Belligerent Fuzzy GE-Filters on GE-Algebras

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1. Introduction

In 1966, Diego [1] introduced the concept of Hilbert algebras. Diego discussed several properties of Hilbert algebras and deductive systems. More studies on the ideas of Hilbert algebras and deductive systems were performed by Busneag [2, 3]. Bandaru discussed a new algebraic structure, called GE-algebra [4] as a generalization of Hilbert algebra. Filters, upper sets, and congruence kernels in GE-algebra are considered, and the congruence kernel of transitive GE-algebra is characterized. The concept of belligerent GE-filter in GE-algebra is introduced by Bandaru [5]. Relationships between a belligerent GE-filter and a GE-filter are established, and the necessary and sufficient conditions on which a GE-filter to be a belligerent GE-filter are presented. Further, different properties of the union and the product of GE-algebras are investigated.


Motivated by the above results, in this research, the concept of belligerent fuzzy GE-filter of GE-algebra is introduced. The relationship between a fuzzy GE-filter of GE-algebra and a belligerent fuzzy GE-filter of GE-algebra is given. Further, it is shown that a finite product (union) of belligerent fuzzy GE-filters of GE-algebras is a belligerent fuzzy GE-filter of the finite product (union) of GE-algebras.

2. Preliminaries

This section deals on definitions and basic results that we use in the sequel such as the concepts on GE-algebras, GE-filters of GE-algebras, belligerent GE-filters, and fuzzy filters on a given set.

Definition 1 (see [13]). Hilbert algebra is an algebra $\langle G, \cdot, 1 \rangle$ of type $(2, 0)$ such that the following axioms hold, for all $a, b, c \in G$.

\begin{align*}
(1) \quad a \cdot (b \cdot a) &= 1, \\
(2) \quad (a \cdot (b \cdot c)) \cdot ((a \cdot b) \cdot (a \cdot c)) &= 1, \\
(3) \quad \text{if } (a \cdot b) = (b \cdot a) = 1, \text{ then } a &= b.
\end{align*}

Definition 2 (see [4]). GE-algebra is a non-empty set $G$ with a constant $1$ and a binary operation $\cdot$ satisfying axioms:
(1) \((a \ast a) = 1\),
(2) \((1 \ast a) = a\),
(3) \((a \ast (b \ast c)) = a \ast ((b \ast (a \ast c)))\), for all \(a, b, c \in G\).

**Definition 3** (see [4]). GE-algebra \((G, \ast, 1)\) is said to be transitive if it satisfies
\[(a \ast b) \leq (c \ast a) \ast (c \ast b),\] for all \(a, b, c \in G\).

**Theorem 1** (see [4]). In a transitive GE-algebra \((G, \ast, 1)\), for all \(a, b, c \in G\), the following conditions hold
\[
\begin{align*}
(1) & \quad a \leq b \implies (c \ast a) \leq (c \ast b) \\
(2) & \quad (a \ast b) \leq (b \ast c) \ast (a \ast c) \\
(3) & \quad ((a \ast b) \ast b) \ast c \leq (a \ast c) \\
(4) & \quad a \leq b \implies c \ast (a \ast b) \leq c \ast (a \ast b).
\end{align*}
\]

**Definition 4** (see [4]). GE-algebra \((G, \ast, 1)\) is said to be commutative if it satisfies
\[(a \ast b) \ast b = (b \ast a) \ast a,\] for all \(a, b \in G\).

**Theorem 2** (see [4]). Every commutative GE-algebra is a generalized Hilbert algebra.

**Definition 5** (see [4]). A subset \(S\) of GE-algebra \(G\) is called a GE-filter of \(G\) if it satisfies the following:
\[
\begin{align*}
(1) & \quad 1 \in S \\
(2) & \quad \text{if } a \ast b \in S \text{ and } a \in S, \text{ then } b \in S.
\end{align*}
\]

**Theorem 3** (see [4]). Let \(S\) be a filter of \(G\). If \(a \leq b\) and \(a \in S\), then \(b \in S\).

**Theorem 4** (see [4]). A non-empty subset \(S\) of GE-algebra \(G\) is a filter of \(G\) if and only if it satisfies \(1 \in S\), \((a \ast (b \ast c)) \in S\) and \((b \ast c) \in S\) implies that \((a \ast c) \in S\) for all \(a, b, c \in G\).

**Definition 6** (see [5]). A subset \(S\) of GE-algebra \(G\) is called a belligerent GE-filter of \(G\) if it satisfies \(1 \in S\) and
\[a \ast (b \ast c) \in S \text{ and } (a \ast b) \in S \implies (a \ast c) \in S,\] for all \(a, b, c \in G\).

**Definition 7** (see [5]). GE-algebra \(G\) is said to be left exchangeable if \(a \ast (b \ast c) = b \ast (a \ast c)\) for all \(a, b, c \in G\).

**Theorem 5** (see [5]). Let \(S\) be a GE-filter of a transitive and left exchangeable GE-algebra \(G\). If \(S\) satisfies the condition, \(a \ast (a \ast b) \in S\) implies that \((a \ast b) \in S\) for all \(a, b \in G\), then \(S\) is a belligerent GE-filter of \(G\).

Recall that, for any set \(G\) a function, \(\zeta: G \longrightarrow ([0, 1], \land, \lor)\) is called a fuzzy subset of \(G\) [6], where \([0, 1]\) is a unit interval, \(\delta \land x = \min(\delta, x)\) and \(\delta \lor x = \max(\delta, x)\) for all \(\delta, x \in [0, 1]\). A fuzzy subset \(\nu\) of \(G\) is proper if it is a nonconstant function. A fuzzy subset \(\nu\) such that \(\nu(a) = 0\) for all \(a \in G\) is an improper fuzzy subset. Let \(\zeta: G \longrightarrow ([0, 1], \land , \lor)\). For every \(\delta \in [0, 1]\), the level subset \(\zeta_{\delta}\) of \(G\) is \(\zeta_{\delta} = \{a \in G | \delta \leq \zeta(a)\}\).

**Definition 8** (see [14, 15]). Let \(y \in G\), \(0(\alpha \leq 1)\). A fuzzy point \(y_{\alpha}\) of \(G\) is a fuzzy subset that is defined as
\[y_{\alpha}(b) = \begin{cases} 
\alpha, & \text{if } b = y, \\
0, & \text{otherwise}.
\end{cases}\]

**Definition 9** (see [16]). A fuzzy subset \(\zeta\) of a bounded lattice \(G\) is said to be a fuzzy ideal of \(G\), if for all \(a, b \in G\),
\[
\begin{align*}
(1) & \quad \zeta(0) = 1 \\
(2) & \quad \zeta(a \ast b) \geq \zeta(a) \lor \zeta(b) \\
(3) & \quad \zeta(a \lor b) \geq \zeta(a) \land \zeta(b).
\end{align*}
\]
In [16], Swamy and Raju discussed that, a fuzzy subset \(\zeta\) of a bounded lattice \(G\) is a fuzzy ideal if and only if \(\zeta(0) = 1\) and \(\zeta(a \lor b) = \zeta(a) \lor \zeta(b)\), for all \(a, b \in G\).

**Theorem 6** (see [16]). Let \(\zeta\) be a fuzzy subset of \(G\), then \(\zeta\) is a fuzzy ideal of \(G\) if and only if, for any \(a \in [0, 1]\), \(\zeta_{a}\) is an ideal of \(G\).

**Definition 10** (see [17]). A fuzzy subset \(\zeta\) of a bounded lattice \(G\) is said to be a fuzzy filter of \(G\), if for all \(a, b \in G\),
\[
\begin{align*}
(1) & \quad \zeta(1) = 1 \\
(2) & \quad \zeta(a \ast b) \geq \zeta(a) \lor \zeta(b) \\
(3) & \quad \zeta(a \lor b) \geq \zeta(a) \land \zeta(b).
\end{align*}
\]

### 3. Belligerent Fuzzy GE-Filters

In this topic, the concept of belligerent fuzzy GE-filter of GE-algebra is introduced. The relationship between a fuzzy GE-filter of GE-algebra and a belligerent fuzzy GE-filter of GE-algebra is given.

Some concepts of about fuzzy GE-filters of GE-algebras are recalled.

**Definition 11** (see [12]). Let \(\zeta\) be a fuzzy subset of a GE-algebra \(G\), then \(\zeta\) is called a fuzzy GE-filter on \(G\) if the following conditions hold for any \(x, y \in G\):
\[
\begin{align*}
(1) & \quad \zeta(1) = 1 \\
(2) & \quad \zeta(y) \geq \zeta(x) \land \zeta(y).
\end{align*}
\]

**Lemma 1** (see [12]). A fuzzy GE-filter \(\zeta\) on a GE-algebra \(G\) satisfies the following if \(x \leq y\), then \(\zeta(x) \leq \zeta(y)\) for all \(x, y \in G\).

**Theorem 7** (see [12]). Let \(\zeta\) be a fuzzy subset of \(G\) such that \(\zeta(1) = 1\) and \(\zeta(b \ast c) \geq \zeta(a) \land \zeta(b \ast (b \ast c))\). Then, \(\zeta\) is a fuzzy GE-filter.
Definition 12. Let G be GE-algebra and μ be a fuzzy subset on G, then ζ is said to be a belligerent fuzzy GE-filler on G if for any a, b ∈ G,

1. ζ(1) = 1,
2. ζ(a ∗ c) ≥ ζ(a ∗ (b ∗ c)) ∧ ζ(a ∗ b).

Example 1. Consider a set G = {a, b, c, d, e, f, 1} and define a binary operation * on G as in Table 1.

It is easy to check that G is GE-algebra. Now, define a fuzzy subset ζ on G by ζ(1) = 1, ζ(a) = ζ(b) = ζ(f) = 0.9, and ζ(c) = ζ(d) = ζ(e) = 0.8. It is easy to verify that ζ is a belligerent fuzzy GE-filler on G.

The following results show that the relation between belligerent fuzzy GE-filler and fuzzy GE-filter.

Theorem 8. ζ is a belligerent fuzzy GE-filter of GE-algebra G if and only if, ∀α ∈ [0, 1], ζα is a belligerent GE-filter.

Proof. Suppose ζ is a belligerent fuzzy GE-filter of GE-algebra G, then ζ(1) = 1. This implies 1 ∈ ζ1 ⊆ ζα, whereα ∈ [0, 1]. Let a, b, c ∈ G such that a ∗ (b ∗ c) ∈ ζα and (a ∗ b) ∈ ζα, then ζ((a ∗ (b ∗ c)) ≥ α and ζ((a ∗ b)) ≥ α, so that ζ(a ∗ c) ≥ ζ(a ∗ (b ∗ c)) ∧ ζ(a ∗ b) ≥ α; thus, (a ∗ c) ∈ ζα. Hence ζα is belligerent GE-filter.

Conversely, suppose that ζα is a belligerent GE-filter of GE-algebra G. Since 1 ∈ ζ1 for all α ∈ [0, 1], 1 ∈ ζa. Hence, ζ(1) = 1. Let a, b, c ∈ G such that ζ((a ∗ (b ∗ c)) ∧ ζ(a ∗ b) = α for all α ∈ [0, 1]. Clearly, ζ((a ∗ (b ∗ c)) ≥ α and ζ((a ∗ b)) ≥ α. So, a ∗ (b ∗ c) ∈ ζα and (a ∗ b) ∈ ζα which shows that (a ∗ b) ∈ ζα. Consequently, ζ(a ∗ b) ≥ α = ζ((a ∗ (b ∗ c)) ∧ ζ(1)) ∧ ζ(a ∗ b). Hence ζ is a belligerent fuzzy GE-filter.

Example 2. Consider set G and the binary operation * on G given in the above examples (3) and (5). Define a fuzzy GE-filter ζ on G by ζ(1) = 1, ζ(b) = 0.8, ζ(a) = ζ(c) = ζ(d) = ζ(e) = ζ(f) = 0.6. Since ζ((d ∗ (c ∗ f)) ∧ ζ((d ∗ (c ∗ f)) = ζ(1) ∧ ζ(1) = 1. This implies μ is not a belligerent fuzzy GE-filter on G.

Let G be GE-algebra, and ζ be any fuzzy subset of G. The following equation is considered as

ζ((a ∗ b) ∗ (a ∗ c)) ≥ ζ((a ∗ (b ∗ c))), for any a, b, c ∈ G. (5)

Any fuzzy GE-filter cannot satisfy equation (5), so we look at the following examples:

Table 1: A binary operation * defined on a set G.

<table>
<thead>
<tr>
<th>*</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
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<th>f</th>
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<td>e</td>
<td>e</td>
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<tr>
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<td>1</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>f</td>
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<td>b</td>
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<td>d</td>
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<tr>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>e</td>
<td>f</td>
</tr>
</tbody>
</table>

Example 3. Consider the GE-algebra G in Example 1 and fuzzy GE-filters ζ on G as ζ(1) = 1 = ζ(b), ζ(a) = ζ(c) = ζ(d) = ζ(e) = ζ(f) = 0.5 does not satisfies equation (1), since ζ((d ∗ (c ∗ f)) = ζ(1) = 1 and ζ((d ∗ c) ∗ (d ∗ f)) = ζ(f) = ζ(1) ≤ 0.5. Thus ζ((d ∗ c) ∗ (d ∗ f)) ≥ ζ((d ∗ (c ∗ f)) = ζ((1) = 0.5). Hence ζ is not a belligerent fuzzy GE-filter.

Theorem 10. Let ζ be a fuzzy GE-filter of GE-algebra G such that ζ(a ∗ (b ∗ c)) ≥ ζ((a ∗ b) ∗ c), then ζ is a belligerent fuzzy GE-filter of G.

Proof. Since ζ is a fuzzy GE-filter of G, ζ(1) = 1 and ζ((a ∗ (b ∗ c)) ≥ ζ((a ∗ b) ∗ c). Also, from the assumption we have ζ((a ∗ b) ∗ (c ∗ a)) ∧ ζ(1) ≥ ζ((a ∗ b) ∗ c) ∧ ζ(a ∗ c) so that ζ is a belligerent fuzzy GE-filter of G.
\( (a \ast ((a \ast b) \ast (a \ast c))) \). From the left exchangeable property of \( G \), we get \( \zeta((a \ast ((a \ast b) \ast (a \ast c))) = \zeta((a \ast (a \ast ((a \ast b) \ast c)) \). Now, applying the given condition in the theorem and the left exchangeable property of \( G \) yields \( \zeta((a \ast (a \ast ((a \ast b) \ast c)) \leq \zeta((a \ast ((a \ast b) \ast c)) = \zeta((a \ast (a \ast b) \ast c)). \)

**Theorem 13.** Let \( \zeta \) be a fuzzy GE-filter of a transitive and left exchangeable GE-algebra \( G \) such that \( \zeta((a \ast b) \geq \zeta((a \ast (a \ast b)), \) then \( \zeta \) is a belligerent fuzzy GE-filter of \( G \).

**Proof.** Suppose the conditions in the theorem hold, then by Theorem 12, we get \( \zeta((a \ast b) \ast (a \ast c)) \geq \zeta((a \ast (b \ast c)) \) and hence, \( \zeta((a \ast (b \ast c)) \geq \zeta((a \ast (b \ast c)) \). However, we give the conditions that \( \zeta((a \ast (b \ast c)) \geq \zeta((a \ast (b \ast c)) \). This implies \( \zeta((a \ast (b \ast c)) \geq \zeta((a \ast (b \ast c)). \)

**Lemma 2.** If \( \zeta \) is a fuzzy GE-filter of GE-algebra, then \( \zeta(1, \zeta(u)) = 1 \) for all \( a \in G \).

\[ \zeta(1, \zeta(u)) = 1 \] for all \( a \in G \).

**Theorem 14.** Let \( \zeta \) be a belligerent fuzzy GE-filter on \( G \), then \( \zeta(u) \) is a fuzzy GE-filter on \( G \).

**Proof.** Let \( u, a, b \in G \). Since \( \zeta \) is a belligerent fuzzy GE-filter on \( G \), we obtain that \( \zeta(u \ast b) \geq \zeta((u \ast (a \ast b)) \ast (u \ast a)), i.e., \zeta(u \ast (a \ast b)) \geq \zeta((u \ast a) \ast (a \ast b)). \) So, \( \zeta(u) \) is a fuzzy GE-filter on \( G \).

We suggest the conditions under which a fuzzy GE-filter can be a belligerent fuzzy GE-filter.

**Theorem 15.** Let \( \zeta \) be a fuzzy subset on \( G, \zeta(1) = 1 \) and for any \( u \in G, \zeta(u) \) is a fuzzy GE-filter on \( G \). Then, \( \zeta \) is a belligerent fuzzy GE-filter on \( G \).

**Proof.** Since \( \zeta(u) \) is a fuzzy GE-filter on \( G \), for all \( a \in G, \) we get

\[ \zeta(a \ast b) \geq \zeta((a \ast b) \ast (a \ast b)). \]

So that \( \zeta(a \ast b) \geq \zeta((a \ast b) \ast (a \ast b)) \). Thus, \( \zeta \) is a belligerent fuzzy GE-filter on \( G \).

**Corollary 2.** Let \( \zeta \) be a fuzzy GE-filter and \( \zeta(u) \) be a fuzzy GE-filter of \( G \), then \( \zeta \) is a belligerent fuzzy GE-filter on \( G \).

**Theorem 16.** Let \( \zeta \) be a belligerent fuzzy GE-filter on \( G \), then fuzzy subset \( \zeta(u) \), where \( u \in G \) is the smallest fuzzy GE-filter on \( G \) such that \( \zeta \subseteq \zeta(u) \).

**Proof.** From Theorem 14, \( \zeta(u) \) is a fuzzy GE-filter on \( G \). For any \( u, a \in G \), we obtain that \( a \leq (u \ast a) \) and hence, \( \zeta(a) \leq \zeta((u \ast a)) \). Hence \( \zeta \subseteq \zeta(u) \).

**Theorem 17.** Let \( h: G_{1} \longrightarrow G_{2} \) be a GE-morphism from GE-filters \( G_{1} \) and \( G_{2} \) be a belligerent fuzzy GE-filter of \( G_{2} \), then \( h^{-1}(\zeta) \) is a belligerent fuzzy GE-filter of \( G_{1} \).

**Proof.** Suppose the conditions hold. Since \( h^{-1}(\zeta)(a) = \zeta((h)(a)) \) for all \( a \in G_{1} \) and \( h \) is a morphism, \( h^{-1}(\zeta)(1) = \zeta((h)(1)) = (\zeta(h)) \) and \( h^{-1}(\zeta)(a \ast b) \leq h^{-1}(\zeta)(a) \ast h^{-1}(\zeta)(b) \).

\[ h^{-1}(\zeta)(a \ast b) \leq h^{-1}(\zeta)(a) \ast h^{-1}(\zeta)(b) \]

so that \( h^{-1}(\zeta) \) is a belligerent fuzzy GE-filter of \( G_{1} \).

**4. Direct Product of Finite Belligerent Fuzzy GE-Filters of GE-Algebras**

In this section, it is discussed that the finite product (union) of belligerent fuzzy GE-filters of GE-algebras becomes a belligerent fuzzy GE-filter of the finite product (union) of GE-algebras. Further the sufficient and necessary conditions on which the finite product of a belligerent GE-filter of a finite product of GE-algebras becomes a belligerent fuzzy GE-filter is given.

**Definition 13.** Let \( \zeta_{1}: G_{1} \longrightarrow [0,1] \) and \( \zeta_{2}: G_{2} \longrightarrow [0,1] \) be fuzzy subsets of GE-algebras \( G_{1} \) and \( G_{2} \), respectively. Then the direct product of fuzzy subsets \( \zeta_{1} \) and \( \zeta_{2} \) is denoted by \( (\zeta_{1} \times \zeta_{2}): (G_{1} \times G_{2}) \longrightarrow [0,1] \) and is defined as

\[ (\zeta_{1} \times \zeta_{2})(a_{1}, a_{2}) = \zeta_{1}(a_{1}) \land \zeta_{2}(a_{2}), \]

for any \( a_{1} \in G_{1} \) and \( a_{2} \in G_{2} \).
**Definition 14.** Let \( \zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_n \) be \( n \) fuzzy subsets of GE-algebras \( G_1, G_2, G_3, \ldots, G_n \), respectively. Then the direct product of fuzzy subsets of GE-algebras is denoted by 
\[
\zeta_1 \times \zeta_2 \times \ldots \times \zeta_n \colon G_1 \times G_2 \times G_3 \times \ldots \times G_n \rightarrow [0,1],
\]
and is defined as 
\[
\zeta_1 \times \zeta_2 \times \ldots \times \zeta_n(a_1, a_2, a_3, \ldots, a_n) = \zeta_1(a_1) \land \zeta_2(a_2) \land \ldots \land \zeta_n(a_n) \text{ for any } a_i \in G_1, a_2 \in G_2, \ldots, a_n \in G_n.
\]

(11)

**Theorem 18.** Let \( \zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_n \) be \( n \) (bellergent) fuzzy GE-filters of GE-algebras \( G_1, G_2, G_3, \ldots, G_n \), respectively. Then \( \zeta_1 \times \zeta_2 \times \ldots \times \zeta_n \) is (bellergent) fuzzy GE-filter of \( G_1 \times G_2 \times G_3 \times \ldots \times G_n \).

**Proof.** Suppose that the hypotheses hold and let 
\[
a_1, b_1, c_1 \in G_1; a_2, b_2, c_2 \in G_2, \ldots, a_n, b_n, c_n \in G_n.
\]
Define a binary operation \( \ast \) on \( G_1 \times G_2 \times \ldots \times G_n \) by 
\[
(a_1, a_2, \ldots, a_n) \ast (b_1, b_2, \ldots, b_n) = (a_1 \ast b_1, a_2 \ast b_2, \ldots, a_n \ast b_n),
\]
where \( \ast, \ast_2, \ldots, \ast_n \) are the binary operations on \( G_1, G_2, \ldots, G_n \), respectively. Since \( \zeta_i \) is fuzzy GE-filter for each \( i, 1 \leq i \leq n \), it is trivial to show that 
\[
(\zeta_1 \times \zeta_2) \times \ldots \times \zeta_n((a_1, a_2, \ldots, a_n) \ast (b_1, b_2, \ldots, b_n)) \supseteq (\zeta_1 \times \zeta_2) \times \ldots \times \zeta_n((a_1, a_2, \ldots, a_n)) \times \zeta_n((b_1, b_2, \ldots, b_n)).
\]
From the given hypothesis \( \zeta \) is a bellergent fuzzy GE-filter for each \( i, 1 \leq i \leq n \). So,
\[
\zeta_1(a_1 \ast b_1) \geq \zeta_1(a_1 \ast (c_1 \ast b_1)) \land \zeta_1(c_1 \ast b_1),
\]
\[
\zeta_2(a_2 \ast b_2) \geq \zeta_2(a_2 \ast (c_2 \ast b_2)) \land \zeta_2(c_2 \ast b_2),
\]
\[
\ldots
\]
\[
\zeta_n(a_n \ast b_n) \geq \zeta_n(a_n \ast (c_n \ast b_n)) \land \zeta_n(c_n \ast b_n).
\]
After some steps it follows that 
\[
(\zeta_1 \times \zeta_2) \times \ldots \times \zeta_n((a_1, a_2, \ldots, a_n) \ast (b_1, b_2, \ldots, b_n)) \supseteq (\zeta_1 \times \zeta_2) \times \ldots \times \zeta_n((a_1, a_2, \ldots, a_n)) \times \zeta_n((b_1, b_2, \ldots, b_n)).
\]
Therefore, \( \zeta_1 \times \zeta_2 \times \ldots \times \zeta_n \) is a bellergent fuzzy GE-filter of \( G_1 \times G_2 \times \ldots \times G_n \).

**Example 5.** Let \( G_1 = \{1, a, b, c, d\} \) and \( G_2 = \{1, a, b, c, d, e\} \) be GE-algebras where the binary operations \( \ast_1 \) and \( \ast_2 \) are defined by Tables 2 and 3.

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Now, define \( \zeta_1 : G_1 \rightarrow [0,1] \) by \( \zeta_1(1) = \zeta_1(b) = 1 \), \( \zeta_1(a) = \zeta_1(c) = \zeta_1(d) = 0.3 \) and \( \zeta_2 : G_2 \rightarrow [0,1] \) by \( \zeta_2(1) = \zeta_2(e) = \zeta_2(a) = \zeta_2(b) = \zeta_2(c) = \zeta_2(d) = 0.4 \). Clearly, \( \zeta_1 \) and \( \zeta_2 \) are fuzzy GE-filters of \( G_1 \) and \( G_2 \), respectively. Now, we define \( \zeta_1 \times \zeta_2 : (G_1 \times G_2) \rightarrow [0,1] \) by \( (\zeta_1 \times \zeta_2)(1,1) = (\zeta_1 \times \mu_2)(1,1) = (\zeta_1 \times \zeta_2)(1,2) = (\zeta_1 \times \zeta_2)(2,1) = (\zeta_1 \times \zeta_2)(2,2) = 1 \).

(15)

Theorem 19. If \( \zeta_1 \) and \( \zeta_2 \) are (bellergent) fuzzy GE-filters of \( G_1 \) and \( G_2 \), respectively, then the union \( (\zeta_1 \cup \zeta_2) \) is a (bellergent) fuzzy GE-filter of \( G_1 \cup G_2 \).

**Proof.** Clearly, \( (\zeta_1 \cup \zeta_2)(1) = \zeta_1(1) \lor \zeta_2(1) = 1 \lor 1 = 1 \). Recall that for any \( a, b \in (G_1 \cup G_2) \), define a binary operation \( \ast \) on \( G_1 \cup G_2 \) as 
\[
(a \ast b) = \begin{cases} 
\ast 1, & \text{if } a, b \in G_1, \\
\ast 2, & \text{if } a, b \in G_2, \\
b & \text{if } a \text{ and } b \text{ are not belong to the same GE-algebra.}
\end{cases}
\]

Now,
$(\xi_1 \cup \xi_2)(b) = \xi_1(b) \lor \xi_2(b) \geq (\xi_1(a \ast b) \land \xi_1(a))$

$\lor (\xi_2(a \ast b) \land \xi_2(a))$

$= (\xi_1 \cup \xi_2)(a \ast b) \land (\xi_1 \cup \xi_2)(a)$. (16)

Hence, $(\xi_1 \cup \xi_2)$ is a fuzzy GE-filter of $(G_1 \cup G_2)$. Next, assume that $\xi_1$ and $\xi_2$ be belligerent fuzzy GE-filters of $G_1$ and $G_2$, respectively. Then

$$(\xi_1 \cup \xi_2)(a \ast c) = (\xi_1(a \ast c) \lor \xi_2(a \ast c))$$

${}\geq (\xi_1(a \ast (b \ast c)) \lor \xi_1(a \ast b))$

$\lor (\xi_2(a \ast (b \ast c)) \lor \xi_2(a \ast b))$

$= (\xi_1 \lor \xi_2)(a \ast (b \ast c)) \land (\xi_1 \lor \xi_2)(a \ast b)$

$= (\xi_1 \lor \xi_2)(a \ast (b \ast c)) \land (\xi_1 \lor \xi_2)(a \ast (b \ast c))$. (17)

so that $\xi_1 \lor \xi_2$ is a belligerent fuzzy GE-filter of $G_1 \cup G_2$. □

**Theorem 20.** Let $\xi_1, \xi_2, \xi_3, \ldots, \xi_n$ be n fuzzy subsets of GE-algebras $G_1, G_2, G_3, \ldots, G_n$, respectively, and let $t \in [0, 1]$, then $(\xi_1 \times \xi_2 \times \ldots \times \xi_n)_t = (\xi_1)_t \times (\xi_2)_t \times \ldots \times (\xi_n)_t$.

Proof. Let $t \in [0, 1]$ and for any element

$$(a_1, a_2, a_3, \ldots, a_n) \in (\xi_1 \times \xi_2 \times \ldots \times \xi_n)_t$$

$= (\xi_1)_t \times (\xi_2)_t \times \ldots \times (\xi_n)_t$.

Hence $(\xi_1 \times \xi_2 \times \ldots \times \xi_n)_t = (\xi_1)_t \times (\xi_2)_t \times \ldots \times (\xi_n)_t$. □

**Theorem 21.** Let $\xi_1, \xi_2, \xi_3, \ldots, \xi_n$ be n fuzzy subsets of GE-algebra $G_1 \times G_2 \times G_3 \times \ldots \times G_n$ if and only if for any $t \in [0, 1]$, $(\xi_1 \times \xi_2 \times \ldots \times \xi_n)_t$ is a fuzzy GE-filter of GE-algebra $(G_1 \times G_2 \times G_3 \times \ldots \times G_n)$.

Proof. Suppose that $(\xi_1 \times \xi_2 \times \ldots \times \xi_n)_t$ is a fuzzy GE-filter of GE-algebra $(G_1 \times G_2 \times G_3 \times \ldots \times G_n)$. Since $(\xi_1 \times \xi_2 \times \ldots \times \xi_n)(1) = (\xi_1)(1) \land (\xi_2)(1) \land \ldots \land (\xi_n)(1) = 1$. This implies 1 $\in (\xi_1 \times \xi_2 \times \ldots \times \xi_n)_t$. If $a \in (\xi_1 \times \xi_2 \times \ldots \times \xi_n)_t$ and $b \in (\xi_1 \times \xi_2 \times \ldots \times \xi_n)_t$, then $(\xi_1 \times \xi_2 \times \ldots \times \xi_n)(a \ast b) \leq t$, $(\xi_1 \times \xi_2 \times \ldots \times \xi_n)(a) \leq t$.

And, therefore, $(\xi_1 \times \xi_2 \times \ldots \times \xi_n)(a \ast b) \leq t$. Thus, $y \in (\xi_1 \times \xi_2 \times \ldots \times \xi_n)_t$. Conversely, suppose that $\xi_1 \times \xi_2 \times \ldots \times \xi_n$ is a fuzzy GE-filter. Clearly $\xi_1 \times \xi_2 \times \ldots \times \xi_n(1) = 1$. Let $(\xi_1 \times \xi_2 \times \ldots \times \xi_n)(a \ast b) \land (\xi_1 \times \xi_2 \times \ldots \times \xi_n)(a) \leq t$. This implies $(\xi_1 \times \xi_2 \times \ldots \times \xi_n)(a \ast b) \land (\xi_1 \times \xi_2 \times \ldots \times \xi_n)(a) \leq t$. Thus, $y \in (\xi_1 \times \xi_2 \times \ldots \times \xi_n)_t$. □

**5. Conclusion and Future Work**

In this paper, the concept of belligerent fuzzy GE-filters is introduced. We investigate the relationships between fuzzy GE-filter and a belligerent fuzzy GE-filter. Additionally, prove that that the finite product (union) of belligerent fuzzy GE-filters of GE-algebras becomes a belligerent fuzzy GE-filter of the finite product (union) of GE-algebras. We hope in the future, we study prominent fuzzy GE-filters, imploring fuzzy GE-filters, soft GE-filter, and Soft GE-ideals of GE-algebras.

**Data Availability**

No data were used.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**References**


