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Research Article

On the Elementary Symmetric Polynomials and the Zeros of Legendre Polynomials

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In this paper, we seek to present some new identities for the elementary symmetric polynomials and use these identities to construct new explicit formulas for the Legendre polynomials. First, we shed light on the variable nature of elementary symmetric polynomials in terms of repetition and additive inverse by listing the results related to these. Sequentially, we have proven an important formula for the Legendre polynomials, in which the exponent moves one walk instead of twice as known. The importance of this formula appears throughout presenting Vieta's formula for the Legendre polynomials in terms of their zeros and the results mentioned therein. We provide new identities for the elementary symmetric polynomials of the zeros of the Legendre polynomials. Finally, we propose the relationship between elementary symmetric polynomials for the zeros of $P_n(x)$ and the zeros of $P_{n-1}(x)$.

1. Introduction

In the last few decades, the importance of symmetric polynomials has emerged in many branches of pure and applied mathematics, such as representation theory [1], algebraic combinatorics [2], and numerical analysis [3–5]. The symmetric polynomials have several types, for instance, monomial, complete, and the elementary, see [6–9] and references therein. Throughout the current paper, we will focus on the elementary type. Let us first recall the fundamental concepts and well-known results that we will use subsequently, for further details see [6, 10] and references therein.

For positive integer n, the elementary symmetric polynomial (for short, ESP) of degree k denoted by $e_k^{(n)}(x_1, x_2, \ldots, x_n)$ is defined as the sum of all possible products of k variables of $x_1, x_2, \ldots, x_n$; that is,

$$e_k^{(n)}(x_1, x_2, \ldots, x_n) = \sum_{1 \le \ell_1 < \ell_2 < \cdots < \ell_k \le n} \prod_{i=1}^{k} x_{\ell_i}, \tag{1}$$

with $e_0^{(n)}(x_1, x_2, \ldots, x_n) = 1$ and $e_k^{(n)}(x_1, x_2, \ldots, x_n) = 0$ for $k > n$ or $k < 0$. It is easy to see that $e_k^{(n)}(x_1, x_2, \ldots, x_n)$ involves $\binom{n}{k}$ terms. For example, the ESPs for $n = 3$ are given by the following:

$$\begin{aligned}
e_0^{(3)}(x_1, x_2, x_3) &= 1, \\
e_1^{(3)}(x_1, x_2, x_3) &= x_1 + x_2 + x_3, \\
e_2^{(3)}(x_1, x_2, x_3) &= x_1x_2 + x_1x_3 + x_2x_3, \\
e_3^{(3)}(x_1, x_2, x_3) &= x_1x_2x_3.
\end{aligned} \tag{2}$$

The generating function of the ESP, $e_k^{(n)}(\mathbf{x})$, where $\mathbf{x}$ denotes the $n-$ tuple of the variables $(x_1, x_2, \ldots, x_n)$ is given by the following:

$$\prod_{i=1}^{n} (1 + x_i\lambda) = \sum_{k=0}^{n} e_k^{(n)}(\mathbf{x})\lambda^k. \tag{3}$$

Notice that, from the generation function (3), we have the following recurrence relation:

$$e_k^{(n)}(\mathbf{x}) = e_k^{(n-1)}(x_1, x_2, \ldots, x_{n-1}) + x_n e_{k-1}^{(n-1)}(x_1, x_2, \ldots, x_{n-1}), \quad (4)$$

for all $n \geq 1$ and $k = 1, 2, \ldots, n$ with $e_0^{(n-1)}(\cdot) = 1$, see [10]. Moreover, we can deduce the following:

$$e_k^{(n)}(-\mathbf{x}) = (-1)^k e_k^{(n)}(\mathbf{x}). \quad (5)$$

The orthogonal polynomials such as Hermite, Laguerre, and Jacobi have great importance in applied sciences, for instance, in quantum mechanics, engineering, and computational mathematics (e.g., [11–15] and references therein). The Legendre polynomials are a particular case of the Jacobi polynomials that can be applied to reveal Schoenberg's representation of positive definite functions on the unit sphere, see [11]. Now, we shall go through some known results of the Legendre polynomials that we ought to use later on, see [16–24] for further details. The Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$ can be defined as the coefficients of β^n in the following generating function, that is,

$$\frac{1}{\sqrt{1 - 2x\beta + \beta^2}} = \sum_{n=0}^{\infty} P_n(x)\beta^n, \quad (6)$$

where $\beta \in (-1, 1)$. Favard's theorem ([19], Theorem 4.5.1) showed that the Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$ satisfies the following three-term recurrence relation:

$$P_n(x) = xP_{n-1}(x) + \left(1 - \frac{1}{n}\right)\left[xP_{n-1}(x) - P_{n-2}(x)\right], \quad (7)$$

for all $n = 2, 3, 4, \ldots$, with $P_0(x) = 1$ and $P_1(x) = x$, see [25]. Also, the Legendre polynomials $P_n(x)$ satisfies the following:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[P_n(x)\right] = nP_{n-1}(x) + x\frac{\mathrm{d}}{\mathrm{d}x}\left[P_{n-1}(x)\right], \quad (8)$$

for all $n = 1, 2, \ldots$, see [22]. In addition to this, the Legendre polynomials have many explicit formulas, for instance the following representation:

$$P_n(x) = 2^{-n}\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k}\binom{2n-2k}{n} x^{n-2k}, \quad (9)$$

where $\lfloor \cdot \rfloor$ denotes the floor function, see [22]. The orthogonality property for a sequence of the Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$ on the closed interval $[-1, 1]$ is given by the following:

$$\int_{-1}^{1} P_n(x)P_m(x)\mathrm{d}x = \frac{2}{2n+1}\delta_{nm}, \quad (10)$$

where δ_{nm} denotes the Kronecker delta which is equal to 1 if $n = m$, and 0 otherwise, see [26].

The current paper is organized as follows: In Section 2, we introduce new identities for the elementary symmetric polynomials. Section 3 is mainly devoted to the present new formulas for the Legendre polynomials in terms of their zeros. In Section 4, novel identities for the elementary symmetric polynomials of the zeros of the Legendre polynomials are presented. In Section 5, some illustrative examples are given.

2. New Identities for Elementary Symmetric Polynomials

In this section, we are mainly concerned with introducing the main results related to the ESPs that we will use subsequently.

Lemma 1. *For $k \in \mathbb{Z}^+ \cup \{0\}$ and for any even positive integer n, we have the following:*

$$e_k^{(n)}(x_1, x_2, \ldots, x_{n/2}, -x_1, -x_2, \ldots, -x_{n/2}) = \begin{cases} 0, & \text{if } k \equiv 0\,(\mathrm{mod}2); \\ (-1)^{k/2} e_{k/2}^{(n/2)}\left(x_1^2, x_2^2, \ldots, x_{n/2}^2\right), & \text{if } k \equiv 0\,(\mathrm{mod}2). \end{cases} \quad (11)$$

Proof: Following the generating function (3), we have as follows:

$$\begin{aligned}\sum_{k=0}^{n} e_k^{(n)}(x_1, x_2, \ldots, x_{n/2}, -x_1, -x_2, \ldots, -x_{n/2})\lambda^k &= \prod_{i=1}^{n/2}(1 + x_i\lambda)\prod_{i=1}^{n/2}(1 - x_i\lambda) = \prod_{i=1}^{n/2}\left(1 - x_i^2\lambda^2\right) \\ &= \sum_{k=0}^{n/2}(-1)^k e_k^{(n/2)}\left(x_1^2, x_2^2, \ldots, x_{n/2}^2\right)\lambda^{2k}.\end{aligned} \quad (12)$$

By comparing the coefficients on both sides, we complete the proof.

Moreover, we present the following result that studies the ESPs for another special case of the variables $x_1, x_2, \ldots, x_n$. □

Lemma 2. *For $k \in \mathbb{Z}^+ \cup \{0\}$ and for any odd positive integer $n > 1$, we have*

$$e_k^{(n)}\left(x_1, x_2, \ldots, x_{(n-1)/2}, 0, -x_1, -x_2, \ldots, -x_{(n-1)/2}\right) = \begin{cases} 0, & \text{if } k \equiv 0\,(\mathrm{mod}2); \\ (-1)^{k/2} e_{k/2}^{((n+1)/2)}\left(0, x_1^2, x_2^2, \ldots, x_{(n-1)/2}^2\right), & \text{if } k \equiv 0\,(\mathrm{mod}2). \end{cases} \quad (13)$$

Proof: Following the generating function (3), we have as follows:

$$\begin{aligned}&\sum_{k=0}^{n} e_k^{(n)}\left(x_1, x_2, \ldots, x_{(n-1)/2}, 0, -x_1, -x_2, \ldots, -x_{(n-1)/2}\right)\lambda^k \\ &= \prod_{i=1}^{(n-1)/2}(1 + x_i\lambda)\left(1 + x_{(n+1)/2}\lambda\right)\Big|_{x_{(n+1)/2}=0}\prod_{i=1}^{(n-1)/2}(1 - x_i\lambda) \\ &= \prod_{i=1}^{(n+1)/2}\left(1 - x_i^2\lambda^2\right)\Big|_{x_{(n+1)/2}=0} \\ &= \sum_{k=0}^{(n+1)/2}(-1)^k e_k^{(n+1)/2}\left(0, x_1^2, x_2^2, \ldots, x_{(n-1)/2}^2\right)\lambda^{2k}.\end{aligned} \quad (14)$$

By comparing the coefficients on both sides, we obtain (13). This completes the proof.

The following results give a closed form of the ESPs for non-distinct variables. Notice that since $e_k^{(n)}(\cdot) = 0$ for $k > n$, then the generating function of the ESP (3) is equivalent to the following:

$$\prod_{i=1}^{n}(1 + x_i\lambda) = \sum_{k=0}^{\infty} e_k^{(n)}(\mathbf{x})\lambda^k. \tag{15}$$

□

Lemma 3. *For $k \in \mathbb{Z}^+ \cup \{0\}$ and n a positive integer, we have as follows:*

$$\begin{aligned} & e_k^{(2n)}(x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n) \\ & = \sum_{i=0}^{k} e_i^{(n)}(x_1, x_2, \ldots, x_n) e_{k-i}^{(n)}(x_1, x_2, \ldots, x_n). \end{aligned} \tag{16}$$

Proof: Following the generating function (3), we have as follows:

$$\begin{aligned} & \sum_{k=0}^{2n} e_k^{(2n)}(x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n)\lambda^k \\ & = \prod_{i=1}^{n}(1 + x_i\lambda) \prod_{i=1}^{n}(1 + x_i\lambda) \\ & = \left[\prod_{i=1}^{n}(1 + x_i\lambda)\right]^2 \\ & = \left[\sum_{k=0}^{n} e_k^{(n)}(x_1, x_2, \ldots, x_n)\lambda^k\right]^2. \end{aligned} \tag{17}$$

By considering (15) and applying the Cauchy product ([27], Definition 9.4.6.), we deduce the following:

$$\begin{aligned} & \sum_{k=0}^{2n} e_k^{(2n)}(x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n)\lambda^k \\ & = \sum_{k=0}^{2n}\left[\sum_{i=0}^{k} e_i^{(n)}(x_1, x_2, \ldots, x_n) e_{k-i}^{(n)}(x_1, x_2, \ldots, x_n)\right]\lambda^k. \end{aligned} \tag{18}$$

This completes the proof.

Now, we seek to generalize Lemma 3 in the following result. □

Theorem 1. *For $k \in \mathbb{Z}^+ \cup \{0\}$ and n, ℓ positive integers, we have as follows:*

$$\begin{aligned} & e_k^{(n+\ell)}(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_\ell) \\ & = \sum_{i=0}^{k} e_i^{(n)}(x_1, x_2, \ldots, x_n) e_{k-i}^{(\ell)}(y_1, y_2, \ldots, y_\ell). \end{aligned} \tag{19}$$

Proof: Following the generating function (3), we have as follows:

$$\begin{aligned} & \sum_{k=0}^{n+\ell} e_k^{(n+\ell)}(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_\ell)\lambda^k \\ & = \prod_{i=1}^{n}(1 + x_i\lambda) \prod_{i=1}^{\ell}(1 + y_i\lambda) \\ & = \left(\sum_{k=0}^{n} e_k^{(n)}(x_1, x_2, \ldots, x_n)\lambda^k\right)\left(\sum_{k=0}^{\ell} e_i^{(\ell)}(y_1, y_2, \ldots, y_\ell)\lambda^i\right). \end{aligned} \tag{20}$$

By considering (15) and applying the Cauchy product ([27], Definition 9.4.6.), we deduce the following:

$$\begin{aligned} & \sum_{k=0}^{n+\ell} e_k^{(n+\ell)}(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_\ell)\lambda^k \\ & = \sum_{k=0}^{n+\ell}\left[\sum_{i=0}^{k} e_i^{(n)}(x_1, x_2, \ldots, x_n) e_{k-i}^{(\ell)}(y_1, y_2, \ldots, y_\ell)\right]\lambda^k. \end{aligned} \tag{21}$$

This completes the proof. □

3. More Formulas for the Legendre Polynomials

Throughout the current section, we will introduce some results related to the Legendre polynomials. We start this section by giving a simple proof for an equivalent explicit formula for the Legendre polynomials.

Lemma 4. *Consider $\{P_n\}_{n=0}^{\infty}$ to be a sequence of the Legendre polynomials. Then, the explicit formula (9) coincides with the following formula:*

$$P_n(x) = 2^n \sum_{k=0}^{n} \binom{n}{k} \binom{\frac{n+k-1}{2}}{n} x^k. \tag{22}$$

Proof: Let $a_k^n = 2^{-n}\binom{n}{k}\binom{2n-2k}{n}$ and $b_m^n = 2^n\binom{n}{m}\binom{(n+m-1)/2}{n}$. Then, we have the following:

$$b_{n-2k+1}^n = 2^n \binom{n}{n-2k+1}\binom{n-k}{n}, \tag{23}$$

since $\binom{n-k}{n} = 0$ for all $k \geq 1$. Hence $b_{n-2k+1}^n = 0$ for $k = 1, 2, \ldots, \lfloor n/2 \rfloor - 1$. Now, it is sufficient to show that $b_{n-2k}^n = a_k^n$ for all $k = 0, 1, \ldots, \lfloor n/2 \rfloor$.

$$
\begin{aligned}
b_{n-2k}^{n} &= 2^{n}\binom{n}{n-2k}\binom{\frac{2n-2k-1}{2}}{n} \\
&= \frac{(2n-2k-1)(2n-2k-3)\cdots(-2k+1)}{(n-2k)!(2k)!} \\
&= \frac{(2n-2k-1)(2n-2k-3)\cdots 3\cdot 1}{(n-2k)!}\,\frac{(-1)(-3)\cdots(-2k-1)(-2k+1)}{(2k)!} \\
&= \frac{(2n-2k)!}{2^{n-k}(n-k)!}\,\frac{1}{(n-2k)!(2k)!}(-1)^{k}\frac{(2k)!}{2^{k}k!} \\
&= 2^{-n}(-1)^{k}\frac{n!}{(n-k)!k!}\,\frac{(2n-2k)!}{n!(n-2k)!} \\
&= 2^{-n}\binom{n}{k}\binom{2n-2k}{n} = a_{k}^{n}.
\end{aligned}
\tag{24}
$$

This fact completes the proof.

It is easy to see that, according to formula (22), the Legendre polynomials take the following expression:

$$P_{n}(x) = \alpha_{n,0} + \alpha_{n,1}x + \alpha_{n,2}x^{2} + \cdots + \alpha_{n,n}x^{n}, \tag{25}$$

where

$$\alpha_{n,k} = 2^{n}\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}, \tag{26}$$

for all $k = 0, 1, 2, \ldots, n$.

Now, it is the time to give the following result which reformulates the Legendre polynomial explicit formula (22). □

Lemma 5. *Consider $\{P_{n}\}_{n=0}^{\infty}$ to be a sequence of the Legendre polynomials. Then, the explicit formula (22) is equivalent to the following:*

$$P_{n}(x) = \sum_{\substack{k=0 \\ k\equiv 0(\mathrm{mod}2)}}^{n} (-1)^{k/2}\frac{(2n-k-1)!!(k-1)!!}{k!(n-k)!}x^{n-k}, \tag{27}$$

where $(\cdot)!!$ denotes the double factorial, with $(-1)!! = 1$ and $(0)!! = 0$.

Proof: The coefficient of x^{n-k} in equation (22) is given by the following:

$$
\begin{aligned}
\alpha_{n,n-k} &= 2^{n}\binom{n}{n-k}\binom{\frac{2n-k-1}{2}}{n} \\
&= \frac{(2n-k-1)(2n-k-3)\cdots(-k+1)}{(n-k)!k!}.
\end{aligned}
\tag{28}
$$

Assume that $\widehat{\alpha}_{n,k} = (2n-k-1)(2n-k-3)\cdots(-k+1)$, then we have two cases to consider based on k:

(i) If $k \equiv 0(\mathrm{mod}2)$, then $\widehat{\alpha}_{n,k}$ will vanish since it contains 0 as a factor; hence, so the coefficients in (27) equals to zero.

(ii) If $k \equiv 0(\mathrm{mod}2)$, then we have the following:

$$
\begin{aligned}
\widehat{\alpha}_{n,k} &= (2n-k-1)\cdots(3)\cdot(1)\cdot(-1)\cdot(-3)\cdots(-k+1) \\
&= (2n-k-1)!!(-1)^{k/2}(k-1)!!.
\end{aligned}
\tag{29}
$$

This fact completes the proof.

Notice that following Lemma 5, we conclude that $\lceil n/2 \rceil$ coefficients in (25) are equals to zero, where $\lceil \cdot \rceil$ denotes the ceiling function. Now, we state the following result, which constructs the Legendre polynomials in terms of their zeros. □

Theorem 2. *For any positive integer n, consider $x_1, x_2, \ldots, x_n$ to be the zeros of Legendre polynomials $P_n(x)$, then*

$$P_n(x) = 2^{-n+1}\binom{2n-1}{n}\sum_{k=0}^{n}(-1)^{n-k}e_{n-k}^{(n)}(x_1, x_2, \ldots, x_n)x^k. \tag{30}$$

Proof: . The leading coefficient $\alpha_{n,n}$ in formula (25) is given by the following:

$$\begin{aligned}\alpha_{n,n} &= 2^n\binom{\frac{2n-1}{2}}{n}\\ &= 2^n\frac{((2n-1)/2)((2n-2)/2)((2n-3)/2)((2n-4)/2)\cdots(2/2)(1/2)}{n!(n-1)!}\\ &= 2^{-n+1}\binom{2n-1}{n}.\end{aligned} \tag{31}$$

Now, we can rewrite formula (25) as follows:

$$P_n(x) = 2^{-n+1}\binom{2n-1}{n}q(x), \tag{32}$$

where $q(x)$ be a monic polynomial of degree n. Since $x_1, x_2, \ldots, x_n$ be the zeros of the Legendre polynomial of degree n, then the polynomial $q(x)$ has n distinct linear factors. Therefore, we can write the polynomial $q(x)$ as follows:

$$q(x) = \prod_{i=1}^{n}(x - x_i). \tag{33}$$

By applying Vieta's theorem [28], we deduce the following:

$$q(x) = \sum_{k=0}^{n}(-1)^{n-k}e_{n-k}^{(n)}(\mathbf{x})x^k, \tag{34}$$

which completes the proof.

On the other hand, for any Legendre polynomial $P_n(x)$, there is a negative zero with the same absolute value for every positive zero. So , it is clear that $e_1^{(n)}(x_1, x_2, \ldots, x_n) = 0$ for all $n \geq 2$. □

4. On the Zeros of Legendre Polynomials

In this section, we present the results related to the zeros of the Legendre polynomials. Based on Lemma 5 and Theorem 2, we state the following result that gives a closed-form for the ESPs of Legendre's zeros.

Corollary 1. *For any positive integers n and k. Consider $x_1, x_2, \ldots, x_n$ to be the zeros of the Legendre polynomial $P_n(x)$, then*

$$\begin{aligned}&e_k^{(n)}(x_1, x_2, \ldots, x_n)\\ &= (-1)^{\lceil 3k/2 \rceil}2^{n-1}\binom{n}{k}\frac{(2n-k-1)!!(k-1)!!(n-1)!}{(2n-1)!},\end{aligned} \tag{35}$$

with $e_0^{(n)}(x_1, x_2, \ldots, x_n) = 1$.

The Legendre polynomial $P_n(x)$ is an even function when n is even; moreover, it has n zeros, half of the zeros are negative and the other half is positive. On the contrary, when n is odd, we find that the polynomial is an odd function and has zero at $x = 0$ and half of the remaining zeros are negative and the other half positive. Based on the type of degree of the polynomial being even or odd and applying Lemmas 1 and 2, respectively, we conclude the following corollaries.

Corollary 2. *For any positive even integer n. The Legendre polynomial $P_n(x)$ is given by the following:*

$$P_n(x) = 2^{-n+1}\binom{2n-1}{n}\sum_{\substack{k=0\\ k\equiv 0(\mathrm{mod}2)}}^{n}(-1)^{3k/2}e_{k/2}^{(n/2)}\left(x_1^2, x_2^2, \ldots, x_{n/2}^2\right)x^{n-k}, \tag{36}$$

for all positive zeros $x_1, x_2, \ldots, x_{n/2}$.

Corollary 3. *For any positive odd integer n. The Legendre polynomial $P_n(x)$ is given by the following:*

$$P_n(x) = 2^{-n+1}\binom{2n-1}{n}\sum_{\substack{k=0\\ k\equiv 0(\mathrm{mod}2)}}^{n}(-1)^{3k/2}e_{k/2}^{((n+1)/2)}\left(0, x_1^2, x_2^2, \ldots, x_{(n-1)/2}^2\right)x^{n-k}, \tag{37}$$

for all positive zeros $x_1, x_2, \ldots, x_{(n-1)/2}$.

It is clear that Corollaries 2 and 3 reduce the computational cost, since in the case of odd k the coefficient becomes zero.

Based on the orthogonality of the Legendre polynomials, we provide the following result.

Lemma 6. *For any positive integer n, we have the following:*

$$\begin{aligned}&\int_{-1}^{1}[P_n(x)]^2\mathrm{d}x\\ &= \left[2^{-n+(3/2)}\binom{2n-1}{n}\right]^2\sum_{\substack{k=0\\ k\equiv 0(\mathrm{mod}2)}}^{2n}\frac{(-1)^{k/2}e_{k/2}^{(n)}\left(x_1^2, x_2^2, \ldots, x_n^2\right)}{2n-k+1},\end{aligned} \tag{38}$$

for all zeros $x_1, x_2, \ldots, x_n$ *of* $P_n(x)$.

Proof: Following Theorem 2, we deduce the following:

$$[P_n(x)]^2 = \Upsilon_n \left[\sum_{k=0}^{n} (-1)^k e_k^{(n)}(x_1, x_2, \ldots, x_n) x^{n-k}\right]^2, \quad (39)$$

where $\Upsilon_n = \left[2^{-n+1}\binom{2n-1}{n}\right]^2$.

By considering (15) and applying the Cauchy product, we deduce the following:

$$[P_n(x)]^2 = \Upsilon_n \sum_{k=0}^{2n} (-1)^k \left[\sum_{i=0}^{k} e_i^{(n)}(x_1, x_2, \ldots, x_n) \times e_{k-i}^{(n)}(x_1, x_2, \ldots, x_n)\right] x^{2n-k}. \quad (40)$$

By applying Lemma 3, we obtain the following:

$$[P_n(x)]^2 = \Upsilon_n \sum_{k=0}^{2n} (-1)^k e_k^{(2n)}(x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n) x^{2n-k}. \quad (41)$$

By integrating both sides on the interval $[-1, 1]$, we conclude as follows:

$$\begin{aligned}\int_{-1}^{1} [P_n(x)]^2 \mathrm{d}x = \Upsilon_n \sum_{k=0}^{2n} \bigg[&\frac{(-1)^k}{2n-k+1} e_k^{(2n)}(x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n) \\ &+ \frac{1}{2n-k+1} e_k^{(2n)}(x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n)\bigg]. \end{aligned} \quad (42)$$

Therefore, by considering the nature of Legendre's zeros, we obtain (38) by expanding the previous summation on the right-hand side and then applying Lemma 1. This completes the proof.

Furthermore, we can deduce the following result. □

Corollary 4. *For any positive integer n. Consider the Legendre polynomial* $P_n(x)$ *with zeros* $x_1, x_2, \ldots, x_n$, *then*

$$\sum_{\substack{k=0 \\ k \equiv 0 \,(\mathrm{mod}2)}}^{2n} \frac{(-1)^{k/2} e_{k/2}^{(n)}(x_1^2, x_2^2, \ldots, x_n^2)}{2n-k+1} = \frac{2}{2n+1}\left[2^{n-1}\frac{n!(n-1)!}{(2n-1)!}\right]^2. \quad (43)$$

Similarly, following the orthogonality of the Legendre polynomials with distinct degrees, we deduce the following result.

Corollary 5. *Let* n, m *be positive integers with* $n \neq m$. *Consider the Legendre polynomial* $P_n(x)$ *with zeros* $x_1, x_2, \ldots, x_n$ *arranged in ascending order, namely,* $x_1 < x_2 < \ldots < x_n$ *and similarly* $P_n(x)$ *with zeros* $y_1, y_2, \ldots, y_m$ *with* $y_1 < y_2 < \cdots < y_m$, *then*

$$\sum_{\substack{k=0 \\ k \equiv 0 \,(\mathrm{mod}2)}}^{n+m} \frac{(-1)^{k/2}}{n+m-k+1} e_{k/2}^{(\lceil (n+m)/2 \rceil)}\left(x_1^2, x_2^2, \ldots, x_{\lceil n/2 \rceil}^2, y_1^2, y_2^2, \ldots, y_{\lceil m/2 \rceil}^2\right) = 0. \quad (44)$$

The following result gives a relation between the ESPs for the zeros of Legendre polynomials $P_n(x)$ and $P_{n-1}(x)$.

Theorem 3. *Let n be a positive integer. Consider the Legendre polynomial* $P_n(x)$ *with zeros* $x_1, x_2, \ldots, x_n$ *and* $P_{n-1}(x)$ *with zeros* $y_1, y_2, \ldots, y_{n-1}$, *then*

$$e_k^{(n)}(x_1, x_2, \ldots, x_n) = \frac{n(2n-k-1)}{(2n-1)(n-k)} e_k^{(n-1)}(y_1, y_2, \ldots, y_{n-1}), \quad (45)$$

for all $k = 0, 1, 2, \ldots, n-1$.

Proof: Following the recurrence relation (8), we obtain the following:

$$\begin{aligned}&\frac{2n-1}{n}\sum_{k=0}^{n-1} (-1)^k (n-k) e_k^{(n)}(\mathbf{x}) x^{n-k-1} \\ &= n\sum_{k=0}^{n-1} (-1)^k e_k^{(n-1)}(\mathbf{y}) x^{n-k-1} + \sum_{k=0}^{n-2} (-1)^k (n-k-1) e_k^{(n-1)}(\mathbf{y}) x^{n-k-1},\end{aligned} \quad (46)$$

where $\mathbf{y}$ denotes the $(n-1)$– tuple $(y_1, y_2, \ldots, y_{n-1})$. Based on the definition of ESPs, we can rewrite the last term in (46) as follows:

$$\sum_{k=0}^{n-2} (-1)^k (n-k-1) e_k^{(n-1)} (\mathbf{y}) x^{n-k-1} = \sum_{k=0}^{n-1} (-1)^k (n-k-1) e_k^{(n-1)} (\mathbf{y}) x^{n-k-1}. \tag{47}$$

Therefore, by comparing the coefficients of x^{n-k-1}, we obtain the following:

$$\frac{(2n-1)(n-k)}{n} e_k^{(n)} (\mathbf{x}) = n e_k^{(n-1)} (\mathbf{y}) + (n-k-1) e_k^{(n-1)} (\mathbf{y}), \tag{48}$$

which completes the proof.

Following Theorem 3, it is easy to construct the Legendre polynomial $P_n(x)$ via the zeros of $P_{n-1}(x)$, looking forward the following corollary. □

Corollary 6. *Let n be a positive integer, with $n > 1$. Assume that the Legendre polynomial $P_{n-1}(x)$ has zeros $y_1, y_2, \ldots, y_{n-1}$, then*

$$P_n(x) = 2^{-n+1} \left(\frac{n}{2n-1}\right) \binom{2n-1}{n} \sum_{k=1}^{n} (-1)^{n-k} \left(\frac{n+k-1}{k}\right) e_{n-k}^{(n-1)} (y_1, y_2, \ldots, y_{n-1}) x^k + 2^n \binom{\frac{n-1}{2}}{n}. \tag{49}$$

It is easy to see that if n is odd, then the last term of (49) will vanish.

5. Numerical Examples

Throughout this section, we shall give some examples to illustrate the results are posted in Sections 3 and 4. Let us start with the following example that illustrates the idea of Theorem 2.

Example 1. The zeros of $P_3(x)$ are $x_1 = \sqrt{3/5}, x_2 = 0$ and $x_3 = -\sqrt{3/5}$, then following (2), we have the following:

$$\begin{aligned} e_0^{(3)}(x_1, x_2, x_3) &= 1, \\ e_1^{(3)}(x_1, x_2, x_3) &= 0, \\ e_2^{(3)}(x_1, x_2, x_3) &= -\frac{3}{5}, \\ e_3^{(3)}(x_1, x_2, x_3) &= 0. \end{aligned} \tag{50}$$

Therefore, by applying Theorem 2, then we obtain as follows:

$$\begin{aligned} P_3(x) &= 2^{-2} \binom{5}{3} \left[-e_3^{(3)}(x_1, x_2, x_3) + e_2^{(3)}(x_1, x_2, x_3) x - e_1^{(3)}(x_1, x_2, x_3) x^2 + e_0^{(3)}(x_1, x_2, x_3) x^3\right] \\ &= \left(\frac{1}{4}\right)(10)\left[-\frac{3}{5}x + x^3\right]. \end{aligned} \tag{51}$$

Moreover, we deduce that $P_3(x) = (1/2)(5x^3 - 3x)$.

The following example to illustrates the idea of Corollaries 2 and 3.

Example 2. The zeros of Legendre polynomial $P_4(x)$ are $x_1 = \sqrt{(15 + 2\sqrt{30})/35}, x_2 = \sqrt{(15 - 2\sqrt{30})/35}, x_3 = -\sqrt{(15 + 2\sqrt{30})/35}$ and $x_4 = -\sqrt{(15 - 2\sqrt{30})/35}$, then following Corollary 2, we have the following:

$$P_4(x) = \frac{35}{8} \left[\Lambda_{4,0} x^4 + \Lambda_{4,2} x^2 + \Lambda_{4,4}\right], \tag{52}$$

where $\Lambda_{4,1} = \Lambda_{4,3} = 0$. Moreover, we obtain as follows:

$$\begin{aligned} \Lambda_{4,0} &= e_0^{(2)}\left(x_1^2, x_2^2\right) = 1, \\ \Lambda_{4,2} &= e_1^{(2)}\left(x_1^2, x_2^2\right) = -\frac{30}{35}, \\ \Lambda_{4,4} &= e_2^{(2)}\left(x_1^2, x_2^2\right) = \frac{105}{(35)^2}. \end{aligned} \tag{53}$$

Thus, we conclude that $P_4(x) = (1/8)(35x^4 - 30x^2 + 3)$.

Following Theorem 3 and Corollary 6, we give the next example.

Example 3. The zeros of the Legendre polynomial $P_2(x)$ are $x_1 = 1/\sqrt{3}$ and $x_2 = -(1/\sqrt{3})$. Thus, following Corollary 6 we can construct the Legendre polynomial $P_3(x)$ as follows:

$$\begin{aligned} P_3(x) &= 2^{-2}\left(\frac{3}{5}\right)\binom{5}{3}\left[3e_2^{(2)}\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)x - 2e_1^{(2)}\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)x^2 + \frac{5}{3}e_0^{(2)}\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)x^3\right] \\ &= \frac{3}{2}\left(\frac{5}{3}x^3 - x\right), \end{aligned} \tag{54}$$

therefore, $P_3(x) = (1/2)(5x^3 - 3x)$.

6. Conclusion

Throughout this paper, we provided some results related to the elementary symmetric polynomials such as Lemmas 1–3. The importance of the results mentioned in Section 2 appears by providing some new results related to zeros of the Legendre polynomials that have been presented in Section 3. Also, we constructed the Legendre polynomials via their zeros, and we presented the closed form of Vieta's formula. By Theorem 3, we were able to construct $P_n(x)$ via the zeros of $P_{n-1}(x)$ for all $n \geq 2$, which reduce the computational costs for constructing the Legendre polynomials.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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