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*Research Article*

# Belligerent Fuzzy GE-Filters on GE-Algebras

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In this paper, the concept of belligerent fuzzy GE-filter of GE-algebra is introduced. The relationship between a fuzzy GE-filter of GE-algebra and a belligerent fuzzy GE-filter of GE-algebra is given. Further, it is shown that a finite product (union) of belligerent fuzzy GE-filters of GE-algebras is a belligerent fuzzy GE-filter of the finite product (union) of GE-algebras.

## 1. Introduction

In 1966, Diego [1] introduced the concept of Hilbert algebras. Diego discussed several properties of Hilbert algebras and deductive systems. More studies on the ideas of Hilbert algebras and deductive systems were performed by Busneag [2, 3]. Bandaru discussed a new algebraic structure, called GE-algebra [4] as a generalization of Hilbert algebra. Filters, upper sets, and congruence kernels in GE-algebra are considered, and the congruence kernel of transitive GE-algebra is characterized. The concept of belligerent GE-filter in GE-algebra is introduced by Bandaru [5]. Relationships between a belligerent GE-filter and a GE-filter are established, and the necessary and sufficient conditions on which a GE-filter to be a belligerent GE-filter are presented. Further, different properties of the union and the product of GE-algebras are investigated.

Zadeh, in 1965, studied the notion of fuzzy sets [6]. Following this, Rosenfeld in 1971, used this theory and has formulated the concept of a fuzzy subgroup of a group [7]. Since then, many researchers have been studying several algebraic structures including fuzzy subalgebras, fuzzy subgroups, fuzzy ideals, and fuzzy filters. Jun and Hong in [8] studied the concept of a fuzzy deductive system in Hilbert algebra. Akram and Zham [9] studied sensible fuzzy ideals of BCK-algebras with respect to a t-conorm. Akram and Dar [10] investigated $T$-fuzzy ideals in BCI-algebras. Additionally, Jun, Akram, and Pasha [11] presented intuitionistic fuzzy quasi-associative ideals in BCI-algebras. Recently, Alemayehu and EShetie [12] studied Fuzzy ideals and fuzzy filters of GE-algebras.

Motivated by the above results, in this research, the concept of belligerent fuzzy GE-filter of GE-algebra is introduced. The relationship between a fuzzy GE-filter of GE-algebra and a belligerent fuzzy GE-filter of GE-algebra is given. Further, it is shown that a finite product (union) of belligerent fuzzy GE-filters of GE-algebras is a belligerent fuzzy GE-filter of a finite product (union) of GE-algebras.

## 2. Preliminaries

This section deals on definitions and basic results that we use in the sequel such as the concepts on GE-algebras, GE-filters of GE-algebras, belligerent GE-filters, and fuzzy filters on a given set.

*Definition 1* (see [13]). Hilbert algebra is an algebra $(G, *, 1)$ of type $(2, 0)$ such that the following axioms hold, for all $a, b, c \in G$.

(1) $a * (b * a) = 1$,
(2) $(a * (b * c)) * ((a * b) * (a * c)) = 1$,
(3) if $(a * b) = (b * a) = 1$, then $a = b$.

*Definition 2* (see [4]). GE-algebra is a non-empty set $G$ with a constant 1 and a binary operation $*$ satisfying axioms:

(1) $(a * a) = 1$,
(2) $(1 * a) = a$ ,
(3) $a * (b * c) = a * (b * (a * c))$, for all $a, b, c \in G$.

*Definition 3* (see [4]). GE-algebra $(G, *, 1)$ is said to be transitive if it satisfies

$$(a * b) \leq (c * a) * (c * b), \tag{1}$$

for all $a, b, c \in G$.

**Theorem 1** (see [4]). *In a transitive GE-algebra $(G, *, 1)$, for all $a, b, c \in G$, the following conditions hold*

*(1) $a \leq b implies (c * a) \leq (c * b)$*
*(2) $(a * b) \leq (b * c) * (a * c)$*
*(3) $((a * b) * b) * c \leq (a * c)$*
*(4) $a \leq band b \leq cimplies a \leq c$.*

*Definition 4* (see [4]). GE-algebra $(G, *, 1)$ is said to be commutative if it satisfies

$$(a * b) * b = (b * a) * a, \tag{2}$$

for all $a, b \in G$.

**Theorem 2** (see [4]). *Every commutative GE-algebra is a generalized Hilbert algebra.*

*Definition 5* (see [4]). A subset $S$ of GE-algebra $G$ is called a GE-filter of $G$ if it satisfies the following:

(1) $1 \in S$
(2) if $a * b \in S$ and $a \in S$, then $b \in S$.

**Theorem 3** (see [4]). *Let $S$ be a filter of $G$. If $a \leq b$ and $a \in S$, then $b \in S$.*

**Theorem 4** (see [4]). *A non-empty subset $S$ of GE-algebra $G$ is a filter of $G$ if and only if it satisfies $1 \in S$, $a * (b * c) \in S$ and $b \in S$ implies that $(a * c) \in S$ for all $a, b, c \in G$.*

*Definition 6* (see [5]). A subset $S$ of GE-algebra $G$ is called a belligerent GE-filter of $G$ if it satisfies $1 \in S$ and

$$a * (b * c) \in S \text{ and } (a * b) \in S \Rightarrow (a * c) \in S, \tag{3}$$

for all $a, b, c \in G$.

*Definition 7* (see [5]). GE-algebra $G$ is said to be left exchangeable if $a * (b * c) = b * (a * c)$ for all $a, b, c \in G$.

**Theorem 5** (see [5]). *Let $S$ be a GE-filter of a transitive and left exchangeable GE-algebra $G$. If $S$ satisfies the condition, $a * (a * b) \in S$ implies that $(a * b) \in S$ for all $a, b \in G$, then $S$ is a belligerent GE-filter of $G$.*

Recall that, for any set $G$ a function, $\zeta\colon G \longrightarrow ([0,1], \wedge, \vee)$ is called a fuzzy subset of $G$ [6], where $[0,1]$ is a unit interval, $\delta \wedge \nu = \min\{\delta, \nu\}$ and $\delta \vee \nu = \max\{\delta, \nu\}$ for all $\delta, \nu \in [0,1]$. A fuzzy subset $\nu$ of $G$ is proper if it is a nonconstant function. A fuzzy subset $\nu$ such that $\nu(a) = 0$ for all $a \in G$ is an improper fuzzy subset. Let $\zeta\colon G \longrightarrow ([0,1]$. For every $\delta \in [0,1]$, the level subset $\zeta$ of $G$ is $\zeta_\delta = |a \in G | \delta \leq \zeta(a)\}$.

*Definition 8* (see [14, 15]). Let $y \in G$, $0 \langle \alpha \leq 1$. A fuzzy point $y_\alpha$ of $G$ is a fuzzy subset that is defined as

$$y_\alpha(b) = \begin{cases} \alpha, & \text{if } b = y, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

*Definition 9* (see [16]). A fuzzy subset $\zeta$ of a bounded lattice $G$ is said to be a fuzzy ideal of $G$, if for all $a, b \in G$,

(1) $\zeta(0) = 1$
(2) $\zeta(a \wedge b) \geq \zeta(a) \vee \zeta(b)$
(3) $\zeta(a \vee b) \geq \zeta(a) \wedge \zeta(b)$.

In [16], Swamy and Raju discussed that, a fuzzy subset $\zeta$ of a bounded lattice $G$ is a fuzzy ideal if and only if $\zeta(0) = 1$ and $\zeta(a \vee b) = \zeta(a) \wedge \zeta(b)$, for all $a, b \in G$.

**Theorem 6** (see [16]). *Let $\zeta$ be a fuzzy subset of $G$, then $\zeta$ is a fuzzy ideal of $G$ if and only if, for any $\alpha \in [0,1]$, $\zeta_\alpha$ is an ideal of $G$.*

*Definition 10* (see [17]). A fuzzy subset $\zeta$ of a bounded lattice $G$ is said to be a fuzzy filter of $G$, if for all $a, b \in G$,

(1) $\zeta(1) = 1$
(2) $\zeta(a \wedge b) \geq \zeta(a) \vee \zeta(b)$
(3) $\zeta(a \vee b) \geq \zeta(a) \wedge \zeta(b)$.

## 3. Belligerent Fuzzy GE-Filters

In this topic, the concept of belligerent fuzzy GE-filter of GE-algebra is introduced. The relationship between a fuzzy GE-filter of GE-algebra and a belligerent fuzzy GE-filter of GE-algebra is given.

Some concepts of about fuzzy GE-filters of GE-algebras are recalled.

*Definition 11* (see [12]). Let $\zeta$ be a fuzzy subset of a GE-algebra $G$, then $\zeta$ is called a fuzzy GE-filter on $G$ if the following conditions hold for any $x, y \in G$:

(1) $\zeta(1) = 1$,
(2) $\zeta(y) \geq \zeta(x * y) \wedge \zeta(x)$.

**Lemma 1** (see [12]). *A fuzzy GE-filter $\zeta$ on a GE-algebra $G$ satisfies the following if $x \leq y$, then $\zeta(x) \leq \zeta(y)$ for all $x, y \in G$.*

**Theorem 7** (see [12]). *Let $\zeta$ be a fuzzy subset of $G$ such that $\zeta(1) = 1$ and $\zeta(b * c) \geq \zeta(a) \wedge \zeta(a * (b * c))$. Then, $\zeta$ is a fuzzy GE-filter.*

*Definition 12.* Let $G$ be GE-algebra and $\mu$ be a fuzzy subset on $G$, then $\zeta$ is said to be a belligerent fuzzy GE-filter on $G$ if for any $a, b \in G$.

(1) $\zeta(1) = 1$,

(2) $\zeta(a * c) \geq \zeta(a * (b * c)) \wedge \zeta(a * b)$.

*Example 1.* Consider a set $G = \{a, b, c, d, e, f, 1\}$ and define a binary operation $*$ on $G$ as in Table 1.

It is easy to check that $G$ is GE-algebra. Now, define a fuzzy subset $\zeta$ on $G$ by $\zeta(1) = 1, \zeta(a) = \zeta(b) = \zeta(f) = 0.9$, and $\zeta(c) = \zeta(d) = \zeta(e) = 0.8$. It is easy to verify that $\zeta$ is a belligerent fuzzy GE-filter on $G$.

The following results show that the relation between belligerent fuzzy GE-filter and fuzzy GE-filter.

Table 1: A binary operation * defined on a set G.

| * | a | b | c | d | e | f | 1 |
|---|---|---|---|---|---|---|---|
| a | 1 | 1 | c | e | e | 1 | 1 |
| b | a | 1 | d | d | d | f | 1 |
| c | 1 | b | 1 | 1 | 1 | 1 | 1 |
| d | a | 1 | 1 | 1 | 1 | z | 1 |
| e | a | b | 1 | 1 | 1 | 1 | 1 |
| f | a | b | e | d | e | 1 | 1 |
| 1 | a | b | c | d | e | f | 1 |

**Theorem 8.** *$\zeta$ is a belligerent fuzzy GE-filter of GE-algebra $G$ if and only if, $\forall \alpha \in [0, 1]$, $\zeta_\alpha$ is a belligerent GE-filter.*

*Proof.* Suppose $\zeta$ be a belligerent fuzzy GE-filter of GE-algebra $G$, then $\zeta(1) = 1$. This implies $1 \in \zeta_1 \subseteq \zeta_\alpha$, where$\alpha \in [0, 1]$. Let $a, b, c \in G$ such that $a * (b * c) \in \zeta_\alpha$ and $(a * b) \in \zeta_\alpha$, then $\zeta(a * (b * c)) \geq \alpha$ and $\zeta(a * b) \geq \alpha$, so that $\zeta(a * c) \geq \zeta(a * (b * c)) \wedge \zeta(a * b) \geq \alpha$; thus, $(a * c) \in \zeta_\alpha$. Hence $\zeta_\alpha$ is belligerent GE-filter.

Conversely, suppose that $\zeta_\alpha$ be a belligerent GE-filter of GE-algebra $G$. Since $1 \in \zeta_\alpha$ for all $\alpha \in [0, 1]$, $1 \in \zeta_1$. Hence, $\zeta(1) = 1$. Let $a, b, c \in G$ such that $\zeta(a * (b * c)) \wedge \zeta(a * b) = \alpha$ for all $\alpha \in [0, 1]$. Clearly, $\zeta(a * (b * c)) \geq \alpha$ and $\zeta(a * b) \geq \alpha$. So, $a * (b * c) \in \zeta_\alpha$ and $(a * b) \in \zeta_\alpha$ which shows that $(a * b) \in \zeta_\alpha$. Consequently, $\zeta(a * b) \geq \alpha = \zeta(a * (b * c)) \wedge \zeta(a * b)$. Hence $\zeta$ is a belligerent fuzzy GE-filter. □

**Corollary 1.** *$K$ is a belligerent GE-filter of GE-algebra $G$ if and only if $\chi_K$ is a belligerent fuzzy GE-filter.*

**Theorem 9.** *Let $\zeta$ be a belligerent fuzzy GE-filter of GE-algebra $G$, then $\zeta$ is a fuzzy GE-filter.*

*Proof.* Since $\zeta(1) = 1$ and $\zeta$ is a belligerent fuzzy GE-filter of $G$, we have $\zeta(1 * b) \geq \zeta(1 * (a * b)) \wedge \zeta(1 * a)$ which means $\zeta(b) \geq \zeta(a * b) \wedge \zeta(a)$. So, $\zeta$ is a fuzzy GE-filter. But the converse of this theorem may not be true. □

*Example 2.* Consider set $G$ and the binary operation $*$ on $G$ given in the above examples (3) and (5). Define a fuzzy GE-filter $\zeta$ on $G$ by $\zeta(1) = 1, \zeta(b) = 0.8$, $\zeta(a) = \zeta(c) = \zeta(d) = \zeta(e) = \zeta(f) = 0.6$. Since $\zeta(d * f) = \zeta(f) = 0.6 \leq \zeta(d * (c * f)) \wedge \zeta(d * c) = \zeta(d * (1)) \wedge \zeta(1) = \zeta(1) \wedge \zeta(1) = 1 \wedge 1 = 1$. This implies $\mu$ is not a belligerent fuzzy GE-filter on $G$.

Let $G$ be GE-algebra, and $\zeta$ be any fuzzy subset of $G$. The following equation is considered as

$$\zeta((a * b) * (a * c)) \geq \zeta(a * (b * c)) \text{foranya}, b, c \in G. \quad (5)$$

Any fuzzy GE-filter cannot satisfy equation (5), so we look at the following examples:

*Example 3.* Consider the GE-algebra $G$ in Example 1 and fuzzy GE-filters $\zeta$ on $G$ as $\zeta(1) = 1 = \zeta(b)$, $\zeta(a) = \zeta(c) = \zeta(d) = \zeta(e) = \zeta(f) = 0.5$ does not satisfies equation (1), since $\zeta(d * (c * f)) = \zeta(d * 1) = \zeta(1) = 1$ and $\zeta((d * c) * (d * f)) = \zeta(1 * f) = \zeta(f) = 0.5$. Thus $\zeta(d * (c * f)) \geq \zeta((d * c) * (d * f))$.

The next result shows that the condition on which a fuzzy GE-filter to be a belligerent fuzzy GE-filter.

**Theorem 10.** *Let $\zeta$ be a fuzzy GE-filter of GE-algebra $G$ such that $\zeta((a * b) * (a * c)) \geq \zeta(a * (b * c))$, then $\zeta$ is a belligerent fuzzy GE-filter of $G$.*

*Proof.* Since $\zeta$ is a fuzzy GE-filter of $G$, $\zeta(1) = 1$ and $\zeta(a * c) \geq \zeta((a * b) * (a * c)) \wedge \zeta(a * b)$. Also, from the assumption we have $\zeta((a * b) * (a * c)) \wedge \zeta(a * b) \geq \zeta(a * (b * c)) \wedge \zeta(a * b)$ so that $\zeta$ is a belligerent fuzzy GE-filter of $G$. □

**Corollary 2.** *In belligerent GE-algebra, the trivial fuzzy GE-filter is a belligerent fuzzy GE-filter.*

GE-algebra $G$ is said to be left exchangeable and transitive if $a * (b * c) = b * (a * c)$ and $(a * b) \leq (c * a) * (c * b)$ for all $a, b, c \in G$, respectively [4, 5].

**Theorem 11.** *The trivial fuzzy GE-filter of transitive GE-algebra $G$ is a belligerent fuzzy GE-filter.*

*Proof.* Let $\zeta$ be the trivial fuzzy GE-filter on transitive GE-algebra $G$. Let $x, y, z \in G$, then $\zeta(x * z) = 1 = 1 \wedge 1 = \zeta(x * (y * z)) \wedge \zeta(x * y)$. Hence, $\zeta$ is a belligerent fuzzy GE-filter.

The fuzzy subset given in Theorem 7 is not a belligerent fuzzy GE-filter. For if we consider set $G$ and the binary operation $*$ on $G$ given in the above example and define a fuzzy subset $\zeta$ on $G$ by $\zeta(1) = \zeta(f) = 1$, $\zeta(a) = \zeta(b) = \zeta(c) = \zeta(d) = \zeta(f) = 0.7$. Since $\zeta(c * b) = \zeta(b) = 0.7 \leq 1 = 1 \wedge 1 = \zeta(1) \wedge \zeta(c * 1) = \zeta(c * d) \wedge \zeta(c * (d * b))$. Hence, $\zeta$ is not belligerent fuzzy GE-filter on $G$. □

**Theorem 12.** *Let $\zeta$ be a fuzzy GE-filter on $G$ such that $\zeta(a * b) \geq \zeta(a * (a * b))$. If $G$ is transitive and left exchangeable, then $\zeta((a * b) * (a * c)) \geq \zeta(a * (b * c))$.*

*Proof.* Assume that $G$ is transitive and left exchangeable and $\zeta$ be a fuzzy GE-filter on $G$ such that $\zeta(a * b) \geq \zeta(a * (a * b))$. From the transitivity property of $G$, we get $\zeta(b * c) \leq \zeta((a * b) * (a * c))$ and $\zeta(a * (b * c)) \leq \zeta$

$(a*((a*b)*(a*c)))$. From the left exchangeable property of $G$, we get $\zeta(a*((a*b)*(a*c))) = \zeta(a*(a*((a*b)*c)))$. Now, applying the given condition in the theorem and the left exchangeable property of $G$ yields $\zeta(a*(a*((a*b)*c))) \leq \zeta(a*((a*b)*c)) = \zeta((a*b)*(a*c))$. □

**Theorem 13.** *Let $\zeta$ be a fuzzy GE-filter of a transitive and left exchangeable GE-algebra $G$ such that $\zeta(a*b) \geq \zeta(a*(a*b))$, then $\zeta$ is a belligerent fuzzy GE-filter of $G$.*

*Proof.* Suppose the conditions in the theorem hold, then, by Theorem 12 , we get $\zeta((a*b)*(a*c)) \geq \zeta(a*(b*c))$ and, hence, from Theorem 10, we conclude that $\zeta$ is a belligerent fuzzy GE-filter of $G$.

For a point $u$ and a non-empty fuzzy subset $\zeta$ of GE-algebra $G$, define a fuzzy subset $\zeta_u$ on $G$ as

$$\zeta_u(a) = \zeta(u*a) \text{for all } a \in G. \tag{6}$$

□

**Lemma 2.** *If $\zeta$ is a fuzzy GE-filter of GE-algebra, then $\zeta_u(1) = \zeta_u(u) = 1$.*

If $\zeta$ is a fuzzy GE-filter of $G$, then fuzzy subset $\zeta_u$ may not be a fuzzy GE-filter.

*Example 4.* Let $G = \{a, b, c, d, e, f, 1\}$ be a GE-algebra of Example 1. A fuzzy subset $\zeta$ of $G$, such as $\zeta(1) = \zeta(b) = 1$ and $\zeta(a) = \zeta(c) = \zeta(d) = \zeta(e) = \zeta(f) = 0.4$, is a fuzzy GE-filter of $G$ but $\zeta_d$ is not a fuzzy GE-filter. Since $\zeta_d(c*a) = \zeta(d*(c*a)) = \zeta(d*1) = \zeta(1) = 1$, $\zeta_d(c) = \zeta(d*c) = \zeta(1) = 1$, and $\zeta_d(a) = \zeta(d*a) = \zeta(a) = 0.4$. This implies $\zeta_d(c*a) \wedge \zeta_d(c) \rangle \zeta_d(a)$. Thus, $\zeta_d$ is not a fuzzy GE-filter of $G$.

However, we give the conditions that $\zeta_d$ is a fuzzy GE-filter of $G$ in the following result.

**Theorem 14.** *Let $\zeta$ be a belligerent fuzzy GE-filter on $G$, then $\zeta_u$ is a fuzzy GE-filter on $G$.*

*Proof.* Let $u, a, b \in G$. Since $\zeta$ is a belligerent fuzzy GE-filter on $G$, we obtain that $\zeta(u*b) \geq \zeta(u*(a*b)) \wedge \zeta(u*a)$, i.e., $\zeta_u(b) \geq \zeta_u(a*b) \wedge \zeta_u(a)$. So, $\zeta_u$ is a fuzzy GE-filter on $G$.

We suggest the conditions under which a fuzzy GE-filter can be a belligerent fuzzy GE-filter. □

**Theorem 15.** *Let $\zeta$ be a fuzzy subset on $G$, $\zeta(1) = 1$ and for any $u \in G$, $\zeta_u$ be a fuzzy GE-filter on $G$. Then, $\zeta$ is a belligerent fuzzy GE-filter on $G$.*

*Proof.* Since $\zeta_u$ is a fuzzy GE-filter on $G$, for all $a \in G$, we get

$$\zeta_a(c) \geq \zeta_a(b*c) \wedge \zeta_a(b). \tag{7}$$

So that $\zeta(a*c) \geq \zeta(a*(b*c)) \wedge \zeta(a*b)$. Thus, $\zeta$ is a belligerent fuzzy GE-filter on $G$. □

**Corollary 2.** *Let $\zeta$ be a fuzzy GE-filter and $\zeta_u$ be a fuzzy GE-filter of $G$ for every $u \in G$, then $\zeta$ is a belligerent fuzzy GE-filter on $G$.*

**Theorem 16.** *Let $\zeta$ be a belligerent fuzzy GE-filter on $G$, then fuzzy subset $\zeta_u$, where $u \in G$ is the smallest fuzzy GE-filter on $G$ such that $\zeta \subseteq \zeta_u$.*

*Proof.* From Theorem 14, $\zeta_u$ is a fuzzy GE-filter on $G$. For any $u, a \in G$, we obtain that $a \leq (u*a)$ and hence, $\zeta(a) \leq \zeta(u*a) = \zeta_u(a)$. Hence $\zeta \subseteq \zeta_u$. Let $\eta$ be a fuzzy GE-filter on $G$ such that $\zeta \subseteq \eta$ and $\eta(u) = 1$. Hence $\zeta \subseteq \zeta_u$. Now, for any $a \in G$, $\zeta_u(a) = \zeta(u*a) \subseteq \eta(u*a) = \eta(u*a) \wedge \eta(u) \leq \eta(a)$. This implies $\zeta_u$ is the least fuzzy GE-filter of $G$. □

**Theorem 17.** *Let $h: G_1 \longrightarrow G_2$ be a GE-morphism from GE-filters $G_1$ to $G_2$ and $\zeta$ be a belligerent fuzzy GE-filter of $G_2$, then $h^{-1}(\zeta)$ is a belligerent fuzzy GE-filter of $G_1$.*

*Proof.* Suppose the conditions hold. Since $h^{-1}(\zeta)(a) = \zeta(h(a))$ for all $a \in G_1$ and $h$ is a morphism, $h^{-1}(\zeta)(1) = \zeta(h(1)) = \zeta(1)$ and

$$\begin{aligned} h^{-1}(\zeta)(a*b) \wedge h^{-1}(\mu)(a) &= \zeta(h(a*b)) \wedge \zeta(h(a)) \\ &= \zeta(h(a)*h(b)) \wedge \zeta(h(a)) \leq \zeta(h(b)) \\ &= h^{-1}(\zeta)(b). \end{aligned} \tag{8}$$

Hence, $h^{-1}(\zeta)$ is a fuzzy GE-filter of $G_1$. Also,

$$\begin{aligned} h^{-1}(\zeta)(a*c) &= \zeta(h(a*c)) \\ &= \zeta(h(x)*h(z)) \geq \zeta(h(x)*(h(y) \\ &\quad *h(z))) \wedge \zeta(h(x)*h(y)) \\ &= \zeta(h(a)*h(b*c)) \wedge \zeta(h(a*b)) \\ &= \zeta(h(a)*h(b*c)) \wedge \zeta(h(x*y)) \\ &= \zeta(h(x*(y*z))) \wedge \zeta(h(x*y)) \\ &= h^{-1}(\zeta)(a*(b*c)) \wedge h^{-1}(\zeta)(a*b), \end{aligned} \tag{9}$$

so that $h^{-1}(\zeta)$ is a belligerent fuzzy GE-filter of $G_1$. □

## 4. Direct Product of Finite Belligerent Fuzzy GE-Filters of GE-Algebras

In this section, it is discussed that the finite product (union) of belligerent fuzzy GE-filters of GE-algebras becomes a belligerent fuzzy GE-filter of the finite product (union) of GE-algebras. Further the sufficient and necessary conditions on which the finite product of a belligerent GE-filter of a finite product of GE-algebras becomes a belligerent fuzzy GE-filter is given.

*Definition 13.* Let $\zeta_1: G_1 \longrightarrow [0,1]$ and $\zeta_2: G_2 \longrightarrow [0,1]$ be fuzzy subsets of GE-algebras $G_1$ and $G_2$, respectively. Then the direct product of fuzzy subsets $\zeta_1$ and $\zeta_2$ is denoted by $(\zeta_1 \times \zeta_2): (G_1 \times G_2) \longrightarrow [0,1]$ and is defined as

$$(\zeta_1 \times \zeta_2)(a_1, a_2) = \zeta_1(a_1) \wedge \zeta_2(a_2), \tag{10}$$

for any $a_1 \in G_1$ and $a_2 \in G_2$.

*Definition 14.* Let $\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_n$ be $n$ fuzzy subsets of GE-algebras $G_1, G_2, G_3, \ldots, G_n$, respectively. Then the direct product of finite fuzzy subsets of GE-algebras is denoted by $\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n$ and is defined as

$$\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n\colon G_1 \times G_2 \times G_3 \times \cdots \times G_n \longrightarrow [0,1], \tag{11}$$

$\zeta_1 \times \zeta_2 \times \cdots \times \zeta_n(a_1, a_2, a_3, \ldots, a_n) = \zeta_1(a_1) \wedge \zeta_2(a_2) \wedge \ldots \wedge \zeta_n(a_n)$ for any $a_1 \in G_1, a_2 \in G_2, \ldots, a_n \in G_n$.

**Theorem 18.** *Let $\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_n$ be $n$ (belligerent) fuzzy GE-filters of GE-algebras $G_1, G_2, G_3, \ldots, G_n$, respectively. Then $\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n$ is (belligerent) fuzzy GE-filter of $G_1 \times G_2 \times G_3 \times \cdots \times G_n$.*

*Proof.* Suppose that the assumptions hold and let $a_1, b_1, c_1 \in G_1; a_2, b_2, c_2 \in G_2, \ldots, a_n, b_n, c_n \in G_n$. Define a binary operation $\circledast$ on $G_1 \times G_2 \times \cdots \times G_n$ by

$$(a_1, a_2,, \ldots, a_n) \circledast (b_1, b_2, \ldots, b_n) = (a_1 *_1 b_1, a_2 *_2 b_2, \ldots, a_n *_n b_n), \tag{12}$$

where $*_1, *_2, \ldots, *_n$ are the binary operations on $G_1, G_2, \ldots, G_n$, respectively. Since $\zeta_i$ is fuzzy GE-filter for each $i, 1 \leq i \leq n$, it is trivial to show that $(\zeta_1 \times \zeta_2) \times \cdots \times \zeta_n(a_1, a_2, \ldots, a_n) \geq (\zeta_1 \times \zeta_2) \times \cdots \times \zeta_n((b_1, b_2, \ldots, b_n) \circledast (a_1, a_2, \ldots, a_n)) \wedge (\zeta_1 \times \zeta_2) \times \cdots \times \zeta_n(b_1, b_2, \ldots, b_n)$. From the given hypothesis $\zeta_i$ is a belligerent fuzzy GE-filter for each $i$, $1 \leq i \leq n$. So,

$$\begin{aligned} &\zeta_1(a_1 *_1 b_1) \geq \zeta_1(a_1 *_1 (c_1 *_1 b_1)) \wedge \zeta_1(a_1 *_1 c_1),\\ &\zeta_2(a_2 *_2 b_2) \geq \zeta_2(a_2 *_2 (c_2 *_2 b_2)) \wedge \zeta_2(a_2 *_2 c_2),\\ &\quad\vdots\\ &\zeta_n(a_n *_n b_n) \geq \zeta_n(a_n *_n (c_n *_n b_n)) \wedge \zeta_n(a_n *_n c_n). \end{aligned} \tag{13}$$

After some steps it follows that

$$\begin{aligned} &(\zeta_1 \times \zeta_2) \times \cdots \times \zeta_n((a_1, a_2, \ldots, a_n) \circledast (b_1, b_2, \ldots, b_n))\\ &\quad \geq (\zeta_1 \times \zeta_2) \times \cdots \times \zeta_n((a_1, a_2, \ldots, a_n)\\ &\quad \circledast ((c_1, c_2, \ldots, c_n) \circledast (b_1, b_2, \ldots, b_n)))\\ &\quad \wedge (\zeta_1 \times \zeta_2) \times \cdots \times \zeta_n((a_1, a_2, \ldots, a_n) \circledast (c_1, c_2, \ldots, c_n)). \end{aligned} \tag{14}$$

Therefore, $(\zeta_1 \times \zeta_2) \times \cdots \times \zeta_n$ is a belligerent fuzzy GE-filter of $(G_1 \times G_2) \times \cdots G_n$. □

*Example 5.* Let $G_1 = \{1, a, b, c, d\}$ and $G_2 = \{1, a, b, c, d, e\}$ be GE-algebras where the binary operations $*_1$ and $*_2$ are defined by Tables 2 and 3.

Then, $(G_1 \times G_2) = \{(1,1), (1,a), (1,b), (1,c), (1,d), (1,e), (a,1), (a,e), (a,a), (a,b), (a,c), (a,d), (b,1), (b,e), (b,a), (b,b), (b,c), (b,d), (c,1), (c,e), (c,a), (c,b), (c,c), (c,d), (d,1), (d,e), (d,a), (d,b), (d,c), (d,d)\}$ is GE-algebra with point wise operation.

Table 2

| $*_1$ | 1 | $w$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|---|
| 1 | 1 | $w$ | $x$ | $y$ | $z$ |
| $w$ | 1 | 1 | $x$ | 1 | 1 |
| $x$ | 1 | $z$ | 1 | $y$ | $z$ |
| $y$ | 1 | $w$ | $x$ | 1 | $w$ |
| $z$ | 1 | 1 | $x$ | 1 | 1 |

Table 3: A binary operation * 2 defined on a set G2.

| $*_2$ | 1 | $r$ | $w$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|---|---|
| 1 | 1 | $r$ | $w$ | $x$ | $y$ | $z$ |
| $r$ | 1 | 1 | 1 | $x$ | $y$ | $z$ |
| $w$ | 1 | $r$ | 1 | $y$ | $y$ | $y$ |
| $x$ | 1 | 1 | $w$ | 1 | 1 | 1 |
| $y$ | 1 | $r$ | 1 | 1 | 1 | 1 |
| $z$ | 1 | $r$ | 1 | 1 | 1 | 1 |

Now, define $\zeta_1\colon G_1 \longrightarrow [0,1]$ by $\zeta_1(1) = \zeta_1(b) = 1$, $\zeta_1(a) = \zeta_1(c) = \zeta_1(d) = 0.3$ and $\zeta_2\colon G_2 \longrightarrow [0,1]$ by $\zeta_2(1) = \zeta_2(e) = \zeta_2(a) = 1, \zeta_2(b) = \zeta_2(c) = \zeta_2(d) = 0.4$. Clearly, $\zeta_1$ and $\zeta_2$ are fuzzy GE-filters of $G_1$ and $G_2$, respectively. Now, we define $(\zeta_1 \times \zeta_2)\colon (G_1 \times G_2) \longrightarrow [0,1]$ by $(\zeta_1 \times \zeta_2)(1,1) = (\zeta_1 \times \mu_2)(1,e) = (\zeta_1 \times \zeta_2)(1,a) = (\zeta_1 \times \zeta_2)\ (b,1) = (\zeta_1 \times \zeta_2)(b,e) = (\zeta_1 \times \zeta_2)(b,a) = 1$,

$$\begin{aligned} (\zeta_1 \times \zeta_2)(1,b) &= (\zeta_1 \times \zeta_2)(1,c)\\ &= (\zeta_1 \times \zeta_2)(1,d)\\ &= (\zeta_1 \times \zeta_2)(b,b)\\ &= (\zeta_1 \times \zeta_2)(b,c)\\ &= (\zeta_1 \times \zeta_2)(b,d)\\ &= 0.4. \end{aligned} \tag{15}$$

$(\zeta_1 \times \zeta_2)(a,1) = (\zeta_1 \times \zeta_2)\quad (a,e) = (\zeta_1 \times \zeta_2)(a,b) = (\zeta_1 \times \zeta_2)(a,c) = (\zeta_1 \times \zeta_2)(a,d) = (\zeta_1 \times \zeta_2)(c,1) = (\zeta_1 \times \zeta_2)(c,e) = (\zeta_1 \times \zeta_2)(c,b) = (\zeta_1 \times \zeta_2)(c,c) = (\zeta_1 \times \zeta_2)(c,d) = (\zeta_1 \times \zeta_2)(d,1) = (\zeta_1 \times \zeta_2)(d,e) = (\zeta_1 \times \zeta_2)(d,b) = (\zeta_1 \times \zeta_2)(d,c) = (\zeta_1 \times \zeta_2)(d,d) = 0.3$. Then, clearly, $(\zeta_1 \times \zeta_2)$ is a fuzzy GE-filter of $(G_1 \times G_2)$.

**Theorem 19.** *If $\zeta_1$ and $\zeta_2$ are (belligerent) fuzzy GE-filters of $G_1$ and $G_2$, respectively, then the union $(\zeta_1 \cup \zeta_2)$ is a (belligerent) fuzzy GE-filter of $(G_1 \cup G_2)$.*

*Proof.* Clearly, $(\zeta_1 \cup \zeta_2)(1) = \zeta_1(1) \vee \zeta_2(1) = 1 \vee 1 = 1$. Recall that for any $a, b \in (G_1 \cup G_2)$, define a binary operation $*$ on $(G_1 \cup G_2)$ as

$$(a * b) = \begin{cases} a *_1 b, & \text{if } a, b \in G_1,\\ a *_2 b, & \text{if } a, b \in G_2,\\ b & \text{if } a \text{ and } b \text{ are not belong to the same GE} - \text{algebra}. \end{cases}$$

Now,

$$\begin{aligned}(\zeta_1 \cup \zeta_2)(b) &= \zeta_1(b) \vee \zeta_2(b) \geq (\zeta_1(a * b) \wedge \zeta_1(a)) \\ &\quad \vee (\zeta_2(a * b) \wedge \zeta_2(a)) \\ &= (\zeta_1 \vee \zeta_2)(a * b) \wedge (\zeta_1 \vee \zeta_2)(a) \\ &= (\zeta_1 \cup \zeta_2)(a * b) \wedge (\zeta_1 \cup \zeta_2)(a).\end{aligned} \tag{16}$$

Hence, $(\zeta_1 \cup \zeta_2)$ is a fuzzy GE-filter of $(G_1 \cup G_2)$. Next, assume that $\zeta_1$ and $\zeta_2$ be belligerent fuzzy GE-filters of $G_1$ and $G_2$, respectively. Then

$$\begin{aligned}(\zeta_1 \cup \zeta_2)(a * c) &= \zeta_1(a *_1 c) \vee \zeta_2(a *_2 c) \\ &\geq (\zeta_1(a *_1 (b *_1 c)) \wedge \zeta_1(a *_1 b)) \\ &\quad \vee (\zeta_2(a *_2 (b *_2 c)) \wedge \zeta_2(a *_2 b)) \\ &= (\zeta_1 \vee \zeta_2)(a * (b * c)) \wedge (\zeta_1 \vee \zeta_2)(a * b) \\ &= (\zeta_1 \cup \zeta_2)(a * (b * c)) \wedge (\zeta_1 \cup \zeta_2)(a * b),\end{aligned} \tag{17}$$

so that $\zeta_1 \cup \zeta_2$ is a belligerent fuzzy GE-filter of $G_1 \cup G_2$. □

**Theorem 20.** *Let $\zeta_1, \zeta_2, \zeta_2, \ldots, \zeta_n$ be n fuzzy subsets of GE-algebras $G_1, G_2, G_3, \ldots, G_n$, respectively, and lett $\in [0, 1]$, then $(\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)_t = (\zeta_1)_t \times (\zeta_2)_t \times (\zeta_2)_t \times \cdots \times (\zeta_n)_t$.*

*Proof.* Let $t \in [0, 1]$ and for any element

$$\begin{aligned}&(a_1, a_2, a_3, \ldots, a_n) \in (\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n)_t \\ &\Leftrightarrow (\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n)(a_1, a_2, a_3, \ldots, a_n) \geq t \\ &\Leftrightarrow \zeta_1(a_1) \wedge \zeta_2(a_2) \wedge \zeta_3(a_3) \wedge \ldots \wedge \zeta_n(a_n) \geq t \\ &\Leftrightarrow \zeta_1(a_1) \geq t, \zeta_2(a_2) \geq t, \zeta_3(a_3) \geq t, \ldots, \zeta_n(a_n) \geq t \\ &\Leftrightarrow a_1 \in (\zeta_1)_t, a_2 \in (\zeta_2)_t, a_3 \in (\zeta_3)_t, \ldots, a_n \in (\zeta_n)_t \\ &\Leftrightarrow (a_1, a_2, a_3, \ldots, a_n) \in (\zeta_1)_t \times (\zeta_2)_t \times (\zeta_3)_t \times \cdots \times (\zeta_n)_t.\end{aligned} \tag{18}$$

Hence $(\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)_t = (\zeta_1)_t \times (\zeta_2)_t \times (\zeta_2)_t \times \cdots \times (\zeta_n)_t$. □

**Theorem 21.** *Let $\zeta_1, \zeta_2, \zeta_2, \ldots, \zeta_n$ be n fuzzy subsets of GE-algebra $G_1, G_2, G_3, \ldots, G_n$, respectively. Then $\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n$ is a fuzzy GE-filter of a GE-algebra $G_1 \times G_2 \times G_3 \times \cdots \times G_n$ if and only if for any $t \in [0, 1]$, $(\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)_t$ is a GE-filter of a GE-algebra $(G_1 \times G_2) \times G_3 \times \cdots \times G_n$.*

*Proof.* Suppose that $(\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)$ is a fuzzy GE-filter of GE-algebra $(G_1 \times G_2), G_3 \times \cdots \times G_n$. Since $(\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)(1) = (\zeta_1(1) \wedge \zeta_2(1) \wedge \zeta_2(1) \wedge \ldots \wedge \zeta_n)(1) = 1$. This implies $1 \in (\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)_t$.

If $(a * b) \in (\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)_t$ and $a \in (\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)_t$, then $(\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)(a * b) \geq t$, $(\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)(a) \geq t$. This implies $(\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)(a * b) \wedge (\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)(a) \geq t$. This implies $(\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)(b) \geq t$. Thus, $y \in (\mu_1 \times \mu_2 \times \zeta_3 \times \cdots \times \zeta_n)_t$. Thus, $(\zeta_1 \times \zeta_2) \times \zeta_3 \times \cdots \times \zeta_n$ is a GE-filter of $(G_1 \times G_2) \times G_3 \times \cdots \times G_n$.

Conversely, suppose that $(\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n)_t$ is a GE-filter. Clearly $(\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n)(1) = 1$. Let $(\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n)(a * b) \wedge (\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n)(a) = t$. This implies $(\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n)(a * b) \geq t$, $(\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n)(a) \geq t$. This implies $a * b \in (\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)_t$ and $a \in (\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)_t$. Thus, $b \in (\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)_t$. This implies $(\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n)(b) \geq t = (\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n)\ (a * b) \wedge (\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n)(a)$.

Thus, $(\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n)(a * b) \wedge (\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n)(a)$ is a fuzzy GE-filter. □

**Theorem 22.** *Let $\zeta_1, \zeta_2, \zeta_2, \ldots, \zeta_n$ be n fuzzy subsets of GE-algebra $G_1, G_2, G_3, \ldots, G_n$, respectively. Then, $(\zeta_1 \times \zeta_2 \times \zeta_3 \times \cdots \times \zeta_n)$ is a belligerent fuzzy GE-filter of GE-algebra $(G_1 \times G_2) \times G_3 \times \cdots \times G_n$ if and only if for any $t \in [0, 1]$, $(\zeta_1 \times \zeta_2 \times \zeta_2 \times \cdots \times \zeta_n)_t$ is belligerent GE-filter of GE-algebra $(G_1 \times G_2) \times G_3 \times \cdots \times G_n$.*

*Proof.* The proof of it is straight forward by Theorems 8 and 18. □

## 5. Conclusion and Future Work

In this paper, the concept of belligerent fuzzy GE-filters is introduced. We investigate the relationships between fuzzy GE-filter and a belligerent fuzzy GE-filter. Additionally prove that that the finite product (union) of belligerent fuzzy GE-filters of GE-algebras becomes a belligerent fuzzy GE-filter of the finite product (union) of GE-algebras. We hope in the future, we study prominent fuzzy GE-filters, imploring fuzzy GE-filters, soft GE-filter, and Soft GE-ideals of GE-algebras.

## Data Availability

No data were used.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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