Hindawi
Journal of Mathematics
Volume 2022, Article ID 4279702, 19 pages
https://doi.org/10.1155/2022/4279702

Research Article

Computing Connection-Based Topological Indices of Sudoku Graphs

Muhammad Basit Ali,[1] Ebenezer Bonyah ,[2] and Muhammad Javaid [1]

[1]*Department of Mathematics, School of Science, University of Management and Technology, Lahore 54770, Pakistan*
[2]*Department of Mathematics Education, Akenten Appiah-Menka University of Skills Training and Entrepreneurial Development, Kumasi 00233, Ghana*

Correspondence should be addressed to Ebenezer Bonyah; ebbonya@yahoo.com

Received 3 March 2022; Accepted 25 March 2022; Published 28 April 2022

Academic Editor: Gohar Ali

A topological index (TI) is a function that associates a numeric number to the under-study molecular graph. The first TI based on distance was presented by Wiener to find the boiling point of paraffins. In 1972, to compute the total π-electron energy of a molecule, the degree-based indices are used. In this paper, we obtained first and second Zagreb connection, modified first, second, third, and fourth Zagreb connection, first, second, third, and fourth multiplicative Zagreb connection, and modified first, second, and third multiplicative Zagreb connection indices for Sudoku graphs.

1. Introduction

The field of graph theory is comprehensively growing and assuming an exceptional part in the subject of cheminformatics that is the combination of chemistry, mathematics, and information technology which studies the diverse chemical formation and their physicochemical properties. In particular, Graph theory is used in a branch of mathematical chemistry which is known as chemical graph theory [17, 29]. Chemical graph theory gives a stage to concentrate on the physicochemical properties of the molecular graph with the assistance of topological indices (TIs). TIs are the numerical numbers which are associated with different chemical structures of molecular graphs and anticipate the structural, toxicological, biological, and physicochemical properties of the chemical compounds existing in the molecular graphs [31]. Moreover, TIs are same for the isomorphic graphs. Up till now, many researchers have worked on TIs, see [21, 32]. A remarkable utilization of these TIs has been generally displayed in the investigation of qualitative structural activity and property relationships (QSAR and QSPR). Todeschini and Consonni showed that the connection numbers are richly utilized in the subject of cheminformatics, see [13, 18, 26].

There are different types of TIs such as distance-based TIs, degree-based TIs, and polynomial-related TIs. Among these, the degree-based topological indices are of remarkable importance and expect to be a key part in chemical graph theory and particularly in chemistry [16]. This class is further grouped into two subclasse degrees and connection-based TIs. These TIs were characterized by Gutman and Trinajsti c [14] to derive the whole π-electron energy of alternant hydrocarbons. For more review, we refer to [7, 9, 25]. Gutman, Rucic, and Furtula also worked on various TIs, see [12, 15]. In 2003, Nikoli c [24] described modified Zagreb indices. In 2011, Hao [19] utilized modified approach and accomplished the trial of Zagreb and modified Zagreb indices. Dhanalakshmi [7] derived many results for modified and multiplicative ZIs of graph obtained by different operators. In 2013, Das [8] derived upper bounds for the multiplicative Zagreb indices of operations on graphs. Many researchers work on multiplicative Zagreb indices [10, 11] and sum-multiplicative Zagreb indices [27, 33].

Connection number is a number of such vertices which are on distance 2 from a specific vertex. Trinajstic [1] studied the modification of first Zagreb indices. Tang [28] fostered certain Zagreb connection indices (ZCIs). Moreover, Ali [2] derived modified ZCIs of T-sum graphs. In recent times, Liu [23] computed ZCIs of molecular graphs based on operations. Cao [4] obtained ZCIs to work out both exact and upper bounds of some product-related graphs. Moreover, Javaid et al. [22] found out novel connection-based ZIs of numerous wheel related graphs, Javaid et al. [30] also derived connection-based multiplicative Zagreb Indices of dendrimer nanostars, and Haoer [20] derived the multiplicative leap ZIs.

In this paper, we constructed the partitions of the edges of the Sudoku graphs with respect to their connection numbers (see Table 1) and degree-connection numbers (see Table 2) of their end vertices. Moreover, the partition of the vertex set of the Sudoku graph is obtained with respect to the degree and connection number considering at the same time (see Table 3). Finally, we obtained first and second Zagreb connection, modified first, second, third, and fourth Zagreb connection, first, second, third, and fourth multiplicative Zagreb connection, and modified first, second, and third multiplicative Zagreb connection indices that are computed for Sudoku graphs. This paper is composed as follows: Section 2 presents a few significant definitions which are required to lay out the concept of the present study. Section 3 is about the construction of Sudoku graph, Section 4 and Section 5 consist of the main results, and Section 6 covers the conclusion.

2. Connection Number-Based Topological Indices

Let $\kappa = (V(\kappa), E(\kappa))$ be a graph, where $V(\kappa)$ and $E(\kappa)$ denote the set of vertices and edges accordingly; for a vertex $x \in V(\kappa)$, the degree $d_\kappa(x)$ is the number of those vertices which are distance at one from x, and connection number $\lambda_\kappa(x)$ is the number of such vertices which are at distance two from vertex x. For more graph-theoretic notions, see [3, 5, 6].

Definition 1 (see [23]). Let κ be a graph, then the first Zagreb connection index is

$$\widehat{Z}C_1(\kappa) = \sum_{v \in V(\kappa)} [\lambda_\kappa(v)]^2. \quad (1)$$

Definition 2 (see [23]). Let κ be a graph, then the second Zagreb connection index is

$$\widehat{Z}C_2(\kappa) = \sum_{uv \in E(\kappa)} [\lambda_\kappa(u) \times \lambda_\kappa(v)], \quad (2)$$

where $\lambda_\kappa(u)$ and $\lambda_\kappa(v)$ represent the connection number of the vertices u and v for the edge uv.

Definition 3 (see [23]). Let κ be a graph, then the modified first Zagreb connection index, modified second Zagreb connection index, modified third Zagreb connection index, and modified fourth Zagreb connection are

(i) $(a)\ \widehat{Z}C_1^*(\kappa) = \sum_{uv \in E(\kappa)} [\lambda_\kappa(u) + \lambda_\kappa(v)]$

(ii) $(b)\ \widehat{Z}C_2^*(\kappa) = \sum_{uv \in E(\kappa)} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)]$

$(c)\ \widehat{Z}C_3^*(\kappa) = \sum_{uv \in E(\kappa)} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(u)]$

$(d)\ \widehat{Z}C_4^*(\kappa) = \sum_{uv \in E(\kappa)} [d_\kappa(u)\lambda_\kappa(v) \times d_\kappa(v)\lambda_\kappa(u)] = \sum_{uv \in E(\kappa)} [d_\kappa(u)\lambda_\kappa(u) \times d_\kappa(v)\lambda_\kappa(v)]$

where $\lambda_\kappa(u)$ and $\lambda_\kappa(v)$ represent the connection number of the vertices u and v for the edge uv.

Definition 4 (see [23]). Let κ be a graph, then first multiplicative Zagreb connection index ($M\widehat{Z}C_1(\kappa)$), second multiplicative Zagreb connection index ($M\widehat{Z}C_2(\kappa)$), third multiplicative Zagreb connection index ($M\widehat{Z}C_3(\kappa)$), and fourth multiplicative Zagreb connection index ($M\widehat{Z}C_4(\kappa)$) are as follows:

(i) $(a)\ M\widehat{Z}C_1(\kappa) = \prod_{v \in V(\kappa)} [\lambda_\kappa(v)]^2$

$(b)\ M\widehat{Z}C_2(\kappa) = \prod_{uv \in E(\kappa)} [\lambda_\kappa(u) \times \lambda_\kappa(v)]$

$(c)\ M\widehat{Z}C_3(\kappa) = \prod_{uv \in V(\kappa)} [d_\kappa(u) \times \lambda_\kappa(v)]$

$(d)\ M\widehat{Z}C_4(\kappa) = \prod_{uv \in E(\kappa)} [\lambda_\kappa(u) + \lambda_\kappa(v)]$

Definition 5 (see [23]). For a graph κ, modified first multiplicative Zagreb connection index ($M\widehat{Z}C_1^*(\kappa)$), modified second multiplicative Zagreb connection index ($M\widehat{Z}C_2^*(\kappa)$), and modified third multiplicative Zagreb connection index ($M\widehat{Z}C_3^*(\kappa)$) are as follows:

$(a)\ M\widehat{Z}C_1^*(\kappa) = \prod_{uv \in E(\kappa)} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)]$

$(b)\ M\widehat{Z}C_2^*(\kappa) = \prod_{uv \in E(\kappa)} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]$

$(c)\ M\widehat{Z}C_3^*(\kappa) = \prod_{uv \in E(\kappa)} [d_\kappa(u)\lambda_\kappa(v) \times d_\kappa(v)\lambda_\kappa(u)] = \prod_{uv \in E(\kappa)} [d_\kappa(u)\lambda_\kappa(u) \times d_\kappa(v)\lambda_\kappa(v)]$

3. Construction of Sudoku Graph

For some integral value $n \geq 2$, Sudoku graph ($SK_{n\times n}$) is a graphical representation of the Sudoku puzzle in the form of a $n \times n$ grid consisting of n^2 cells, where each cell is an individual graph of a nodes with four vertices of degree 5, four vertices of degree 7, and one vertex of degree 8. Moreover, for Sudoku graphs $\kappa \cong SK_{n\times n}$, $|V(\kappa)| = 9n^2$ are the number of vertices, and $|E(\kappa)| = n(34n - 6)$ are the number of edges, respectively. For $n = 3$, the Sudoku graph $SK_{3\times 3}$ is shown in Figure 1.

In Table 1, we computed the results for the connection number-based partition of edges for $n \geq 3$ of Sudoku graph $SK_{n\times n}$. In Table 2 and Table 3, we also computed the results for the degree-connection number-based partition of edges and vertices for $n \geq 3$ of Sudoku graph $SK_{n\times n}$.

4. Connection Number-Based Zagreb Indices

This section deals with the results of the connection number-based Zagreb indices of the Sudoku graphs.

Theorem 1. *Let $\kappa \cong SK_{n\times n}$ be a Sudoku graph, then its first Zagreb connection index $\widehat{Z}C_1(\kappa)$ is*

$$\widehat{Z}C_1(\kappa) = 2192n^2 - 3382n + 932. \quad (3)$$

Table 1: Connection number-based partition of edges of $SK_{n\times n}$.

$E^c_{\lambda(n_1),\lambda(n_2)}$	$\lvert E^c_{\lambda(n_1),\lambda(n_2)}\rvert$	$E^c_{\lambda(n_1),\lambda(n_2)}$	$\lvert E^c_{\lambda(n_1),\lambda(n_2)}\rvert$
$E^c_{5,6}$	12	$E^c_{6,9}$	24
$E^c_{9,11}$	16	$E^c_{11,14}$	8
$E^c_{11,6}$	16	$E^c_{6,14}$	12
$E^c_{6,6}$	12	$E^c_{11,5}$	8
$E^c_{11,11}$	4	$E^c_{10,13}$	$2(8n-16)$
$E^c_{13,15}$	$2(8n-16)$	$E^c_{14,15}$	$8(n-1)$
$E^c_{9,10}$	$8(2n-3)$	$E^c_{9,13}$	$2(8n-16)$
$E^c_{9,15}$	$2(8n-16)$	$E^c_{9,14}$	$4n-8$
$E^c_{9,9}$	$4n-8$	$E^c_{10,14}$	$8p-16$
$E^c_{13,14}$	$8n-16$	$E^c_{16,16}$	$26n^2-110n+116$
$E^c_{12,16}$	$8(n-2)^2$	$E^c_{11,13}$	8
$E^c_{15,16}$	$8n-16$	$E^c_{14,16}$	$4n-8$
$E^c_{10,10}$	$4(n-3)$	$E^c_{13,13}$	$4(n-3)$
$E^c_{15,15}$	$4(n-3)$		

Table 2: Degree-connection number-based partition of edges of $SK_{n\times n}$.

$E^{\lambda(u),\lambda(v)}_{d(u),d(v)}$	$\lvert E^{\lambda(u),\lambda(v)}_{d(u),d(v)}\rvert$	$E^{\lambda(u),\lambda(v)}_{d(u),d(v)}$	$\lvert E^{\lambda(u),\lambda(v)}_{d(u),d(v)}\rvert$	$E^{\lambda(u),\lambda(v)}_{d(u),d(v)}$	$\lvert E^{\lambda(u),\lambda(v)}_{d(u),d(v)}\rvert$
$E^{13,14}_{8,8}$	$8(n-2)$	$E^{6,9}_{7,6}$	16	$E^{15,16}_{7,7}$	$8(n-2)$
$E^{10,9}_{6,8}$	$8(n-2)$	$E^{5,11}_{5,8}$	8	$E^{15,15}_{7,7}$	$4(n-3)$
$E^{10,13}_{6,8}$	$16(n-2)$	$E^{5,6}_{5,8}$	4	$E^{9,10}_{6,6}$	8
$E^{10,14}_{6,8}$	$8(n-2)$	$E^{6,6}_{7,7}$	4	$E^{10,10}_{6,6}$	$4(n-3)$
$E^{9,9}_{7,8}$	$4(n-2)$	$E^{6,6}_{7,8}$	8	$E^{11,13}_{8,8}$	8
$E^{9,13}_{7,8}$	$8(n-2)$	$E^{6,14}_{7,7}$	8	$E^{13,13}_{8,8}$	$4(n-3)$
$E^{9,15}_{7,7}$	$8(n-2)$	$E^{6,11}_{7,8}$	8	$E^{14,16}_{8,8}$	$4(n-2)$
$E^{9,14}_{8,8}$	$4(n-2)$	$E^{9,6}_{6,8}$	8	$E^{16,16}_{8,8}$	$[\{(2n-5)^2-1\}/2+(2n-4)^2]$
$E^{9,13}_{8,8}$	$8(n-2)$	$E^{9,11}_{6,8}$	16	$E^{16,16}_{7,8}$	$16(n-2)^2$
$E^{13,15}_{8,7}$	$16(n-2)$	$E^{6,11}_{8,8}$	8	$E^{16,12}_{8,8}$	$4(n-2)^2$
$E^{15,9}_{7,8}$	$8(n-2)$	$E^{11,11}_{8,8}$	4	$E^{16,12}_{7,8}$	$4(n-2)^2$
$E^{15,14}_{7,8}$	$8(n-2)$	$E^{11,14}_{8,7}$	8	$E^{16,16}_{7,7}$	$[(2n-5)^2-1]$
$E^{10,9}_{6,7}$	$8(n-2)$	$E^{6,14}_{8,7}$	4		
$E^{5,6}_{5,7}$	8	$E^{14,15}_{7,7}$	8		

Table 3: Degree-connection number-based partition of vertices of $SK_{n\times n}$.

$V^{\lambda(u)}_{d(u)}$	$\lvert V^{\lambda(u)}_{d(u)}\rvert$	$V^{\lambda(u)}_{d(u)}$	$\lvert V^{\lambda(u)}_{d(u)}\rvert$
V^5_5	4	V^{13}_8	$8(n-2)$
V^6_7	8	V^{14}_7	4
V^6_8	4	V^{14}_8	$4(n-2)$
V^9_6	8	V^{15}_7	$8(n-2)$
V^9_7	$4(n-2)$	V^{16}_7	$4(n-2)^2$
V^9_8	$4(n-2)$	V^{16}_8	$4(n-2)^2$
V^{10}_6	$8(n-2)$	V^{12}_8	$(n-2)^2$
V^{11}_8	8		

proof. By Definition 1,

$$\widehat{Z}C_1(\kappa)=\sum_{v\in V(\kappa)}\left[\lambda_\kappa(v)\right]^2. \tag{4}$$

We consider the partition of the vertices of the Sudoku graphs with respect to their connection numbers,

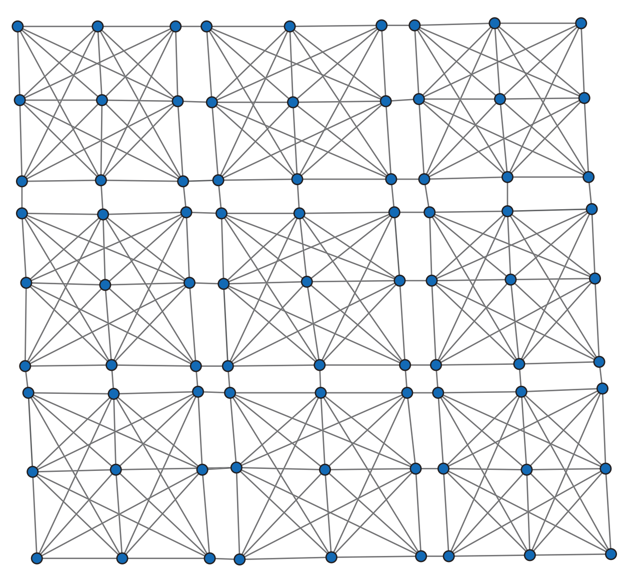

Figure 1: Sudoku graph. $SK_{3\times 3}$.

$$\begin{aligned}
V_1 &= \{x \in \nu(\kappa)\colon \lambda_\kappa(x) = 5\}\\
V_2 &= \{x \in \nu(\kappa)\colon \lambda_\kappa(x) = 6\}\\
V_3 &= \{x \in \nu(\kappa)\colon \lambda_\kappa(x) = 9\}\\
V_4 &= \{x \in \nu(\kappa)\colon \lambda_\kappa(x) = 10\}\\
V_5 &= \{x \in \nu(\kappa)\colon \lambda_\kappa(x) = 11\}\\
V_6 &= \{x \in \nu(\kappa)\colon \lambda_\kappa(x) = 12\}\\
V_7 &= \{x \in \nu(\kappa)\colon \lambda_\kappa(x) = 13\}\\
V_8 &= \{x \in \nu(\kappa)\colon \lambda_\kappa(x) = 14\}\\
V_9 &= \{x \in \nu(\kappa)\colon \lambda_\kappa(x) = 15\}\\
V_{10} &= \{x \in \nu(\kappa)\colon \lambda_\kappa(x) = 16\},
\end{aligned} \tag{5}$$

such that $|V_1| = 4$, $|V_2| = 12$, $|V_3| = 8(n-1)$, $|V_4| = 8(n-2)$, $|V_5| = 8$, $|V_6| = (n-2)^2$, $|V_7| = 8(n-2)$, $|V_8| = 4(n-1)$, $|V_9| = 8(n-2)$, and $|V_{10}| = 8(n-2)^2$.

$$\begin{aligned}
\widehat{Z}C_1(\kappa) &= \sum_{v\in V_1(\kappa)}[\lambda_\kappa(v)]^2 + \sum_{v\in V_2(\kappa)}[\lambda_\kappa(v)]^2 + \sum_{v\in V_3(\kappa)}[\lambda_\kappa(v)]^2 + \sum_{v\in V_4(\kappa)}[\lambda_\kappa(v)]^2 + \sum_{v\in V_5(\kappa)}[\lambda_\kappa(v)]^2\\
&\quad + \sum_{v\in V_6(\kappa)}[\lambda_\kappa(v)]^2 + \sum_{v\in V_7(\kappa)}[\lambda_\kappa(v)]^2 + \sum_{v\in V_8(\kappa)}[\lambda_\kappa(v)]^2 + \sum_{v\in V_9(\kappa)}[\lambda_\kappa(v)]^2 + \sum_{v\in V_{10}(\kappa)}[\lambda_\kappa(v)]^2\\
\widehat{Z}C_1(\kappa) &= |V_1|(5)^2 + |V_2|(6)^2 + |V_3|(9)^2 + |V_4|(10)^2 + |V_5|(11)^2 + |V_6|(12)^2 + |V_7|(13)^2\\
&\quad + |V_8|(14)^2 + |V_9|(15)^2 + |V_{10}|(16)^2\\
\widehat{Z}C_1(\kappa) &= 4(5)^2 + 12(6)^2 + 8(n-1)(9)^2 + 8(n-2)(10)^2 + 8(11)^2 + (n-2)^2(12)^2 + 8(n-2)(13)^2\\
&\quad + 4(n-1)(14)^2 + 8(n-2)(15)^2 + 8(n-2)^2(16)^2.
\end{aligned} \tag{6}$$

By performing some calculation, we get

$$\widehat{Z}C_1(\kappa) = 2192n^2 - 3382n + 932. \tag{7}$$

□

Theorem 2. *Let $\kappa \cong SK_{n\times n}$ be a Sudoku graph. Then, its second Zagreb connection index is given as $\widehat{Z}C_2(\kappa) = 8192n^2 - 13756n + 4204$.*

Proof. By Definition 2 and Table 1,

$$\begin{aligned}
\widehat{Z}C_2(\kappa) &= \sum_{uv\in E(\kappa)}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\\
\widehat{Z}C_2(\kappa) &= ZC_2(\kappa) = 12[5\times 6] + 16[9\times 11] + 16[11\times 6] + 12[6\times 6] + 4[11\times 11] + 2(8n-16)[13\times 15]\\
&\quad + 8(2n-3)[9\times 10] + 2(8n-16)[9\times 15] + 4n - 8[9\times 9] + 8n - 16[13\times 14] + 8(n-2)^2[12\times 16]\\
&\quad + (8n-16)[15\times 16] + 4(n-3)[10\times 10] + 4(n-3)[15\times 15] + 24[6\times 9]\\
&\quad + 8[11\times 14] + 12[6\times 14] + 8[11\times 5] + 2(8n-16)[10\times 13] + 8(n-1)[14\times 15] + 2(8n-16)[9\times 13]\\
&\quad + 4n - 8[9\times 14] + 8(n-2)[10\times 14] + 26n^2 - 110n + 116[16\times 16] + 8[11\times 13] + 4(n-2)[14\times 16]\\
&\quad + 4(n-3)[13\times 13].
\end{aligned} \tag{8}$$

By performing some calculations, we get

$$\widehat{Z}C_2(\kappa) = 8192n^2 - 13756n + 4204. \tag{9}$$

□

Theorem 3. *Let $\kappa \cong SK_{n\times n}$ be a Sudoku graph. Then, its modified first Zagreb connection index, modified second Zagreb connection index, modified third Zagreb connection index, and the modified fourth Zagreb connection index are as follows:*

(a) $\widehat{Z}C_1^*(\kappa) = 1056n^2 - 1164n + 180$
(b) $\widehat{Z}C_2^*(\kappa) = 8016n^2 - 10388n + 2808$
(c) $\widehat{Z}C_3^*(\kappa) = 8000n^2 - 8812n + 980$
(d) $\widehat{Z}C_4^*(\kappa) = 470016n^2 - 853316n + 259092$

Proof.
(a). By Definition 3 and Table 1,

$$\begin{aligned}\widehat{Z}C_1^*(\kappa) &= \sum_{uv\in E(\kappa)}[\lambda_\kappa(u)+\lambda_\kappa(v)]\\ \widehat{Z}C_1^*(\kappa) &= 12[5+6]+16[9+11]+16[11+6]+12[6+6]+4[11+11]+2(8n-16)[13+15]\\ &+8(2n-3)[9+10]+2(8n-16)[9+15]+(4n-8)[9+9]+(8n-16)[13+14]+8(n-2)^2[12+16]\\ &+8n-16[15+16]+4(n-3)[10+10]\\ &+4(n-3)[15+15]+24[6+9]+8[11+14]+12[6+14]+8[11+5]\\ &+2(8n-16)[10+13]+8(n-1)[14+15]+2(8n-16)[9+13]+(4n-8)[9+14]+8(n-2)[10+14]\\ &+26n^2-110n+116[16+16]+8[11+13]+4(n-2)[14+16]+4(n-3)[13+13].\end{aligned}\tag{10}$$

By performing some calculations, we get

$$\widehat{Z}C_1^*(\kappa) = 1056n^2 - 1164n + 180.\tag{11}$$

(b). As we know from Definition 3,

$$\begin{aligned}\widehat{Z}C_2^*(\kappa) &= \sum_{uv\in E(\kappa)}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]\\ \widehat{Z}C_2^*(\kappa) &= \sum_{uv\in E_{8,8}^{13,14}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]+\sum_{uv\in E_{6,8}^{10,9}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]\\ &+\sum_{uv\in E_{6,8}^{10,13}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]+\sum_{uv\in E_{6,8}^{10,14}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]\\ &+\sum_{uv\in E_{7,8}^{9,9}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]+\sum_{uv\in E_{7,8}^{9,13}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]\\ &+\sum_{uv\in E_{7,7}^{9,15}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]+\sum_{uv\in E_{8,8}^{9,14}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]\\ &+\sum_{uv\in E_{8,8}^{9,13}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]+\sum_{uv\in E_{8,7}^{13,15}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]\\ &+\sum_{uv\in E_{7,8}^{15,9}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]+\sum_{uv\in E_{7,8}^{15,14}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]\\ &+\sum_{uv\in E_{6,7}^{10,9}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]+\sum_{uv\in E_{5,7}^{5,6}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]\\ &+\sum_{uv\in E_{7,6}^{6,9}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]+\sum_{uv\in E_{5,8}^{5,11}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]\\ &+\sum_{uv\in E_{5,8}^{5,6}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]+\sum_{uv\in E_{7,7}^{6,6}}[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)]\end{aligned}$$

$$\begin{aligned}
&+\sum_{uv\in E_{7,8}^{6,6}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]+\sum_{uv\in E_{7,7}^{6,14}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&+\sum_{uv\in E_{7,8}^{6,11}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]+\sum_{uv\in E_{6,8}^{9,6}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&+\sum_{uv\in E_{6,8}^{9,11}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]+\sum_{uv\in E_{8,8}^{6,11}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&+\sum_{uv\in E_{8,8}^{11,11}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]+\sum_{uv\in E_{8,7}^{11,14}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&+\sum_{uv\in E_{8,7}^{6,14}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]+\sum_{uv\in E_{7,7}^{14,15}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&+\sum_{uv\in E_{7,7}^{15,16}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]+\sum_{uv\in E_{7,7}^{15,15}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&+\sum_{uv\in E_{6,6}^{9,10}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]+\sum_{uv\in E_{6,6}^{10,10}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&+\sum_{uv\in E_{8,8}^{11,13}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]+\sum_{uv\in E_{8,8}^{13,13}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&+\sum_{uv\in E_{8,8}^{14,16}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]+\sum_{uv\in E_{7,8}^{16,16}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&+\sum_{uv\in E_{8,8}^{16,12}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]+\sum_{uv\in E_{7,8}^{16,12}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&+\sum_{uv\in E_{7,7}^{16,16}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]+\sum_{uv\in E_{8,8}^{16,16}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&+\left|E_{7,7}^{9,15}\right|[(7\times 15)+(7\times 9)]+\left|E_{8,8}^{9,14}\right|[(8\times 14)+(8\times 9)]+\left|E_{8,8}^{9,13}\right|[(8\times 13)+(8\times 9)]+\left|E_{8,7}^{13,15}\right|[(8\times 15)+(7\times 13)]\\
&+\left|E_{7,8}^{15,9}\right|[(7\times 9)+(8\times 15)]\\
&+\left|E_{7,8}^{15,14}\right|[(7\times 14)+(8\times 15)]+\left|E_{6,7}^{10,9}\right|[(6\times 9)+(7\times 10)]+\left|E_{5,7}^{5,6}\right|[(5\times 6)+(5\times 7)]+\left|E_{7,6}^{6,9}\right|[(7\times 9)+(6\times 6)]\\
&+\left|E_{6,8}^{9,6}\right|[(6\times 6)+(8\times 9)]+\left|E_{6,8}^{9,11}\right|[(6\times 11)+(8\times 9)]+\left|E_{8,8}^{6,11}\right|[(8\times 11)+(8\times 6)]\\
&+\left|E_{8,8}^{11,11}\right|[(8\times 11)+(8\times 11)]+\left|E_{8,7}^{11,14}\right|[(8\times 14)+(7\times 11)]+\left|E_{8,7}^{6,14}\right|[(8\times 14)+(7\times 6)]\\
&+\left|E_{8,8}^{13,13}\right|[(8\times 13)+(8\times 13)]+\left|E_{8,8}^{16,14}\right|[(8\times 14)+(8\times 16)]+\left|E_{7,7}^{16,16}\right|[(7\times 16)+(8\times 16)]+\left|E_{8,8}^{16,12}\right|[(8\times 12)+(8\times 16)]\\
&+\left|E_{7,8}^{16,12}\right|[(7\times 12)+(8\times 16)]+\left|E_{7,7}^{16,16}\right|[(7\times 16)+(7\times 16)]+\left|E_{8,8}^{16,16}\right|[(8\times 16)+(8\times 16)].
\end{aligned}\tag{12}$$

By using degree-connection number-based partition of edges of $SK_{n\times n}$ as given in Table 2,

$$\begin{aligned}
\widehat{Z}C_2^*(\kappa)={}&8(n-2)[216]+8(n-2)[134]+16(n-2)[158]+8(n-2)[164]+4(n-2)[135]+8(n-2)[163]\\
&+8(n-2)[168]+4(n-2)[184]+8(n-2)[176]+16(n-2)[211]+8(n-2)[183]+8(n-2)[218]\\
&+8(n-2)[124]+8[65]+16[99]\\
&+8[95]+4[70]+4[84]+8[90]+8[140]+8[125]+8[108]+16[138]+8[136]\\
&+4[176]+8[189]+4[154]+
\end{aligned}$$

$$+8[203]+8(n-2)[217]+4(n-3)[210]+8[114]$$
$$+4(n-3)[120]+8[192]\times 4(n-3)[208]+4(n-2)[240]+16(n-2)^2[240]$$
$$+4(n-2)^2[224]+4(n-2)^2[212]+\left\{(2n-5)^2-1\right\}[224]+\left\{\frac{(2n-5)^2-1}{2}+(2n-4)^2\right\}[256]. \quad (13)$$

By performing some calculations, we get

$$\widehat{Z}C_2^*(\kappa)=8016n^2-10388n+2808. \quad (14)$$

(c). As we know from Definition 3,

$$\widehat{Z}C_3^*(\kappa)=\sum_{uv\in E(\kappa)}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]$$
$$\begin{aligned}\widehat{Z}C_3^*(\kappa)=&\sum_{uv\in E_{8,8}^{13,14}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]+\sum_{uv\in E_{6,8}^{10,9}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]\\
&+\sum_{uv\in E_{6,8}^{10,13}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]+\sum_{uv\in E_{6,8}^{10,14}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]\\
&+\sum_{uv\in E_{7,8}^{9,9}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]+\sum_{uv\in E_{7,8}^{9,13}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]\\
&+\sum_{uv\in E_{7,7}^{9,15}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]+\sum_{uv\in E_{8,8}^{9,14}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]\\
&+\sum_{uv\in E_{8,8}^{9,13}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]+\sum_{uv\in E_{8,7}^{13,15}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]\\
&+\sum_{uv\in E_{7,8}^{15,9}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]+\sum_{uv\in E_{7,8}^{15,14}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]\\
&+\sum_{uv\in E_{6,7}^{10,9}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]+\sum_{uv\in E_{5,7}^{5,6}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]\\
&+\sum_{uv\in E_{7,6}^{6,9}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]+\sum_{uv\in E_{5,8}^{5,11}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]\\
&+\sum_{uv\in E_{5,8}^{5,6}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]+\sum_{uv\in E_{7,7}^{6,6}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]\\
&+\sum_{uv\in E_{7,8}^{6,6}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]+\sum_{uv\in E_{7,7}^{6,14}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]\\
&+\sum_{uv\in E_{7,8}^{6,11}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]+\sum_{uv\in E_{6,8}^{9,6}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]\\
&+\sum_{uv\in E_{6,8}^{9,11}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]+\sum_{uv\in E_{8,8}^{6,11}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]\\
&+\sum_{uv\in E_{8,8}^{11,11}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]+\sum_{uv\in E_{8,7}^{11,14}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]\\
&+\sum_{uv\in E_{8,7}^{6,14}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]+\sum_{uv\in E_{7,7}^{14,15}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]\\
&+\sum_{uv\in E_{7,7}^{15,16}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]+\sum_{uv\in E_{7,7}^{15,15}}\left[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)\right]\end{aligned}$$

$$+\sum_{uv\in E_{6,6}^{9,10}}[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)]+\sum_{uv\in E_{6,6}^{10,10}}[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)]$$
$$+\sum_{uv\in E_{8,8}^{11,13}}[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)]+\sum_{uv\in E_{8,8}^{13,13}}[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)]$$
$$+\sum_{uv\in E_{8,8}^{14,16}}[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)]+\sum_{uv\in E_{7,8}^{16,16}}[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)] \quad (15)$$
$$+\sum_{uv\in E_{8,8}^{16,12}}[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)]+\sum_{uv\in E_{7,8}^{16,12}}[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)]$$
$$+\sum_{uv\in E_{7,7}^{16,16}}[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)]+\sum_{uv\in E_{8,8}^{16,16}}[d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)].$$

By using degree-connection number-based partition of edges of $SK_{n\times n}$ as given in Table 2,

$$\begin{aligned}\widehat{Z}C_3^*(\kappa) = {} & 8(n-2)[216]+8(n-2)[132]+16(n-2)[164]+8(n-2)[172]+4(n-2)[135] \\ & +8(n-2)[167]+8(n-2)[183]+4(n-2)[184]+8(n-2)[176]+16(n-2)[209]+8(n-2)[177] \\ & +8(n-2)[217]+8(n-2)[123]+8[67]+16[96]+8[113]+4[73]+4[84]+8[90]+8[140]+8[130] \\ & +8[102]+16[142]+8[136]+4[176]+8[186]+4[146]+8[203]+8(n-2)[217]+4(n-3)[210] \\ & +8[114]+4(n-3)[120]+8[192]+4(n-3)[208]+4(n-2)[240]+16(n-2)^2[240]+4(n-2)^2[224] \\ & +4(n-2)^2[208]+[224]^{\{(2n-5)^2-1\}}+\left\{\frac{(2n-5)^2-1}{2}+(2n-4)^2\right\}[256].\end{aligned} \quad (16)$$

By performing some calculations, we get

$$\widehat{Z}C_3^*(\kappa)=8000n^2-8812n+980. \quad (17)$$

(d). As we know from Definition 3,

$$\widehat{Z}C_4^*(\kappa)=\sum_{uv\in E(\kappa)}[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]$$
$$\widehat{Z}C_4^*(\kappa)=\sum_{uv\in E_{8,8}^{13,14}}[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]+\sum_{uv\in E_{6,8}^{10,9}}[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]$$
$$+\sum_{uv\in E_{6,8}^{10,13}}[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]+\sum_{uv\in E_{6,8}^{10,14}}[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]$$
$$+\sum_{uv\in E_{7,8}^{9,9}}[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]+\sum_{uv\in E_{7,8}^{9,13}}[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]$$
$$+\sum_{uv\in E_{7,7}^{9,15}}[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]+\sum_{uv\in E_{8,8}^{9,14}}[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]$$
$$+\sum_{uv\in E_{8,8}^{9,13}}[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]+\sum_{uv\in E_{8,7}^{13,15}}[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]$$

$$
\begin{aligned}
&+ \sum_{uv\in E_{7,8}^{15,9}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{7,8}^{15,14}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{6,7}^{10,9}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{5,7}^{5,6}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{7,6}^{6,9}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{5,8}^{5,11}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{5,8}^{5,6}} \left[d_\kappa(u)\lambda_\kappa(v)\times d_\kappa(v)\lambda_\kappa(u)\right] + \sum_{uv\in E_{7,7}^{6,6}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{7,8}^{6,6}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{7,7}^{6,14}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{7,8}^{6,11}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{6,8}^{9,6}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{7,8}^{6,6}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{7,7}^{6,14}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{7,8}^{6,11}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{6,8}^{9,6}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{6,8}^{9,11}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{8,8}^{6,11}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{8,8}^{11,11}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{8,7}^{11,14}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{8,7}^{6,14}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{7,7}^{14,15}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{7,7}^{15,16}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{7,7}^{15,15}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{8,7}^{6,14}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{7,7}^{14,15}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{7,7}^{15,16}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{7,7}^{15,15}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{8,7}^{6,14}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{7,7}^{14,15}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{7,7}^{15,16}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{7,7}^{15,15}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{6,6}^{9,10}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{6,6}^{10,10}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{8,8}^{11,13}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{8,8}^{13,13}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{8,8}^{14,16}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{7,8}^{16,16}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{8,8}^{16,12}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{7,8}^{16,12}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&+ \sum_{uv\in E_{7,7}^{16,16}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right] + \sum_{uv\in E_{8,8}^{16,16}} \left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right].
\end{aligned}
\tag{18}
$$

By using degree-connection number-based partition of edges of $SK_{n\times n}$ as given in Table 2,

$$\begin{aligned}\widehat{Z}C_4^*(\kappa) &= 8(n-2)[11648] + 8(n-2)[4320] + 16(n-2)[6240] + 8(n-2)[6720] + 4(n-2)[4536]\\&+ 8(n-2)[6552] + 8(n-2)[6615] + 4(n-2)[8064] + 8(n-2)[7488] + 16(n-2)[10920]\\&+ 8(n-2)[7560] + 8(n-2)[11760] + 8(n-2)[3780] + 8[1050] + 16[2268] + 8[2200] + 4[1200] + 4[1764]\\&+ 8[2016] + 8[4116] + 8[3696] + 8[2592] + 16[4752] + 8[4224] + 4[7744] + 8[2464] + 4[4704]\\&+ 8[10290] + 8(n-2)[11760] + 4(n-3)[11025] + 8[3240] + 4(n-3)[3600] + 8[9152]\\&+ 4(n-3)[10816] + 4(n-2)[14336] + 16(n-2)^2[14336]\\&+ 4(n-2)^2[12288] + 4(n-2)^2[10752] + \left\{(2n-5)^2 - 1\right\}[12544] + \left\{\frac{(2n-5)^2-1}{2} - (2n-4)^2\right\}[16384].\end{aligned} \tag{19}$$

By performing some calculations, we get

$$\widehat{Z}C_4^*(\kappa) = 470016n^2 - 853316n + 259092. \tag{20}$$

□

5. Connection Number-Based Multiplicative Zagreb Indices

This section deals with the results of the connection number-based multiplicative Zagreb indices of the Sudoku graphs.

Theorem 4. *Let $\kappa \cong SK_{n\times n}$ be a Sudoku graph. Then, its first multiplicative Zagreb connection index, second multiplicative Zagreb connection index, third multiplicative Zagreb connection index, and fourth multiplicative Zagreb connection index are as follows:*

(a) $M\widehat{Z}C_1(\kappa) = 2^{68n^2-248n+256} \times 3^{2n^2+40n-32} \times 5^{32n-56} \times 7^{8(n-1)} \times 11^{16} \times 13^{16(n-2)}$

(b) $M\widehat{Z}C_2(\kappa) = 2^{256n^2-944n+980} \times 3^{8n^2+144n-136} \times 5^{104n-188} \times 7^{32n-36} \times 11^{64} \times 13^{64(n-2)}$

(c) $M\widehat{Z}C_3(\kappa) = 2^{49n^2-112n+104} \times 3^{n^2+28n-24} \times 5^{16n-24} \times 7^{4n^2} \times 11^8 \times 13^{8(n-2)}$

(d) $M\widehat{Z}C_4(\kappa) = 2^{146n^2-470n+456} \times 3^{64n-96} \times 5^{12(n+3)} \times 7^{8n^2-48n+32} \times 11^{16(n-1)} \times 13^{4(n-3)}. \times 17^{16} \times 19^{8(2n-3)} \times 23^{20(n-2)} \times 29^{8(n-1)} \times 31^{8(n-2)}$

Proof

(a) By Definition 4,

$$\begin{aligned}M\widehat{Z}C_1(\kappa) &= \prod_{v\in V(\kappa)} [\lambda_\kappa(v)]^2\\M\widehat{Z}C_1(\kappa) &= \prod_{v\in V_1(\kappa)} [\lambda_\kappa(v)]^2 \times \prod_{v\in V_2(\kappa)} [\lambda_\kappa(v)]^2 \times \prod_{v\in V_3(\kappa)} [\lambda_\kappa(v)]^2 \times \prod_{v\in V_4(\kappa)} [\lambda_\kappa(v)]^2 \times \prod_{v\in V_5(\kappa)} \times \prod_{v\in V_6(\kappa)} [\lambda_\kappa(v)]^2\\&\times \prod_{v\in V_7(\kappa)} [\lambda_\kappa(v)]^2 \times \prod_{v\in V_8(\kappa)} [\lambda_\kappa(v)]^2 \times \prod_{v\in V_9(\kappa)} [\lambda_\kappa(v)]^2 \times \prod_{v\in V_{10}(\kappa)} [\lambda_\kappa(v)]^2.\end{aligned} \tag{21}$$

By using the partition of the vertices of the Sudoku graphs with respect to their connection numbers, given in Theorem 1, we get

$$\begin{aligned}M\widehat{Z}C_1(\kappa) &= (5)^{2|V_1|} \times (6)^{2|V_2|} \times (9)^{2|V_3|} \times (10)^{2|V_4|} \times (11)^{2|V_5|} \times (12)^{2|V_6|} \times (13)^{2|V_7|} \times (14)^{2|V_8|} \times (15)^{2|V_9|} \times (16)^{2|V_{10}|}\\M\widehat{Z}C_1(\kappa) &= (5)^{2\times(4)} \times (6)^{2\times(12)} \times (9)^{2\times 8(n-1)} \times (10)^{2\times 8(n-2)} \times (11)^{2\times 8} \times (12)^{2\times(n-2)^2} \times (13)^{2\times 8(n-2)} \times (14)^{2\times 4(n-1)}\\&\times (15)^{2\times 8(n-2)} + \times (16)^{2\times 8(n-2)^2}.\end{aligned} \tag{22}$$

By performing some calculations, we get $M\widehat{Z}C_1(\kappa) = 2^{68n^2-248n+256}\times 3^{2n^2+40n-32}\times 5^{32n-56}\times 7^{8(n-1)}\times 11^{16}\times 13^{16(n-2)}$.

(b) As we know from Definition 4,

$$\begin{aligned}
M\widehat{Z}C_2(\kappa) &= \prod_{uv\in E(\kappa)}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\\
M\widehat{Z}C_2(\kappa) &= \prod_{uv\in E^c_{5,6}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{9,11}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{11,6}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{6,6}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\\
&\times\prod_{uv\in E^c_{11,11}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{13,15}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{9,10}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{9,15}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\\
&\times\prod_{uv\in E^c_{9,9}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{13,14}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{12,16}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{15,16}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\\
&\times\prod_{uv\in E^c_{10,10}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{15,15}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{6,9}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\\
&\times\prod_{uv\in E^c_{11,14}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{6,14}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{11,5}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\\
&\times\prod_{uv\in E^c_{10,13}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{14,15}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{9,13}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\\
&\times\prod_{uv\in E^c_{9,14}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{10,14}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E_{16,16}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\\
&\times\prod_{uv\in E^c_{11,13}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{14,16}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{13,13}}[\lambda_\kappa(u)\times\lambda_\kappa(v)].
\end{aligned}\tag{23}$$

By using connection-based partition of edges of $SK_{n\times n}$ given in Table 1,

$$\begin{aligned}
M\widehat{Z}C_2(\kappa) &= [5\times 6]^{12}\times[9\times 11]^{16}\times[11\times 6]^{16}\times[6\times 6]^{12}\times[11\times 11]^{4}\times[13\times 15]^{2(8n-16)}\\
&\times[9\times 10]^{8(2n-3)}\times[9\times 15]^{2(8n-16)}\times[9\times 9]^{4n-8}\times[13\times 14]^{8n-16}\times[12\times 16]^{8(n-2)^2}\\
&\times[15\times 16]^{8n-16}\times[10\times 10]^{4(n-3)}\times[15\times 15]^{4(n-3)}\times[6\times 9]^{24}\times[11\times 14]^{8}\times[6\times 14]^{12}\\
&\times[11\times 5]^{8}\times[10\times 13]^{2}(8n-16)\times[14\times 15]^{8(n-1)}\times[9\times 13]^{2(8n-16)}\times[9\times 14]^{4n-8}\\
&\times[10\times 14]^{8(n-2)}\times[16\times 16]^{26n^2-110n+116}\times[11\times 13]^{8}\times[14\times 16]^{4(n-2)}\times[13\times 13]^{4(n-3)}.
\end{aligned}\tag{24}$$

By performing some calculations, we get

$$\begin{aligned}
M\widehat{Z}C_2(\kappa) &= 2^{256n^2-944n+980}\times 3^{8n^2+144n-136}\times 5^{104n-188}\\
&\times 7^{32n-36}\times 11^{64}\times 13^{64(n-2)}.
\end{aligned}\tag{25}$$

(c) As we know from Definition 4,

$$\begin{aligned}
M\widehat{Z}C_3(\kappa) &= \prod_{v\in V(\kappa)}[d_\kappa(v)\times\lambda_\kappa(v)]\\
M\widehat{Z}C_3(\kappa) &= \prod_{v\in V_1(\kappa)}[d_\kappa(v)\times\lambda_\kappa(v)]\times\prod_{v\in V_2(\kappa)}[d_\kappa(v)\times\lambda_\kappa(v)]\times\prod_{v\in V_3(\kappa)}[d_\kappa(v)\times\lambda_\kappa(v)]
\end{aligned}$$

$$\times \prod_{v\in V_4(\kappa)}[d_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{v\in V_5(\kappa)}[d_\kappa(v)\times\lambda_\kappa(v)]\times\prod_{v\in V_6(\kappa)}[d_\kappa(v)\times\lambda_\kappa(v)]\times\prod_{v\in V_7(\kappa)}[d_\kappa(v)\times\lambda_\kappa(v)]\times\prod_{v\in V_8(\kappa)}[d_\kappa(v)\times\lambda_\kappa(v)]$$
$$\times \prod_{v\in V_9(\kappa)}[d_\kappa(v)\times\lambda_\kappa(v)]\times\prod_{v\in V_{10}(\kappa)}[d_\kappa(v)\times\lambda_\kappa(v)]\times\prod_{v\in V_{11}(\kappa)}[d_\kappa(v)\times\lambda_\kappa(v)]\times\prod_{v\in V_{12}(\kappa)}[d_\kappa(v)\times\lambda_\kappa(v)]$$
$$\times \prod_{v\in V_{13}(\kappa)}[d_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{v\in V_{14}(\kappa)}[d_\kappa(v)\times\lambda_\kappa(v)]\times\prod_{v\in V_{15}(\kappa)}[d_\kappa(v)\times\lambda_\kappa(v)]. \tag{26}$$

By using degree-connection number-based partition of vertices of $SK_{n\times n}$ as given in Table 3,

$$M\widehat{Z}C_3(\kappa) = (5\times5)^4\times(6\times7)^8\times(6\times8)^4\times(9\times6)^8\times(9\times7)^{4(n-2)}\times(9\times8)^{4(n-2)}\times(10\times6)^{8(n-2)}$$
$$\times(11\times8)^8\times(13\times8)^{8(n-2)}\times(14\times7)^4\times(14\times8)^{4(n-2)}\times(15\times7)^{8(n-2)}\times(16\times7)^{4(n-2)^2}$$
$$\times(16\times8)^{4(n-2)^2}\times(12\times8)^{(n-2)^2}. \tag{27}$$

By performing some calculations, we get

$$M\widehat{Z}C_3(\kappa) = 2^{49n^2-112n+104}\times3^{n^2+28n-24}\times5^{16n-24}\times7^{4n^2}\times11^8\times13^{8(n-2)}. \tag{28}$$

(d) As we know from Definition 4,

$$M\widehat{Z}C_4(\kappa) = \prod_{uv\in E^c_{5,6}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{9,11}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{11,6}}[\lambda_\kappa(u)+\lambda_\kappa(v)]$$
$$\times\prod_{uv\in E^c_{6,6}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{11,11}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{13,15}}[\lambda_\kappa(u)+\lambda_\kappa(v)]$$
$$\times\prod_{uv\in E^c_{9,10}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{9,15}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{9,9}}[\lambda_\kappa(u)+\lambda_\kappa(v)]$$
$$\times\prod_{uv\in E^c_{13,14}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{12,16}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{15,16}}[\lambda_\kappa(u)+\lambda_\kappa(v)]$$
$$\times\prod_{uv\in E^c_{10,10}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{15,15}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{6,9}}[\lambda_\kappa(u)+\lambda_\kappa(v)]$$
$$\times\prod_{uv\in E^c_{11,14}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{6,14}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{11,5}}[\lambda_\kappa(u)+\lambda_\kappa(v)]$$
$$\times\prod_{uv\in E^c_{10,13}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{14,15}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{9,13}}[\lambda_\kappa(u)\times\lambda_\kappa(v)]\times\prod_{uv\in E^c_{9,14}}[\lambda_\kappa(u)+\lambda_\kappa(v)]$$
$$\times\prod_{uv\in E^c_{10,14}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{16,16}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{11,13}}[\lambda_\kappa(u)+\lambda_\kappa(v)]\times\prod_{uv\in E^c_{14,16}}[\lambda_\kappa(u)+\lambda_\kappa(v)]$$
$$\times\prod_{uv\in E^c_{13,13}}[\lambda_\kappa(u)+\lambda_\kappa(v)]. \tag{29}$$

By using connection number-based partition of edges of $SK_{n\times n}$ as given in Table 1,

$$M\widehat{Z}C_4(\kappa) = [5+6]^{12} \times [9+11]^{16} \times [11+6]^{16} \times [6+6]^{12} \times [11+11]^{4} \times [13+15]^{2(8n-16)} \times [9+10]^{8(2n-3)} \times [9+15]^{2(8n-16)} \times [9+9]^{4n-8} \times [13+14]^{8n-16} \times [12+16]^{8(n-2)^2} \times [15+16]^{8n-16} \times [10+10]^{4(n-3)} \times [15+15]^{4(n-3)} \times [6+9]^{24} \times [11+14]^{8} \times [6+14]^{12} \times [11+5]^{8} \times [10+13]^{2}(8n-16) \times [14+15]^{8(n-1)} \times [9+13]^{2(8n-16)} \times [9+14]^{4n-8} \times [10+14]^{8(n-2)} \times [16+16]^{26n^2-110n+116} \times [11+13]^{8} \times [14+16]^{4(n-2)} \times [13+13]^{4(n-3)}. \tag{30}$$

By performing some calculations, we get

$$M\widehat{Z}C_4(\kappa) = 2^{146n^2-470p+456} \times 3^{64n-96} \times 5^{12(n+3)} \times 7^{8n^2-48n+32} \times 11^{16(n-1)} \times 13^{4(n-3)} \times 17^{16} \times 19^{8(2n-3)} + \times 23^{20(n-2)} \times 29^{8(n-1)} \times 31^{8(n-2)}. \tag{31}$$

□

Theorem 5. *Let $\kappa \cong SK_{n\times n}$ be a Sudoku graph. Then, the modified first multiplicative Zagreb connection index, modified second multiplicative Zagreb connection index, and modified third multiplicative Zagreb connection index are as follows:*

(**a**) $M\widehat{Z}C_1^*(\kappa) = 2^{160n^2-472n+440} \times 3^{16n^2+60} \times 5^{16n^2-48n+80} \times 7^{8n^2-16n+32} \times 11^{8(n+1)} \times 13^{4(n-1)} \times 17^{8} \times 19^{16} \times 23^{4(n+2)} \times 29^{8} \times 31^{16(n-2)} \times 41^{8(n-2)} \times 53^{4(n-2)^2} \times 61^{8(n-2)} \times 67^{8(n-2)} \times 79^{16n-32} \times 109^{8n-16} \times 163^{8(n-2)} \times 211^{16(n-2)}$,

(**b**) $M\widehat{Z}C_2^*(\kappa) = 2^{168n^2-504n+556} \times 3^{16n^2+16n-52} \times 5^{16n^2-48n+48} \times 7^{8n^2-16n+16} \times 11^{32n-60} \times 13^{4n^2-12n+12} \times 17^{8} \times 19^{16\{n-2\}} \times 23^{4(n-2)} \times 29^{8} \times 31^{16n-24} \times 41^{24(n-2)} \times 43^{8(n-2)} \times 51^{8} \times 57^{8} \times 59^{8(n-2)} \times 67^{8} \times 71^{16} \times 73^{12} \times 113^{8} \times 167^{8}(n-2)$,

(**c**) $M\widehat{Z}C_3^*(\kappa) = 2^{376n^2-992n+932} \times 3^{8n^2+192n-184} \times 5^{104n-168} \times 7^{28n^2+4n-8} \times 11^{64} \times 13^{64(n-2)}$.

Proof

(a) By Definition 5,

$$M\widehat{Z}C_1^*(\kappa) = \prod_{uv\in E(\kappa)} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)]$$

$$\begin{aligned}M\widehat{Z}C_1^*(\kappa) = &\prod_{uv\in E_{8,8}^{13,14}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)] \times \prod_{uv\in E_{6,8}^{10,9}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)]\\
&\times \prod_{uv\in E_{6,8}^{10,13}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)] \times \prod_{uv\in E_{6,8}^{10,14}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)]\\
&\times \prod_{uv\in E_{7,8}^{9,9}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)] \times \prod_{uv\in E_{7,8}^{9,13}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)]\\
&\times \prod_{uv\in E_{7,7}^{9,15}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)] \times \prod_{uv\in E_{8,8}^{9,14}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)]\\
&\times \prod_{uv\in E_{8,8}^{9,13}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)] \prod_{uv\in E_{8,7}^{13,15}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)]\\
&\times \prod_{uv\in E_{7,8}^{15,9}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)] \times \prod_{uv\in E_{7,8}^{15,14}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)]\\
&\times \prod_{uv\in E_{6,7}^{10,9}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)] \times \prod_{uv\in E_{5,7}^{5,6}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)]\\
&\times \prod_{uv\in E_{7,6}^{6,9}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)] \times \prod_{uv\in E_{5,8}^{5,11}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)]\\
&\times \prod_{uv\in E_{5,8}^{5,6}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)] \times \prod_{uv\in E_{7,7}^{6,6}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)]\\
&\times \prod_{uv\in E_{7,8}^{6,6}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)] \times \prod_{uv\in E_{7,7}^{6,14}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)]\end{aligned}$$

$$\begin{aligned}
&\times \prod_{uv\in E_{7,8}^{6,11}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\times \prod_{uv\in E_{6,8}^{9,6}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&\times \prod_{uv\in E_{6,8}^{9,11}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\times \prod_{uv\in E_{8,8}^{6,11}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&\times \prod_{uv\in E_{8,8}^{11,11}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\times \prod_{uv\in E_{8,7}^{11,14}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&\times \prod_{uv\in E_{8,7}^{6,14}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\times \prod_{uv\in E_{7,7}^{14,15}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&\times \prod_{uv\in E_{7,7}^{15,16}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\times \prod_{uv\in E_{7,7}^{15,15}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&\times \prod_{uv\in E_{6,6}^{9,10}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\times \prod_{uv\in E_{6,6}^{10,10}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&\times \prod_{uv\in E_{8,8}^{11,13}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\times \prod_{uv\in E_{8,8}^{13,13}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&\times \prod_{uv\in E_{8,8}^{14,16}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\times \prod_{uv\in E_{7,8}^{16,16}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&\times \prod_{uv\in E_{6,6}^{9,10}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\times \prod_{uv\in E_{6,6}^{10,10}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&\times \prod_{uv\in E_{8,8}^{11,13}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\times \prod_{uv\in E_{8,8}^{13,13}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&\times \prod_{uv\in E_{8,8}^{14,16}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\times \prod_{uv\in E_{7,8}^{16,16}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&\times \prod_{uv\in E_{8,8}^{16,12}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\times \prod_{uv\in E_{7,8}^{16,12}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\\
&\times \prod_{uv\in E_{7,7}^{16,16}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right]\times \prod_{uv\in E_{8,8}^{16,16}}\left[d_\kappa(u)\lambda_\kappa(v)+d_\kappa(v)\lambda_\kappa(u)\right].
\end{aligned} \tag{32}$$

By using degree-connection number-based partition of edges of $SK_n \times n$ as given in Table 2, we get

$$\begin{aligned}
M\widehat{Z}C_1^*(\kappa) = {} & [216]^{8(n-2)}\times[134]^{8(n-2)}\times[158]^{16(n-2)}\times[164]^{8(n-2)}\times[135]^{4(n-2)}\times[163]^{8(n-2)}\times[168]^{8(n-2)}\\
&\times[184]^{4(n-2)}\times[176]^{8(n-2)}\times[211]^{16(n-2)}\times[183]^{8(n-2)}\times[218]^{8(n-2)}\times[124]^{8(n-2)}\times[65]^{8}\times[99]^{16}\\
&\times[95]^{8}\times[70]^{4}\times[84]^{4}\times[90]^{8}\times[140]^{8}\times[125]^{8}\times[108]^{8}\times[138]^{16}\times[136]^{8}\times[176]^{4}\\
&\times[189]^{8}\times[154]^{4}\times\\
&\times[203]^{8}\times[217]^{8(n-2)}\times[210]^{4(n-3)}\times[154]^{8}\times[120]^{4(n-3)}\times[192]^{8}\times[208]^{4(n-3)}\times[240]^{4(n-2)}\\
&\times[240]^{16(n-2)^2}\times[224]^{4(n-2)^2}\times[212]^{4(n-2)^2}\times[224]^{\{(2n-5)^2-1\}}\times[256]^{\left\{\frac{(2n-5)^2-1}{2}-(2n-4)^2\right\}}.
\end{aligned} \tag{33}$$

By performing some calculations, we get

$$M\widehat{Z}C_1^*(\kappa) = 2^{160n^2-472n+440} \times 3^{16n^2+60} \times 5^{16n^2-48n+80} \times 7^{8n^2-16n+32} \times 11^{8(n+1)} \times 13^{4(n-1)} \times 17^8 \times 19^{16}$$
$$\times 23^{4(n+2)} \times 29^8 \times 31^{16(n-2)} \times 41^{8(n-2)} \times 53^{4(n-2)^2} \times 61^{8(n-2)} \times 67^{8(n-2)} \times 79^{16n-32} \times 109^{8n-16} \times 163^{8(n-2)} \times 211^{16(n-2)}. \quad (34)$$

(b) As we know from Definition 5,

$$M\widehat{Z}C_2^*(\kappa) = \prod_{uv\in E(\kappa)} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]$$

$$\begin{aligned}
M\widehat{Z}C_2^*(\kappa) = &\prod_{uv\in E_{8,8}^{13,14}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{6,8}^{10,9}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{6,8}^{10,13}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{6,8}^{10,14}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{7,8}^{9,9}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{7,8}^{9,13}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{7,7}^{9,15}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{8,8}^{9,14}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{8,8}^{9,13}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{8,7}^{13,15}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{7,8}^{15,9}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{7,8}^{15,14}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{6,7}^{10,9}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{5,7}^{5,6}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{7,6}^{6,9}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{5,8}^{5,11}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{5,8}^{5,6}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)] \times \prod_{uv\in E_{7,7}^{6,6}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{6,7}^{10,9}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{5,7}^{5,6}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{7,6}^{6,9}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{5,8}^{5,11}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{5,8}^{5,6}} [d_\kappa(u)\lambda_\kappa(v) + d_\kappa(v)\lambda_\kappa(u)] \times \prod_{uv\in E_{7,7}^{6,6}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{7,8}^{6,6}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{7,7}^{6,14}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{7,8}^{6,11}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{6,8}^{9,6}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{6,8}^{9,11}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{8,8}^{6,11}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{8,8}^{11,11}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{8,7}^{11,14}} [d_\kappa(u)\lambda_\kappa(u) + d_\kappa(v)\lambda_\kappa(v)]
\end{aligned}$$

$$\begin{aligned}
&\times \prod_{uv\in E_{8,7}^{6,14}} [d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{7,7}^{14,15}} [d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{7,7}^{15,16}} [d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{7,7}^{15,15}} [d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{6,6}^{9,10}} [d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{6,6}^{10,10}} [d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{8,8}^{11,13}} [d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{8,8}^{13,13}} [d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{8,8}^{14,16}} [d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{7,8}^{16,16}} [d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{8,8}^{16,12}} [d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{7,8}^{16,12}} [d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{7,7}^{16,16}} [d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{8,8}^{16,16}} [d_\kappa(u)\lambda_\kappa(u)+d_\kappa(v)\lambda_\kappa(v)].
\end{aligned} \tag{35}$$

By using degree-connection number-based partition of edges of $SK_{n\times n}$ as given in Table 2,

$$\begin{aligned}
M\widehat{Z}C_2^*(\kappa) ={}& [216]^{8(n-2)} \times [132]^{8(n-2)} \times [164]^{16(n-2)} \times [172]^{8(n-2)} \times [135]^{4(n-2)} \times [167]^{8(n-2)} \times [168]^{8(n-2)}\\
&\times [184]^{4(n-2)} \times [176]^{8(n-2)} \times [209]^{16(n-2)} \times [177]^{8(n-2)} \times [217]^{8(n-2)} \times [123]^{8(n-2)} \times [67]^{8} \times [96]^{16} \times [113]^{8}\\
&\times [73]^{4} \times [84]^{4} \times [90]^{8} \times [140]^{8} \times [130]^{8} \times [102]^{8} \times [142]^{16} \times [136]^{8} \times [176]^{4} \times [186]^{8} \times [146]^{4} \times\\
&\times [203]^{8} \times [217]^{8(n-2)} \times [210]^{4(n-3)} \times [114]^{8} \times [120]^{4(n-3)} \times [192]^{8} \times [208]^{4(n-3)}\\
&\times [240]^{4(n-2)} \times [240]^{16(n-2)^2} \times [224]^{4(n-2)^2} \times [208]^{4(n-2)^2} \times [224]^{\{(2n-5)^2-1\}} \times [256]^{\left\{\frac{(2n-5)^2-1}{2} - (2n-4)^2\right\}}.
\end{aligned} \tag{36}$$

By performing some calculations, we get

$$\begin{aligned}
M\widehat{Z}C_2^*(\kappa) ={}& 2^{168n^2-504n+556} \times 3^{16n^2+16n-52} \times 5^{16n^2-48n+48} \times 7^{8n^2-16n+16} \times 11^{32n-60} \times 13^{4n^2-12n+12} \times 17^{8} \times 19^{16\{n-2\}}\\
&\times 23^{4(n-2)} \times 29^{8} \times 31^{16n-24} \times 41^{24(n-2)} \times 43^{8(n-2)} \times 51^{8} \times 57^{8} \times 59^{8(n-2)} \times 67^{8} \times 71^{16} \times 73^{12} \times 113^{8} \times 167^{8(n-2)}.
\end{aligned} \tag{37}$$

(c) As we know from Definition 5,

$$
\begin{aligned}
M\widehat{Z}C_3^*(\kappa) &= \prod_{uv\in E(\kappa)}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
M\widehat{Z}C_3^*(\kappa) &= \prod_{uv\in E_{8,8}^{13,14}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{6,8}^{10,9}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{6,8}^{10,13}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{6,8}^{10,14}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{7,8}^{9,9}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{7,8}^{9,13}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{7,7}^{9,15}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{8,8}^{9,14}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{8,8}^{9,13}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{8,7}^{13,15}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{7,8}^{15,9}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{7,8}^{15,14}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{6,7}^{10,9}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{5,7}^{5,6}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{7,6}^{6,9}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{5,8}^{5,11}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{5,8}^{5,6}}\left[d_\kappa(u)\lambda_\kappa(v)\times d_\kappa(v)\lambda_\kappa(u)\right]\times\prod_{uv\in E_{7,7}^{6,6}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
M\widehat{Z}C_3^*(\kappa) &= \prod_{uv\in E_{8,8}^{13,14}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{6,8}^{10,9}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{6,8}^{10,13}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{6,8}^{10,14}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{7,8}^{9,9}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{7,8}^{9,13}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{7,7}^{9,15}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{8,8}^{9,14}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{8,8}^{9,13}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{8,7}^{13,15}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{7,8}^{15,9}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{7,8}^{15,14}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{6,7}^{10,9}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{5,7}^{5,6}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{7,6}^{6,9}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{5,8}^{5,11}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{5,8}^{5,6}}\left[d_\kappa(u)\lambda_\kappa(v)\times d_\kappa(v)\lambda_\kappa(u)\right]\times\prod_{uv\in E_{7,7}^{6,6}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{7,8}^{6,6}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times{}_{uv\in E_{7,7}^{6,14}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{7,8}^{6,11}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{6,8}^{9,6}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\\
&\quad\times\prod_{uv\in E_{6,8}^{9,11}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]\times\prod_{uv\in E_{8,8}^{6,11}}\left[d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)\right]
\end{aligned}
$$

$$\begin{aligned}
&\times \prod_{uv\in E_{8,8}^{11,11}} [d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{8,7}^{11,14}} [d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{8,7}^{6,14}} [d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{7,7}^{14,15}} [d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{7,7}^{15,16}} [d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{7,7}^{15,15}} [d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{6,6}^{9,10}} [d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{6,6}^{10,10}} [d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{8,8}^{11,13}} [d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{8,8}^{13,13}} [d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{8,8}^{14,16}} [d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{7,8}^{16,16}} [d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{8,8}^{16,12}} [d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{7,8}^{16,12}} [d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)]\\
&\times \prod_{uv\in E_{7,7}^{16,16}} [d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)] \times \prod_{uv\in E_{8,8}^{16,16}} [d_\kappa(u)\lambda_\kappa(u)\times d_\kappa(v)\lambda_\kappa(v)].
\end{aligned} \tag{38}$$

By using degree-connection number based partition of edges of $SK_{n\times n}$ as given in Table 2,

$$\begin{aligned}
M\widehat{Z}C_3^*(\kappa) = {} & [11648]^{8(n-2)}\times[4320]^{8(n-2)}\times[6240]^{16(n-2)}\times[6720]^{8(n-2)}\times[4536]^{4(n-2)}\times[6552]^{8(n-2)}\\
&\times[6615]^{8(n-2)}\times[8064]^{4(n-2)}\times[7488]^{8(n-2)}\times[10920]^{16(n-2)}\times[7560]^{8(n-2)}\times[11760]^{8(n-2)}\times[3780]^{8(n-2)}\\
&\times[1050]^{8}\times[2268]^{16}\times[2200]^{8}\times[1200]^{4}\times[1764]^{4}\times[2016]^{8}\times[4116]^{8}\times[3696]^{8}\times[2592]^{8}\times[4752]^{16}\\
&\times[4224]^{8}\times[7744]^{4}\times[8624]^{8}\times[4704]^{4}\times\\
&\times[10290]^{8}\times[11760]^{8(n-2)}\times[11025]^{4(n-3)}\\
&\times[3240]^{8}\times[3600]^{4(n-3)}\times[9152]^{8}\times[10816]^{4(n-3)}\times[14336]^{4(n-2)}\times[14336]^{16(n-2)^2}\times[12288]^{4(n-2)^2}\\
&\times[10752]^{4(n-2)^2}\times[12544]^{\{(2n-5)^2-1\}}\times[16384]^{\left\{\frac{(2n-5)^2-1}{2}-(2n-4)^2\right\}}.
\end{aligned} \tag{39}$$

By performing some calculations, we get

$$\begin{aligned}
M\widehat{Z}C_3^*(\kappa) = {} & 2^{376n^2-992n+932}\times 3^{8n^2+192n-184}\times 5^{104n-168}\\
&\times 7^{28n^2+4n-8}\times 11^{64}\times 13^{64(n-2)}.
\end{aligned} \tag{40}$$

6. Conclusion

In this paper, we constructed the partitions of the edges of the Sudoku graphs with respect to their connection numbers (see Table 1) and degree-connection numbers (see Table 2) of their end vertices. Moreover, the partition of the vertex set of the Sudoku graph is obtained with respect to the degree and connection number considering at the same time (see Table 3). Finally, we obtained first and second Zagreb connection, modified first, second, third, and fourth Zagreb connection, first, second, third, and fourth multiplicative Zagreb connection, and modified first, second, and third multiplicative Zagreb connection indices that are computed for Sudoku graphs.

Data Availability

All the data are included within this paper. However, the reader may contact the corresponding author for more details of the data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] A. Ali and N. Trinajstić, "A novel/old modification of the first Zagreb index," *Molecular informatics*, vol. 37, pp. e1800008–7, 2018.

[2] U. Ali, M. Javaid, and A. Kashif, "Modified Zagreb connection indices of the T-sum graphs," *Main Group Metal Chemistry*, vol. 43, no. 1, pp. 43–55, 2020.

[3] F. K. Bell, "A note on the irregularity of graphs," *Linear Algebra and Its Applications*, vol. 161, pp. 45–54, 1992.

[4] J. Cao, U. Ali, M. Javaid, and C. Huang, "Zagreb connection indices of molecular graphs based on operations," *Complexity*, vol. 2020, 2020.

[5] De Caen and D. Dominique, "An upper bound on the sum of squares of degrees in a graph," *Discrete Mathematics*, vol. 185, no. 1–3, pp. 245–248, 1998.

[6] S. M. Cioab, "Sums of powers of the degrees of a graph," *Discrete Mathematics*, vol. 306, no. 16, pp. 1959–1964, 2006.

[7] K. Dhanalakshmi, J. A. Jerline, and L. B. M. Raj, "Modified and multiplicative Zagreb indices on graph operators," *J. Comput. Math. Sci.*, vol. 7, no. 4, pp. 225–232, 2016.

[8] K. C. Das, A. Yurttas, M. Togan, A. S. Cevik, and I. N. Cangul, "The multiplicative Zagreb indices of graph operations," *Journal of Inequalities and Applications*, vol. 90, no. 1, pp. 1–14, 2013.

[9] K. C. Das, "Maximizing the sum of the squares of the degrees of a graph," *Discrete Mathematics*, vol. 285, no. 1–3, pp. 57–66, 2004.

[10] M. Eliasi, A. Iranmanesh, and I. Gutman, "Multiplicative versions of first Zagreb index," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 68, pp. 217–230, 2012.

[11] M. Eliasi and D. Vukicevic, "Comparing the multiplicative Zagreb indices," *MATCH Commun. Math. Comput. Chem*, vol. 69, pp. 765–773, 2013.

[12] B. Furtula and I. Gutman, "A forgotten topological index," *Journal of Mathematical Chemistry*, vol. 53, no. 4, pp. 1184–1190, 2015.

[13] H. Gonzalez-Diaz, S. Vilar, L. Santana, and E. Uriarte, "Medicinal chemistry and bioinformatics - current trends in drugs discovery with networks topological indices," *Current Topics in Medicinal Chemistry*, vol. 7, no. 10, pp. 1015–1029, 2007.

[14] I. Gutman and N. Trinajstić, "Graph theory and molecular orbitals. Total φ-electron energy of alternant hydrocarbons," *Chemical Physics Letters*, vol. 17, no. 4, pp. 535–538, 1972.

[15] I. Gutman, B. Ruscic, N. Trinajsti, and C. F. Wilson, "Graph theory and molecular orbitals. XII. Acyclic polyenes," *The Journal of Chemical Physics*, vol. 62, no. 9, pp. 3399–3405, 1975.

[16] I. Gutman, "Degreebased topological indices," *Croatica Chemica Acta*, vol. 86, no. 4, p. 351361, 2013.

[17] I. Gutman and O. E. Polansky, "Chemical graphs," in *Mathematical Concepts in Organic Chemistry*, pp. 19–22, Springer, Berlin, 1986.

[18] L. H. Hall and L. B. Kier, "Molecular connectivity in chemistry and drug research," in *Medicinal Chemistry Series*, G. deStevens, Ed., vol. 14, pp. 1–257, 1976.

[19] J. Hao, "Theorems about Zagreb indices and modified Zagreb indices," *MATCH Commun. Math. Comput. Chem*, vol. 65, pp. 659–670, 2011.

[20] R. S. Haoer, M. A. Mohammed, T. Selvarasan, N. Chidambaram, and N. Devadoss, "Multiplicative leap Zagreb indices of T-thorny graphs," *Eurasian Chemical Communications*, vol. 2, no. 8, pp. 841–846, 2020.

[21] S. Hayat and M. Imran, "Computation of topological indices of certain networks," *Applied Mathematics and Computation*, vol. 240, p. 213228, 2014.

[22] M. Javaid, U. Ali, and K. Siddiqui, "Novel connection based Zagreb indices of several wheel-related graphs," *Computational Journal of Combinatorial Mathematics*, vol. 1, pp. 1–28, 2021.

[23] J. B. Liu, Z. Raza, and M. Javaid, "Zagreb connection numbers for cellular neural networks," *Discrete Dynamics in Nature and Society*, vol. 2020, 2020.

[24] S. Nikoli, G. Kova, Evi cevic, A. Mili, and N. Trinajsti, "Modified zagreb indices," *Croatica Chemica Acta*, vol. 76, pp. 113–124, 2003.

[25] U. N. Peled, R. Petreschi, and A. Sterbini, "(n,e)-Graphs with maximum sum of squares of degrees∗," *Journal of Graph Theory*, vol. 31, no. 4, pp. 283–295, 1999.

[26] R. Todeschini and V. Consonni, *Molecular descriptors for chemoinformatics,* vol. 2, Wiley-VCH Verlag GmbH, Weinheim, Germany.

[27] R. Todescini and V. Consonni, "New local vertex invariants and molecular descriptors based on functions of the vertex degrees," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 64, pp. 359–372, 2010.

[28] J. H. Tang, U. Ali, M. Javaid, and K. Shabbir, "Zagreb connection indices of subdivision and semi-total point operations on graphs," *Journal of Chemistry*, vol. 2019, 2019.

[29] N. Trinajstic, *Chemical Graph Theory*, CRC Press, Boca Raton, 1983.

[30] A. Sattar, M. Javaid, and E. Bonyah, "Connection-based multiplicative zagreb indices of dendrimer nanostars," *Journal of Mathematics*, vol. 2021, 2021.

[31] H. Wiener, "Structural determination of paraffin boiling points," *Journal of the American Chemical Society*, vol. 69, pp. 17–20, 1947.

[32] W. Gao, M. Younas, A. Farooq, A. Mahboob, and W. Nazeer, "M-polynomials and degree-based topological indices of the crystallographic structure of molecules," *Biomolecules*, vol. 8, no. 4, p. 107, 2018.

[33] K. Xu and K. C. Das, "Trees, unicyclic and bicyclic graphs extremal with respect to multiplicative sum Zagreb index," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 68, pp. 257–272, 2012.