Research Article

An Effective New Iterative Method to Solve Conformable Cauchy Reaction-Diffusion Equation via the Shehu Transform

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For the first time, we establish a new procedure by using the conformable Shehu transform (CST) and an iteration method for solving fractional-order Cauchy reaction-diffusion equations (CRDEs) in the sense of conformable derivative (CD). We call this recommended method the conformable Shehu transform iterative method (CSTIM). To evaluate the efficacy and consistency of CSTIM for conformable partial differential equations (PDEs), the absolute errors of four CRDEs are reviewed graphically and numerically. Furthermore, graphical significances are correspondingly predicted for several values of fractional-order derivatives. The results and examples establish that our new method is unpretentious, accurate, valid, and capable. CSTIM does not necessarily use He’s polynomials and Adomian polynomials when solving nonlinear problems, so it has a strong advantage over the homotopy analysis and Adomian decomposition methods. The convergence and absolute error analysis of the series solutions is also offered.

1. Introduction

Fractional calculus is well used for proceedings with any order of derivatives or integrals. It is the preference of integer derivatives and integrals. There are innumerable categories of definitions for fractional-order derivatives. Although the most frequent fractional derivatives, for instance, Riemann–Liouville, Caputo derivative, and Caputo–Fabrizio derivative, are utilized by numerous researchers for studying numerical solutions of distributed order differential equations [1], mathematical modeling of COVID-19 [2], analysis of cancer tumor [3], analytical studies on the physical phenomena [4], and analysis of different diseases [5, 6] and viruses [7], but there are several dissertations in the literature that state that these operators have a bit of a limitation [8, 9].

In 2014 [10], Khalil et al. offered a novel operator, so-called conformable derivative (CD), which contains several of the axioms that the existing operators do not satisfy [8, 9]. The CD is relatively similar to the derivative in the classical limit method, and it is rather uncomplicated to manage. Consequently, it has been acknowledged so swiftly and has been the subject of numerous dissertations for scientists [11–13].

In the disciplines of science and engineering, we find natural and physical events that, when characterized by mathematical models, happen to be differential equations (DEs). For instance, the equations of Kaup–Kupershmidt (KK), Burgers–Huxley equation, and the deflection of a beam are characterized by DEs [14–16]. Consequently, the solutions of DEs are required. Many DEs arising in applications are so intricate that it is sometimes impractical to have close-form solutions. Numerical methods provide a powerful alternative tool for solving the DEs under the given initial conditions. For example, in the past couple of years, various methods have been presented to solve DEs containing the Elzaki residual power series method [17], the Laplace decomposition method [18], the fractional
natural decomposition approach [19], the variational iteration technique [20], the operational matrix scheme [21], the homotopy analysis approach [22], the Sumudu decomposition method [23], the Lie-group scheme [24], and the residual power series method [25].

The reaction-diffusion equations (RDEs) explain numerous nonlinear structures in physics, chemistry, ecology, and biology [26–28]. The RDE is described in the following procedure:

\[ \frac{\partial \xi (v, \tau)}{\partial \tau} = \Delta \xi + \varphi (\xi, \nabla \xi; \Omega, \tau). \]  

(1)

The expression of \( \Delta \xi \) is diffusion term and \( \varphi (\xi, \nabla \xi; \Omega, \tau) \) is the reaction function.

In this study, we discussed the conformable CRDEs of one-dimensional systems, which is described as follows:

\[ T^\lambda \xi (v, \tau) = \sigma D_{\tau} \xi (v, \tau) + \gamma (v, \tau) \xi (v, \tau), \quad 0 < \lambda \leq 1, \]  

(2)

where \( T^\lambda \) is the CD.

If \( \lambda = 1 \), then, it converts into a classical RDE; consequently, \( \xi (v, \tau) \) is the concentration, \( \gamma (v, \tau) \) is the reaction parameter, and \( \sigma = 0 \) is the diffusion coefficient, by means of the preliminary and boundary conditions:

\[ \xi (v, 0) = \theta (v), \quad v \in \mathbb{R}, \]  

(3)

\[ \xi (0, \tau) = \theta_1 (\tau), \]  

\[ D_{\tau} \xi (0, \tau) = \theta_2 (\tau), \quad \tau \in \mathbb{R}^+. \]  

(4)

Equations (2) and (3) are named as the characteristic Cauchy prototype in the domain \( \Omega = \mathbb{R} \times \mathbb{R}^+ \), and equations (2) and (4) are termed as the noncharacteristic Cauchy equation in the domain \( \Omega = \mathbb{R}^+ \times \mathbb{R} \).

In this study, we recognized an effectual CSTIM, which is the collaboration of the ST in the sense of CD and the iterative method familiarized by Daftardar-Gejji and Jafari [29] for gaining approximate and closed-form solutions to CRDEs. The superiority of this new method, which we planned, is that it makes the calculation effortless and extremely precise, allowing us to approximate the definite result.

The subsequent organization of this work is structured as follows. In Section 2, we recall some straightforward definitions and consequences dealing with the CD and ST. The fundamental recommendation beyond the CSTIM with convergence and absolute error analysis for conformable CRDEs is demonstrated in Section 3. In Section 4, we demonstrated numerical examples of CRDEs to exemplify the competency, potential, and straightforwardness of the new method. Lastly, in Section 5, the consequences are gathered in the conclusions.

### 2. Fundamental Concepts

In this section, we review some necessary definitions, theorems, and mathematical preliminaries concerning CD and ST that will be used in this work.

**Definition 1** (see [30]). For given a function \( \xi: [0, \infty) \rightarrow \mathbb{R} \), the CD of \( \xi \) of order \( \lambda \) is formulated as

\[ T^\lambda \xi (\tau) = \lim_{\varepsilon \rightarrow 0} \frac{\xi (\tau + \varepsilon \tau^{1-\lambda}) - \xi (\tau)}{\varepsilon}, \]  

(5)

for \( \tau > 0, \) and \( \lambda \in (0, 1] \). If \( \xi \) is \( \xi \)-differentiable in some \((0, Y)\), \( Y > 0 \), and \( \lim_{\tau \rightarrow Y^{-0} -} (T^\lambda \xi) (\tau) \) occur subsequently, it becomes

\[ (T^\lambda \xi) (0) = \lim_{\tau \rightarrow Y^{-0} -} (T^\lambda \xi) (\tau). \]  

(6)

**Definition 2** (see [31]). Shenu transform (ST) is a new integral transformation which is defined for function of exponential order. We take a function in the set \( \eta \) defined by

\[ \eta = \{ \xi (\tau) : \exists \Xi_1, \Xi_2 > 0 \text{ s.t.} |\xi (\tau)| < \Xi_1 e^{(|\tau|/\Xi_2)}, \text{ if } \tau \in [0, \infty) \}. \]  

(7)

The ST which is represented by \( \varrho(\cdot) \) for a function \( \xi (\tau) \) is defined as

\[ \varrho(\xi (\tau)) = \tilde{\varrho} (\vartheta, \omega) = \int_0^\infty \xi (\tau) e^{- (\omega \tau/\vartheta)} d\tau, \quad \tau > 0. \]  

(8)

The ST of a function \( \xi (\tau) \) is \( \tilde{\varrho} (\vartheta, \omega) \). Then, \( \xi (\tau) \) is called the inverse of \( \varrho(\cdot) \), which is expressed as \( \varrho^{-1} (\tilde{\varrho} (\vartheta, \omega)) = \xi (\tau) \), for \( \tau \geq 0 \). \( \varrho^{-1} \) is called the inverse ST.

**Definition 3** (see [32]). Let \( 0 < \lambda \leq 1 \) and \( \xi: [0, \infty) \rightarrow \mathbb{R} \) be a real valued function. Then, the conformable Shehu transform (CST) of order \( \lambda \) is defined by

\[ \varrho_\lambda [\xi] = \tilde{\varrho} (\vartheta, \omega) = \int_0^\infty e^{(\omega \tau/\vartheta \lambda)} \xi (\tau) d\tau. \]  

(9)

**Definition 4** (see [33]). The Mittag–Leffler function is defined as follows:

\[ E_\lambda [\tau] = \sum_{a=0}^{\infty} \frac{\tau^a}{\Gamma (a\lambda + 1)}, \quad \lambda \in \mathbb{C}, \text{Re}(\lambda) > 0. \]  

(10)

**Theorem 1** (see [34]). Let \( 0 < \lambda \leq 1 \) and \( \xi_1 \) and \( \xi_2 \) be \( \lambda \)-differentiable at a point \( \tau > 0 \). Then,

1. \( T^\lambda (\chi_1 \xi_1 + \chi_2 \xi_2) = \chi_1 T^\lambda (\xi_1) + \chi_2 T^\lambda (\xi_2), \quad \forall \chi_1, \chi_2 \in \mathbb{R} \),
2. \( T^\lambda (e^\tau) = \tau e^{\tau-1}, \quad \forall \chi \in \mathbb{R} \),
3. \( T^\lambda (c) = 0, \quad c \in \mathbb{R} \),
4. \( T^\lambda (\xi_1 \xi_2) = \xi_1 T^\lambda (\xi_2) + \xi_2 T^\lambda (\xi_1) \),
5. \( T^\lambda (\xi_1/\xi_2) = (\xi_2 T^\lambda (\xi_1) - \xi_1 T^\lambda (\xi_2))/ (\xi_2)^2 \).

In addition, if \( \xi \) is differentiable, then \( T^\lambda (\xi) (\tau) = \tau^{1-\lambda} (d\xi (d\tau)/d\tau) (\tau) \).

**Theorem 2** (see [32]). Let \( \xi: [0, \infty) \rightarrow \mathbb{R} \) be a real valued map and \( 0 < \lambda \leq 1 \); then, we have
\[ \theta_1 \left[ T_n^1 \xi (v, r) \right] = \frac{\lambda}{\omega} \theta_1 \left[ \xi (v, r) \right] - \xi (v, 0). \] (11)

**Theorem 3** (see [32]). Let \( m, w \) and \( c \in \mathbb{R} \) and \( 0 < \lambda \leq 1 \). We have the following:

1. \( \theta_1 [c] = c (\omega/\lambda) \)
2. \( \theta_1 [r^1] = \bar{Y} (\omega, \omega) = \lambda^{w/\lambda} (\omega/\lambda)^{w/\lambda+1} (1 + (\omega/\lambda)) \)
3. \( \theta_1 [\sin (m r^1/\lambda)] = \bar{Y} (\omega, \omega) = (\omega^2/\lambda^2 + m^2 r^2), (\omega/\lambda) > 0 \)
4. \( \theta_1 [\cos (m r^1/\lambda)] = \bar{Y} (\omega, \omega) = (\omega^2/\lambda^2 - m^2 r^2), (\omega/\lambda) > 0 \)
5. \( \theta_1 [\sinh (m r^1/\lambda)] = \bar{Y} (\omega, \omega) = (\omega^2/\lambda^2 - m^2 r^2), (\omega/\lambda) > 0 \)
6. \( \theta_1 [\cosh (m r^1/\lambda)] = \bar{Y} (\omega, \omega) = (\omega^2/\lambda^2 - m^2 r^2), (\omega/\lambda) > 0 \)

In Section 3, we present a novel computational methodology for solving conformable CRDEs utilizing the CST and iteration method. The convergence and absolute error analysis of the series solutions obtained through the mentioned method are also implemented.

### 3. Methodology of the CSTIM with the Convergence and Error Analysis

In this section, we illustrate the basic idea of CSTIM to solve conformable CRDEs.

#### 3.1. Methodology of the CSTIM

We contemplate the following equation in the common operator system by means of the preliminary condition to demonstrate the straightforward idea of the CSTIM for the CRDEs:

\[ \begin{align*}
T^1_n \xi (v, r) + \Theta (v, r) + \mathcal{N} \xi (v, r) = \Phi (v, r), \\
\xi (v, 0) = \theta (v), \quad 0 < \lambda \leq 1,
\end{align*} \] (12)

where \( T^1_n \) is the CD, \( \Theta \) is a linear operator, \( \mathcal{N} \) is a nonlinear operator, and \( \Phi (v, r) \) is a continuous function. Employing CST at both sides of equation (12), we obtain as follows:

\[ \theta_1 \left[ T^1_n \xi (v, r) + \Theta (v, r) + \mathcal{N} \xi (v, r) \right] = \theta_1 [\Phi (v, r)]. \] (13)

By consuming the linear axiom of CST, we get as follows:

\[ \theta_1 \left[ T^1_n \xi (v, r) \right] + \theta_1 [\Theta (v, r)] + \theta_1 [\mathcal{N} \xi (v, r)] = \theta_1 [\Phi (v, r)]. \] (14)

Utilizing the following definition of CST

\[ \theta_1 \left[ T^1_n \xi (v, r) \right] = \frac{\omega}{\lambda} \theta_1 \left[ \xi (v, r) \right] - \xi (v, 0), \] (15)

and using equations (15) in (14),

\[ \frac{\omega}{\lambda} \theta_1 \left[ \xi (v, r) \right] - \xi (v, 0) + \theta_1 [\Theta (v, r)] + \theta_1 [\mathcal{N} \xi (v, r)] = \theta_1 [\Phi (v, r)]. \] (16)

By simplifying equation (16), we obtain as follows:

\[ \theta_1 [\xi (v, r)] = \frac{\omega}{\lambda} \theta_1 [\xi (v, r)] - \xi (v, 0) + \theta_1 [\Phi (v, r)] \]

\[ - \frac{\omega}{\lambda} \theta_1 [\mathcal{N} \xi (v, r)] - \theta_1 [\theta_1 [\mathcal{N} \xi (v, r)]] \] (17)

By utilizing the inverse CST on both sides of equation (17), we obtain as

\[ \xi (v, r) = \theta_1^{-1} \left[ \frac{\omega}{\lambda} \theta_1 (v, r) + \theta_1 [\Phi (v, r)] \right] \]

\[ - \theta_1^{-1} \left[ \frac{\omega}{\lambda} \theta_1 [\mathcal{N} \xi (v, r)] \right] - \theta_1^{-1} \left[ \theta_1 [\mathcal{N} \xi (v, r)] \right] \]. (18)

Next, assume the following:

\[ \Theta (v, r) = \frac{\omega}{\lambda} \theta_1 (v, r) + \theta_1 [\Phi (v, r)], \]

\[ \mathcal{N} (\xi (v, r)) = \theta_1^{-1} \left[ \frac{\omega}{\lambda} \theta_1 [\mathcal{N} \xi (v, r)] \right], \] (19)

\[ \mathcal{N} (\xi (v, r)) = \theta_1^{-1} \left[ \frac{\omega}{\lambda} \theta_1 [\mathcal{N} \xi (v, r)] \right]. \] (20)

As a result, equation (18) can be written as

\[ \xi (v, r) = \Theta (v, r) + \mathcal{N} (\xi (v, r)) + \mathcal{N} (\mathcal{N} (\xi (v, r))), \]

where \( \Theta (v, r) \) is a recognized function and \( \mathcal{N} \) and \( \Psi \) are specified nonlinear and linear operators of \( \xi (v, r) \) correspondingly. The outcome of equation (20) can be represented in the expansion form:

\[ \xi (v, r) = \sum_{k=0}^{\infty} \xi_k (v, r). \] (21)

Also, we take

\[ \Psi \left( \sum_{k=0}^{\infty} \xi_k (v, r) \right) = \sum_{k=0}^{\infty} \Psi (\xi_k (v, r)). \] (22)

The nonlinear operator \( \mathcal{N} \) is written as [29]

\[ \mathcal{N} \left( \sum_{k=0}^{\infty} \xi_k (v, r) \right) = \mathcal{N} (\xi_0 (v, r)) + \sum_{k=0}^{\infty} \left\{ \mathcal{N} \left( \sum_{\beta=0}^{k-1} \xi_\beta (v, r) \right) \right\} \] (23)

Therefore, equation (20) can be characterized as this arrangement:

\[ \sum_{k=0}^{\infty} \xi_k (v, r) = \Theta (v, r) + \sum_{k=0}^{\infty} \Psi (\xi_k (v, r)) + \mathcal{N} (\xi_0 (v, r)) \]

\[ + \sum_{k=0}^{\infty} \left\{ \mathcal{N} \left( \sum_{\beta=0}^{k-1} \xi_\beta (v, r) \right) \right\}. \] (24)

Defining the recurrence relation by using equation (24), we set
\[\xi_0(v, \tau) = \Theta(v, \tau),\]
\[\xi_1(v, \tau) = \Psi(\xi_0(v, \tau)) + N(\xi_0(v, \tau)),\]
\[\xi_{k+1}(v, \tau) = \Psi(\xi_k(v, \tau)) + N\left(\xi_0(v, \tau) + \xi_1(v, \tau) + \ldots + \xi_k(v, \tau)\right) - N(\xi_0(v, \tau) + \xi_1(v, \tau) + \ldots + \xi_{k-1}(v, \tau)).\] (25)

So, we have the following:
\[\xi_1(v, \tau) + \xi_2(v, \tau) + \ldots + \xi_{k+1}(v, \tau) = \Psi(\xi_0(v, \tau) + \xi_1(v, \tau) + \ldots + \xi_k(v, \tau)) + N\left(\xi_0(v, \tau) + \xi_1(v, \tau) + \ldots + \xi_k(v, \tau)\right).\] (26)

Namely,
\[\sum_{a=0}^{\infty} \xi_a(v, \tau) = \Theta(v, \tau) + \Psi\left(\sum_{a=0}^{\infty} \xi_a(v, \tau)\right) + N\left(\sum_{a=0}^{\infty} \xi_a(v, \tau)\right).\] (27)

The kth approximate solution of equation (12) is given by
\[\xi_k(v, \tau) = \xi_0(v, \tau) + \xi_1(v, \tau) + \ldots + \xi_{k-1}(v, \tau).\] (28)

3.2. Convergence and Absolute Error Analysis of CSTIM.
In this section, we illustrate the convergence and absolute error analysis of CSTIM for the conformable CRDEs.

The following theorem explains and governs the condition for the convergence of the expansion solution.

**Theorem 4.** Let \(Y\) be a Banach space; then, the expansion result of \(\xi(v, \tau)\) is convergent if \(\exists \xi \in (0, 1)\) s.t. \(\|\xi_\alpha(v, \tau)\| \leq \xi \|\xi_{\alpha-1}(v, \tau)\|, \forall \alpha \in \mathbb{N}\).

**Proof.** Consider
\[S_k(v, \tau) = \xi_0(v, \tau) + \xi_1(v, \tau) + \xi_2(v, \tau) + \ldots + \xi_k(v, \tau).\] (29)

It is essential to validate that the series of kth partial sums \(\{S_k\}\) is a Cauchy series in the Banach space. For this, we have
\[\|S_{k+1}(v, \tau) - S_k(v, \tau)\| \leq \xi \|S_k(v, \tau)\| \leq \xi^2 \|S_{k-1}(v, \tau)\| \leq \cdots \leq \xi^k \|\xi_\alpha(v, \tau)\|.\] (30)

For every \(k, \ell \in \mathbb{N}\) and \(k \leq \ell\) and by using equation (30) and the triangle inequality, we obtain
\[\|S_k(v, \tau) - S_\ell(v, \tau)\| \leq \left(\frac{\xi^{\ell+1}}{1 - \xi}\right)\|\xi_\alpha(v, \tau)\|.\] (31)

Since \(0 < \xi \leq 1,\) then \(1 - \xi^{\ell+1} \leq 1.\) Therefore, from inequality (31), we have
\[\|S_k(v, \tau) - S_\ell(v, \tau)\| \leq \frac{\xi^{\ell+1}}{1 - \xi}\|\xi_\alpha(v, \tau)\|.\] (32)

So, \(\xi_0(v, \tau)\) is bounded. Hence, we have
\[\lim_{x \to \infty} \left\|S_k(v, \tau) - S_\ell(v, \tau)\right\| = 0.\] (33)

Therefore, \(\{S_k\}\) is a Cauchy series in Banach space, so the expansion solution of equation (12) is convergent.

We propose an absolute error study of the planned scheme in the successive theorem.

**Theorem 5.** Let \(\xi(v, \tau)\) be the approximate solution of the truncated finite series \(\xi(v, \tau) = \sum_{a=0}^{n} \xi_a(v, \tau).\) Suppose that it is reasonable to acquire a real number \(\Xi \in (0, 1)\) s.t. \(\|\xi_{\alpha+1}(v, \tau)\| \leq \Xi \|\xi_\alpha(v, \tau)\|, \forall \alpha \in \mathbb{N}\). Moreover, the greatest absolute error is
\[\left\|\xi(v, \tau) - \sum_{a=0}^{\delta} \xi_a(v, \tau)\right\| \leq \left(\frac{\Xi^{\delta+1}}{1 - \Xi}\right)\|\xi_\alpha(v, \tau)\|.\] (34)

**Proof.** Let the series \(\sum_{a=0}^{\delta} \xi_a(v, \tau)\) be finite. Then,
\[\left\|\xi(v, \tau) - \sum_{a=0}^{\delta} \xi_a(v, \tau)\right\| \leq \sum_{a=0}^{\delta} \|\xi_a(v, \tau)\| \leq \sum_{a=0}^{\delta} \Xi^a \|\xi_\alpha(v, \tau)\| \leq \Xi^{\delta+1} \left(1 + \Xi + \Xi^2 + \Xi^3 + \cdots\right)\|\xi_\alpha(v, \tau)\| \leq \left(\frac{\Xi^{\delta+1}}{1 - \Xi}\right)\|\xi_\alpha(v, \tau)\|.\] (35)
and the proof is completed.

The applicability of the recommended method is examined in Section 4.

4. Numerical Examples and Concluding Remarks

In this section, four problems of CRDEs are established to illustrate the performance and appropriateness of the recommended method.

Example 1. Consider the following Cauchy reaction-diffusion equation [35]:

\[ T^\omega \xi (v, r) = \xi_{\tau r} (v, r) - \xi (v, r), \quad 0 < \lambda \leq 1, \quad (36) \]

subject to the initial condition,

\[ \xi (v, 0) = e^{-\gamma} + \nu. \quad (37) \]

Applying CST on both sides of equation (36), we obtain

\[ \omega_1 \left[ T^\omega \xi (v, r) \right] = \omega_1 \left[ \xi_{\tau r} (v, r) - \xi (v, r) \right]. \quad (38) \]

By using the linear property of CST, the above equation becomes as

\[ \omega_1 \left[ T^\omega \xi (v, r) \right] = \omega_1 \left[ \xi_{\tau r} (v, r) \right] - \omega_1 \left[ \xi (v, r) \right]. \quad (39) \]

The CD definition of ST implies that

\[ \omega_1 \left[ \xi (v, r) \right] = \omega \xi (v, 0) + \omega \omega_1 \left[ \xi_{\tau r} (v, r) \right] - \omega \omega_1 \left[ \xi (v, r) \right]. \quad (40) \]

By applying the inverse CST on both sides of equation (40), we reach to

\[ \xi (v, r) = \omega_1^{-1} \left[ \omega \xi (v, 0) + \omega \omega_1 \left[ \xi_{\tau r} (v, r) \right] - \omega \omega_1 \left[ \xi (v, r) \right] \right] - \omega_1^{-1} \left[ \omega \omega_1 \left[ \xi (v, r) \right] \right]. \quad (41) \]

According to the CSTIM, we have

\[ \xi (v, r) = \sum_{n=0}^{\infty} \xi_n (v, r), \]

\[ \Psi \left[ \xi (v, r) \right] = \omega_1^{-1} \left[ \omega \omega_1 \left[ \xi_{\tau r} (v, r) \right] \right] - \omega_1^{-1} \left[ \omega \omega_1 \left[ \xi (v, r) \right] \right]. \quad (42) \]

Using equations (42) in (41), it becomes

\[ \sum_{n=0}^{\infty} \xi_n (v, r) = \omega_1^{-1} \left[ \omega \xi (v, 0) \right] + \omega_1^{-1} \left[ \omega \omega_1 \left[ \xi_{\tau r} (v, r) \right] \right] - \omega_1^{-1} \left[ \omega \omega_1 \left[ \xi (v, r) \right] \right] - \omega_1^{-1} \left[ \omega \omega_1 \left[ \xi (v, r) \right] \right]. \quad (43) \]

By using the initial value in the above equation, we obtain

\[ \sum_{n=0}^{\infty} \xi_n (v, r) = \omega_1^{-1} \left[ \omega \xi (v, 0) \right] + \omega_1^{-1} \left[ \omega \omega_1 \left[ \xi_{\tau r} (v, r) \right] \right] - \omega_1^{-1} \left[ \omega \omega_1 \left[ \xi (v, r) \right] \right]. \]

By iteration, the following results are obtained from equation (44):

\[ \xi_0 (v, r) = e^{-\gamma} + \nu, \]

\[ \xi_1 (v, r) = -\frac{\nu \lambda}{\lambda}, \]

\[ \xi_2 (v, r) = \frac{\nu \lambda}{2 \lambda}, \]

\[ \xi_3 (v, r) = \frac{\nu \lambda}{6 \lambda}, \]

\[ \xi_4 (v, r) = \frac{\nu \lambda}{24 \lambda}, \]

\[ \xi_5 (v, r) = \frac{\nu \lambda}{120 \lambda}. \]

Therefore, we have the 5th approximate solution of the problem equations (36) and (37) as follows:

\[ \xi^{(5)} (v, r) = e^{-\gamma} + \nu \left( 1 - \frac{\lambda}{\lambda} + \frac{\lambda}{2 \lambda} - \frac{\lambda}{3 \lambda} + \frac{\lambda}{4 \lambda} - \frac{\lambda}{5 \lambda} \right). \quad (46) \]

When \( \lambda = 1 \), equation (46) becomes as

\[ \xi^{(5)} (v, r) = e^{-\gamma} + \nu \left( 1 - \frac{\lambda}{\lambda} + \frac{\lambda}{2 \lambda} - \frac{\lambda}{3 \lambda} + \frac{\lambda}{4 \lambda} - \frac{\lambda}{5 \lambda} \right). \quad (47) \]

Equation (47) matches with first six terms of \( e^{-\gamma} + \nu E(-\tau) \), so the exact solution of equations (36) and (37) equals to \( \xi (v, r) = e^{-\gamma} + \nu E(-\tau) \).

Figure 1 depicts the performance of the 5th approximate and exact solutions of equation (36) and (37) for different values of \( \lambda \), when \( \nu = 1 \), in the interval \( r \in [0, 1] \). Undoubtedly, the results in instances of fractional values of \( \lambda \) converge to the results in case of \( \lambda = 1 \). As well, the approximate solutions are consistent with the exact solution at \( \lambda = 1 \) and this confirms the efficiency and exactness of the recommended method.

Figure 2 demonstrates the absolute errors in the interval \( r \in [0, 1] \) over the 5th-approximation and accurate solutions of equations (36) and (37) at \( \lambda = 1 \) when \( \nu = 1 \), attained by means of CSTIM. From the figure, it can be predicted that the approximate solution is very close to the exact solution and this confirms the efficacy of the CSTIM.

Error functions are accessible to perceive the accuracy and applicability of the scheme. To demonstrate the
precision and ability of CSTIM, we introduced an absolute error function. Table 1 displays the absolute errors at reasonable introduced grid points in the interval [0, 1] amongst the 5 th approximate and exact solutions of equations (36) and (37) at $\lambda = 1$, when $\nu = 0.5$, attained by means of CSTIM. Table 1 shows that the approximate solution is quite near to the precise solution, indicating that the proposed strategy is effective.

**Example 2.** Consider the following Cauchy reaction-diffusion equation [28]:

$$T_{\tau}^{\lambda} \xi (\nu, \tau) = \xi_{\nu\nu} (\nu, \tau) - (1 + 4 \nu^4) \xi (\nu, \tau), \quad 0 < \lambda \leq 1,$$

subject to the initial condition,

$$\xi (\nu, 0) = e^{\nu^2}.$$

Applying CST on both sides of equation (48), we obtain

$$\theta_1 \left[ T_{\tau}^{\lambda} \xi (\nu, \tau) \right] = \theta_1 \left[ \xi_{\nu\nu} (\nu, \tau) - (1 + 4 \nu^4) \xi (\nu, \tau) \right].$$

By using the linear property of CST, the above equation becomes

By applying the inverse CST on both sides of equation (52), we reach to

$$\xi (\nu, \tau) = \theta_1^{-1} \left[ \frac{\omega}{\lambda} \xi (\nu, 0) \right] + \theta_1^{-1} \left[ \frac{\omega}{\lambda} \xi_{\nu\nu} (\nu, \tau) \right] - \theta_1^{-1} \left[ \frac{\omega}{\lambda} \left[ (1 + 4 \nu^4) \xi (\nu, \tau) \right] \right].$$

According to the CSTIM, we have

$$\xi (\nu, \tau) = \sum_{a=0}^{\infty} \xi_a (\nu, \tau),$$

$$\Psi [\xi (\nu, \tau)] = \theta_1^{-1} \left[ \frac{\omega}{\lambda} \xi (\nu, 0) \right] + \theta_1^{-1} \left[ \frac{\omega}{\lambda} \xi_{\nu\nu} (\nu, \tau) \right] - \theta_1^{-1} \left[ \frac{\omega}{\lambda} \left[ (1 + 4 \nu^4) \xi (\nu, \tau) \right] \right].$$

Using equation (54) in (53), it becomes

$$\sum_{a=0}^{\infty} \xi_a (\nu, \tau) = \theta_1^{-1} \left[ \frac{\omega}{\lambda} \xi (\nu, 0) \right] + \theta_1^{-1} \left[ \frac{\omega}{\lambda} \left[ \mathcal{D}_{\nu\nu} \sum_{a=0}^{\infty} \xi_a (\nu, \tau) \right] \right] - \theta_1^{-1} \left[ \frac{\omega}{\lambda} \left[ (1 + 4 \nu^4) \sum_{a=0}^{\infty} \xi_a (\nu, \tau) \right] \right].$$

By using the initial value in the above equation, we obtain

$$\sum_{a=0}^{\infty} \xi_a (\nu, \tau) = e^{-1} \left[ \frac{\omega}{\lambda} e^{\nu^2} \right] + e^{-1} \left[ \frac{\omega}{\lambda} e^{\nu^2} \mathcal{D}_{\nu\nu} \sum_{a=0}^{\infty} \xi_a (\nu, \tau) \right] - e^{-1} \left[ \frac{\omega}{\lambda} e^{\nu^2} \left[ (1 + 4 \nu^4) \sum_{a=0}^{\infty} \xi_a (\nu, \tau) \right] \right].$$

By iteration, the following results are obtained from equation (56):
Example 3. Consider the following Cauchy reaction-diffusion equation [36]:

\[ T^{\lambda}_{\tau} \xi (v, \tau) = \xi_{\nu\nu} (v, \tau) - (4v^4 - 2\tau + 2) \xi (v, \tau), \quad 0 < \lambda \leq 1, \]

subject to the initial condition,

\[ \xi (v, 0) = e^{\omega^2}. \]  \hspace{1cm} (61)

Applying CST on both sides of equation (60), we obtain

\[ \varphi_{h} \left[ T^{\lambda}_{\tau} \xi (v, \tau) \right] = \varphi_{1} \left[ \xi_{\nu\nu} (v, \tau) - (4v^4 - 2\tau + 2) \xi (v, \tau) \right]. \]  \hspace{1cm} (62)

By using the linear property of CST, the above equation becomes as

\[ \varphi_{1} \left[ T^{\lambda}_{\tau} \xi (v, \tau) \right] = \varphi_{1} \left[ \xi_{\nu\nu} (v, \tau) \right] - \varphi_{1} \left[ (4v^4 - 2\tau + 2) \xi (v, \tau) \right]. \]  \hspace{1cm} (63)

The CD definition of ST implies that

\[ \xi_{\nu\nu} = \frac{\partial^2 \xi}{\partial \tau^2}, \]

Therefore, we have the 5th approximate solution of the problem equations (48) and (49) as follows:

\[ \xi^{(5)} (v, \tau) = e^{\omega^2} \left( 1 + \frac{r^2}{\lambda^2} + \frac{r^{4\lambda}}{24\lambda^2} \right). \]  \hspace{1cm} (58)

When \( \lambda = 1 \), equation (58) becomes as

\[ \xi^{(5)} (v, \tau) = e^{\omega^2} \left( 1 + \frac{\tau^2}{2!} + \frac{\tau^4}{4!} \right). \]  \hspace{1cm} (59)

Equation (59) matches with first six terms of \( e^{\omega^2} E (\tau) \), so the exact solution of equations (48) and (49) equals to \( \xi (v, \tau) = e^{\omega^2} E (\tau) \).

Figure 3 depicts the performance of the 5th approximate and exact solutions of equations (48) and (49) for different values of \( \lambda \), when \( v = 1 \), in the interval \( \tau \in [0, 1] \). Definitely, the results in instances of fractional values of \( \lambda \) converge to the results in case of \( \lambda = 1 \). As well, the approximate solutions are consistent with the exact solution at \( \lambda = 1 \), and this once more confirms the efficiency and exactness of the recommended method.

Figure 4 demonstrates the absolute errors in the interval \([0, 1]\) over the 5th-approximation and accurate solutions of equations (48) and (49) at \( \lambda = 1 \) when \( v = 1 \), attained by means of CSTIM. From the figure, it can be predicted that the approximate solution is close to the exact one, which confirms the efficacy of the CSTIM.

Table 2 displays the absolute errors at reasonable introduced grid points in the interval \([0, 1]\) amongst the 5th-approximate and exact solutions of equation (48) and (49) at \( \lambda = 1 \) when \( v = 0.5 \), attained by means of CSTIM. Table 2 shows that the approximate solution is very near to the exact solution, indicating that the proposed plan is appropriate.

Table 2: The absolute errors for Example 2 at \( \lambda = 1 \) with \( v = 0.5 \).

| \( \tau \) | \( |\xi - \xi^{(5)}| \) |
|---|---|
| 0.08 | 4.7289616666700115 \times 10^{-10} |
| 0.16 | 3.061777098523066 \times 10^{-8} |
| 0.24 | 3.528519882944891 \times 10^{-7} |
| 0.32 | 0.000000200604438021026 |
| 0.40 | 0.000000774392467887999 |
| 0.48 | 0.0000023402073148215408 |

Figure 3: Approximate solutions for different values.

Figure 4: The absolute errors of Example 2.
According to the CSTIM, we have

\[ \xi_{\alpha}(v, \tau) = D_{\alpha \alpha} \left[ \frac{\partial^2}{\partial \tau^2} \xi_{\alpha}(v, \tau) \right] \]

By applying the inverse CST on both sides of equation (64), we reach to

\[ \xi(v, \tau) = \sum_{\alpha=0}^{\infty} \xi_{\alpha}(v, \tau), \]

\[ \Psi[\xi(v, \tau)] = \sum_{\alpha=0}^{\infty} \xi_{\alpha}(v, \tau), \]

Using equation (66) in (65), it becomes

\[ \sum_{\alpha=0}^{\infty} \xi_{\alpha}(v, \tau) = \sum_{\alpha=0}^{\infty} \xi_{\alpha}(v, \tau) \]

By iteration, the following results are obtained from equation (68):

\[ \xi_{0}(v, \tau) = e^{\tau}, \]
\[ \xi_{1}(v, \tau) = e^{\tau} \frac{\lambda}{\alpha}, \]
\[ \xi_{2}(v, \tau) = e^{\tau} \frac{\lambda^2}{2 \alpha^2}, \]
\[ \xi_{3}(v, \tau) = e^{\tau} \frac{\lambda^3}{6 \alpha^3}, \]
\[ \xi_{4}(v, \tau) = e^{\tau} \frac{\lambda^4}{24 \alpha^4}, \]
\[ \xi_{5}(v, \tau) = e^{\tau} \frac{\lambda^5}{120 \alpha^5}. \]

Therefore, we have the 5th approximate solution of the problem equations (60) and (61) as follows:

\[ \xi^{(5)}(v, \tau) = e^{\tau} \left[ 1 + \frac{\lambda}{2 \alpha^2} + \frac{(\lambda^2)^2}{8 \alpha^4} + \frac{(\lambda^3)^2}{8 \alpha^6} + \frac{(\lambda^4)^2}{8 \alpha^8} + \frac{(\lambda^5)^2}{8 \alpha^{10}} \right]. \]

When \( \lambda = 1 \), equation (70) becomes as

\[ \xi^{(5)}(v, \tau) = e^{\tau} \left[ 1 + \frac{\lambda}{2 \alpha^2} + \frac{(\lambda^2)^2}{8 \alpha^4} + \frac{(\lambda^3)^2}{8 \alpha^6} + \frac{(\lambda^4)^2}{8 \alpha^8} + \frac{(\lambda^5)^2}{8 \alpha^{10}} \right]. \]

Equation (71) matches with first six terms of \( e^{\tau} E(\tau^2) \), so the exact solution of equations (60) and (61) equals to

\[ \xi(v, \tau) = e^{\tau} \left[ 1 + \frac{\lambda}{2 \alpha^2} + \frac{(\lambda^2)^2}{8 \alpha^4} + \frac{(\lambda^3)^2}{8 \alpha^6} + \frac{(\lambda^4)^2}{8 \alpha^8} + \frac{(\lambda^5)^2}{8 \alpha^{10}} \right]. \]

Figure 5 depicts the performance of the 5th approximate and exact solutions of equations (60) and (61) for different values of \( \lambda \), when \( \phi = 1 \), in the interval \( 0 \leq \tau \leq 1 \). Unquestionably, the results in instances of fractional values of \( \lambda \) converge to the results in case of \( \lambda = 1 \). As well, the approximate solutions are consistent to the exact solution at \( \lambda = 1 \), and this once more confirms the efficiency and exactness of the recommended method.

Figure 6 demonstrates the absolute errors in the interval [0, 1] over the 5th approximate and accurate solutions of equations (60) and (61) at \( \lambda = 1 \) when \( \phi = 1 \), attained by means of CSTIM. The approximate solution is predicted to be near to the precise solution in the figure, demonstrating that the CSTIM is efficient.

Table 3 displays the absolute errors at reasonable introduced grid points in the interval [0, 1] amongst the 5th approximate and exact solutions of equations (60) and (61) at \( \lambda = 1 \), when \( \phi = 0.5 \), attained by means of CSTIM. From Table 3, it can be perceived that the approximate solutions are close to the exact solutions, which confirms the efficacy of the recommended method numerically.

**Example 4.** In this example, we consider the nonlinear fractional Cauchy reaction–diffusion equation [37]:

\[ T^{\dagger} \xi(v, \tau) = \xi_{\tau}(v, \tau) - \xi_{\tau}(v, \tau) + \xi(v, \tau) \xi_{\tau}(v, \tau) \]

subject to the initial condition,

\[ \xi(v, 0) = e^{\tau}. \]

Applying CST on both sides of equation (72), we obtain

\[ \Theta_{\tau} \left[ T^{\dagger} \xi(v, \tau) \right] = \Theta_{\tau} \left[ \xi_{\tau}(v, \tau) - \xi_{\tau}(v, \tau) + \xi(v, \tau) \xi_{\tau}(v, \tau) \right] - \xi^{2}(v, \tau) + \xi(v, \tau) \]

By using the linear property of CST, the above equation becomes as
\[ \xi(v, \tau) = \phi_1^{-1} \left( \frac{\omega}{\omega_1} \phi_1 \xi(v, \tau) \right) + \phi_1^{-1} \left( \frac{\omega}{\omega_1} \phi_1 \left[ D_{xy} \xi(v, \tau) - D_y \xi(v, \tau) + \xi(v, \tau) \right] \right) + \phi_1^{-1} \left( \frac{\omega}{\omega_1} \phi_1 \xi(v, \tau) - \xi^3(v, \tau) \right). \] (77)

According to the CST, we have
\[ \Psi[\xi(v, \tau)] = \phi_1^{-1} \left( \frac{\omega}{\omega_1} \phi_1 \left[ D_{xy} \xi(v, \tau) - D_y \xi(v, \tau) + \xi(v, \tau) \right] \right), \]
\[ N[\xi(v, \tau)] = \phi_1^{-1} \left( \frac{\omega}{\omega_1} \phi_1 \left[ \xi(v, \tau) \xi_{xx}(v, \tau) - \xi^3(v, \tau) \right] \right). \] (78)

Using equations (78) in (77), it becomes
\[ \sum_{a=0}^{\infty} \xi_a(v, \tau) = \phi_1^{-1} \left( \frac{\omega}{\omega_1} \phi_1 \left[ D_{xy} \sum_{a=0}^{\infty} \xi_a(v, \tau) \right. \right. \]
\[ - D_x \sum_{a=0}^{\infty} \xi_a(v, \tau) + \sum_{a=0}^{\infty} \xi_a(v, \tau) \left. \right) \left. \right] \]
\[ - \phi_1^{-1} \left[ \frac{\omega}{\omega_1} \phi_1 \left[ \sum_{a=0}^{\infty} \xi_a(v, \tau) \right] D_{xx} \sum_{a=0}^{\infty} \xi_a(v, \tau) \right. \]
\[ - \sum_{a=0}^{\infty} \xi_a(v, \tau) \sum_{a=0}^{\infty} \xi_a(v, \tau) \right]. \] (79)

By iteration, the following results are obtained from equation (79):
\[ \xi_0(v, \tau) = e^r, \]
\[ \xi_1(v, \tau) = \frac{e^r \tau^1}{\lambda}, \]
\[ \xi_2(v, \tau) = \frac{e^r \tau^{2\lambda}}{2\lambda}, \]
\[ \xi_3(v, \tau) = \frac{e^r \tau^{3\lambda}}{6\lambda}, \]
\[ \xi_4(v, \tau) = \frac{e^r \tau^{4\lambda}}{24\lambda}, \]
\[ \xi_5(v, \tau) = \frac{e^r \tau^{5\lambda}}{120\lambda^5}. \] (80)

Therefore, we have the 5th approximate solution of the problem equations (72) and (73) as follows:
the exact solution of equations (72) and (73) equals to

\[ \xi(t, \tau) = e^t \left( 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} \right). \] (81)

When \( \lambda = 1 \), equation (81) becomes as

\[ \xi^{(5)}(t, \tau) = e^t \left( 1 + \tau + \frac{\tau^2}{2!} + \frac{\tau^3}{3!} + \frac{\tau^4}{4!} + \frac{\tau^5}{5!} \right). \] (82)

Equation (82) matches with first six terms of \( e^t E(\tau) \), so the exact solution of equations (72) and (73) equals to \( \xi(t, \tau) = e^t E(\tau) \).

Figure 7 depicts the performance of the 5th-approximate and exact solutions of equations (72) and (73) for different values of \( \lambda \), when \( \nu = 1 \), in the interval \( \tau \in [0, 1] \). Undoubtedly, the results in instances of fractional values of \( \lambda \) converge to the results in case of \( \lambda = 1 \). We get the same results from Example 4 as we found in Examples 1, 2, and 3.

Figure 8 demonstrates the absolute errors in the interval \([0, 1]\) over the 5th-approximation and accurate solutions of equations (72) and (73) at \( \lambda = 1 \) when \( \nu = 1 \), attained by means of CSTIM. One can perceive the equivalent verdicts depicted for Examples 1, 2, and 3.

Table 4 displays the absolute errors at reasonable introduced grid points in the interval \([0, 1]\) amongst the 5th-approximate and exact solutions of equations (72) and (73) at \( \lambda = 1 \), when \( \nu = 0.5 \), attained by means of CSTIM. We get the same outcomes as we found in Examples 1, 2, and 3.

Table 4: The absolute errors for Example 4 at \( \lambda = 1 \) with \( \nu = 0.5 \).

| \( t \) | \( |\xi - \xi^{(5)}|\) |
|-------|----------------|
| 0.08  | 6.072105040999531 \times 10^{-10} |
| 0.16  | 3.931399583656514 \times 10^{-8} |
| 0.24  | 4.530709207806183 \times 10^{-7} |
| 0.32  | 6.00009275812558802681 |
| 0.40  | 6.00009943396112266356 |
| 0.48  | 6.0000948856725528594 |

5. Conclusion

In this study, a novel procedure is designed by a combination of CST and an iterative method. The CSTIM has been effectively applied to obtain approximate and exact solutions for the conformable CRDEs. To assess the effectiveness and reliability of CSTIM for conformable PDEs, the absolute errors of linear and nonlinear problems are studied graphically and numerically, at \( \nu = 1 \) and \( \nu = 0.5 \), respectively, when \( \lambda = 1 \). The interpretation of the 2D plots and tables for different values of \( \lambda \) as well validates that the approximate solution is rapidly convergent to the exact solution. The numerical and graphical consequences confirm that the CSTIM is extremely effective and precise.

The CSTIM distinguishes itself from various other numerical methods in three important aspects. The advantage of this method is that there is no need for any small or large physical parametric assumptions in the problem. Thus, it is applicable to not only weakly but also strongly nonlinear problems, going beyond some of the inherent limitations of the standard perturbation methods. Second, the CSTIM does not need He’s polynomials and Adomian polynomials when solving nonlinear problems. Thus, only a few calculations are required to solve nonlinear fractional-order DEs. As a result, it has a significant advantage over homotopy analysis and Adomian decomposition methods. Finally, unlike the other analytic approximation techniques, the CSTIM can be used to construct expansion solutions for linear and nonlinear fractional-order DEs without perturbation, linearization, or discretization. Therefore, we concluded that the recommended procedure is quick, precise, and easy to implement and yields outstanding results. Consequently, we can further apply this procedure to solve more linear and nonlinear fractional-order DEs.

Data Availability

No data were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.
Authors’ Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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