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Research Article

Giaccardi Inequality for s-Convex Functions in the Second Sense for Isotonic Linear Functionals and Associated Results

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In this paper, a well-known inequality called Giaccardi inequality is established for isotonic linear functionals by applying s-convexity in the second sense, which leads to notable Petrović inequality. As a special case, discrete and integral versions of Giaccardi inequality are derived along with the Petrović inequality as a particular case. In application point of view, newly established inequalities are derived for different time scales.

1. Introduction

Many important areas of mathematics have been developed just because of convex functions. Due to important properties and characterizations of convex functions supreme branches of pure and applied mathematics are studied extensively. A convex function in $\mathbb{R}$ is a function, whose epigraph is a convex set. A convex function is always continuous and has left and right derivatives in the interior of its domain. More generally, a function ψ is convex on an interval U if for any two points x and y of U and any λ, where $0 \leq \lambda \leq 1$, it satisfies the following inequality:

$$\psi(\lambda x + (1-\lambda)y) \leq \lambda\psi(x) + (1-\lambda)\psi(y). \tag{1}$$

Many researchers had generalized the notion of convex functions by inducting different parameters and functions provided the inequality (1) must retain preserved. For instance s-convex functions [1], m-convex functions [2], $(\alpha - m)$-convex functions [3] and h-convex functions [4, 5] can be seen in this regard. One of the important generalizations of convex functions is s-convex functions [6]. The s-convex function in the second sense was introduced by H. Hudzik and L. Maligranda in [1].

Definition 1. Let $s \in (0,1]$ and $U \subseteq \mathbb{R}$ be an interval. A function $\psi\colon U \longrightarrow \mathbb{R}$ is said to be s-convex in the second sense if

$$\psi(\lambda x + \mu y) \leq \lambda^s \psi(x) + \mu^s \psi(y), \tag{2}$$

for all $x, y \in U$ and $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$.

The class of s-convex functions in the second sense is usually denoted by K_s^2.

Example 1 (see [7]). *Let $s \in (0,1)$ and $a, b, c \in \mathbb{R}$. Define the function ψ: $[0,\infty) \longrightarrow \mathbb{R}$* as follows:

$$\psi(\zeta) = \begin{cases} a, & \text{if } \zeta = 0, \\ b\zeta^s + c, & \text{if } \zeta > 0. \end{cases} \tag{3}$$

It can be easily checked.

(i) if $b \geq 0$ and $0 \leq c \leq a$, then $\psi \in K_s^2$

(ii) if $b > 0$ and $c < 0$, then $\psi \notin K_s^2$.

There are many publications in the literature that explore this form of convexity. Theories and inequalities underlying s-convexity by analytical methods are currently being discussed. It can be easily seen that in case $s = 1$, s-convexity is reduced to the ordinary convexity of function ψ.

Let us recall the definition of isotonic linear functionals from [8].

Definition 2. LetEbe a nonempty set andLbe a class of real-valued functions defined onE having the properties:

L1: if $\psi, \phi \in L$, then $a\psi + b\phi \in L$ for all $a, b \in \mathbb{R}$

L2: $1 \in L$, that is, if $\psi(\zeta) = 1$ for $\zeta \in E$, then $\psi \in L$.

An isotonic linear functional is a functional $A\colon L \longrightarrow \mathbb{R}$ that satisfies the following axioms:

A1: $A(a\psi + b\phi) = aA(\psi) + bA(\phi)$ for $\psi, \phi \in L$, $a, b \in \mathbb{R}$

A2: $\psi \in L, \psi(\zeta) \geq 0$ for $\zeta \in E \Rightarrow A(\psi) \geq 0$.

Isotonic linear functionals contribute in a wide area of pure and applied mathematics and play a key role in resolving many problems and inequalities. One can note the common examples of such isotonic functionals A are given in [8] as follows:

$$
\begin{aligned}
&A(\phi) = \sum_{n\in E} p_n \phi(n), \quad \text{where } E \subseteq \{1, 2, 3, \ldots\}, p_n > 0,\\
&\text{or, } A(\phi) = \int_E \phi \mathrm{d}\mu,
\end{aligned}
\tag{4}
$$

where μ is a positive measure on some suitable set E.

Giaccardi inequality for isotonic linear functionals for convex functions is stated in the following theorem.

Theorem 1 (see [8]). *LetL, Asatisfy the conditions L1, L2, and A1, A2. Also, let $x_0 \in E$ and $\phi \in L$ such that $x_0 \neq A(\phi)$ and either $x_0 \leq \phi(\zeta) \leq A(\phi)$ for all $\zeta \in E$ or $A(\phi) \leq \phi(\zeta) \leq x_0$ for all $\zeta \in E$. If ψ is convex on $[x_0, A(\phi)]$ (or on $[A(\phi), x_0]$) and $\psi(\phi) \in L$, then the following is valid:*

$$
A(\psi(\phi)) \leq \frac{\{[A(\phi) - A(1)x_0]\psi(A(\phi)) + [A(1) - 1]A(\phi)\psi(x_0)\}}{[A(\phi) - x_0]}.
\tag{5}
$$

The aim of this paper is to prove the Giaccardi inequality for s-convex functions in the second sense for isotonic linear functionals. The well-known Petrović inequality is derived as a particular case of the newly established Giaccardi inequality. As a special case, discrete and integral versions of these inequalities are derived. As applications on the time scale, Giaccardi and Petrović inequalities are discussed for different time scales.

2. Main Results

In the following theorem, we prove Giaccardi inequality for s -convex functions in the second sense for isotonic linear functionals.

Theorem 2. *LetL ,Asatisfy the conditions L1, L2, and A1, A2, respectively. Also, let $\phi \in L$ such that $x_0 \neq A(\phi)$ and either*

$$
\begin{aligned}
&(x_0 - \phi(\zeta))(\phi(\zeta) - A(\phi)) \leq 0, \quad \text{for all } \zeta \in E,\\
&\text{or, } (x_0 - \phi(\zeta))(\phi(\zeta) - A(\phi)) \geq 0, \quad \text{for all } \zeta \in E.
\end{aligned}
\tag{6}
$$

If ψ is s-convex in second sense on $[x_0, A(\phi)]$ or on $[A(\phi), x_0]$, and $\psi(\phi) \in L$, then,

$$
A(\psi(\phi)) \leq \mathscr{A}\psi(A(\phi)) + \mathscr{B}\psi(x_0),
\tag{7}
$$

where

$$
\begin{aligned}
\mathscr{A} &= \frac{A((\phi(\zeta) - x_0)^s)}{(A(\phi) - x_0)^s},\\
\mathscr{B} &= \frac{A((A(\phi) - \phi(\zeta))^s)}{(A(\phi) - x_0)^s}.
\end{aligned}
\tag{8}
$$

Proof. Let u, v and w be points from the interval having end points x_0 and $A(\phi)$ such that $u < v < w$. It is easy to check that

$$
\begin{aligned}
\lambda &:= \frac{w - v}{w - u} \in (0, 1), \ \mu := \frac{v - u}{w - u} \in (0, 1),\\
\lambda + \mu &= \frac{w - v}{w - u} + \frac{v - u}{w - u} = 1.
\end{aligned}
\tag{9}
$$

If we take $x = u$ and $w = y$, then $\lambda x + \mu y = v$. As ψ is given to be s-convex in second sense so using the value of x and y along with the values of λ and μ defined above in inequality (2), one has

$$
\psi(v) \leq \left(\frac{w - v}{w - u}\right)^s \psi(u) + \left(\frac{v - u}{w - u}\right)^s \psi(w).
\tag{10}
$$

Substituting $u = m$, $v = \phi(\zeta)$ and $w = M$ in above inequality to get

$$
\psi(\phi) \leq \frac{(M - \phi(\zeta))^s}{(M - m)^s}\psi(m) + \frac{(\phi(\zeta) - m)^s}{(M - m)^s}\psi(M).
\tag{11}
$$

Now setting $m = x_0$ and $M = A(\phi)$ (or) $m = A(\phi)$ and $M = x_0$ to get

$$
\psi(\phi) \leq \frac{(\phi(\zeta) - x_0)^s}{(A(\phi) - x_0)^s}\psi(A(\phi)) + \frac{(A(\phi) - \phi(\zeta))^s}{(A(\phi) - x_0)^s}\psi(x_0).
\tag{12}
$$

That is,

$$
\frac{(\phi(\zeta) - x_0)^s}{(A(\phi) - x_0)^s}\psi(A(\phi)) + \frac{(A(\phi) - \phi(\zeta))^s}{(A(\phi) - x_0)^s}\psi(x_0) - \psi(\phi) \geq 0.
\tag{13}
$$

Applying the property A2 of the isotonic linear functional A, one has

$$
A\left(\frac{(\phi(\zeta) - x_0)^s}{(A(\phi) - x_0)^s}\psi(A(\phi)) + \frac{(A(\phi) - \phi(\zeta))^s}{(A(\phi) - x_0)^s}\psi(x_0) - \psi(\phi)\right) \geq 0.
\tag{14}
$$

Using the linearity of the functional A, one get

$$A\left(\frac{(\phi(\zeta)-x_0)^s}{(A(\phi)-x_0)^s}\psi(A(\phi))\right)+A\left(\frac{(A(\phi)-\phi(\zeta))^s}{(A(\phi)-x_0)^s}\psi(x_0)\right)\geq A(\psi(\phi)). \tag{15}$$

A simplification of the inequality (15) leads us to the required result. □

Remark 1. If we take $s=1$ in Theorem 2, then one gets Giaccardi's inequality for isotonic linear functional stated in [8] (*as given in Theorem*1).

In the following, a discrete version of the above theorem has been derived.

Theorem 3. *Let$x_1, x_2, \ldots, x_n \in I$, where I is an interval,p_1, $p_2, \ldots, p_n$ be positive numbers and $x_0, \tilde{x}_n{:}\ = \sum_{i=1}^n p_i x_i \in I$ such that*

$$(x_i-x_0)(\tilde{x}_n-x_i)\geq 0, \text{or}\,(x_i-x_0)(\tilde{x}_n-x_i)\leq 0, \tag{16}$$

where $\tilde{x}_n \neq x_0$. If ψ is s-convex in second sense on I, then

$$\sum_{i=1}^n p_i\psi(x_i)\leq \mathscr{A}'\psi(\tilde{x}_n)+\mathscr{B}'\psi(x_0), \tag{17}$$

where

$$\mathscr{A}'=\frac{\sum_{i=1}^n p_i(x_i-x_0)^s}{(\tilde{x}_n-x_0)^s}, \mathscr{B}'=\frac{\sum_{i=1}^n p_i(\tilde{x}_n-x_i)^s}{(\tilde{x}_n-x_0)^s}. \tag{18}$$

Proof. Let $E=\{1,2,\ldots,n\}$, $L=\{\phi\colon E\longrightarrow I|\phi(i)=x_i, i\in E\}$ and

$$A(\phi)=\sum_{i\in E}p_i\phi(i)=\sum_{i=1}^n p_i x_i. \tag{19}$$

Then, L satisfies conditions L1, L2 and A satisfies conditions A1, A2 of Definition 2. Substituting above values of A and ϕ in Theorem 2, we get

$$(\phi(\zeta)-x_0)(A(\phi)-\phi(\zeta))=(x_i-x_0)(\tilde{x}_n-x_i)\geq 0\text{or}\leq 0,$$
$$A(\phi)=\sum_{i=1}^n p_i x_i\neq x_0. \tag{20}$$

Finally, inequality (7) becomes our required result.

The following result provides us the integral version of Theorem 2. □

Theorem 4. *Let (Ω,Λ,ρ) be a measurable space, where $\rho(\Omega)$ is positive finite measure. Also let $\phi\colon \Omega\longrightarrow I$ be a measurable function and x_0, $\int_\Omega\phi(\zeta)d\rho\in I$ such that $\int_\Omega\phi(\zeta)d\rho\neq x_0$ and*

$$(\phi(\zeta)-x_0)\left(\int_\Omega\phi(\zeta)\mathrm{d}\rho-\phi(\zeta)\right)\geq 0 \text{ or } \leq 0. \tag{21}$$

If ψ is s-convex on second sense on I, then an inequality

$$\int_\Omega\psi(\phi)\mathrm{d}\rho\leq\mathscr{C}\psi\left(\int_\Omega\phi(\zeta)\mathrm{d}\rho\right)+\mathscr{D}\psi(x_0), \tag{22}$$

is valid, provided the integrals exist and

$$\mathscr{C}:=\frac{\int_\Omega(\phi(\zeta)-x_0)^s\mathrm{d}\rho}{\left(\int_\Omega\phi(\zeta)\mathrm{d}\rho-x_0\right)^s},$$
$$\mathscr{D}:=\frac{\int_\Omega\left(\int_\Omega\phi(\zeta)\mathrm{d}\rho-\phi(\zeta)\right)^s\mathrm{d}\rho}{\left(\int_\Omega\phi(\zeta)\mathrm{d}\rho-x_0\right)^s}. \tag{23}$$

Proof. Assume that

$$L=\left\{\phi\colon\ \Omega\longrightarrow I|\int_\Omega\phi(\zeta)\mathrm{d}\rho\text{ exists}\right\},$$
$$A(\phi)=\int_\Omega\phi(\zeta)\mathrm{d}\rho. \tag{24}$$

Then, L satisfies conditions L1, L2 and A satisfies conditions A1, A2 of Definition 1. Substituting above values of $A(\phi)$ in Theorem 2, we get

$$(\phi(\zeta)-x_0)(A(\phi)-\phi(\zeta))=(\phi(\zeta)-x_0)\left(\int_\Omega\phi(\zeta)\mathrm{d}\rho-\phi(\zeta)\right)$$
$$\geq 0 \text{ or } \leq 0,$$
$$A(\phi)=\int_\Omega\phi(\zeta)\mathrm{d}\rho\neq x_0. \tag{25}$$

Ultimately, inequality (7) becomes (22) as our required result.

In the following theorem, Petrović's inequality for s -convex functions in second sense for isotonic linear functionals is derived. □

Theorem 5. *Letψ be s -convex in second sense on $[0, A(\phi)]$ or on $[A(\phi), 0]$, then the inequality,*

$$A(\psi(\phi))\leq\frac{A(\phi^s)}{A^s(\phi)}\psi(A(\phi))+\frac{A((A(\phi)-\phi(\zeta))^s)}{A^s(\phi)}\psi(0), \tag{26}$$

holds for either $0\leq\phi(\zeta)\leq A(\phi)$ Thwor all $\zeta\in E$ or $A(\phi)\leq\phi(\zeta)\leq 0$ for all. $\zeta\in E$.

Proof. Put $x_0=0$ in (7) to get (26). □

Remark 2. If we put$s=1$in Theorem 5, one get Petrović' inequality for isotonic linear functionals stated in [8].

The famous Giaccardi's inequality given in [9] can be deduced directly from Theorem 2 as stated in the following corollary.

Corollary 1. *Let $x_1, x_2, \ldots, x_n \in I$, where I is an interval, p_1, $p_2, \ldots, p_n$ be positive numbers and $x_0, \tilde{x}_n{:}\ = \sum_{i=1}^n p_i$ $x_i \in I$ such that condition (16) is valid. If ψ is a convex on I*

$$\sum_{i=1}^{n} p_i\psi(x_i) \leq \sum_{i=1}^{n} p_i\left(\frac{x_i - x_0}{\tilde{x}_n - x_0}\right)\psi(\tilde{x}_n) + \left(\frac{\sum_{i=1}^{n} p_i - 1}{\tilde{x}_n - x_0}\right)\tilde{x}_n\psi(x_0). \tag{27}$$

Proof. Take $s = 1$ in (17) to get (27).

In the following corollary, Petrović's inequality for s–convex functions has been deduced. □

Corollary 2. *Let* $x_1, x_2, \ldots, x_n \in [0, a]$ *(or* $[a, 0]$*),* $p_1, p_2, \ldots, p_n$ *be positive numbers such that,*

$$\tilde{x}_n \leq x_i \text{ or } x_i \leq \tilde{x}_n, \quad \text{for } i = 1, 2, \ldots, n, \tag{28}$$

where $\tilde{x}_n := \sum_{i=1}^{n} p_i x_i \neq 0$. *If* ψ *is s-convex in second sense on* $[0, a]$ *(or)* $[a, 0]$*, then*

$$\sum_{i=1}^{n} p_i\psi(x_i) \leq \sum_{i=1}^{n} p_i\left(\frac{x_i}{\tilde{x}_n}\right)^s\psi(\tilde{x}_n) + \sum_{i=1}^{n} p_i\left(1 - \left(\frac{x_i}{\tilde{x}_n}\right)^s\right)\psi(0). \tag{29}$$

Proof. Put $x_0 = 0$ in (17) to get (29).

In the following corollary, Petrović's inequality [9] has been derived. □

Corollary 3. *Let* $x_1, x_2, \ldots, x_n \in [0, a]$ *(or* $[a, 0]$*)* $, p_1, p_2, \ldots, p_n$ *be positive numbers such that condition (28) is valid. If* ψ *is convex on* $[0, a]$ *(or* $[a, 0]$*), then*

$$\sum_{i=1}^{n} p_i\psi(x_i) \leq \psi(\tilde{x}_n) + \left(\sum_{i=1}^{n} p_i - 1\right)\psi(0). \tag{30}$$

Proof. Put $x_0 = 0$ and $s = 1$ in (17) to get (30). □

Corollary 4. *Let* (Ω, Λ, ρ) *be a measurable space, where* $\rho(\Omega)$ *is a positive finite measure. Also let* $\phi\colon \Omega \longrightarrow I$ *be a measurable function and* $x_0, \int_\Omega \phi(\zeta)d\rho \in I$ *such that* $\int_\Omega \phi(\zeta)d\rho \neq x_0$ *and condition (21) is valid. If* ψ *is convex on I, then the following is valid:*

$$\int_\Omega \psi(\phi)\mathrm{d}\rho \leq \frac{\int_\Omega (\phi(\zeta) - x_0)\mathrm{d}\rho}{\left(\int_\Omega \phi(\zeta)\mathrm{d}\rho - x_0\right)}\psi\left(\int_\Omega \phi(\zeta)\mathrm{d}\rho\right) + \frac{\int_\Omega\left(\int_\Omega \phi(\zeta)\mathrm{d}\rho - \phi(\zeta)\right)\mathrm{d}\rho}{\left(\int_\Omega \phi(\zeta)\mathrm{d}\rho - x_0\right)}\psi(x_0), \tag{31}$$

Provided the integrals exist.

Proof. It is a simple consequence of Theorem 4 for $s = 1$. □

Corollary 5. *Let* (Ω, Λ, ρ) *be a measurable space, where* $\rho(\Omega)$ *is positive finite measure. Also let* $\phi\colon \Omega \longrightarrow I$ *be a measurable function and* $x_0, \int_\Omega \phi(\zeta)d\rho \in I$ *such that* $\int_\Omega \phi(\zeta) d\rho \neq 0$ *and*

$$\int_\Omega \phi(\zeta)\mathrm{d}\rho \leq \phi(\zeta) \text{or} \int_\Omega \phi(\zeta)\mathrm{d}\rho \geq \phi(\zeta). \tag{32}$$

If ψ *is s-convex in second sense on I, then the following is valid:*

$$\int_\Omega \psi(\phi)\mathrm{d}\rho \leq \frac{\int_\Omega (\phi(\zeta))^s\mathrm{d}\rho}{\left(\int_\Omega \phi(\zeta)\mathrm{d}\rho\right)^s}\psi\left(\int_\Omega \phi(\zeta)\mathrm{d}\rho\right) + \frac{\int_\Omega\left(\int_\Omega \phi(\zeta)\mathrm{d}\rho - \phi(\zeta)\right)^s\mathrm{d}\rho}{\left(\int_\Omega \phi(\zeta)\mathrm{d}\rho\right)^s}\psi(0), \tag{33}$$

Provided the integrals exist.

Proof. Put $x_0 = 0$ in (22) to get (33).

The following result holds as the integral version of the famous Petrović's inequality. □

Corollary 6. *Let* (Ω, Λ, ρ) *be a measurable space, where* $\rho(\Omega)$ *is a positive finite measure. Also let* $\phi\colon \Omega \longrightarrow I$ *be a measurable function and* $\int_\Omega \phi(\zeta)d\rho \in I$ *such that (32) is valid. If* ψ *is a convex on I, then the following is valid:*

$$\int_\Omega \psi(\phi)\mathrm{d}\rho \leq \psi\left(\int_\Omega \phi(\zeta)\mathrm{d}\rho\right) + \frac{\int_\Omega\left(\int_\Omega \phi(\zeta)\mathrm{d}\rho - \phi(\zeta)\right)\mathrm{d}\mu}{\left(\int_\Omega \phi(\zeta)\mathrm{d}\rho\right)}\psi(0). \tag{34}$$

Provided the integrals exist.

Proof. Put $s = 1$ in Corollary 5 to get the required result. □

3. Applications in Time Scale Calculus

The theory of time scale was introduced by S. Hilger in his Ph.D. Thesis in 1988 (see [10]). A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$, and we usually denote it by the symbol $\mathbb{T}$. The real numbers, the integers, the nonnegative integers, and the natural numbers are examples of time scales. In this section, we give Giaccardi and Petrović inequalities for s-convex functions in the second sense for isotonic linear functional defined on different time-scales taken from the literature (for example, see [11]).

(a) For $p, q \in \mathbb{T} = \mathbb{Z}$ with $p < q$, suppose L consists of all real-valued functions defined on $[p, q - 1] \cap \mathbb{Z}$ and $A(\phi) = \sum_{\zeta=p}^{q-1} \phi(\zeta)$. Let $\phi \in L$ such that the assumptions of Theorem 2 are satisfied. If ψ is an s-convex in second sense on $[x_0, \infty)$ (or $(-\infty, x_0]$) such that $\psi(\phi) \in L$, then

$$\sum_{\zeta=p}^{q-1}\psi(\phi(\zeta))\leq\frac{\sum_{\zeta=p}^{q-1}(\phi(\zeta)-x_0)^s}{\left(\sum_{\zeta=p}^{q-1}\phi(\zeta)-x_0\right)^s}\psi\left(\sum_{\zeta=p}^{q-1}\phi(\zeta)\right)+\frac{\sum_{\zeta=p}^{q-1}\left(\sum_{\zeta=p}^{q-1}\phi(\zeta)-\phi(\zeta)\right)^s}{\left(\sum_{\zeta=p}^{q-1}\phi(\zeta)-x_0\right)^s}\psi(x_0). \tag{35}$$

Under the same conditions, Petrović's inequality for s-convex in second sense for time-scale $\mathbb{T}$ can be derived by putting $x_0=0$ in (35), that is,

$$\sum_{\zeta=p}^{q-1}\psi(\phi(\zeta))\leq\frac{\sum_{\zeta=p}^{q-1}(\phi(\zeta))^s}{\left(\sum_{\zeta=p}^{q-1}\phi(\zeta)\right)^s}\psi\left(\sum_{\zeta=p}^{q-1}\phi(\zeta)\right)+\frac{\sum_{\zeta=p}^{q-1}\left(\sum_{\zeta=p}^{q-1}\phi(\zeta)-\phi(\zeta)\right)^s}{\left(\sum_{\zeta=p}^{q-1}\phi(\zeta)\right)^s}\psi(0). \tag{36}$$

(b) For $h>0$ and $p,q\in\mathbb{T}=h\mathbb{Z}$ with $p<q$, suppose L consists of all real-valued functions defined on $[p,q-h]\cap h\mathbb{Z}$ and $A(\phi)=h\sum_{\zeta=p/h}^{q/h-1}\phi(\zeta)$. Also let $\phi\in L$ such that the assumptions of Theorem 2 are satisfied. If ψ is s-convex in second sense on $[x_0,\infty)$ (or $(-\infty,x_0]$) such that $\psi(\phi)\in L$, then

$$\sum_{\zeta=p/h}^{q/h-1}\psi(\phi(\zeta))\leq\frac{\sum_{\zeta=p/h}^{q/h-1}(\phi(\zeta)-x_0)^s}{\left(h\sum_{\zeta=p/h}^{q/h-1}\phi(\zeta)-x_0\right)^s}\psi\left(h\sum_{\zeta=p/h}^{q/h-1}\phi(\zeta)\right)+\frac{\sum_{\zeta=p/h}^{q/h-1}\left(h\sum_{\zeta=p/h}^{q/h-1}\phi(\zeta)-\phi(\zeta)\right)^s}{\left(h\sum_{\zeta=p/h}^{q/h-1}\phi(\zeta)-x_0\right)^s}\psi(x_0). \tag{37}$$

Under the same conditions, Petrović's inequality for s-convex functions in second sense for time-scale $\mathbb{T}$ can be derived by putting $x_0=0$ in (37), that is,

$$\sum_{\zeta=p/h}^{q/h-1}\psi(\phi(\zeta))\leq\frac{\sum_{\zeta=p/h}^{q/h-1}(\phi(\zeta))^s}{\left(h\sum_{\zeta=p/h}^{q/h-1}\phi(\zeta)\right)^s}\psi\left(h\sum_{\zeta=p/h}^{q/h-1}\phi(\zeta)\right)+\frac{\sum_{\zeta=p/h}^{q/h-1}\left(h\sum_{\zeta=p/h}^{q/h-1}\phi(\zeta)-\phi(\zeta)\right)^s}{\left(h\sum_{\zeta=p/h}^{q/h-1}\phi(\zeta)\right)^s}\psi(0). \tag{38}$$

(c) For p, $q\in\mathbb{T}=\mathbb{R}$ with $p<q$, suppose

$$L=C([p,q],\mathbb{R})\text{ and }A(\phi)=\int_p^q\phi(\zeta)d\zeta. \tag{39}$$

Also, let $\phi\in L$ such that the assumptions of Theorem 2 are satisfied. If ψ is s-convex in second sense on $[x_0,\infty)$ (or $(-\infty,x_0]$) such that $\psi(\phi)\in L$, then

$$\int_p^q\psi(\phi(\zeta))\mathrm{d}\zeta\leq\frac{\int_p^q(\phi(\zeta)-x_0)^s\mathrm{d}\zeta}{\left(\int_p^q\phi(\zeta)\mathrm{d}\zeta-x_0\right)^s}\psi\left(\int_p^q\phi(\zeta)\mathrm{d}\zeta\right)+\frac{\int_p^q\left(\int_p^q\phi(\zeta)\mathrm{d}\zeta-\phi(\zeta)\right)^s\mathrm{d}\zeta}{\left(\int_p^q\phi(\zeta)\mathrm{d}\zeta-x_0\right)^s}\psi(x_0). \tag{40}$$

Under the same conditions, Petrović's inequality for s-convex functions in the second sense for time-scale $\mathbb{T}$ can be derived by putting $x_0=0$ in (40), that is,

$$\int_p^q\psi(\phi(\zeta))\mathrm{d}\zeta\leq\frac{\int_p^q(\phi(\zeta))^s\mathrm{d}\zeta}{\left(\int_p^q\phi(\zeta)\mathrm{d}\zeta\right)^s}\psi\left(\int_p^q\phi(\zeta)\mathrm{d}\zeta\right)+\frac{\int_p^q\left(\int_p^q\phi(\zeta)\mathrm{d}\zeta-\phi(\zeta)\right)^s\mathrm{d}\zeta}{\left(\int_p^q\phi(\zeta)\mathrm{d}\zeta\right)^s}\psi(0). \tag{41}$$

(d) For $p, q \in \mathbb{T}$ with $p<q$, suppose

$$E=[p,q)\cap\mathbb{T},\ L=C_{rd}([p,q),\mathbb{R}),\quad A(\phi)=\int_p^q \phi(\zeta)\Delta\zeta, \tag{42}$$

where C_{rd} represents rd-continuous functions [12] and the integral is the Cauchy delta time-scale integral [13]. Also, let $\phi \in L$ such that the assumptions of Theorem 2 are satisfied. If ψ is s-convex in second sense on $[x_0,\infty)$ (or $(-\infty,x_0]$) such that $\psi(\phi)\in L$, then

$$\int_p^q \psi(\phi(\zeta))\Delta\zeta \le \frac{\int_p^q (\phi(\zeta)-x_0)^s\Delta\zeta}{\left(\int_p^q \phi(\zeta)\Delta\zeta - x_0\right)^s}\psi\left(\int_p^q \phi(\zeta)\Delta\zeta\right) + \frac{\int_p^q\left(\int_p^q \phi(\zeta)\Delta\zeta-\phi(\zeta)\right)^s\Delta\zeta}{\left(\int_p^q \phi(\zeta)\Delta\zeta - x_0\right)^s}\psi(x_0). \tag{43}$$

Under the same conditions, Petrović's inequality for s-convex functions in second sense for time-scale $\mathbb{T}$ can be derived by putting $x_0=0$ in (43), that is,

$$\int_p^q \psi(\phi(\zeta))\Delta\zeta \le \frac{\int_p^q (\phi(\zeta))^s\Delta\zeta}{\left(\int_p^q \phi(\zeta)\Delta\zeta\right)^s}\psi\left(\int_p^q \phi(\zeta)\Delta\zeta\right) + \frac{\int_p^q\left(\int_p^q \phi(\zeta)\Delta\zeta-\phi(\zeta)\right)^s\Delta\zeta}{\left(\int_p^q \phi(\zeta)\Delta\zeta\right)^s}\psi(0). \tag{44}$$

(e) For $p, q \in \mathbb{T}$ with $p<q$, suppose

$$E=(p,q]\cap\mathbb{T},\ L=C_{ld}((p,q],\mathbb{R}),\ A(\phi)=\int_p^q \phi(\zeta)\nabla\zeta, \tag{45}$$

where C_{ld} represents ld-continuous functions [14] and the integral is the Cauchy nabla time-scale integral [15]. Also, let $\phi \in L$ such that the assumptions of Theorem 2 are satisfied. If ψ is s-convex in second sense on $[x_0,\infty)$ (or $(-\infty,x_0]$) such that $\psi(\phi)\in L$, then,

$$\int_p^q \psi(\phi(\zeta))\nabla\zeta \le \frac{\int_p^q (\phi(\zeta)-x_0)^s\nabla\zeta}{\left(\int_p^q \phi(\zeta)\nabla\zeta - x_0\right)^s}\psi\left(\int_p^q \phi(\zeta)\nabla\zeta\right) + \frac{\int_p^q\left(\int_p^q \phi(\zeta)\nabla\zeta-\phi(\zeta)\right)^s\nabla\zeta}{\left(\int_p^q \phi(\zeta)\nabla\zeta - x_0\right)^s}\psi(x_0). \tag{46}$$

Under the same assumptions, Petrović's inequality for s-convex functions in second sense for time-scale $\mathbb{T}$ can be derived by putting $x_0=0$ in (46), that is,

$$\int_p^q \psi(\phi(\zeta))\nabla\zeta \le \frac{\int_p^q (\phi(\zeta))^s\nabla\zeta}{\left(\int_p^q \phi(\zeta)\nabla\zeta\right)^s}\psi\left(\int_p^q \phi(\zeta)\nabla\zeta\right) + \frac{\int_p^q\left(\int_p^q \phi(\zeta)\nabla\zeta-\phi(\zeta)\right)^s\nabla\zeta}{\left(\int_p^q \phi(\zeta)\nabla\zeta\right)^s}\psi(0). \tag{47}$$

(f) For $p, q \in \mathbb{T}$ with $p<q$, suppose

$$E=[p,q]\cap\mathbb{T},\ L=C([p,q],\mathbb{R}),\ A(\phi)=\int_p^q \phi(\zeta)\diamond_\alpha\zeta, \tag{48}$$

where the integral is the Cauchy α-diamond time-scale integral [16]. Also, let $\phi\in L$ such that the assumptions of Theorem 2 are satisfied. If ψ is s-convex in second sense on $[x_0,\infty)$ (or $(-\infty,x_0]$) such that $\psi(\phi)\in L$, then

$$\int_p^q \psi(\phi(\zeta))\diamond_\alpha\zeta \le \frac{\int_p^q (\phi(\zeta)-x_0)^s\diamond_\alpha\zeta}{\left(\int_p^q \phi(\zeta)\diamond_\alpha\zeta - x_0\right)^s}\psi\left(\int_p^q \phi(\zeta)\diamond_\alpha\zeta\right) + \frac{\int_p^q\left(\int_p^q \phi(\zeta)\diamond_\alpha\zeta-\phi(\zeta)\right)^s\diamond_\alpha\zeta}{\left(\int_p^q \phi(\zeta)\diamond_\alpha\zeta - x_0\right)^s}\psi(x_0). \tag{49}$$

Under the same assumptions, Petrović's inequality for s-convex functions in second sense for given time-scale can be derived by putting $x_0=0$ in (49), that is,

$$\int_p^q \psi(\phi(\zeta))+_\alpha\zeta \leq \frac{\int_p^q (\phi(\zeta))^s +_\alpha\zeta}{\left(\int_p^q \phi(\zeta)+_\alpha\zeta\right)^s}\psi\left(\int_p^q \phi(\zeta)+_\alpha\zeta\right) + \frac{\int_p^q \left(\int_p^q \phi(\zeta)+_\alpha\zeta - \phi(\zeta)\right)^s +_\alpha\zeta}{\left(\int_p^q \phi(\zeta)+_\alpha\zeta\right)^s}\psi(0). \tag{50}$$

(g) Assume that $\mathbb{T}_1, \mathbb{T}_2, \ldots, \mathbb{T}_n$ are time scales and p_i, $q_i \in \mathbb{T}_i$ such that $p_i < q_i$, for $i = 1, 2, \ldots, n$. Also, suppose

$$E \subset \left([p_1, q_1) \cap \mathbb{T}_1\right) \times \cdots \times \left([p_n, q_n) \cap \mathbb{T}_n\right), \tag{51}$$

be Jordan Δ-measurable and let L be the set of all bounded Δ-integrable functions from E to $\mathbb{R}$. Moreover, let

$$A(\phi) = \int_E \phi(\zeta)\Delta\zeta, \tag{52}$$

where the integral is the multiple Riemann delta time-scale integral [11]. Also, let $\phi \in L$ such that the assumptions of Theorem 2 are satisfied. If ψ is s-convex in second sense on $[x_0, \infty)$ (or $(-\infty, x_0]$) such that $\psi(\phi) \in L$, then

$$\int_E \psi(\phi)\Delta\zeta \leq \frac{\int_E (\phi(\zeta) - x_0)^s \Delta\zeta}{\left(\int_E \phi(\zeta)\Delta\zeta - x_0\right)^s}\psi\left(\int_E \phi(\zeta)\Delta\zeta\right) + \frac{\int_E \left(\int_E \phi(\zeta)\Delta\zeta - \phi(\zeta)\right)^s \Delta\zeta}{\left(\int_E \phi(\zeta)\Delta\zeta - x_0\right)^s}\psi(x_0). \tag{53}$$

Under the same assumptions, Petrović's inequality for s-convex functions in second sense for a given time-scale that can be derived by putting $x_0 = 0$ in (53), that is

$$\int_E \psi(\phi)\Delta\zeta \leq \frac{\int_E (\phi(\zeta))^s \Delta\zeta}{\left(\int_E \phi(\zeta)\Delta\zeta\right)^s}\psi\left(\int_E \phi(\zeta)\Delta\zeta\right) + \frac{\int_E \left(\int_E \phi(\zeta)\Delta\zeta - \phi(\zeta)\right)^s \Delta\zeta}{\left(\int_E \phi(\zeta)\Delta\zeta\right)^s}\psi(0). \tag{54}$$

(h) Under the time-scale defined in the above-given clause 7, let E be Lebesgue Δ-measurable, L be the set of all Δ-integrable functions from E to $\mathbb{R}$ and $A(\phi)$, for $\phi \in L$, be multiple Lebesgue delta time-scale integral (see [17]). If the assumptions of Theorem 2 are satisfied under these conditions, then for s-convex function in the second sense ψ on $[x_0, \infty)$ (or $(-\infty, x_0]$), the inequalities (53) and (54) are valid.

4. Discussion

The most famous Giaccardi and Petrović inequalities for s-convex functions in the second sense for isotonic linear functionals are derived. It has been shown that these new findings generate many classical results for different particular cases of the isotonic linear functionals. While understanding the significance of time-scale calculus, these inequalities have been derived for various time-scale integrals. It is important to note that the results derived in this paper for isotonic linear functional can be particularized for other branches of science.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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