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Research Article

The Number of Perfect Matchings in Hexagons on the Torus by Pfaffians

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Let G be a (molecular) graph. A perfect matching of G is defined as a set of edges which are independent and cover every vertex of G exactly once. In the article, we present the formula on the number of the perfect matchings of two types of hexagons on the torus by Pfaffians.

1. Introduction

A hexagonal system, or benzenoid system [1], or honeycomb lattice, is a finite connected subgraph of the infinite hexagonal lattice without cut vertices or non-hexagonal interior faces. Hansen and Zheng [2] and Gutman et al. [3] considered the problem about the normal components of benzenoid systems, respectively.

A perfect matching of a (molecular) graph G is defined as a set of edges which are independent and cover every vertex of G exactly once. It is also named as Kekulé structure in organic chemistry and closed-packed dimer in statistical physics, respectively. Denote the number of perfect matchings of G by $\Phi(G)$. In the 1930s, Cyvin and Lovász et al. [4–6] firstly focused on the problems about the perfect matchings of a graph for two different and unrelated purposes. The number of perfect matchings or Kekulé structures had been widely used in various problems in the chemical fields [4, 7–9]. Cyvin et al. [10–14] discussed the enumeration of Kekulé structures for all kinds of benzenoids. Qian and Zhang presented the number of Kekulé structures for capped armchair nanotubes and capped zigzag nanotubes by transfer matrix method in [15, 16], respectively. Yan et al. [17] further discussed the problem about the perfect matchings for one type of hexagons on a cylinder (cf. Figure 1(a)). Li and Zhang also obtained the number of perfect matchings for two types of hexagons on the cylinder and the Möbius strip in [18] (cf. Figure 1).

Let $G=(V(G),E(G))$ and $H=(V(H),E(H))$ be two given graphs. Let $G\times H$ denote the Cartesian product of G with H, where $V(G\times H)=V(G)\times V(H)$, $E(G\times H)=\{\{(u_1,u_2),(v_1,v_2)\}|u_1=v_1$ and $\{u_2,v_2\}\in E(H)$, or $u_2=v_2$ and $\{u_1,v_1\}\in E(G)\}$. A hexagonal cylinder, which is denoted by HS, of length $2m$ and breadth n is a graph obtained from the Cartesian product of a 2 m-cycle $(x_1u_1x_2u_2\cdots x_mu_m)$ and an n-path $(12\cdots n)$ by deleting the set of edges

$$\left\{\{(x_i,2s+1),(x_i,2s+2)\}\middle|0\le s\le\left\lceil\frac{n-1}{2}-1\right\rceil,1\le i\le m\right\}\cup\left\{\{(u_i,2s),(u_i,2s+1)\}\middle|0\le s\le\left\lfloor\frac{n-1}{2}\right\rfloor,1\le i\le m\right\}. \quad (1)$$

Let $x_1u_1x_2u_2\cdots x_mu_m$ and $y_1v_1y_2v_2\cdots y_mv_m$ indicate the vertices of two cycles on the upper and lower boundaries of HS, respectively, where x_i corresponds to y_i and u_i corresponds to $v_i\,(i=12,\cdots,m)$ (cf. Figure 2). Without loss of generality, suppose further that both u_i and v_i are the vertices of degree 3. The graphs $H_{2m,n,r}$, $0\le r\le\lfloor m/2\rfloor$ and $i=12,\cdots,m$, are from HS by adding

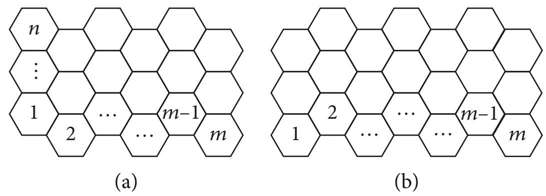


Figure 1: Two types of hexagons.

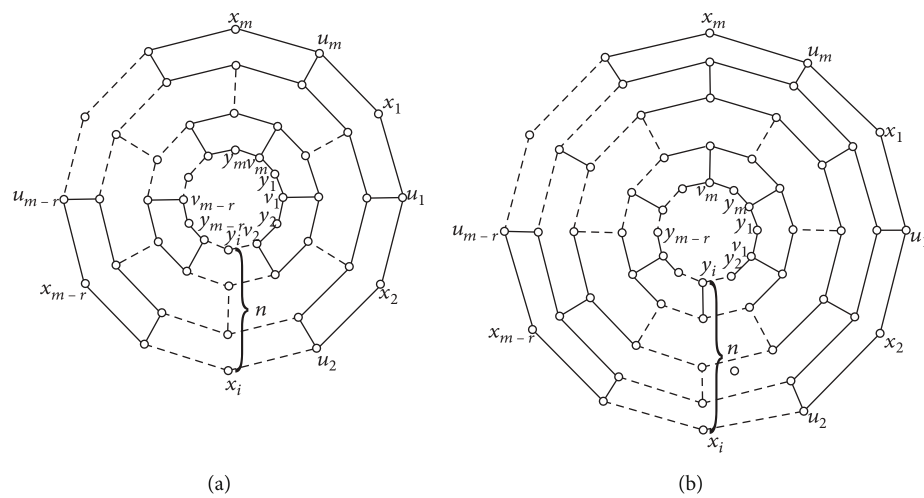


Figure 2: Plane spanning subgraphs of $H_{2m,n,r}$: (a) n is even and (b) n is odd.

all edges $x_i y_{i-r}$, where $i-r$ is modulo m (cf. Figure 2). Clearly, $H_{2m,n,r}$ has a natural embedding on a torus.

The enumerations of perfect matchings of $H_{2m,n,r}$ are considered by Wu [19] and Klein [20], where Klein gave the formula of the number of perfect matchings of $H_{2m,n,r}$ by transfer matrix method when both n and r are odd or even. In the present article, we consider further the problem involving perfect matchings of $H_{2m,n,r}$ by crossing orientation given by Tesler and the plane model of a graph.

2. Pfaffian Orientation

Let $G=(V(G),E(G))$ be an undirected graph, where $V(G)=\{1,2,3,\cdots,2p\}$ is the vertex set of G. Let each edge $\{i,j\}$ of a graph G have a weight denoted by ω_{ij}. And the weight $\omega_{ij}=1$ for all edges in each unweighted graph. Suppose that $\vec{G}$ is an arbitrary orientation of G. If the direction of an edge $\{i,j\}$ in $\vec{G}$ is from the vertex i to vertex j, then (i,j) denotes the arc of $\vec{G}$ and the set of all the arcs of it is denoted by $E(\vec{G})$. Denote the *skew adjacency matrix* of $\vec{G}$ by $A(\vec{G})$. And it is defined as

$$A\left(\vec{G}\right)=\left(a_{ij}\right)_{2p\times 2p}, \tag{2}$$

where

$$a_{ij}=\begin{cases}\omega_{ij}, & \text{if } (i,j)\in E\left(\vec{G}\right);\\ -\omega_{ij}, & \text{if } (j,i)\in E\left(\vec{G}\right);\\ 0, & \text{otherwise.}\end{cases} \tag{3}$$

Suppose that $\mathbf{M}=\{\{j_1,j_1'\},\cdots,\{j_p,j_p'\}\}$ is a perfect matching. Then the signed weight of the perfect matching $\mathbf{M}$ is

$$\omega_{\mathrm{M}}=\operatorname{sgn}\begin{pmatrix}1 & 2 & \cdots & 2p-1 & 2p\\ j_1 & j_1' & \cdots & j_p & j_p'\end{pmatrix}\cdot a_{j_1j_1'}\cdots a_{j_pj_p'}, \tag{4}$$

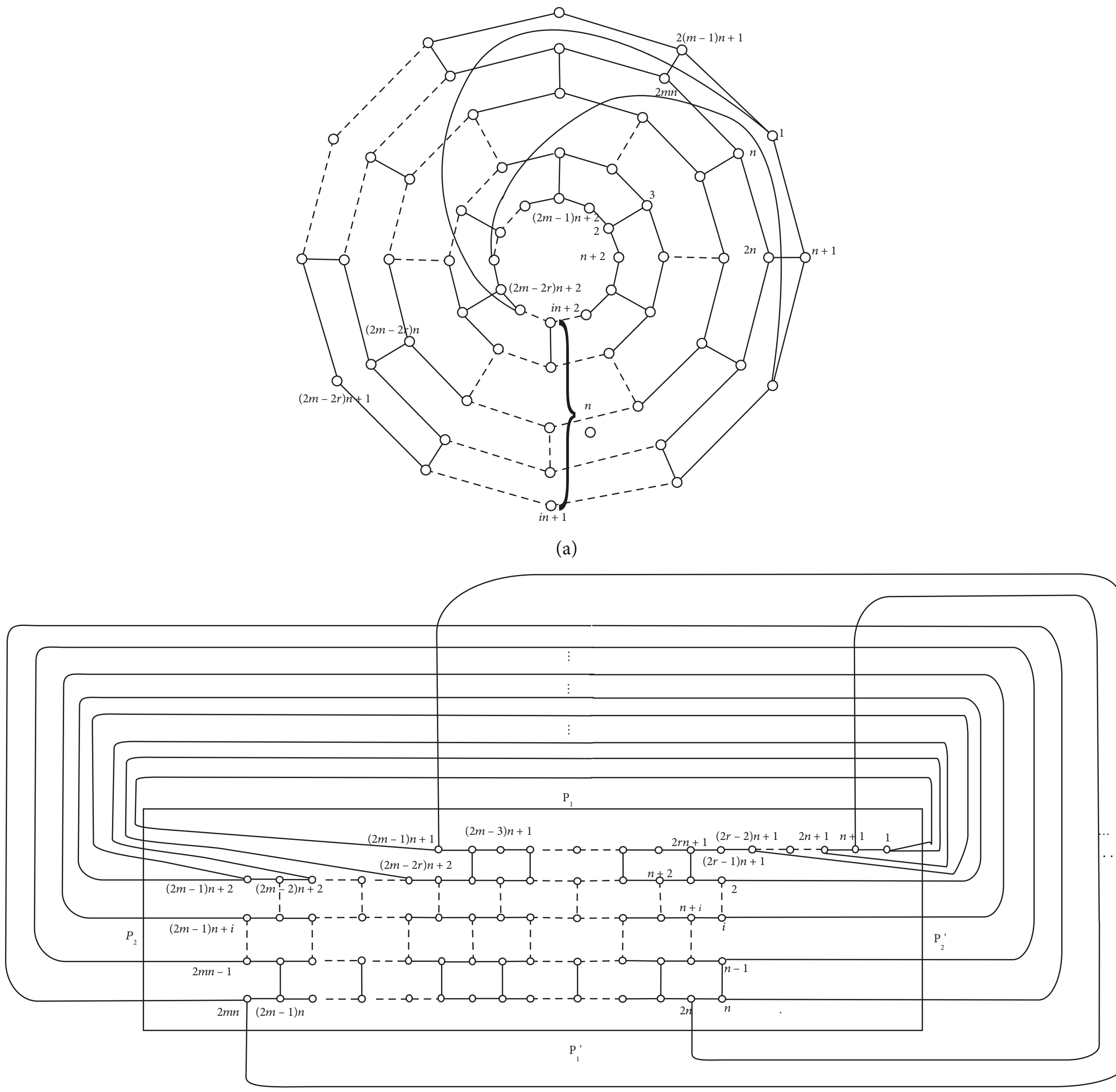
where

$$\operatorname{sgn}\begin{pmatrix}1 & 2 & \cdots & 2p-1 & 2p\\ j_1 & j_1' & \cdots & j_p & j_p'\end{pmatrix}=\begin{cases}1, & \text{if the permutation is even};\\ -1, & \text{if the permutation is odd.}\end{cases} \tag{5}$$

One can define the Pfaffian of the matrix A as

$$\operatorname{Pf} A=\sum_{\mathrm{M}}\omega_{\mathrm{M}}, \tag{6}$$

where $\mathbf{M}$ is over all perfect matchings of G.



(a)

(b)

Figure 3: $H_{2m,n,r}$ embedded on a torus and a corresponding plane model, where n is odd.

Theorem 1 ([21]). *If* $A = (a_{ij})_{2p\times 2p}$ *is a skew symmetric matrix, then*

$$\det(A) = (Pf\,A)^2. \tag{7}$$

For the perfect matching **M**, denote its signed weight by ω_M. One can think the sign of ω_M is just *the sign of the perfect matching***M.** For an orientation $\vec{G}$ of a graph, if the signs of its all perfect matchings are the same, then the orientation is *a Pfaffian orientation* of G. If a graph has a Pfaffian orientation, then the graph is named to be *Pfaffian*.

Theorem 2 ([6]). *Let a graph G be Pfaffian. If* $\vec{G}$ *is a Pfaffian orientation of G, then*

$$\Phi(G) = \left|Pf\,A\left(\vec{G}\right)\right| = \sqrt{\det\left(A\left(\vec{G}\right)\right)}. \tag{8}$$

Kasteleyn [22] presented the ***Pfaffian orientations for planar graphs*** *and also interpreted that the perfect matchings of a graph which is embedded on a surface with genus g can be computed as a linear combination of* 4^g *Pfaffians of the graph. Gallucio and Loebl [23], and Tesler [24] proved Kasteleyn's conclusion, independently. There are*

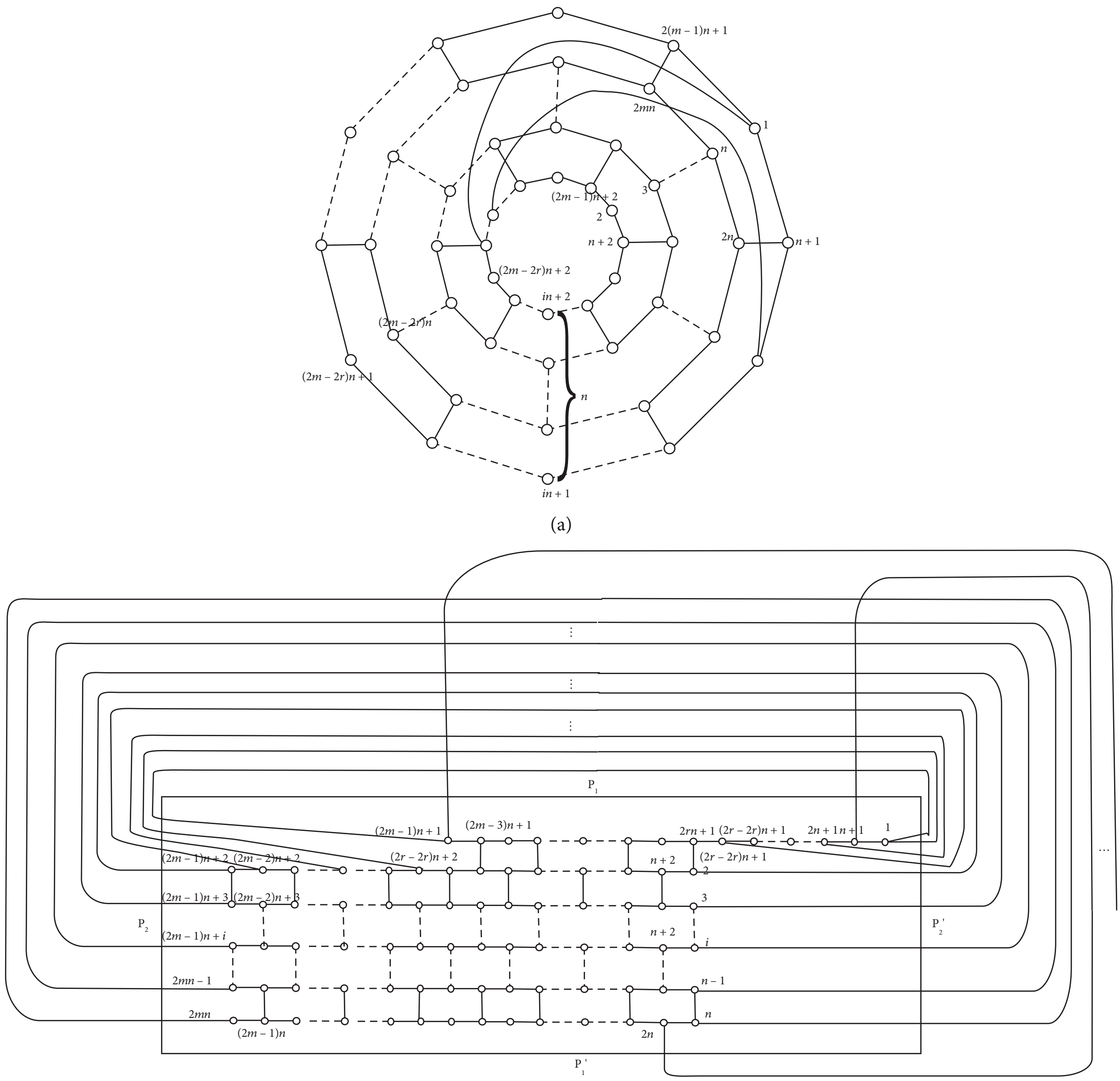


Figure 4: $H_{2m,n,r}$ embedded on a torus and a corresponding plane model, where n is even.

some related results about using Pfaffian to enumerate perfect matchings in the references [17, 25–31].

3. Plane Model on $H_{2m,n,r}$

Let P denote a 4-polygon with four sides p_1, p_2, p_1', p_2'. Suppose that G is a graph embedded on a torus. A drawing of a graph G is defined as a *plane model* of G, in which if all the edges of the graph G could be divided into E_0, E_1 and E_2, and not only the subgraph induced by E_0 is a spanning plane graph, which wholly contained inside P, but also the edges in E_1 and E_2 from the sides p_j to p_j' of P do not cross.

Noting that $H_{2m,n,r}$ is a graph which may be embedded on a torus, now we can give a plane model of $H_{2m,n,r}$ by the definition above-mentioned such that the edges of $H_{2m,n,r}$ are separated into E_0, E_1 and E_2. Moreover, the subgraph induced by E_0 is a spanning plane graph. The spanning plane graph is wholly contained inside P, and its edges in E_1 and E_2 from the sides p_j and p_j' of P do not cross (cf. Figure 2). If n is odd, a plane model of $H_{2m,n,r}$ is shown as in Figure 3(b). If n is even, a plane model of $H_{2m,n,r}$ is shown as in Figure 4(b). It is obvious that every edge in E_1 crosses each every in E_2 exactly once. For simplicity, suppose that $1, 2, \cdots, 2mn$ are $2mn$ vertices of $H_{2m,n,r}$ and $E(H_{2m,n,r})$ denote the edge set. Then E_0, E_1 and E_2 are represented as the followings:

(1) If n is even, then we have

$$
\begin{gathered}
E_1 = \{\{(2t-1)n+1, 2tn\} | t = 1, 2, \cdots, m\};\\
E_2 = \{\{(2t-2)n+1, (2m-2r+2t-2)n+2\} | t = 1, 2, \cdots, r\}\\
\cup\{\{h, (2m-1)n+h\} | h = 1, 2, \cdots, n\};\\
E_0 = E(H_{2m,n,r})/(E_1 \cup E_2).
\end{gathered} \tag{9}
$$

(2) If n is even, then we have

$$
\begin{gathered}
E_1 = \{\{(2t-1)n+1, 2tn\} | t = 1, 2, \cdots, m\};\\
E_2 = \{\{(2t-2)n+1, (2m-2r+2t-1)n+2\} | t = 1, 2, \cdots, r\}\\
\cup\{\{h, (2m-1)n+h\} | h = 1, 2, \cdots, n\};\\
E_0 = E(H_{2m,n,r})/(E_1 \cup E_2)
\end{gathered} \tag{10}
$$

An orientation of a graph embedded on a torus in a plane model is **the crossing orientation** if it conforms to the rule of cross orientation in reference [24]. For the graph G which is embedded on torus, let $X(x_1, x_2)$ be its skew adjacent matrix, with the edges in E_0 having weight 1 and the edges in E_j having weight x_j $(j = 1, 2)$. A formula computing the number for perfect matchings in G was presented by Tesler [24]:

$$
\Phi(G) = \left|\frac{1}{2}[\mathrm{Pf}\, X(1,1) + \mathrm{Pf}\, X(-1,1) + \mathrm{Pf}\, X(1,-1) - \mathrm{Pf}\, X(-1,-1)]\right|. \tag{11}
$$

Theorem 3 ([24]).

(a) *A graph may be oriented such that every perfect matching* $\mathbf{M}$ *has sign*

$$
\omega_M = \omega_0(-1)^{\kappa(M)}, \tag{12}
$$

where the constant $\omega_0 = \pm 1$ could be explained as the sign of a perfect matching without crossing edges if the edges exist, and $\kappa(M)$ denotes the number of times edges in $\mathbf{M}$ cross.

(b) *An orientation of a graph satisfies (a) if and only if the orientation is a crossing orientation*

According to the crossing orientation by Tesler, one crossing orientation of $H_{2m,n,r}$ where n is even or n is odd is indicated in Figures 5 and 6. Figure 5(a) and Figure 5(b) give an orientations of all the edges $E_0 \cup E_1$ and $E_0 \cup E_2$ when n is even, respectively. Figures 6(a) and 6(b) give the orientations of the edges $E_0 \cup E_1$ and $E_0 \cup E_2$ when n is odd, respectively.

4. The Sign Weights of Perfect Matchings of $H_{2m,n,r}$

Let the weight of edge of E_0 in $H_{2m,n,r}$ be 1. Suppose that x and y are the weight of edge of E_1 (Referring to Figures 5(a) and 6(a)) and E_2 (Referring to Figures 5(b) and 6(b)) in $H_{2m,n,r}$, respectively. Let $X(x, y)$ $(x, y = \pm 1)$ be the skew adjacency matrix of $H_{2m,n,r}$ if n is even and $Y(x, y)$ $(x, y = \pm 1)$ the skew adjacency matrix of $H_{2m,n,r}$ if n is odd. To determine the sign of Pfaffians Pf $X(x, y)$ and Pf $Y(x, y)$, the perfect matchings in $H_{2m,n,r}$ are distinguished into four classes. The perfect matchings belonging to class 1 are those that have odd number of edges both in E_1 and in E_2. The perfect matchings belonging to class 2.

have odd number of edges in E_1 and even number of edges in E_2. The perfect matchings belonging to class 3 have even number of edges in E_1 and odd number of edges in E_2. The perfect matchings belonging to class 4 have even number of edges both in E_1 and in E_2. Clearly, κ(M) is always even, where the perfect matching $\mathbf{M}$ lies in three classes other than class 1.

It is convenient to consider the case when $y = \pm 1$. For simplicity, let x and x^2 denote the odd power and even power, respectively. In light of the method deciding the sign of perfect matchings by Lu, Zhang and Lin [29], we also can obtain the sings of all the perfect matchings in $H_{2m,n,r}$. If $y = 1$, the signs of perfect matchings in three classes other than class 1 are always positive by Theorem 3. If $y = -1$, the signs of perfect matchings in three classes other than class 3 are always positive by Theorem 3 because that the perfect matching belonging to class 1 contains an odd of number edges in E_2 and κ(M) is always even. Therefore, we can decide the sign weights of all the perfect matchings of $H_{2m,n,r}$ shown as in Table 1.

From Table 1, we can observe that the signs of perfect matchings belonging to classes 3 and 4 when $y = 1$ are always positive and the signs of perfect matchings in classes 1 and 2 when $y = 1$ are always positive. Then by formula (11) we have the following result.

Lemma 4. *The number for perfect matchings in $H_{2m,n,r}$ is equal to the sum of the number for perfect matchings belonging to classes 3 and 4 when $y = 1$ and that belonging to classes 1 and 2 when $y = -1$.*

5. Perfect Matchings of $H_{2m,n,r}$

Suppose that V is a skew block circulant matrix or block circulant matrix as follows:

$$
V = \begin{pmatrix}
V_0 & V_1 & \cdots & V_{m-1}\\
-V_{m-1} & V_0 & \cdots & V_{m-2}\\
\vdots & \vdots & \ddots & \vdots\\
-V_1 & -V_2 & \cdots & V_0
\end{pmatrix} \tag{13}
$$

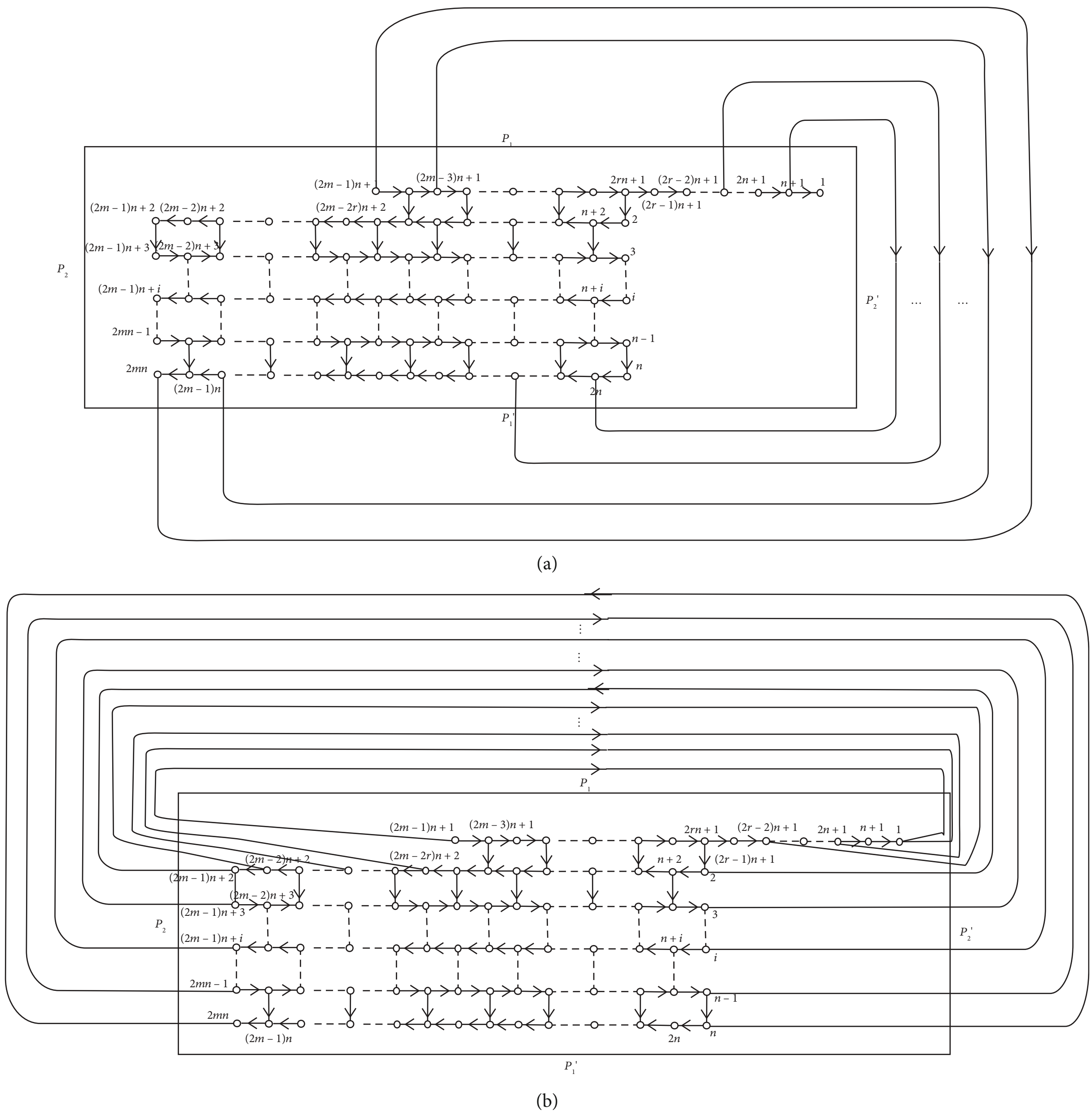


Figure 5: A crossing orientation of $H_{2m,n,r}$, where n is even.

or

$$V=\begin{pmatrix} V_0 & V_1 & \cdots & V_{m-1} \\ V_{m-1} & V_0 & \cdots & V_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ V_1 & V_2 & \cdots & V_0 \end{pmatrix}. \tag{14}$$

Then its determinant

$$\det(V)=\prod_{t=0}^{m-1}\det(J_t), \tag{15}$$

where

$$J_t = V_0+\omega_t V_1+\omega_t^{\,2} V_2+\cdots+\omega_t^{\,m-1}V_{m-1}$$
$$\omega_t=\begin{cases}\cos\dfrac{2t\pi}{m}+i\sin\dfrac{2t\pi}{m}, & \text{if } V \text{ is a block circulant matrix;}\\ \cos\dfrac{(2t+1)\pi}{m}+i\sin\dfrac{(2t+1)\pi}{m}, & \text{if } V \text{ is a skew block circulant matrix.}\end{cases} \tag{16}$$

Let $\vec{H}_{m,2n,r}$ be a crossing orientation of the plane model of $H_{2m,n,r}$ as shown in Figures 5 and 6. We can obtain the Pfafian of the matrix corresponding $\vec{H}_{m,2n,r}$ by formula (11) and Theorem 1. Consequently, an expression

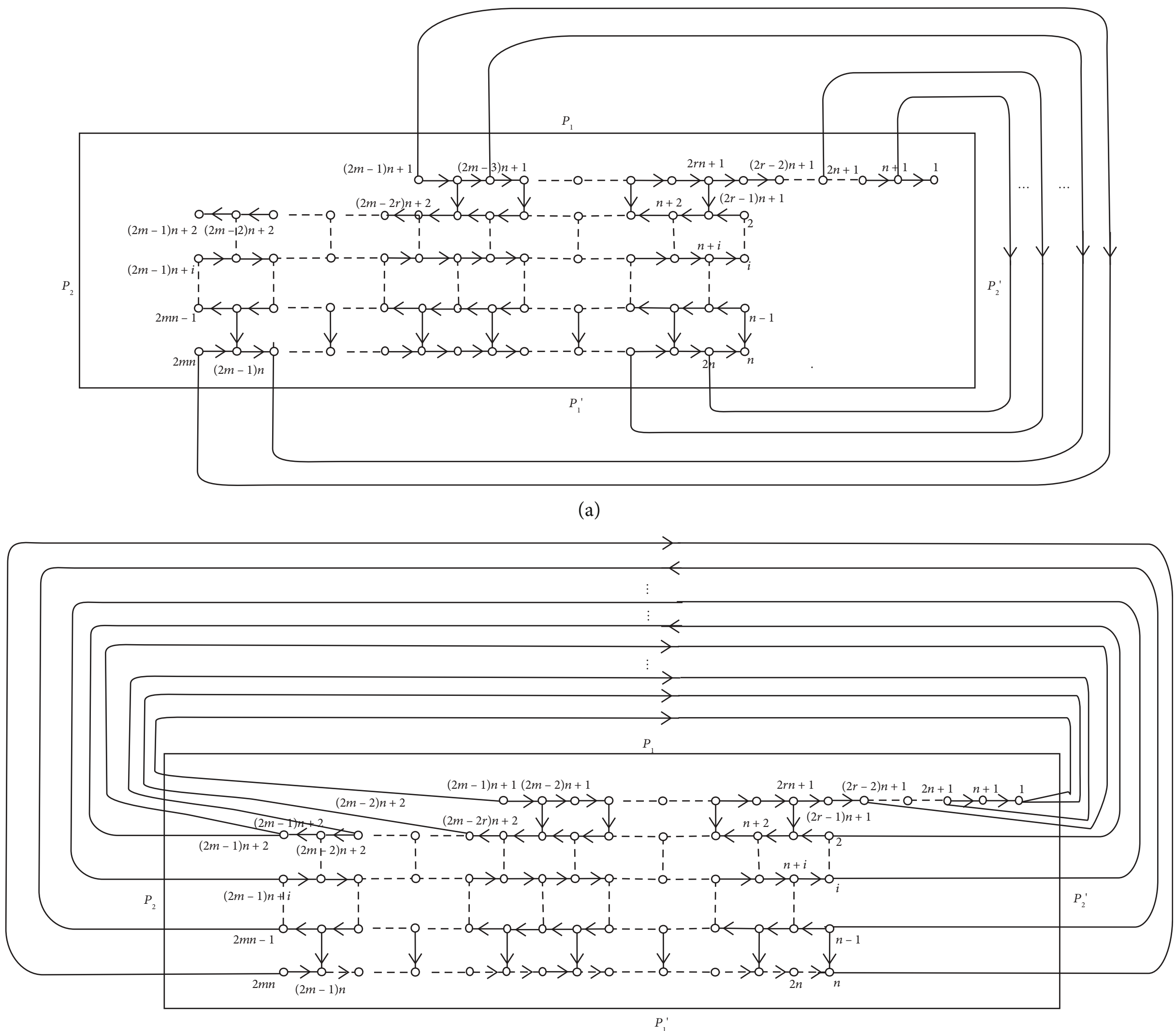


Figure 6: A crossing orientation of $H_{2m,n,r}$, where n is odd.

Table 1: The sign weights of perfect matchings in $H_{2m,n,r}$.

Classs	The signs of the perfect matchings	
	$x, y = 1$	$x, y = -1$
1	$-x$	x
2	x	x
3	x^2	$-x^2$
4	x^2	x^2

computing the number for perfect matchings in $\overrightarrow{H}_{m,2n,r}$ can be obtained by formula (11).

Theorem 5. *If n is even, then*

$$\Phi(H_{2m,n,r}) = f(x,1) + f(x,-1), \tag{17}$$

where $f(x,1)$ and $f(x,-1)$ denote the sum of coefficients of even terms in $Pf\, X(x,1) = \sqrt{\det(X(x,1))}$ and the sum of coefficients of odd terms in $Pf\, X(x,-1) = \sqrt{\det(X(x,-1))}$, respectively; $\varphi_t = (2t+1)\pi/m$ and $\theta_t = 2t\pi/m$;

$$\det(X(x,1)) = \prod_{t=0}^{m-1} \left\{(-1)^{n/2} 2^n (\cos\varphi_t - 1)^n - 2^{n/2}\left[(-1)^{n/2} + 1\right](\cos\varphi_t - 1)^{n/2} \left[(-1)^{n/2}(\cos r\varphi_t + i\sin r\varphi_t) + (\cos r\varphi_t - i\sin r\varphi_t)\right]x + x^2\right\} \tag{18}$$

and

$$\det(X(x,-1)) = \prod_{t=0}^{m-1} \left\{(-1)^{n/2} 2^n (\cos\theta_t - 1)^n - 2^{n/2}\left[(-1)^{n/2} + 1\right](\cos\theta_t - 1)^{n/2} \left[(-1)^{n/2}(\cos r\theta_t + i\sin r\theta_t) + (\cos r\theta_t - i\sin r\theta_t)\right]x + x^2\right\}. \tag{19}$$

Proof. Recall that $\overrightarrow{H}_{m,2n,r}$ is a crossing orientation in the plane model of $H_{2m,n,r}$. Then the elements of $X(\overrightarrow{H}_{m,2n,r})$ which can be read off from Figure 5 have the following form when n is even:

$$X\left(\overrightarrow{H}_{m,2n,r}\right) = X(x,y) = \left(X_{ij}(x,y)\right), \tag{20}$$

where $X_{ij}(x,y)$ is the $n\times n$ matrix. If $i\le j$,

$$X_{ij}(x,y) = \begin{cases} A(x), & \text{if } j=i, i=1,2,\cdots,m-1,m\,;\\ B, & \text{if } j=i+1, i=1,2,\cdots,m-2,m-1\,;\\ C, & \text{if } j=i+r, i=1,2,\cdots,m-r-1,m-r\,;\\ C(\mathrm{y}), & \text{if } j=i+m-r, i=1,2,\cdots,r-1,r\,;\\ B(\mathrm{y}), & \text{if } i=1, j=m\,;\\ 0_{2n}, & \text{others.}\end{cases} \tag{21}$$

If $i>j$, then $X_{ij}=-X_{ji}^T$ (X_{ji}^T is the transpose of X_{ij}). Let A_i $(i=1,2,3,4)$ be an $n\times n$ matrix as follows:

$$A_1 = \begin{pmatrix} 0&0&0&0&0&0&\cdots&0&0\\ 0&0&0&0&0&0&\cdots&0&0\\ 0&0&0&1&0&0&\cdots&0&0\\ 0&0&-1&0&0&0&\cdots&0&0\\ 0&0&0&0&0&1&\cdots&0&0\\ 0&0&0&0&-1&0&\cdots&0&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ 0&0&0&0&0&0&\cdots&0&1\\ 0&0&0&0&0&0&\cdots&-1&0 \end{pmatrix}, A_4 = \begin{pmatrix} 0&0&0&\cdots&0&0&x\\ 0&0&1&\cdots&0&0&0\\ 0&-1&0&\cdots&0&0&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\ 0&0&0&\cdots&0&1&0\\ 0&0&0&\cdots&-1&0&0\\ -x&0&0&\cdots&0&0&0 \end{pmatrix}, A_2 = \begin{pmatrix} -1&0&\cdots&0&0\\ 0&1&\cdots&0&0\\ \vdots&\vdots&\ddots&\vdots&\vdots\\ 0&0&\cdots&-1&0\\ 0&0&\cdots&0&1 \end{pmatrix}, A_3=-A_2. \tag{22}$$

Then the matrices $A(x), B, C, B(y)$ and $C(y)$ can be expressed by the following forms:

$$A(x)=\begin{pmatrix} A_1 & A_2\\ A_3 & A_4\end{pmatrix}, B=\left(\begin{array}{ccccc|ccccc} 0&0&L&0&0&0&0&L&0&0\\ M&M&O&M&M&M&M&O&0&0\\ 0&0&L&0&0&0&0&L&0&0\\ \hline -1&0&L&0&0&0&0&L&0&0\\ 0&1&L&0&0&0&0&L&0&0\\ M&M&O&M&M&M&M&O&M&M\\ 0&0&L&-1&0&0&0&M&0&0\\ 0&0&L&0&1&0&0&L&0&0 \end{array}\right),$$

$$C=\left(\begin{array}{cccc|cccc} 0&0&L&0&0&0&L&0\\ -1&0&L&0&0&0&L&0\\ M&M&O&M&M&M&O&M\\ 0&0&L&0&0&0&L&0\\ \hline 0&0&L&0&0&0&L&0\\ M&M&O&M&M&M&O&M\\ 0&0&L&0&0&0&L&0 \end{array}\right), C(y)=\left(\begin{array}{cccc|cccc} 0&-y&L&0&0&0&L&0\\ 0&0&L&0&0&0&L&0\\ M&M&O&M&M&M&O&M\\ 0&0&L&0&0&0&L&0\\ \hline 0&0&L&0&0&0&L&0\\ M&M&O&M&M&M&O&M\\ 0&0&L&0&0&0&L&0 \end{array}\right) \tag{23}$$

and

$$B(y)=\left(\begin{array}{ccccc|ccccc} 0&0&L&0&0&-y&0&L&0&0\\ 0&0&L&0&0&0&y&L&0&0\\ M&M&O&M&M&M&M&O&M&M\\ 0&0&L&0&0&0&0&L&-y&0\\ 0&0&L&0&0&0&0&L&0&y\\ \hline 0&0&L&0&0&0&0&L&0&0\\ M&M&O&M&M&M&M&O&M&M\\ 0&0&L&0&0&0&0&L&0&0 \end{array}\right) \tag{24}$$

Let 0_{2n} be a $2n\times 2n$ matrix. If $y=1$, then the matrix of $\overrightarrow{H}_{m,2n,r}=X(x,y)$ is a skew block circulant matrix, i.e.,

$$X(x,1)=\operatorname{scirc}\left(A(x), B, 0_{2n}, \overset{r-2}{\cdots}, 0_{2n}, C, 0_{2n},\cdots,0_{2n}, C^T, 0_{2n}, \overset{r-2}{\cdots}, 0_{2n}, B^T\right). \tag{25}$$

If $y=-1$, then the skew adjacency matrix of $\overrightarrow{H}_{m,2n,r}=X(x,y)$ is a block circulant matrix, i.e.,

$$X(x,-1)=\operatorname{scirc}\left(A(x), B, 0_{2n}, \overset{r-2}{\cdots}, 0_{2n}, C, 0_{2n},\cdots,0_{2n}, -C^T, 0_{2n}, \overset{r-2}{\cdots}, 0_{2n}, -B^T\right). \tag{26}$$

Therefore by formula (15) we obtain that

$$\det\left(B(x,y)\right) = \prod_{t=0}^{m-1}\det(J_t), \tag{27}$$

where

$$J_t = A(x) + \omega_t B - \omega_t^{-1} B^T + \omega_t^{\,r} C - \omega_t^{-r} C^T \tag{28}$$

$$\omega_t = \begin{cases} \cos\dfrac{2t\pi}{m} + i\sin\dfrac{2t\pi}{m}, & \text{if } y = -1\,; \\ \cos\dfrac{(2t+1)\pi}{m} + i\sin\dfrac{(2t+1)\pi}{m}, & \text{if } y = 1. \end{cases} \tag{29}$$

Now we turn to calculate the determinant det (J_t) of J_t. By formula (28), we have

$$\det(J_t) = \begin{vmatrix} J_1 & J_2 \\ J_3 & J_4 \end{vmatrix}, \tag{30}$$

where

$$\begin{aligned} J_1 &= \begin{pmatrix} 0 & \omega_t^{-r} & 0 & 0 & \cdots & 0 & x \\ -\omega_t^{-r} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}, J_4 \\ &= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & x \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ -x & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, J_1 \\ &= \begin{pmatrix} -1+\omega_t^{-1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1-\omega_t^{-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1+\omega_t^{-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1-\omega_t^{-1} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1+\omega_t^{-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1-\omega_t^{-1} \end{pmatrix} \end{aligned} \tag{31}$$

and

$$J_3 = \begin{pmatrix} 1-\omega_t^{-1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1+\omega_t^{-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1-\omega_t^{-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1+\omega_t^{-1} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1-\omega_t^{-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1+\omega_t^{-1} \end{pmatrix}. \tag{32}$$

It is convenient to calculate that we adjust appropriately the order of labeling for the vertices of $H_{2m,n,r}$ such that the determinant det (J_t) turn out to be the following:

$$\begin{aligned} \det(J_t) &= \begin{vmatrix} 0 & \omega_t^{-r} & & & & & & -1+\omega_t^{-1} \\ -\omega_t^{-r} & 0 & 1-\omega_t^{-1} & & & & & \\ & -1+\omega_t & 0 & 1 & & & & \\ & & -1 & 0 & 1-\omega_t & & & \\ & & & -1+\omega_t^{-1} & 0 & 1 & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & & 0 & -x \\ 1-\omega_t & & & & & & x & 0 \end{vmatrix} \\ &= \left[-x\omega_t^{r} - \left(1-\omega_t^{-1}\right)^{n/2}(1-\omega_t)^{n/2}\right]\left[-x\omega_t^{-r} + \left(1-\omega_t^{-1}\right)^{n/2}(\omega_t-1)^{n/2}\right] \\ &= \left[(-1)^{n/2}\left(\omega_t+\omega_t^{-1}-2\right)^{n/2} - x\omega_t^{r}\right]\left[\left(\omega_t+\omega_t^{-1}-2\right)^{n/2} - x\omega_t^{-r}\right]. \end{aligned} \tag{33}$$

Let $\varphi_t = (2t+1)\pi/m$ and $\theta_t = 2t\pi/m$. Then it is not difficult to obtain the expressions of ω_t, ω_t^{-1} and $\omega_t + \omega_t^{-1}$ when $y = 1$ and $y = -1$ by formula (29). Thus we have the determinant det (J_t) if $y = 1$, i.e.,

$$\begin{aligned} \det(J_t) = {} & (-1)^{n/2} 2^n (\cos\varphi_t - 1)^n \\ & - 2^{n/2}\left[(-1)^{n/2} + 1\right](\cos\varphi_t - 1)^{n/2} \\ & \left[(-1)^{n/2}(\cos r\varphi_t + i\sin r\varphi_t) + (\cos r\varphi_t - i\sin r\varphi_t)\right]x + x^2. \end{aligned} \tag{34}$$

If $y = -1$, then we have

$$\begin{aligned} \det(J_t) = {} & (-1)^{n/2} 2^n (\cos\theta_t - 1)^n \\ & - 2^{n/2}\left[(-1)^{n/2} + 1\right](\cos\theta_t - 1)^{n/2} \\ & \left[(-1)^{n/2}(\cos r\theta_t + i\sin r\theta_t) + (\cos r\theta_t - i\sin r\theta_t)\right]x + x^2. \end{aligned} \tag{35}$$

Consequently, by formula (27) we have

$$\begin{aligned}\det(X(x,1)) &= \prod_{t=0}^{m-1}\det(J_t)\\ &= \prod_{t=0}^{m-1}\{(-1)^{n/2}2^n(\cos\varphi_t-1)^n\\ &\quad -2^{n/2}\left[(-1)^{n/2}+1\right](\cos\varphi_t-1)^{n/2}\left[(-1)^{n/2}(\cos r\varphi_t+i\sin r\varphi_t)\right.\\ &\quad \left.+(\cos r\varphi_t-i\sin r\varphi_t)\right]x+x^2\}\end{aligned} \quad (36)$$

and

$$\begin{aligned}\det(X(x,-1)) &= \prod_{t=0}^{m-1}\det(J_t)\\ &= \prod_{t=0}^{m-1}\{(-1)^{n/2}2^n(\cos\theta_t-1)^n\\ &\quad -2^{n/2}\left[(-1)^{n/2}+1\right](\cos\theta_t-1)^{n/2}\left[(-1)^{n/2}(\cos r\theta_t+i\sin r\theta_t)\right.\\ &\quad \left.+(\cos r\theta_t-i\sin r\theta_t)\right]x+x^2\}.\end{aligned} \quad (37)$$

Note that x and x^2 in Table 1 denote the odd power and even power of x, respectively. Thus the number for perfect matchings in $H_{2m,n,r}$ is equal to the sum of coefficients of even terms in $\mathrm{Pf}\,X(x,1)=\sqrt{\det(X(x,1))}$ and coefficients of odd terms in $\mathrm{Pf}\,X(x,-1)=\sqrt{\det(X(x,-1))}$ by Lemma 4 and Theorem 2. Consequently, we get a expression for the number for perfect matchings in $H_{2m,n,r}$ by $\det(X(x,1))$, $\det(X(x,-1))$ and Theorem 1 when n is even. □

Theorem 6. *If n is odd, then*

$$\Phi(H_{2m,n,r}) = g(x,1)+g(x,-1), \quad (38)$$

where $g(x,1)$ and $g(x,-1)$ denote the sum of coefficients of even terms in $\mathrm{Pf}\,Y(x,1)=\sqrt{\det(Y(x,1))}$ and the sum of coefficients of odd terms in $\mathrm{Pf}\,Y(x,-1)=\sqrt{\det(Y(x,-1))}$, respectively; $\varphi_t=(2t+1)\pi/m$ and $\theta_t=2t\pi/m$;

$$\begin{aligned}\det(Y(x,1)) &= \prod_{t=0}^{m-1}\left\{-2^{n+1/2}(\cos\varphi_t-1)^{n+1/2}+(-2)^{n-1/2}(\cos\varphi_t-1)^{n-1/2}[(\cos r\varphi_t-2\cos(r-1)\varphi_t]x+x^2\right\}\\ \det(Y(x,-1)) &= \prod_{t=0}^{m-1}\left\{-2^{n+1/2}(\cos\theta_t-1)^{n+1/2}+(-2)^{n-1/2}(\cos\theta_t-1)^{n-1/2}[\cos r\theta_t-2\cos(r-1)\theta_t]x+x^2\right\}.\end{aligned} \quad (39)$$

Proof. Let $Y(x,y)=Y(\vec{H}_{2m,n,r})$ be the skew adjacent matrix which elements can be read off from Figure 6. By the same calculation process in Theorem 1, we have the results as follows:

If $y=1$, then $\det(J_t)$ is expressed by

$$\begin{aligned}\det(J_t) &= -2^{n+1/2}(\cos\varphi_t-1)^{n+1/2}\\ &\quad -(-1)^{n+1/2}2^{n-1/2}(\cos\varphi_t-1)^{n-1/2}[\cos r\varphi_t-2\cos(r-1)\varphi_t]x+x^2.\end{aligned} \quad (40)$$

If $y=-1$, then $\det(J_t)$ is expressed by

$$\begin{aligned}\det(J_t) &= -2^{n+1/2}(\cos\theta_t-1)^{n+1/2}\\ &\quad -(-1)^{n+1/2}2^{n-1/2}(\cos\theta_t-1)^{n-1/2}[\cos r\theta_t-2\cos(r-1)\theta_t]x+x^2.\end{aligned} \quad (41)$$

Consequently, by formula (27) we have

$$\det(Y(x,1)) = \prod_{t=0}^{m-1}\det(J_t) = \prod_{t=0}^{m-1}\left\{-2^{n+1/2}(\cos\varphi_t-1)^{n+1/2} \quad -(-1)^{n+1/2}2^{n-1/2}(\cos\varphi_t-1)^{n-1/2}[\cos r\varphi_t-2\cos(r-1)\varphi_t]x+x^2\right\} \quad (42)$$

and

$$\begin{aligned}\det\left(Y(x,-1)\right) &= \prod_{t=0}^{m-1}\det(J_t) \\ &= \prod_{t=0}^{m-1} - 2^{n+1/2}(\cos\theta_t - 1)^{n+1/2} \\ &\quad - (-1)^{n+1/2}2^{n-1/2}(\cos\theta_t - 1)^{n-1/2}[\cos r\theta_t - 2\cos(r-1)\theta_t]x + x^2\}.\end{aligned} \tag{43}$$

Thus we get an expression of the number for perfect matchings in $H_{2m,n,r}$ by $\det\left(Y(x,1)\right)$, $\det\left(Y(x,-1)\right)$ and Theorem 1 when n is odd. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no potential conflicts of interest.

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