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Research Article

On Interpolative Prešić-Type Set-Valued Contractions

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This study aims to present the notions of interpolative Prešić-type set-valued contractions for the set-valued operators defined on product spaces. With the help of these notions, we have studied the existence of fixed points for such set-valued operators. An application of the obtained results is also discussed with the help of graph theory.

1. Introduction and Preliminaries

Banach [1] initiated the study of the existence of fixed points for self-maps defined on a metric space. This study was further strengthened by Kannan and Chatterjea through their fixed point results derived in [2, 3], respectively. Following this study, Nadler [4] proposed a result to ensure the existence of fixed points for set-valued maps. Prešić [5] extended Banach contraction principle to the maps defined on product spaces, that is, $Q\colon R^k \longrightarrow R$, for any fixed $k \in \mathbb{N}$. Afterwards, this result was extended by Ćirić and Prešić [6]. The results of Prešić [5] and Ćirić and Prešić [6] are presented below.

Theorem 1 (see [5]). *Let $Q\colon R^k \longrightarrow R$, for any fixed $k \in \mathbb{N}$, be a map on complete metric space (R, d_R) and satisfies*

$$d_R\left(Q\left(r_1, r_2, \ldots, r_k\right), Q\left(r_2, r_3, \ldots, r_{k+1}\right)\right) \leq \sum_{j=1}^{k} \beta_j d_R\left(r_j, r_{j+1}\right), \tag{1}$$

for each $r_1, r_2, \ldots, r_k, r_{k+1} \in R$, where $\beta_1, \beta_2, \ldots, \beta_k \geq 0$ with $\sum_{j=1}^{k} \beta_j < 1$. Then, there exists a unique fixed point $r \in R$ of Q, that is, $r = Q(\underbrace{r, r, \ldots, r}_{k})$.

Theorem 2 (see [6]). *Let $Q\colon R^k \longrightarrow R$, for any fixed $k \in \mathbb{N}$, be a map on complete metric space (R, d_R) and satisfies*

$$\begin{aligned} &d_R\left(Q\left(r_1, r_2, \ldots, r_k\right), Q\left(r_2, r_3, \ldots, r_{k+1}\right)\right) \\ &\quad \leq \zeta \max\left\{d_R\left(r_i, r_{i+1}\right)\colon i \in \{1, 2, \ldots, k\}\right\}, \end{aligned} \tag{2}$$

for each $r_1, r_2, \ldots, r_k, r_{k+1} \in R$, where $\zeta \in (0, 1)$. Then, there exists a unique fixed point $r \in R$ of Q, that is, $r = Q(\underbrace{r, r, \ldots, r}_{k})$.

Karapınar [7] presented interpolative Kannan contraction by following the Kannan contraction as follows.

A map $Q\colon (R, d_R) \longrightarrow (R, d_R)$ is called an interpolative Kannan contraction [7] if

$$d_R(Qr, Ql) \leq \zeta d_R(r, Qr)^{\vartheta} d_R(l, Ql)^{1-\vartheta}, \tag{3}$$

for each $r, l \in R$ with $r \neq Qr$ and $l \neq Ql$, where $\zeta \in [0, 1)$ and $\vartheta \in (0, 1)$.

The above work of Karapınar [7] is adopted by several researchers; for example, the notions of interpolative Ćirić–Reich–Rus type contractions in Branciari metric spaces, and partial metric spaces are defined by Aydi et al. [8] and Karapınar et al. [9], the notions of interpolative type F-contractions are defined by Mohammadi et al. [10] and Alansari and Ali [11], the notions of interpolative Hardy-Rogers type contractions, and set-valued interpolative Hardy–Rogers type contractions are defined by Karapınar et al. [12] and Debnath and Sen [13], the notion of interpolative Suzuki-type contraction is discussed by Fulga and Yesilkaya [14], and the notion of interpolative proximal contraction is discussed by Altun and Tasdemir [15].

Gaba and Karapınar [16] redefined the notion of interpolative Kannan contraction through modifying exponential powers in a following way.

A map $Q\colon (R,d_R)\longrightarrow (R,d_R)$ is called an $(\zeta,\vartheta_1,\vartheta_2)$-interpolative Kannan contraction if

$$d_R(Qr,Ql)\leq \zeta d_R(r,Qr)^{\vartheta_1}d_R(l,Ql)^{\vartheta_2}, \quad (4)$$

for each $r,l\in R$ with $r\neq Qr$ and $l\neq Ql$, where $\vartheta_1,\vartheta_2\in(0,1)$ with $\vartheta_1+\vartheta_2<1$ and $\zeta\in[0,1)$.

Recently, Alansari and Ali [17] defined the notion of extended interpolative Prešić-type contraction map as follows.

A map $Q\colon R^k\longrightarrow R$, for any fixed $k\in\mathbb{N}$, on a metric space (R,d_R) is called extended interpolative Prešić type contraction if, for each $w_1,w_2,\ldots,w_k$, $p_1,p_2,\ldots,p_k\in R\backslash\text{Fix}(Q)$, we have

$$\begin{aligned}&d_R(Q(w_1,w_2,\ldots,w_k),Q(p_1,p_2,\ldots,p_k))^{\min\{\gamma(w_1,p_1),\gamma(w_2,p_2),\ldots,\gamma(w_k,p_k)\}}\\&\leq \zeta d_R(w_k,Q(w_1,w_2,\ldots,w_k))^{\vartheta_1}d_R(p_k,Q(p_1,p_2,\ldots,p_k))^{\vartheta_2},\end{aligned} \quad (5)$$

where $\gamma\colon R\times R\longrightarrow\mathbb{R}\backslash\{0\}$ is a map, $\vartheta_1,\vartheta_2\in(0,1)$ with $\vartheta_1+\vartheta_2=1$, $\zeta\in[0,1)$, and $\text{Fix}(Q)=\{r\in R\colon r=Q(r,r,\ldots,r)\}$.

The purpose of this study is to extend and redefine the concepts of interpolative Prešić-type contractions by introducing interpolative Prešić-type set-valued contractions for set-valued maps. We will also present a few fixed-point results to study the existence of fixed points of such maps.

The literature of metric fixed-point theory contains several interesting results that are the generalizations of Banach fixed-point theorem, for example, the study of common fixed-point results for two or more maps [18] and the study of the existence of fixed points for the maps defined on generalized metric spaces, such as b-metric space [19], partial metric space [20], dislocated quasi-metric [21], hypergraphical metric space [22], and soft metric space [23, 24].

Before the next section, we recall the Pompeiu–Hausdorff distance. The Pompeiu–Hausdorff distance is a map $H_R\colon CB(R)\times CB(R)\longrightarrow[0,\infty)$ defined by

$$H_R(J,K)=\max\left\{\sup_{j\in J}d_R(j,K),\sup_{k\in K}d_R(k,J)\right\}, \quad (6)$$

where $d_R(k,J)=\inf\{d_R(k,j)\colon j\in J\}$ and $\text{CB}(R)$ represents the collection of all nonvoid closed and bounded subsets of (R,d_R).

2. Main Results

We begin this section with the following definition.

Definition 1. A map $Q\colon R\times R\longrightarrow\text{CB}(R)$ is said to be an interpolative Prešić type-I set-valued contraction if, for all $w_1,w_2,p_1,p_2\in R\backslash\text{Fix}(Q)$, the following inequality exhibits

$$\begin{aligned}&H_R(Q(w_1,w_2),Q(p_1,p_2))^{\min\{\gamma(w_1,p_1),\gamma(w_2,p_2)\}}\\&\leq\zeta\max\{d_R(w_1,p_1),d_R(w_2,p_2)\}^{\vartheta_1}d_R(w_2,Q(w_1,w_2))^{\vartheta_2}d_R(p_2,Q(p_1,p_2))^{\vartheta_3},\end{aligned} \quad (7)$$

where $\gamma\colon R\times R\longrightarrow\mathbb{R}\backslash\{0\}$ is a map, $\vartheta_1,\vartheta_2,\vartheta_3\in(0,1)$ with $\vartheta_1+\vartheta_2+\vartheta_3=1$, $\zeta\in(0,1)$, and $\text{Fix}(Q)=\{r\in R\colon r\in Q(r,r)\}$.

With the help of below stated result, we will study the existence of fixed points for the above map.

Theorem 3. *Let $Q\colon R\times R\longrightarrow CB(R)$ be an interpolative Prešić type-I set-valued contraction map on a complete metric space (R,d_R). Also, consider that*

(i) If $\min\{\gamma(w_1,p_1),\gamma(w_2,p_2)\}=1$, then $\gamma(z_1,z_2)=1$, for all $z_1\in Q(w_1,w_2)$ and $z_2\in Q(p_1,p_2)$

(ii) There exist $w_1,w_2\in R$ with $\min\{\gamma(w_1,w_2),\gamma(w_2,z_1)\}=1$, for all $z_1\in Q(w_1,w_2)$

(iii) For each sequence $\{r_m\}$ in R with $\gamma(r_m,r_{m+1})=1$, $\forall m\geq m_0$, for some natural number m_0, and $r_m\longrightarrow r$, we have $\gamma(r_m,r)=1$, $\forall m\geq m_0$

Then, there exists an element r of R with $r\in Q(r,r)$.

Proof. By hypothesis (ii), we get two points in R, say r_0 and r_1, with

$$\min\{\gamma(r_0,r_1),\gamma(r_1,z)\}=1,\quad\forall z\in Q(r_0,r_1). \quad (8)$$

Let $r_2\in Q(r_0,r_1)$; then, by (7), we obtain

$$
\begin{aligned}
d_R(r_2,Q(r_1,r_2)) &\le H_R(Q(r_0,r_1),Q(r_1,r_2))^{\min\{\gamma(r_0,r_1),\gamma(r_1,r_2)\}} \\
&\le \zeta \max\{d_R(r_0,r_1),d_R(r_1,r_2)\}^{\vartheta_1} d_R(r_1,Q(r_0,r_1))^{\vartheta_2} d_R(r_2,Q(r_1,r_2))^{\vartheta_3}.
\end{aligned} \tag{9}
$$

That is,

$$
d_R(r_2,Q(r_1,r_2))^{1-\vartheta_3} \le \zeta \max\{d_R(r_0,r_1),d_R(r_1,r_2)\}^{\vartheta_1} d_R(r_1,Q(r_0,r_1))^{\vartheta_2}. \tag{10}
$$

Clearly, $d_R(r_1,Q(r_0,r_1)) \le \max\{d_R(r_0,r_1),d_R(r_1,r_2)\}$; thus, by (10), we obtain

$$
d_R(r_2,Q(r_1,r_2))^{1-\vartheta_3} \le \zeta \max\{d_R(r_0,r_1),d_R(r_1,r_2)\}^{\vartheta_1+\vartheta_2}. \tag{11}
$$

As $1-\vartheta_3 = \vartheta_1+\vartheta_2$, thus, by (11), we obtain

$$
d_R(r_2,Q(r_1,r_2)) \le \zeta \max\{d_R(r_0,r_1),d_R(r_1,r_2)\}. \tag{12}
$$

The fact $\zeta \in (0,1)$ yields the existence of some $r_3 \in Q(r_1,r_2)$ satisfying the inequality $d_R(r_2,r_3) \le (1/\sqrt{\zeta}) d_R(r_2,Q(r_1,r_2))$. Thus, by the last two inequalities, we obtain

$$
d_R(r_2,r_3) \le \sqrt{\zeta} \max\{d_R(r_0,r_1),d_R(r_1,r_2)\}. \tag{13}
$$

As $r_2 \in Q(r_0,r_1)$, $r_3 \in Q(r_1,r_2)$, and $\min\{\gamma(r_0,r_1),\gamma(r_1,r_2)\} = 1$, by hypothesis (i), we get $\gamma(r_2,r_3) = 1$. Thus, we say that $\min\{\gamma(r_1,r_2),\gamma(r_2,r_3)\} = 1$. Again, by considering (7), we obtain

$$
\begin{aligned}
d_R(r_3,Q(r_2,r_3)) &\le H_R(Q(r_1,r_2),Q(r_2,r_3))^{\min\{\gamma(r_1,r_2),\gamma(r_2,r_3)\}} \\
&\le \zeta \max\{d_R(r_1,r_2),d_R(r_2,r_3)\}^{\vartheta_1} d_R(r_2,Q(r_1,r_2))^{\vartheta_2} d_R(r_3,Q(r_2,r_3))^{\vartheta_3}.
\end{aligned} \tag{14}
$$

That is,

$$
d_R(r_3,Q(r_2,r_3))^{1-\vartheta_3} \le \zeta \max\{d_R(r_1,r_2),d_R(r_2,r_3)\}^{\vartheta_1+\vartheta_2}. \tag{15}
$$

As $1-\vartheta_3 = \vartheta_1+\vartheta_2$, thus, by (15), we obtain

$$
d_R(r_3,Q(r_2,r_3)) \le \zeta \max\{d_R(r_1,r_2),d_R(r_2,r_3)\}. \tag{16}
$$

As $\zeta \in (0,1)$, then there is some $r_4 \in Q(r_2,r_3)$ such that $d_R(r_3,r_4) \le (1/\sqrt{\zeta}) d_R(r_3,Q(r_2,r_3))$. Thus, we obtain

$$
d_R(r_3,r_4) \le \sqrt{\zeta} \max\{d_R(r_1,r_2),d_R(r_2,r_3)\}. \tag{17}
$$

Continuing in that way, we reach to a sequence $\{r_m\}$ with the facts $r_{m+1} \in Q(r_{m-1},r_m)$ for all $m \in \mathbb{N}$ and

$$
\min\{\gamma(r_{m-1},r_m),\gamma(r_m,r_{m+1})\} = 1, \ \forall m \in \mathbb{N}, \tag{18}
$$

and

$$
d_R(r_{m+1},r_{m+2}) \le \sqrt{\zeta} \max\{d_R(r_{m-1},r_m),d_R(r_m,r_{m+1})\}, \ \forall m \in \mathbb{N}. \tag{19}
$$

For simplicity, we use $d_{R_m} = d_R(r_m,r_{m+1})$ for each $m \in \mathbb{N}\cup\{0\}$. We will show with induction that $d_{R_{m-1}} \le \beta^m M$ for each $m \in \mathbb{N}$, where $\beta = \zeta^{1/4}$ and $M = \max\{d_{R_0}/\beta, d_{R_1}/\beta^2\}$. Trivially, $d_{R_0} \le \beta M$ and $d_{R_1} \le \beta^2 M$. Suppose that

$$
\begin{aligned}
&d_{R_{k-3}} \le \beta^{k-2} M, \\
&d_{R_{k-2}} \le \beta^{k-1} M \ \forall k \ge 3.
\end{aligned} \tag{20}
$$

Then,

$$
\begin{aligned}
d_{R_{k-1}} &\le \beta^2 \max\{d_{R_{k-3}}, d_{R_{k-2}}\} \le \beta^2 \max\{\beta^{k-2}M, \beta^{k-1}M\} \\
&= \beta^k M, \quad \text{for each } k \ge 3.
\end{aligned} \tag{21}
$$

Thus, $d_{R_{m-1}} \le \beta^m M$, for each $m \in \mathbb{N}$. Now, by considering this fact and the triangle inequality, for each $q,n \in \mathbb{N}$ with $q > n$, we obtain

$$
d_R(r_n,r_q) \le \sum_{j=n}^{q-1} d_R(r_j,r_{j+1}) = \sum_{j=n}^{q-1} d_{R_j} \le \sum_{j=n}^{q-1} \beta^{j+1} M. \tag{22}
$$

Hence, the convergence of $\sum_{j=1}^{\infty} \beta^j$, as $\beta \in (0,1)$, and the above inequality yields that $\{r_m\}$ is a Cauchy sequence in R. Now, the completeness of (R,d_R) yields the existence of a point $r^* \in R$ such that $r_m \longrightarrow r^*$. By hypothesis (iii), we get $\gamma(r_m,r^*) = 1$, as $\gamma(r_m,r_{m+1}) = 1$, $\forall m \in \mathbb{N}$ and $r_m \longrightarrow r^*$. Now, we claim $r^* \in Q(r^*,r^*)$. If it is wrong, then, by (7), for each $m \in \mathbb{N}$, we obtain

$$\begin{aligned}H_R(Q(r_m,r_{m+1}),Q(r^*,r^*)) &= H_R(Q(r_m,r_{m+1}),Q(r^*,r^*))^{\min\{\gamma(r_m,r^*),\gamma(r_{m+1},r^*)\}}\\ &\leq \zeta \max\{d_R(r_m,r^*),d_R(r_{m+1},r^*)\}^{\vartheta_1} d_R(r_{m+1},Q(r_m,r_{m+1}))^{\vartheta_2} d_R(r^*,Q(r^*,r^*))^{\vartheta_3}.\end{aligned} \tag{23}$$

That is,

$$\begin{aligned}d_R(r_{m+2},Q(r^*,r^*)) &\leq H_R(Q(r_m,r_{m+1}),Q(r^*,r^*))\\ &\leq \zeta \max\{d_R(r_m,r^*),d_R(r_{m+1},r^*)\}^{\vartheta_1} d_R(r_{m+1},Q(r_m,r_{m+1}))^{\vartheta_2} d_R(r^*,Q(r^*,r^*))^{\vartheta_3}.\end{aligned} \tag{24}$$

By triangle inequality and (24), we obtain

$$\begin{aligned}d_R(r^*,Q(r^*,r^*)) &\leq d_R(r^*,r_{m+2}) + d_R(r_{m+2},Q(r^*,r^*))\\ &\leq d_R(r^*,r_{m+2}) + \zeta \max\{d_R(r_m,r^*),d_R(r_{m+1},r^*)\}^{\vartheta_1} \times d_R(r_{m+1},Q(r_m,r_{m+1}))^{\vartheta_2} d_R(r^*,Q(r^*,r^*))^{\vartheta_3}.\end{aligned} \tag{25}$$

Hence, by applying the limit as $m \longrightarrow \infty$ in (25), we get $d_R(r^*,Q(r^*,r^*)) = 0$. This shows that the claim is true and $r^* \in Q(r^*,r^*)$. □

Example 1. Let R denote the set of all real numbers with a usual metric $d_R(r,l) = |r-l|$ for each $r,l \in R$. Define maps Q: $R \times R \longrightarrow \mathrm{CB}(R)$ and γ: $R \times R \longrightarrow \mathbb{R}\backslash\{0\}$ by

$$Q(r,l) = \begin{cases} \left[0, \dfrac{r+l}{2}\right], & \text{if } r,l \geq 0,\\ 0, & \text{otherwise},\end{cases} \qquad \gamma(r,l) = \begin{cases} 1, & \text{if } r,l \geq 0,\\ 1/4, & \text{otherwise}.\end{cases} \tag{26}$$

The hypotheses of Theorem 3 can be verified on the above defined maps. Hence, there exists an element r of R with $r \in Q(r,r)$.

We now present an interpolative Prešić type-II set-valued contraction map along with fixed-point result.

Definition 2. A map Q: $R \times R \longrightarrow \mathrm{CB}(R)$ is called an interpolative Prešić type-II set-valued contraction if, for each $w_1, w_2, p_1, p_2 \in R\backslash \mathrm{Fix}(Q)$ with $\min\{\gamma(w_1,p_1), \gamma(w_2,p_2)\} \geq 1$, we obtain

$$\begin{aligned}&H_R(Q(w_1,w_2),Q(p_1,p_2))\\ &\leq \zeta \max\{d_R(w_1,p_1),d_R(w_2,p_2)\}^{\vartheta_1} d_R(w_2,Q(w_1,w_2))^{\vartheta_2} d_R(p_2,Q(p_1,p_2))^{\vartheta_3},\end{aligned} \tag{27}$$

where γ: $R \times R \longrightarrow \mathbb{R}$ is a map, $\vartheta_1,\vartheta_2,\vartheta_3 \in (0,1)$ with $\vartheta_1+\vartheta_2+\vartheta_3 = 1$, $\zeta \in (0,1)$, and $\mathrm{Fix}(Q) = \{r \in R: r \in Q(r,r)\}$.

Theorem 4. *Let Q: $R \times R \longrightarrow CB(R)$ be an interpolative Prešić type-II set-valued contraction map on a complete metric space (R,d_R). Also, consider that*

(i) If $\min\{\gamma(w_1,p_1),\gamma(w_2,p_2)\} \geq 1$, then $\gamma(z_1,z_2) \geq 1$, for all $z_1 \in Q(w_1,w_2)$ and $z_2 \in Q(p_1,p_2)$

(ii) There exist $w_1,w_2 \in R$ with $\min\{\gamma(w_1,w_2), \gamma(w_2,z_1)\} \geq 1$ for all $z_1 \in Q(w_1,w_2)$

(iii) For each sequence $\{r_m\}$ in R with $\gamma(r_m,r_{m+1}) \geq 1$, $\forall m \geq m_0$, for some natural number m_0, and $r_m \longrightarrow r$, we have $\gamma(r_m,r) \geq 1$, $\forall m \geq m_0$

Then, there exists an element r of R with $r \in Q(r,r)$.

Proof. Hypothesis (ii) makes sure the existence of two points in R, say r_0 and r_1, that satisfies the following:

$$\min\{\gamma(r_0,r_1),\gamma(r_1,z)\}\geq 1,\ \forall z\in Q(r_0,r_1). \tag{28}$$

By defining one value of $z\in Q(r_0,r_1)$ as $z=r_2$ in the above inequality, we reach to

$$\min\{\gamma(r_0,r_1),\gamma(r_1,r_2)\}\geq 1. \tag{29}$$

Thus, by (27), we obtain

$$\begin{aligned} d_R(r_2,Q(r_1,r_2)) &\leq H_R(Q(r_0,r_1),Q(r_1,r_2))\\ &\leq \zeta\max\{d_R(r_0,r_1),d_R(r_1,r_2)\}^{\vartheta_1}d_R(r_1,Q(r_0,r_1))^{\vartheta_2}d_R(r_2,Q(r_1,r_2))^{\vartheta_3}. \end{aligned} \tag{30}$$

That is,

$$\begin{aligned} d_R(r_2,Q(r_1,r_2))^{1-\vartheta_3} &\leq \zeta\max\{d_R(r_0,r_1),d_R(r_1,r_2)\}^{\vartheta_1}d_R(r_1,Q(r_0,r_1))^{\vartheta_2}\\ &\leq \zeta\max\{d_R(r_0,r_1),d_R(r_1,r_2)\}^{\vartheta_1+\vartheta_2}. \end{aligned} \tag{31}$$

Since $1-\vartheta_3=\vartheta_1+\vartheta_2$, thus, by (31), we obtain

$$d_R(r_2,Q(r_1,r_2))\leq \zeta\max\{d_R(r_0,r_1),d_R(r_1,r_2)\}. \tag{32}$$

From the above inequality and by the fact $(1/\sqrt{\zeta})>1$, there exists some $r_3\in Q(r_1,r_2)$ such that

$$\begin{aligned} d_R(r_2,r_3) &\leq \frac{1}{\sqrt{\zeta}}d_R(r_2,Q(r_1,r_2))\\ &\leq \sqrt{\zeta}\max\{d_R(r_0,r_1),d_R(r_1,r_2)\}. \end{aligned} \tag{33}$$

Since $\min\{\gamma(r_0,r_1),\gamma(r_1,r_2)\}\geq 1$ and $r_2\in Q(r_0,r_1)$, $r_3\in Q(r_1,r_2)$, by hypothesis (i), we get $\gamma(r_2,r_3)\geq 1$. By proceeding the proof on the above steps, we reach to a sequence $\{r_m\}$ of the form $r_{m+1}\in Q(r_{m-1},r_m)$ for all $m\in\mathbb{N}$ and

$$\min\{\gamma(r_{m-1},r_m),\gamma(r_m,r_{m+1})\}\geq 1,\ \forall m\in\mathbb{N}, \tag{34}$$

and

$$d_R(r_{m+1},r_{m+2})\leq\sqrt{\zeta}\max\{d_R(r_{m-1},r_m),d_R(r_m,r_{m+1})\},\ \forall m\in\mathbb{N}. \tag{35}$$

By viewing the above inequality and the proof of the above theorem, we conclude that $\{r_m\}$ is a Cauchy sequence in R, and there exists a point $r^*\in R$ with $r_m\longrightarrow r^*$. From hypothesis (iii), we have $\gamma(r_m,r^*)\geq 1$ for each $m\in\mathbb{N}$. This implies $\min\{\gamma(r_m,r^*),\gamma(r_{m+1},r^*)\}\geq 1,\ \forall m\in\mathbb{N}$. Suppose that $r^*\notin Q(r^*,r^*)$. Then, by (27), for each $m\in\mathbb{N}$, we obtain

$$\begin{aligned} d_R(r_{m+2},Q(r^*,r^*)) &\leq H_R(Q(r_m,r_{m+1}),Q(r^*,r^*))\\ &\leq \zeta\max\{d_R(r_m,r^*),d_R(r_{m+1},r^*)\}^{\vartheta_1}d_R(r_{m+1},Q(r_m,r_{m+1}))^{\vartheta_2}d_R(r^*,Q(r^*,r^*))^{\vartheta_3}. \end{aligned} \tag{36}$$

By triangle inequality and (36), we obtain

$$\begin{aligned} d_R(r^*,Q(r^*,r^*)) &\leq d_R(r^*,r_{m+2})+d_R(r_{m+2},Q(r^*,r^*))\\ &\leq d_R(r^*,r_{m+2})+\zeta\max\{d_R(r_m,r^*),d_R(r_{m+1},r^*)\}^{\vartheta_1}\times d_R(r_{m+1},Q(r_m,r_{m+1}))^{\vartheta_2}d_R(r^*,Q(r^*,r^*))^{\vartheta_3}. \end{aligned} \tag{37}$$

Thus, by taking the limit $m\longrightarrow\infty$ in (37), we get $d_R(r^*,Q(r^*,r^*))=0$. This shows that the supposition is wrong and $r^*\in Q(r^*,r^*)$. □

2.1. Results for Extended Interpolative Prešić Type Set-Valued Operators. This section presents the extensions of the above listed results. Theorems 5 and 6 can be considered as an extended version of Theorems 3 and 4, respectively.

Theorem 5. *Let* $Q: R^k \longrightarrow CB(R)$, *for any fixed* $k \in \mathbb{N}$, *be an extended interpolative Prešić type-I set-valued contraction map on a complete metric space* (R, d_R), *that is, for every* $w_1, w_2, \ldots, w_k,\ p_1, p_2, \ldots, p_k \in R \backslash Fix(Q)$, *we have*

$$\begin{aligned} &H_R(Q(w_1, w_2, \ldots, w_k), Q(p_1, p_2, \ldots, p_k))^{\min\{\gamma(w_1,p_1),\gamma(w_2,p_2),\ldots,\gamma(w_k,p_k)\}} \\ &\leq \zeta \max\{d_R(w_i, p_i): i \in \{1,2,\ldots,k\}\}^{\vartheta_1} \\ &\quad \times d_R(w_k, Q(w_1, w_2, \ldots, w_k))^{\vartheta_2} d_R(p_k, Q(p_1, p_2, \ldots, p_k))^{\vartheta_3}, \end{aligned} \tag{38}$$

where $\gamma: R \times R \longrightarrow \mathbb{R}\backslash\{0\}$ *is a map,* $\vartheta_1, \vartheta_2, \vartheta_3 \in (0,1)$ *with* $\vartheta_1 + \vartheta_2 + \vartheta_3 = 1$, $\zeta \in (0,1)$, *and* $Fix(Q) = \{r \in R: r \in Q(r, r, \ldots, r)\}$. *Also, consider that*

(i) If $\min\{\gamma(w_1,p_1),\gamma(w_2,p_2),\ldots,\gamma(w_k,p_k)\} = 1$, *then* $\gamma(z_1, z_2) = 1$ *for all* $z_1 \in Q(w_1, w_2, \ldots, w_k)$, $z_2 \in Q(p_1, p_2, \ldots, p_k)$.

(ii) There are $w_1, w_2, \ldots, w_k \in R$ *satisfying*

$$\begin{aligned} &\min\{\gamma(w_1,w_2),\gamma(w_2,w_3),\ldots,\gamma(w_k,z)\} \\ &= 1, \forall z \in Q(w_1, w_2, \ldots, w_k). \end{aligned} \tag{39}$$

(iii) For each sequence $\{r_m\}$ *in* R *with* $\gamma(r_m, r_{m+1}) = 1$, $\forall m \geq m_0$, *for some natural number* m_0, *and* $r_m \longrightarrow r$, *we have* $\gamma(r_m, r) = 1$, $\forall m \geq m_0$.

Then, there exists an element r of R with $r \in Q(\underbrace{r, r, \ldots, r}_{k\text{-times}})$.

Proof. Hypothesis (ii) says that there are points $r_1, r_2, \ldots, r_k$ in R satisfying the condition:

$$\begin{aligned} &\min\{\gamma(r_1,r_2),\gamma(r_2,r_3),\ldots,\gamma(r_k,z)\} \\ &= 1, \forall z \in Q(r_1, r_2, \ldots, r_k). \end{aligned} \tag{40}$$

Thus, for $r_{k+1} \in Q(r_1, r_2, \ldots, r_k)$, we obtain

$$\min\{\gamma(r_1,r_2),\gamma(r_2,r_3),\ldots,\gamma(r_k,r_{k+1})\} = 1. \tag{41}$$

Then, by (38), we obtain

$$\begin{aligned} &d_R(r_{k+1}, Q(r_2, r_3, \ldots, r_{k+1})) \leq H_R(Q(r_1, r_2, \ldots, r_k), Q(r_2, r_3, \ldots, r_{k+1}))^{\min\{\gamma(r_1,r_2),\gamma(r_2,r_3),\ldots,\gamma(r_k,r_{k+1})\}} \\ &\leq \zeta \max\{d_R(r_i, r_{i+1}): i \in \{1,2,\ldots,k\}\}^{\vartheta_1} \times d_R(r_k, Q(r_1, r_2, \ldots, r_k))^{\vartheta_2} d_R(r_{k+1}, Q(r_2, r_3, \ldots, r_{k+1}))^{\vartheta_3}. \end{aligned} \tag{42}$$

That is,

$$\begin{aligned} &d_R(r_{k+1}, Q(r_2, r_3, \ldots, r_{k+1}))^{1-\vartheta_3} \\ &\leq \zeta \max\{d_R(r_i, r_{i+1}): i \in \{1,2,\ldots,k\}\}^{\vartheta_1} d_R(r_k, Q(r_1, r_2, \ldots, r_k))^{\vartheta_2}. \end{aligned} \tag{43}$$

Since $d_R(r_k, Q(r_1, r_2, \ldots, r_k)) \leq \max\{d_R(r_i, r_{i+1}): i \in \{1,2,\ldots,k\}\}$, thus, by (43), we obtain

$$\begin{aligned} &d_R(r_{k+1}, Q(r_2, r_3, \ldots, r_{k+1}))^{1-\vartheta_3} \\ &\leq \zeta \max\{d_R(r_i, r_{i+1}): i \in \{1,2,\ldots,k\}\}^{\vartheta_1+\vartheta_2}. \end{aligned} \tag{44}$$

Since $1 - \vartheta_3 = \vartheta_1 + \vartheta_2$, then (44) gives

$$\begin{aligned} &d_R(r_{k+1}, Q(r_2, r_3, \ldots, r_{k+1})) \\ &\leq \zeta \max\{d_R(r_i, r_{i+1}): i \in \{1,2,\ldots,k\}\}. \end{aligned} \tag{45}$$

As $(1/\sqrt{\zeta}) > 1$, thus, there exists $r_{k+2} \in Q(r_2, r_3, \ldots, r_{k+1})$ of a form

$$\begin{aligned} &d_R(r_{k+1}, r_{k+2}) \leq \frac{1}{\sqrt{\zeta}} d_R(r_{k+1}, Q(r_2, r_3, \ldots, r_{k+1})) \\ &\leq \sqrt{\zeta} \max\{d_R(r_i, r_{i+1}): i \in \{1,2,\ldots,k\}\}. \end{aligned} \tag{46}$$

Hypothesis (i) implies that $\gamma(r_{k+1}, r_{k+2}) = 1$, since $\min\{\gamma(r_1,r_2),\gamma(r_2,r_3),\ldots,\gamma(r_k,r_{k+1})\} = 1$ and $r_{k+1} \in Q(r_1, r_2, \ldots, r_k)$, $r_{k+2} \in Q(r_2, r_3, \ldots, r_{k+1})$. By the repeated application of hypothesis (i) and (38), we reach to a sequence $\{r_m\}$ with the facts $r_{m+k} \in Q(r_m, r_{m+1}, \ldots, r_{m+k-1})$ for all $m \in \mathbb{N}$ and

$$\begin{aligned} &\min\{\gamma(r_m,r_{m+1}),\gamma(r_{m+1},r_{m+2}),\ldots,\gamma(r_{m+k-1},r_{m+k})\} \\ &= 1, \forall m \in \mathbb{N}, \end{aligned} \tag{47}$$

and

$$d_R(r_{m+k}, r_{m+k+1}) \leq \sqrt{\zeta} \max\{d_R(r_{m-1+i}, r_{m+i}) : i \in \{1, 2, \ldots, k\}\}, \forall m \in \mathbb{N}. \tag{48}$$

For simplicity, take $d_{R_m} = d_R(r_m, r_{m+1})$ for each $m \in \mathbb{N}$; from (48), we obtain

$$d_{R_{m+k}} \leq \sqrt{\zeta} \max\left\{d_{R_{m-1+i}} : i \in \{1, 2, \ldots, k\}\right\}, \forall m \in \mathbb{N}. \tag{49}$$

Now, we prove by induction that $d_{R_m} \leq \beta^m M$ for each $m \in \mathbb{N}$, where $\beta = \zeta^{1/2k}$ and $M = \max\left\{d_{R_1}/\beta, d_{R_2}/\beta^2, \ldots, d_{R_k}/\beta^k\right\}$. Trivially, $d_{R_i} \leq \beta^i M$, for each $i \in \{1, 2, \ldots, k\}$. Suppose that $d_{R_i} \leq \beta^i M$ for each $i \in \{m, m+1, \ldots, m+k-1\}$ for some given m, as induction hypothesis. Then, by (49), we obtain

$$\begin{aligned} d_{R_{m+k}} &\leq \sqrt{\zeta} \max\left\{d_{R_{m-1+i}} : i \in \{1, 2, \ldots, k\}\right\} \\ &\leq \sqrt{\zeta} \max\left\{\beta^{m-1+i} M : i \in \{1, 2, \ldots, k\}\right\} \\ &= \beta^k \beta^m M \\ &= \beta^{m+k} M. \end{aligned} \tag{50}$$

Hence, it is shown by induction that $d_{R_m} \leq \beta^m M$, for each $m \in \mathbb{N}$. This fact along with triangle inequality yield that

$$d_R(r_n, r_q) \leq \sum_{j=n}^{q-1} d_R(r_j, r_{j+1}) = \sum_{j=n}^{q-1} d_{R_j} \leq \sum_{j=n}^{q-1} \beta^j M, \tag{51}$$

for each $q, n \in \mathbb{N}$ with $q > n$. Hence, the above inequality and the convergence of $\sum_{j=1}^{\infty} \beta^j$ ensure that $\{r_m\}$ is a Cauchy sequence in R. Now, the completeness of (R, d_R) yields the existence of a point $r^* \in R$ with $r_m \longrightarrow r^*$. By hypothesis (iii), we get $\gamma(r_m, r^*) = 1, \forall m \in \mathbb{N}$, as $\gamma(r_m, r_{m+1}) = 1, \forall m \in \mathbb{N}$ and $r_m \longrightarrow r^*$. Now, we claim that $r^* \in Q(r^*, r^*, \ldots, r^*)$. Suppose it is wrong, then, by (38), for each $m \in \mathbb{N}$, we obtain

$$\begin{aligned} &H_R(Q(r_m, r_{m+1}, \ldots, r_{m+k-1}), Q(r^*, r^*, \ldots, r^*)) \\ &= H_R(Q(r_m, r_{m+1}, \ldots, r_{m+k-1}), Q(r^*, r^*, \ldots, r^*))^{\min\{\gamma(r_m, r^*), \gamma(r_{m+1}, r^*), \ldots, \gamma(r_{m+k-1}, r^*)\}} \\ &\leq \zeta \max\{d_R(r_{m+i-1}, r^*) : i \in \{1, 2, \ldots, k\}\}^{\vartheta_1} \\ &\quad \times d_R(r_{m+k-1}, Q(r_m, r_{m+1}, \ldots, r_{m+k-1}))^{\vartheta_2} d_R(r^*, Q(r^*, r^*, \ldots, r^*))^{\vartheta_3}. \end{aligned} \tag{52}$$

That is,

$$\begin{aligned} d_R(r_{m+k}, Q(r^*, r^*, \ldots, r^*)) &\leq H_R(Q(r_m, r_{m+1}, \ldots, r_{m+k-1}), Q(r^*, r^*, \ldots, r^*)) \\ &\leq \zeta \max\{d_R(r_{m+i-1}, r^*) : i \in \{1, 2, \ldots, k\}\}^{\vartheta_1} \\ &\quad \times d_R(r_{m+k-1}, Q(r_m, r_{m+1}, \ldots, r_{m+k-1}))^{\vartheta_2} d_R(r^*, Q(r^*, r^*, \ldots, r^*))^{\vartheta_3}. \end{aligned} \tag{53}$$

By triangle inequality and (53), we obtain

$$\begin{aligned} d_R(r^*, Q(r^*, r^*, \ldots, r^*)) &\leq d_R(r^*, r_{m+k}) + d_R(r_{m+k}, Q(r^*, r^*, \ldots, r^*)) \\ &\leq d_R(r^*, r_{m+k}) + \zeta \max\{d_R(r_{m+i-1}, r^*) : i \in \{1, 2, \ldots, k\}\}^{\vartheta_1} \\ &\quad \times d_R(r_{m+k-1}, Q(r_m, r_{m+1}, \ldots, r_{m+k-1}))^{\vartheta_2} d_R(r^*, Q(r^*, r^*, \ldots, r^*))^{\vartheta_3}. \end{aligned} \tag{54}$$

After applying the limit as $m \longrightarrow \infty$ in (54), we get $d_R(r^*, Q(r^*, r^*, \ldots, r^*)) = 0$. Hence, the claim is true and $r^* \in Q(r^*, r^*, \ldots, r^*)$. □

Theorem 6. *Let* $Q: R^k \longrightarrow CB(R)$, *for any fixed* $k \in \mathbb{N}$, *be an extended interpolative Prešić type-II set-valued contraction map on a complete metric space* (R, d_R); *that is, for every* $w_1, w_2, \ldots, w_k$, $p_1, p_2, \ldots, p_k \in R \backslash Fix(Q)$ *with* $\min\{\gamma(w_1, p_1), \gamma(w_2, p_2), \ldots, \gamma(w_k, p_k)\} \geq 1$, *we have*

$$\begin{aligned}&H_R(Q(w_1, w_2, \ldots, w_k), Q(p_1, p_2, \ldots, p_k))\\&\leq \zeta \max\{d_R(w_i, p_i): i \in \{1, 2, \ldots, k\}\}^{\vartheta_1}\\&\quad\times d_R(w_k, Q(w_1, w_2, \ldots, w_k))^{\vartheta_2} d_R(p_k, Q(p_1, p_2, \ldots, p_k))^{\vartheta_3},\end{aligned} \tag{55}$$

where $\gamma: R \times R \longrightarrow \mathbb{R}$ *is a map,* $\vartheta_1, \vartheta_2, \vartheta_3 \in (0, 1)$ *with* $\vartheta_1 + \vartheta_2 + \vartheta_3 = 1$, $\zeta \in (0, 1)$, *and* $Fix(Q) = \{r \in R: r \in Q(r, r, \ldots, r)\}$. *Also, consider that*

(i) *If* $\min\{\gamma(w_1, p_1), \gamma(w_2, p_2), \ldots, \gamma(w_k, p_k)\} \geq 1$, *then* $\gamma(z_1, z_2) \geq 1$, *for all* $z_1 \in Q(w_1, w_2, \ldots, w_k)$ *and* $z_2 \in Q(p_1, p_2, \ldots, p_k)$.

(ii) *There are* $w_1, w_2, \ldots, w_k \in R$ *satisfying*

$$\begin{aligned}&\min\{\gamma(w_1, w_2), \gamma(w_2, w_3), \ldots, \gamma(w_k, z)\}\\&\geq 1, \forall z \in Q(w_1, w_2, \ldots, w_k),\end{aligned} \tag{56}$$

(iii) *For each sequence* $\{r_m\}$ *in* R *with* $\gamma(r_m, r_{m+1}) \geq 1$, $\forall m \geq m_0$, *for some natural number* m_0, *and* $r_m \longrightarrow r$, *we have* $\gamma(r_m, r) \geq 1$, $\forall m \geq m_0$.

Then, there exists an element r of R with $r \in Q(\underbrace{r, r, \ldots, r}_{k\text{-times}})$.

This result can be proved on the similar steps as the proofs of Theorems 5 and 4 are obtained. By considering $p_1 = w_2$, $p_2 = w_3$, …, $p_{k-1} = w_k$ and denoting $p_k = w_{k+1}$ in Theorems 5 and 6, we get the following results.

Theorem 7. *Let* $Q: R^k \longrightarrow CB(R)$, *for any fixed* $k \in \mathbb{N}$, *be a set-valued map on a complete metric space* (R, d_R) *such that, for every* $w_1, w_2, \ldots, w_k, w_{k+1} \in R \backslash Fix(Q)$, *we have*

$$\begin{aligned}&H_R(Q(w_1, w_2, \ldots, w_k), Q(w_2, w_3, \ldots, w_{k+1}))^{\min\{\gamma(w_1, w_2), \gamma(w_2, w_3), \ldots, \gamma(w_k, w_{k+1})\}}\\&\leq \zeta \max\{d_R(w_i, w_{i+1}): i \in \{1, 2, \ldots, k\}\}^{\vartheta_1}\\&\quad\times d_R(w_k, Q(w_1, w_2, \ldots, w_k))^{\vartheta_2} d_R(w_{k+1}, Q(w_2, w_3, \ldots, w_{k+1}))^{\vartheta_3},\end{aligned} \tag{57}$$

where $\gamma: R \times R \longrightarrow \mathbb{R} \backslash \{0\}$ *is a map,* $\vartheta_1, \vartheta_2, \vartheta_3 \in (0, 1)$ *with* $\vartheta_1 + \vartheta_2 + \vartheta_3 = 1$, $\zeta \in (0, 1)$, *and* $Fix(Q) = \{r \in R: r \in Q(r, r, \ldots, r)\}$. *Also, consider that*

(i) *If* $\min\{\gamma(w_1, w_2), \gamma(w_2, w_3), \ldots, \gamma(w_k, w_{k+1})\} = 1$, *then* $\gamma(z_1, z_2) = 1$, *for all* $z_1 \in Q(w_1, w_2, \ldots, w_k)$ *and* $z_2 \in Q(w_2, w_3, \ldots, w_{k+1})$.

(ii) *There are* $w_1, w_2, \ldots, w_k \in R$ *satisfying*

$$\begin{aligned}&\min\{\gamma(w_1, w_2), \gamma(w_2, w_3), \ldots, \gamma(w_k, z)\}\\&= 1, \forall z \in Q(w_1, w_2, \ldots, w_k),\end{aligned} \tag{58}$$

(iii) *For each sequence* $\{r_m\}$ *in* R *with* $\gamma(r_m, r_{m+1}) = 1$, $\forall m \geq m_0$, *for some natural number* m_0, *and* $r_m \longrightarrow r$, *we have* $\gamma(r_m, r) = 1$, $\forall m \geq m_0$.

Then, there exists an element r of R with $r \in Q(\underbrace{r, r, \ldots, r}_{k\text{-times}})$.

Theorem 8. *Let* $Q: R^k \longrightarrow CB(R)$, *for any fixed* $k \in \mathbb{N}$, *be a set-valued map on a complete metric space* (R, d_R) *such that, for every* $w_1, w_2, \ldots, w_k, w_{k+1} \in R \backslash Fix(Q)$ *with*

$$\min\{\gamma(w_1, w_2), \gamma(w_2, w_3), \ldots, \gamma(w_k, w_{k+1})\} \geq 1, \tag{59}$$

we have

$$\begin{aligned}&H_R(Q(w_1, w_2, \ldots, w_k), Q(w_2, w_3, \ldots, w_{k+1}))\\&\leq \zeta \max\{d_R(w_i, w_{i+1}): i \in \{1, 2, \ldots, k\}\}^{\vartheta_1}\\&\quad\times d_R(w_k, Q(w_1, w_2, \ldots, w_k))^{\vartheta_2} d_R(w_{k+1}, Q(w_2, w_3, \ldots, w_{k+1}))^{\vartheta_3},\end{aligned} \tag{60}$$

where $\gamma: R \times R \longrightarrow \mathbb{R}$ *is a map,* $\vartheta_1, \vartheta_2, \vartheta_3 \in (0, 1)$ *with* $\vartheta_1 + \vartheta_2 + \vartheta_3 = 1$, $\zeta \in (0, 1)$ *and* $Fix(Q) = \{r \in R: r \in Q(r, r, \ldots, r)\}$. *Also, consider that*

(i) *If* $\min\{\gamma(w_1, w_2), \gamma(w_2, w_3), \ldots, \gamma(w_k, w_{k+1})\} \geq 1$, *then* $\gamma(z_1, z_2) \geq 1$ *for all* $z_1 \in Q(w_1, w_2, \ldots, w_k)$ *and* $z_2 \in Q(w_2, w_3, \ldots, w_{k+1})$.

(ii) *There are* $w_1, w_2, \ldots, w_k \in R$ *satisfying*

$$\begin{aligned}&\min\{\gamma(w_1,w_2),\gamma(w_2,w_3),\ldots,\gamma(w_k,z)\}\\&\geq 1,\ \forall z\in Q(w_1,w_2,\ldots,w_k),\end{aligned} \tag{61}$$

(iii) *For each sequence* $\{r_m\}$ *in* R *with* $\gamma(r_m, r_{m+1}) \geq 1$, $\forall m \geq m_0$, *for some natural number* m_0, *and* $r_m \longrightarrow r$, *we have* $\gamma(r_m, r) \geq 1$, $\forall m \geq m_0$.

Then, there exists an element r of R with $r \in Q(\underbrace{r, r, \ldots, r}_{k\text{-times}})$.

3. Application

In this section, we obtain the following application of the above result through a combination of graph theory. In the following, assume that $G_r = (V_e, E_d)$ be a directed graph defined on a metric space (R, d_R) with vertex set $V_e = R$ and edge set $E_d \subset R \times R$ contains all loops, but it has no parallel edge. From Theorem 8, by defining $\gamma(w, r) = 1$ for each $w, r \in R$ with $(w, r) \in E_d$, for otherwise, $\gamma(w, r) = 0$, we get the following result.

Theorem 9. *Let* $Q: R^k \longrightarrow CB(R)$, *for any fixed* $k \in \mathbb{N}$, *be a set-valued map on a complete metric space* (R, d_R) *equipped with the graph* G_r *such that, for every* $w_1, w_2, \ldots, w_k, w_{k+1} \in R \backslash Fix(Q)$, *with*

$$(w_1,w_2),(w_2,w_3),\ldots,(w_k,w_{k+1})\in E_d, \tag{62}$$

we have

$$\begin{aligned}&H_R(Q(w_1,w_2,\ldots,w_k),Q(w_2,w_3,\ldots,w_{k+1}))\\&\leq\zeta\max\{d_R(w_i,w_{i+1}): i\in\{1,2,\ldots,k\}\}^{\vartheta_1}\\&\quad\times d_R(w_k,Q(w_1,w_2,\ldots,w_k))^{\vartheta_2}d_R(w_{k+1},Q(w_2,w_3,\ldots,w_{k+1}))^{\vartheta_3},\end{aligned} \tag{63}$$

where $\vartheta_1, \vartheta_2, \vartheta_3 \in (0,1)$ *with* $\vartheta_1 + \vartheta_2 + \vartheta_3 = 1$, $\zeta \in (0,1)$ *and* $Fix(Q) = \{r \in R: r \in Q(r, r, \ldots, r)\}$. *Also, consider that*

(i) *For all* $w_1, w_2, \ldots, w_k, w_{k+1} \in R$ *with* $(w_1, w_2), (w_2, w_3), \ldots, (w_k, w_{k+1}) \in E_d$, *we have* $(z_1, z_2) \in E_d$, *for all* $z_1 \in Q(w_1, w_2, \ldots, w_k)$ *and* $z_2 \in Q(w_2, w_3, \ldots, w_{k+1})$.

(ii) *There are* $w_1, w_2, \ldots, w_k \in R$ *with*

$$\begin{aligned}&(w_1,w_2),(w_2,w_3),\ldots,(w_{k-1},w_k),(w_k,z)\\&\in E_d,\ \forall z\in Q(w_1,w_2,\ldots,w_k).\end{aligned} \tag{64}$$

(iii) *For each sequence* $\{r_m\}$ *in* R *with* $(r_m, r_{m+1}) \in E_d$, $\forall m \geq m_0$, *for some natural number* m_0, *and* $r_m \longrightarrow r$, *we have* $(r_m, r) \in E_d$, $\forall m \geq m_0$.

Then, there exists an element r of R with $r \in Q(\underbrace{r, r, \ldots, r}_{k\text{-times}})$.

Sarwar et al. [25] studied the existence of the solution of Caputo–Fabrizio fractional derivative of order γ, which is defined as

$$D_t^{\gamma}u(t)=\frac{N(\gamma)}{1-\gamma}\int_0^t u'(\tau)\exp\left[-\frac{\gamma}{1-\gamma}(t-\tau)\right]\mathrm{d}\tau, \tag{65}$$

under boundary condition $u(0) = 0$, where $N(\gamma)$ is a normalization function satisfying $N(0) = N(1) = 1$ and $a \leq t \leq \tau \leq b$, by using an interpolative Dass and Gupta rational-type contraction condition. Through the work of Sarwar et al. [25], it is obvious that the existence of the solution of above defined Caputo–Fabrizio fractional derivative can also be discussed by an interpolative Kannan contraction that is a particular case of our work.

4. Conclusion

The notions of interpolative Prešić-type set-valued contractions for the set-valued operators defined on product spaces along with fixed-point results are presented. These notions can also be considered as an extended version of interpolative Prešić-type contractions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this article and approved the final manuscript.

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