-\lambda n}{(1-\cos\theta)} + \frac{\delta n}{(1-\cos\theta)} - \frac{\gamma n\left(5n^2-1\right)}{2(1-\cos\theta)} \le \frac{(\delta-\lambda)n}{(1-\cos\theta)} - \frac{\gamma n\left(5n^2-1\right)}{2(1-\cos\theta)} \le 0,$$

which opposes the hypothesis. Hence, there is no $z_0 \in U$, such that

$$\left|\varphi\left(z_0\right)\right| = 1. \tag{34}$$

Hence, $|\varphi(z)| < 1$ for all $z \in U$. So, we get

$$\operatorname{Re}\left(\mathscr{L}'(z)\right) > 0. \tag{35}$$

□

**Theorem 2.** *A function $f = s + \overline{v} \in \mathscr{R}_{\mathscr{H}}^{0}(\lambda, \delta, \gamma)$ is close-to-convex in $U$.*

*Proof.* According to Lemma 3, we derive that $\mathscr{L}_\mu = s + \mu v \in \mathscr{R}(\lambda, \delta, \gamma)$ are close-to-convex in $U$, $\forall \mu(|\mu| = 1)$. Therefore, in the light of Theorem 1 and Lemma 1, we get $\mathscr{L}_\mu \in \mathscr{R}_{\mathscr{H}}^{0}(\lambda, \delta, \gamma)$ are close-to-convex in $U$. □

**Theorem 3.** *Let $f = s + \overline{v} \in \mathscr{R}_{\mathscr{H}}^{0}(\lambda, \delta, \gamma)$. Then,*

$$\left|b_j\right| \leq \frac{1}{j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]}, \quad for\ j \geq 2. \tag{36}$$

The equality is satisfied for

$$f(z) = z + \frac{1}{j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]}\overline{z}^j. \tag{37}$$

*Proof.* Let $f = s + \overline{v} \in \mathscr{R}_{\mathscr{H}}^{0}(\lambda, \delta, \gamma)$. Applying the series of $v(z)$, we get

$$\begin{aligned}
&r^{j-1}[j + j\lambda(j-1) + \delta j(j-1)(j-2) + j\gamma(j-1)(j-2)(j-3)]\left|b_j\right| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi}\left|v'\left(re^{i\theta}\right) + \lambda re^{i\theta}v''\left(re^{i\theta}\right) + \delta r^2e^{2i\theta}v'''\left(re^{i\theta}\right) + \gamma r^3e^{3i\theta}v''''\left(re^{i\theta}\right)\right|\mathrm{d}\theta \\
&< \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\left\{s'\left(re^{i\theta}\right) + \lambda re^{i\theta}s''\left(re^{i\theta}\right) + \delta r^2e^{2i\theta}s'''\left(re^{i\theta}\right) + \gamma r^3e^{3i\theta}s''''\left(re^{i\theta}\right)\right\}\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\left\{1 + \sum_{j=2}^{\infty}j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}] \times a_jr^{j-1}e^{i(j-1)\theta}\right\}\mathrm{d}\theta = 1.
\end{aligned} \tag{38}$$

Taking $r \longrightarrow 1-$, we get required result.

**Theorem 4.** *Let for $j \geq 2$, $f = s + \overline{v} \in \mathscr{R}_{\mathscr{H}}^{0}(\lambda, \delta, \gamma)$. Then,*

$$\begin{aligned}
&(1)\ \left|a_j\right| + \left|b_j\right| \leq \frac{2}{j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]}, \\
&(2)\ \left|\left|a_j\right| - \left|b_j\right|\right| \leq \frac{2}{j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]}, \\
&(3)\ \left|a_j\right| \leq \frac{2}{j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]}.
\end{aligned} \tag{39}$$

The equality holds in each case for the function

$$f(z) = z + \sum_{j=2}^{\infty}\frac{2}{j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]}z^j. \tag{40}$$

*Proof.* Let

$$f = s + \overline{v} \in \mathscr{R}_{\mathscr{H}}^{0}(\lambda, \delta, \gamma). \tag{41}$$

Then, from Theorem 1, we have

$$\mathscr{L}_\mu = s + \mu v \in \mathscr{R}(\lambda, \delta, \gamma), \tag{42}$$

for each $\mu$ $(|\mu| = 1)$. Thus, for each $|\mu| = 1$, we have

$$\operatorname{Re}\{\Phi(s, v)\} > 0, \tag{43}$$

for $z \in U$, where

$$\Phi(s, v) = (s + \mu v)' + \lambda z(s + \mu v)'' + \delta z^2(s + \mu v)''' + \gamma z^3(s + \mu v)''''. \tag{44}$$

Then, there exists an analytic function $p$ of the form

$$p(z) = 1 + \sum_{j=1}^{\infty}p_jz^j, \tag{45}$$

with

$$\operatorname{Re}p(z) > 0, \tag{46}$$

such that

$$\left\{\Phi_1(s, v) + \mu\Phi_2(s, v)\right\} = p(z), \tag{47}$$

where

$$\begin{aligned}
\Phi_1(s, v) &= s'(z) + \lambda zs''(z) + \delta z^2s'''(z) + \gamma z^3s''''(z), \\
\Phi_2(s, v) &= \left(v'(z) + \lambda zv''(z) + \delta z^2v'''(z) + \gamma z^3v''''(z)\right).
\end{aligned} \tag{48}$$

Comparing coefficients on both sides of (47), we yield

$$j[1+(j-1)\{\lambda+(j-2)(\delta+\gamma(j-3))\}]\left(a_j+\mu b_j\right)=p_{j-1}, \quad j\geq 2. \tag{49}$$

Since $|p_j|\leq 2$ for $j\geq 1$ and $\mu(|\mu|=1)$ is arbitrary, first part of Theorem 4 is complete. Similarly, we can prove (2) and (3).

Now, we investigate sufficient condition for $f\in\mathscr{R}^0_{\mathscr{H}}(\lambda,\delta,\gamma)$.

**Theorem 5.** *Let $f=s+\overline{v}\in\mathscr{H}^0$ with*

$$\sum_{j=2}^{\infty} j[1+(j-1)\{\lambda+(j-2)(\delta+\gamma(j-3))\}]\left(\left|a_j\right|+\left|b_j\right|\right)\leq 1. \tag{50}$$

Then, $f\in\mathscr{R}^0_{\mathscr{H}}(\lambda,\delta,\gamma)$.

*Proof.* Let $f=s+\overline{v}\in\mathscr{H}^0$. Then by using (50), we have

$$\begin{aligned}
&\operatorname{Re}\left\{s'(z)+\lambda z s''(z)+\delta z^2 s'''(z)+\gamma z^3 s''''(z)\right\}\\
&=\operatorname{Re}\left\{1+\sum_{j=2}^{\infty} j[1+(j-1)\{\lambda+(j-2)(\delta+\gamma(j-3))\}]a_j z^{j-1}\right\}\\
&>1-\sum_{j=2}^{\infty} j[1+(j-1)\{\lambda+(j-2)(\delta+\gamma(j-3))\}]\left|a_j\right|\\
&\geq\sum_{j=2}^{\infty} j[1+(j-1)\{\lambda+(j-2)(\delta+\gamma(j-3))\}]\left|b_j\right|\\
&>\left|\sum_{j=2}^{\infty} j[1+(j-1)\{\lambda+(j-2)(\delta+\gamma(j-3))\}]b_j z^{j-1}\right|\\
&=\left|v'(z)+\lambda z v''(z)+\delta z^2 v'''(z)+\gamma z^3 v''''(z)\right|.
\end{aligned} \tag{51}$$

Hence, $f\in\mathscr{R}^0_{\mathscr{H}}(\lambda,\delta,\gamma)$.

**Corollary 1.** *Let $f=s+\overline{v}\in\mathscr{H}^0$. If*

$$\sum_{j=2}^{\infty} j^2[3-j+\lambda(j-1)]\left(\left|a_j\right|+\left|b_j\right|\right)\leq 2, \tag{52}$$

*then $f\in\mathscr{R}^0_{\mathscr{H}}(\lambda,((\lambda-1)/2),0)$ with $\lambda\geq 1$.*

*Example 1.* By taking $\lambda=0.5$ and $\delta=0.05$, $\gamma=0$ in the light of Theorem 5, then harmonic polynomials

$$\begin{aligned}
f_1(z)&=z+0.15\overline{z}^3,\\
f_1(z)&=z-0.079z^3+0.079\overline{z}^3,
\end{aligned} \tag{53}$$

belong to $\mathscr{R}^0_{\mathscr{H}}(0.5,0.05,0)$.

*Example 2.* By taking $\lambda=3$ and $\delta=1$, $\gamma=0$ in the light of Theorem 5, then harmonic polynomials

$$\begin{aligned}
f_3(z)&=z-\frac{1}{16}\overline{z}^2+\frac{1}{54}\overline{z}^3,\\
f_1(z)&=z-0.079z^3+0.079\overline{z}^3,
\end{aligned} \tag{54}$$

belong to $\mathscr{R}^0_{\mathscr{H}}(3,1,0)$.

*Remark 1.* The results which we obtained above lead to the results of the classes $\mathscr{R}^0_{\mathscr{H}}(0,0,0)$ and $\mathscr{R}^0_{\mathscr{H}}(\lambda,0,0)$ which are defined and studied in [2–4, 6], respectively.

*Remark 2.* The class $\mathscr{R}(\lambda,((\lambda-1)/2),0)$ with $\lambda\geq 1$ is a special case of the class defined in [13] and also our results lead the results of [13].

Now, we investigate convex combinations and convolutions for the class $\mathscr{R}^0_{\mathscr{H}}(\lambda,\delta,\gamma)$.

**Theorem 6.** *The class $\mathscr{R}^0_{\mathscr{H}}(\lambda,\delta,\gamma)$ of harmonic functions is closed under convex combinations.*

*Proof.* Let, for $k=1,2,\ldots,j$ and $f_k=s_k+\overline{v_k}\in\mathscr{R}^0_{\mathscr{H}}(\lambda,\delta,\gamma)$,

$$\sum_{k=1}^{j} t_k=1, \quad 0\leq t_k\leq 1, \tag{55}$$

and convex combination of functions $f_k$ $(k=1,2,\ldots,j)$ can be defined as

$$f(z)=\sum_{k=1}^{j} t_k f_k(z)=s(z)+\overline{v(z)}, \tag{56}$$

where

$$\begin{aligned}
s(z)&=\sum_{k=1}^{j} t_k s_k(z),\\
v(z)&=\sum_{k=1}^{j} t_k v_k(z),
\end{aligned} \tag{57}$$

and $s$ and $v$ are the analytic in $U$ with

$$s(0)=v(0)=s'(0)-1=v'(0)=0. \tag{58}$$

Now,

$$\begin{aligned}
&\operatorname{Re}\left\{s'(z)+\lambda z s''(z)+\delta z^2 s'''(z)+\gamma z^3 s''''(z)\right\}\\
&=\operatorname{Re}\left\{\sum_{i=1}^{j} t_k\left(s_k'(z)+\lambda z s_k''(z)+\delta z^2 s_k'''(z)+\gamma z^3 s_k''''(z)\right)\right\}\\
&>\sum_{k=1}^{j} t_k\left|v_k'(z)+\lambda z v_k''(z)+\delta z^2 v_k'''(z)+\gamma z^3 v_k''''(z)\right|\\
&\geq\left|v'(z)+\lambda z v''(z)+\delta z^2 v'''(z)+\gamma z^3 v''''(z)\right|,
\end{aligned} \tag{59}$$

showing that $f\in\mathscr{R}^0_{\mathscr{H}}(\lambda,\delta,\gamma)$.

A sequence $\{B_j\}_{j=0}^{\infty}$ of nonnegative real numbers is said to be a convex null sequence, if $B_j \longrightarrow 0$ as $j \longrightarrow \infty$ and

$$B_0 - B_1 \geq B_1 - B_2 \geq B_2 - B_3 \geq \cdots \geq B_{j-1} - B_j \cdots \geq 0. \tag{60}$$

We need Lemma 4 and Lemma [14] to prove results for convolution.

**Lemma 4** (see [15]). *If $\{B_j\}_{j=0}^{\infty}$ is a convex null sequence, then*

$$B(z) = \frac{B_0}{2} + \sum_{j=1}^{\infty} B_j z^j, \tag{61}$$

*is analytic and*

$$\mathrm{Re}(B(z)) > 0, \quad \text{in } U. \tag{62}$$

**Lemma 5.** *Let $\mathscr{L} \in \mathscr{R}(\lambda, \delta, \gamma)$. Then,* $\mathrm{Re}(\mathscr{L}(z)/z) > (1/2)$.

*Proof.* Suppose $\mathscr{L} \in \mathscr{R}(\lambda, \delta, \gamma)$ be given by $\mathscr{L}(z) = z + \sum_{j=2}^{\infty} A_j z^j$. Then,

$$\mathrm{Re}\left\{1 + \sum_{j=2}^{\infty} j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]A_j z^{j-1}\right\} > 0, \tag{63}$$

which is equivalent to

$$\mathrm{Re}(p(z)) > \frac{1}{2}, \quad \text{in } U, \tag{64}$$

where

$$p(z) = 1 + \frac{1}{2}\sum_{j=2}^{\infty} j[1 + (j-1)(\lambda + (j-2)\{\delta + \gamma(j-3)\})]A_j z^{j-1}. \tag{65}$$

Now, for $j \geq 2$, we consider a sequence $\{B_j\}_{j=0}^{\infty}$ defined by

$$B_0 = 1, \quad B_{j-1} = \frac{2}{j[1 + (j-1)(\lambda + (j-2)\{\delta + \gamma(j-3)\})]}. \tag{66}$$

Since $\{B_j\}_{j=0}^{\infty}$ is a convex null sequence and using Lemma 4, the function

$$B(z) = 1 + \sum_{j=2}^{\infty} \frac{2}{j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]} z^{j-1}, \tag{67}$$

is analytic and $\mathrm{Re}(B(z)) > (1/2)$ in $U$. Writing

$$\frac{\mathscr{L}(z)}{z} = p(z) * \left(1 + \sum_{j=2}^{\infty} \frac{2}{j[1 + (j-1)\{\lambda + (j-2)(\delta + \gamma(j-3))\}]} z^{j-1}\right), \tag{68}$$

and using Lemma [14], we get $\mathrm{Re}(\mathscr{L}(z)/z) > (1/2)$.

**Lemma 6.** *Let $\mathscr{L}_i \in \mathscr{R}(\lambda, \delta, \gamma)$, for $k = 1, 2$. Then, $\mathscr{L}_1 * \mathscr{L}_2 \in \mathscr{R}(\lambda, \delta, \gamma)$.*

*Proof.* Suppose $\mathscr{L}_1(z) = z + \sum_{j=2}^{\infty} A_j z^j$ and $\mathscr{L}_2(z) = z + \sum_{j=2}^{\infty} B_j z^j$. Then, the convolution of $\mathscr{L}_1(z)$ and $\mathscr{L}_2(z)$ is defined by

$$\mathscr{L}(z) = (\mathscr{L}_1 * \mathscr{L}_2)(z) = z + \sum_{j=2}^{\infty} A_j B_j z^j. \tag{69}$$

Since

$$\begin{aligned}
\mathscr{L}'(z) &= \mathscr{L}_1'(z) * \frac{\mathscr{L}_2(z)}{z}, \\
z\mathscr{L}''(z) &= z\mathscr{L}_1''(z) * \frac{\mathscr{L}_2(z)}{z}, \\
z^2\mathscr{L}'''(z) &= z^2\mathscr{L}_1''(z) * \frac{\mathscr{L}_2(z)}{z}, \\
z^3\mathscr{L}''''(z) &= z^3\mathscr{L}_1''''(z) * \frac{\mathscr{L}_2(z)}{z}.
\end{aligned} \tag{70}$$

Then, we have

$$\begin{aligned}
&\mathscr{L}'(z) + \lambda z\mathscr{L}_1''(z) + \delta z^2\mathscr{L}'''(z) + \gamma z^3\mathscr{L}''''(z) \\
&= \left(\mathscr{L}_1'(z) + \lambda z\mathscr{L}_1''(z) + \delta z^2\mathscr{L}_1''(z) + \gamma z^3\mathscr{L}_1''''(z)\right) * \frac{\mathscr{L}_2(z)}{z}.
\end{aligned} \tag{71}$$

Since $\mathscr{L}_1 \in \mathscr{R}(\lambda, \delta, \gamma)$,

$$\mathrm{Re}\left\{\mathscr{L}_1'(z) + \lambda z\mathscr{L}_1''(z) + \delta z^2\mathscr{L}_1''(z) + \gamma z^3\mathscr{L}_1''''(z)\right\} > 0, \tag{72}$$

and using Lemma 5, $\mathrm{Re}\{\mathscr{L}(z)/z\} > (1/2)$ in $U$.

Now applying Lemma [14] on (71) yields

$$\mathrm{Re}\left\{\mathscr{L}'(z) + \lambda z\mathscr{L}''(z) + \delta z^2\mathscr{L}'''(z) + \gamma z^3\mathscr{L}''''(z)\right\} > 0, \tag{73}$$

in $U$. Thus, $\mathscr{L} = \mathscr{L}_1 * \mathscr{L}_2 \in \mathscr{R}(\lambda, \delta, \gamma)$.

Now, by using Lemma 6, let us prove that the class $\mathscr{R}_{\mathscr{H}}^0(\lambda, \delta, \gamma)$ is closed under convolutions of its members.

**Theorem 7.** *Let $f_k \in \mathscr{R}_{\mathscr{H}}^0(\lambda, \delta, \gamma)$, for $k = 1, 2$. Then,*

$$f_1 * f_2 \in \mathscr{R}_{\mathscr{H}}^0(\lambda, \delta, \gamma). \tag{74}$$

*Proof.* Let

$$f_k = s_k + \overline{v_k} \in \mathscr{R}_{\mathscr{H}}^0(\lambda, \delta, \gamma), \quad k = 1, 2, \tag{75}$$

Then, the convolution of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2 = s_1 * s_2 + \overline{v_1 * v_2}, \tag{76}$$

in order to show that

$$f_1 * f_2 \in \mathscr{R}_{\mathscr{H}}^0(\lambda, \delta, \gamma). \tag{77}$$

We have to prove that

$$\mathcal{L}_\mu = s_1 * s_2 + \mu(\nu_1 * \nu_2) \in \mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma), \tag{78}$$

for each $\mu(|\mu| = 1)$. By Lemma 6, the class $\mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma)$ is closed under convolutions for each $\mu(|\mu| = 1)$, $s_k + \mu\nu_k \in \mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma)$ for $(k = 1, 2)$. Then, both $\mathcal{L}_1$ and $\mathcal{L}_1$ given by

$$\begin{aligned} \mathcal{L}_1 &= (s_1 - \nu_1) * (s_2 - \mu\nu_2), \\ \mathcal{L}_2 &= (s_1 + \nu_1) * (s_2 + \mu\nu_2), \end{aligned} \tag{79}$$

belong to $\mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma)$. Since $\mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma)$ is closed under convex combinations, then

$$\mathcal{L} = \frac{1}{2}(\mathcal{L}_1 + \mathcal{L}_2) = s_1 * s_2 + \mu(\nu_1 * \nu_2), \tag{80}$$

belongs to $\mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma)$. Hence, $\mathcal{R}^0_{\mathcal{H}}(\lambda, \delta, \gamma)$ is closed under convolution.

Goodloe [16] defined the Hadamard product of a harmonic function as follows:

$$f \,\tilde{*}\, \varphi = s * \varphi + \overline{\nu * \varphi}, \tag{81}$$

where $f = s + \overline{\nu} \in \mathcal{H}$ and $\varphi \in \mathcal{A}$. By considering this Hadamard product of a harmonic function, we investigate following result.

**Theorem 8.** *Let $f \in \mathcal{R}^0_{\mathcal{H}}(\lambda, \delta, \gamma)$ and $\varphi \in \mathcal{A}$ be such that $Re(\varphi(z)/z) > (1/2)$, for $z \in U$. Then, $f \,\tilde{*}\, \varphi \in \mathcal{R}^0_{\mathcal{H}}(\lambda, \delta, \gamma)$.*

*Proof.* Suppose

$$f = s + \overline{\nu} \in \mathcal{R}^0_{\mathcal{H}}(\lambda, \delta, \gamma). \tag{82}$$

Then,

$$\mathcal{L}_\mu = s + \mu\nu \in \mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma), \tag{83}$$

for each $\mu(|\mu| = 1)$. Using Theorem 1 and in order to show that $f \,\tilde{*}\, \varphi \in \mathcal{R}^0_{\mathcal{H}}(\lambda, \delta, \gamma)$, we need to show that

$$G = s * \varphi + \mu(\nu * \varphi) \in \mathcal{R}_{\mathcal{H}}(\lambda, \delta, \gamma), \tag{84}$$

for each $\mu(|\mu| = 1)$. Write $G = \mathcal{L}_\mu * \varphi$, and

$$\begin{aligned} &\left(G\prime(z) + \lambda z G''(z) + \delta z^2 G'''(z) + \gamma z^3 G''''(z)\right) \\ &= \left(\mathcal{L}'_\mu(z) + \lambda z \mathcal{L}''_\mu(z) + \delta z^2 \mathcal{L}'''_\mu(z) + \gamma z^3 \mathcal{L}''''_\mu(z)\right) * \frac{\varphi(z)}{z}. \end{aligned} \tag{85}$$

Since $\mathrm{Re}(\varphi(z)/z) > (1/2)$ and

$$\mathrm{Re}\left(\mathcal{L}'_\mu(z) + \lambda z \mathcal{L}''_\mu + \delta z^2 \mathcal{L}'''_\mu(z) + \gamma z^3 \mathcal{L}''''_\mu(z)\right) > 0, \tag{86}$$

in $U$. Lemma [14] proves that $G \in \mathcal{R}^0_{\mathcal{H}}(\lambda, \delta, \gamma)$.

**Corollary 2.** *Let $f \in \mathcal{R}^0_{\mathcal{H}}(\lambda, \delta, \gamma)$ and $\varphi \in \mathcal{K}$. Then, $f \,\tilde{*}\, \varphi \in \mathcal{R}^0_{\mathcal{H}}(\lambda, \delta, \gamma)$.*

*Proof.* Suppose $\varphi \in \mathcal{K}$. Then, $\mathrm{Re}(\varphi(z)/z) > (1/2)$ for $z \in U$. Theorem 8 concludes that $f \,\tilde{*}\, \varphi \in \mathcal{R}^0_{\mathcal{H}}(\lambda, \delta, \gamma)$.

## 3. Conclusion

Various new subclasses of normalized harmonic functions have been defined in open unit disk $U$, satisfying second-order and third-order differential inequalities. In this study, we defined a new class of normalized harmonic functions in open unit disk $U$, satisfying a fourth-order differential inequality. We gave some useful results such that close-to-convexity, coefficient bounds, growth estimates, sufficient coefficient condition, and convolution for the functions belong to this new class of harmonic functions. Further using the concepts of fourth-order differential inequality, all these problems can be studied for classes of meromorphic harmonic functions, Bazilevic harmonic functions, and for $p$ valent harmonic functions as well.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors equally contributed to this study.

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