1. Introduction

Let $R$ be a ring with identity, and $Z(R)$ be the center of $R$. Let $M$ be a unitary $R$-bimodule. An additive mapping $d$ from $R$ into $M$ is said to be a derivation if $d(ab) = d(a)b + ad(b)$ for all $a, b \in R$. The additive mapping $d: R \times R \rightarrow M$ is called a biderivation if it is a derivation in each component; that is, $d(xy, z) = d(x, z)y + xd(y, z)$ and $d(x, yz) = d(x, y)z + yd(x, z)$ are fulfilled for all $x, y, z \in R$. Likewise, one can further develop the definition of triple derivations in an analogous manner. An additive mapping $d: R \times R \times R \rightarrow M$ is called a triple derivation if it is a derivation in each component; that is,

\[ d(xa, y, z) = d(x, y, z)a + xd(a, y, z), \]
\[ d(x, ya, z) = d(x, y, z)a + yd(x, a, z), \]
\[ d(x, y, za) = d(x, y, z)a + zd(x, y, a), \]

for all $x, y, z, a \in R$. The trivial extension $T(R, M)$ of $R$ by $M$ is the ring

\[ T(R, M) = \{(r, m) : r \in R; m \in M\}, \]

with the componentwise addition and the multiplication given by

\[ (r, m)(r', m') = (rr', rm' + mr'), \quad (r, r' \in R; m, m' \in M). \]

Trivial extensions have been extensively studied in algebra and analysis (see, for instance, [1–7]).

Let $R$ and $S$ be rings with identity, $M$ be a unitary $(R, S)$-bimodule, and $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be the triangular ring determined by $R, S$, and $M$ with the usual addition and multiplication of matrices (see [8, 9] for more details of triangular rings). Then, one can easily verify that $M$ can be made into a unitary $R \times S$-bimodule via the scalar multiplications given by

\[ (r, s)m = rm; \]
\[ m(r, s) = ms, \]
\[ ((r, s) \in R \times S, m \in M). \]

Hence, $T(R \times S, M)$ is the trivial extension of $R \times S$ by $M$. It is clear that triangular rings are examples of trivial extensions. In [1], derivations and biderivations of trivial extensions are studied. Besides the above-mentioned work, biderivations in different backgrounds are studied in [10–12], where further references can be found. A natural question arises: What is the representation of $n$-derivations on trivial extensions? Since the study of $n$-derivations is too complex, we study 3-derivations on trivial extensions and conjecture the representation of $n$-derivations.

2. Main Results and Proofs

We first describe triple derivations of the trivial extension $T(R, M)$. 

In this paper, we describe the structure of triple derivations of trivial extensions. We then apply our results to triangular rings.
Theorem 1. Let $d$ be a triple derivation of a trivial extension $T(R,M), T(R,M)$. Then, there exist

(i) triple derivations $\delta$ of $R$ and $\gamma: R \times R \times R \rightarrow M$

(ii) mappings $\alpha_1: R \times M \times M \rightarrow R$, $\alpha_2: R \times R \times M \rightarrow R$, $\alpha_3: R \times M \times M \rightarrow R$, $\alpha_4: M \times M \rightarrow R$,

$\alpha_5: M \times R \times R \rightarrow R$, $\alpha_6: M \times R \times M \rightarrow M$

(iii) mappings $\beta_1: R \times M \times M \rightarrow M$, $\beta_2: R \times R \times M \rightarrow M$, $\beta_3: M \times M \times M \rightarrow M$, $\beta_4: M \times R \times M \rightarrow M$.

(iv) a mapping $f: M \times M \times M \rightarrow R$ which is a bimodule homomorphism in each coordinate

(v) a mapping $g: M \times M \times M \rightarrow M$

Such that

$$d((r_1,m_1), (r_2,m_2), (r_3,m_3)) = (\delta(r_1,r_2,r_3) + \alpha_1(r_1,m_2,m_3))$$

$$+ \cdots + f(m_1,m_2,m_3) \gamma(r_1,r_2,r_3) + \beta_1(r_1,m_2,m_3) + \cdots + g(m_1,m_2,m_3),$$

holds for all $r_1, r_2, r_3 \in R$ and $m_1, m_2, m_3 \in M$.

Proof. Let $r_1, r_2, r_3 \in R$ and $d((r_1,0), (r_2,0), (r_3,0)) = (\delta(r_1,r_2,r_3), \gamma(r_1,r_2,r_3))$. For any $r_1, r_2, r_3, r' \in R$, we have

$$d((r_1,0), (r_2,0), (r_3,0)) = (r_1,0) d((r_2,0), (r_3,0)) + d((r_1,0), (r_2,0), (r_3,0))(r',0)$$

$$= (r_1,0) (\delta(r_2,0,0), \gamma(r_2,0,0)) + (\delta(r_1,0,0), \gamma(r_1,0,0)) (r',0)$$

$$= (r_1,0) (\delta(r_1,0,0), \gamma(r_1,0,0)) + (\delta(r_1,0,0), \gamma(r_1,0,0)) (r',0)$$

Therefore, $\delta$ and $\gamma$ are derivations in the first coordinate. Similarly, we get that $\delta$ and $\gamma$ are derivations in the second and third coordinates. This proves (i).

To prove (ii) and (iii), let $r_1 \in R, m_2, m_3 \in M$ and set

$$d((r_1,0), (0,m_2), (0,m_3)) = (\alpha_1(r_1,m_2,m_3), \beta_1(r_1,m_2,m_3)).$$

Let $r'$ be also in $R$. Then,

$$\alpha_1(r_1,r',m_2,m_3), \beta_1(r_1,r',m_2,m_3)) = d((r_1,0), (0,m_2), (0,m_3))$$

$$= (r_1,0) (\alpha_1(r_1,r',m_2,m_3), \beta_1(r_1,r',m_2,m_3)) + (\alpha_1(r_1,m_2,m_3), \beta_1(r_1,m_2,m_3)) (r',0)$$

$$= (r_1,0) (\alpha_1(r_1,r',m_2,m_3), \beta_1(r_1,r',m_2,m_3)) + (\alpha_1(r_1,m_2,m_3), \beta_1(r_1,m_2,m_3)) (r',0).$$

It follows from the above expression that $\alpha_1, \beta_1$ are derivations in the first coordinate. Similarly, define

$$d((r_1,0), (r_2,0), (0,m_3)) = (\alpha_2(r_1,r_2,m_3), \beta_2(r_1,r_2,m_3)),$$

$$d((r_1,0), (0,m_2), (r_3,0)) = (\alpha_3(r_1,m_2,r_3), \beta_3(r_1,m_2,r_3)),$$

$$d((0,m_1), (r_2,0), (r_3,0)) = (\alpha_4(m_1,m_2,r_3), \beta_4(m_1,m_2,r_3)),$$

$$d((0,m_1), (r_2,0), (r_3,0)) = (\alpha_5(m_1,r_2,r_3), \beta_5(m_1,r_2,r_3)),$$

$$d((0,m_1), (r_2,0), (0,m_3)) = (\alpha_6(m_1,r_2,m_3), \beta_6(m_1,r_2,m_3)).$$

We get that $\alpha_i, \beta_i$ are derivations in the coordinate of the element in $R, i \in \{1, 2, 3, 4, 5, 6\}.$ Moreover, we have
for all $r_1, r' \in R, m_2, m_3 \in M$. On the other hand, we get

\[
\begin{align*}
(a_1(r_1, r'm_2, m_3), \beta_1(r_1, r'm_2, m_3)) &= d((r_1, 0), (0, r'm_2), (0, m_3)) \\
&= d((r_1, 0), (r', 0)(m_2), (0, m_3)) \\
&= (r', 0)d((r_1, 0), (0, m_2), (0, m_3)) + d((r_1, 0), (r', 0), (0, m_1))(0, m_2) \\
&= (r' \alpha_1(r_1, m_2, m_3), r' \beta_1(r_1, m_2, m_3) + \alpha_2(r_1, r', m_3)m_2),
\end{align*}
\]

for all $r_1, r' \in R, m_2, m_3 \in M$. According to the above relations, we obtain that $\alpha_1$ is a bimodule homomorphism in the second coordinate, and

\[
\begin{align*}
\beta_1(r_1, r'm_2, m_3) &= r' \beta_1(r_1, m_2, m_3) + \alpha_2(r_1, r', m_3)m_2, \\
\beta_1(r_1, m_2r', m_3) &= m_2 \alpha_2(r_1, r', m_3) + \beta_1(r_1, m_2, m_3)r', \\
\beta_1(r_1, m_2, m_3') &= \alpha_3(r_1, m_2, m_3') + \alpha_2(r_1, m_2, r')m_3,
\end{align*}
\]

for all $r_1, r' \in R, m_2, m_3 \in M$. Likewise, $\alpha_i$ is a bimodule homomorphism in the coordinate of the element in $M$ and

\[
\begin{align*}
\beta_1(r_1, m_2, m_3') &= r' \beta_1(r_1, m_2, m_3) + \alpha_3(r_1, m_2, r')m_3, \\
\beta_2(r_1, r_2, m_3') &= r' \beta_2(r_1, r_2, m_3) + \delta(r_1, r_2, r')m_3, \\
\beta_2(r_1, r_2, m_2') &= \beta_2(r_1, r_2, m_3) + m_2 \delta(r_1, r_2, r'), \\
\beta_3(r_1, r_2m_3, r_3) &= r' \beta_3(r_1, r_2, m_3) + \delta(r_1, r_2, r')m_2, \\
\beta_3(r_1, m_2r_3, r_3) &= \beta_3(r_1, m_2, m_3) + m_2 \delta(r_1, r_2, r'), \\
\beta_3(r_1, m_2, r_3') &= \beta_3(r_1, m_2, r_3) + m_2 \delta(r_1, r_2, r_3), \\
\beta_4(r'm_1, m_2, r_3) &= r' \beta_4(m_1, m_2, r_3) + \alpha_3(r', m_2, r_3)m_2, \\
\beta_4(m_1', m_2, r_3) &= \beta_4(m_1, m_2, r_3) + m_1 \alpha_3(r', m_2, r_3).
\end{align*}
\]

for all $r_1, r_2, r_3, r' \in R, m_1, m_2, m_3 \in M$. To prove (iv) and (v), let $m_1, m_2, m_3 \in M$ be arbitrary, and assume

\[
d((0, m_1), (0, m_2), (0, m_3)) = (f(m_1, m_2, m_3), g(m_1, m_2, m_3)).
\]
Since

\[(f(m_1 r', m_2, m_3), g(m_1 r', m_2, m_3)) = d((0, m_1 r'), (0, m_3))
= d((r', 0) (0, m_1) (0, m_3))
= (r', 0) (f(m_1, m_2, m_3), g(m_1, m_2, m_3)) + (a_1(r', m_2, m_3), \beta_1(r', m_2, m_3))(0, m_1)
= (f(m_1, m_2, m_3), g(m_1, m_2, m_3)) + (r', 0) (m_1, m_3) + a_1(r', m_2, m_1),\]

(30)

for all \(r' \in R, m_1, m_2, m_3 \in M\), we have that \(f: M \times M \rightarrow R\) is a bimodule homomorphism in the first coordinate, and

\[g(r_1 m_1, m_2, m_3) = r' g(m_1, m_2, m_3) + a_1(r', m_2, m_3) m_1,
\]

\[g(m_1 r', m_2, m_3) = m_1 a_1(r', m_2, m_3) + g(m_1, m_2, m_3)r',\]

(31)

for all \(r' \in R, m_1, m_2, m_3 \in M\). Similarly, we get that \(f: M \times M \rightarrow R\) is a bimodule homomorphism in the second and third coordinates, and

\[d((r_1, m_1), (r_2, m_2), (r_3, m_3)) = (\delta(r_1, r_2, r_3) + a_1(r_1, m_2, m_3)
+ \cdots + g(m_1, m_2, m_3)) + \beta_1(r_1, m_2, m_3) + \cdots + g(m_1, m_2, m_3)),\]

(33)

for all \(r_1, r_2, r_3 \in R\) and \(m_1, m_2, m_3 \in M\). □

According to Theorem 1, we can decompose the triple derivation \(d\) on \(T(R,M)\) into the sum of five triple derivations.

\[D((r_1, m_1), (r_2, m_2), (r_3, m_3)) = (\delta(r_1, r_2, r_3) + a_1(r_1, m_2, m_3) + \beta_3(r_1, m_2, r_3) + \beta_5(m_1, r_2, r_3)),\]

\[F((r_1, m_1), (r_2, m_2), (r_3, m_3)) = (f(m_1, m_2, m_3), 0),\]

\[G((r_1, m_1), (r_2, m_2), (r_3, m_3)) = (a_1(r_1, m_2, m_3) + a_4(m_1, m_2, r_3) + a_6(m_1, r_2, m_3), g(m_1, m_2, m_3)),\]

\[H((r_1, m_1), (r_2, m_2), (r_3, m_3)) = (a_2(r_1, r_2, m_3) + a_3(r_1, m_2, r_3) + a_5(m_1, r_2, r_3), \beta_1(r_1, m_2, m_3)
+ \beta_4(m_1, m_2, r_3) + \beta_6(m_1, r_2, m_3)),\]

\[K((r_1, m_1), (r_2, m_2), (r_3, m_3)) = (0, \gamma(r_1, r_2, r_3)),\]

(34)

are triple derivations of \(T(R,M)\), and \(d = D + F + G + H + K\).

Next, we will use Theorem 1 to study triple derivations of a triangular ring \(T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}\) identified with the trivial extension \(T(R \times S, M)\).

**Corollary 1.** Let \(d\) and \(T(R, M)\) be as above. Then, the mappings

\[D, F, G, H, K: T(R, M) \times T(R, M) \times T(R, M) \rightarrow T(R, M)\]

are triple derivations of \(T(R, M)\).

**Theorem 2.** Let \(d\) be a triple derivation of the trivial extension \(T(R \times S, M)\). Then,

(i) there exist triple derivations \(\delta_1\) of \(R\) and \(\delta_2\) of \(S\) such that, for every \(r_1, r_2, r_3 \in R\) and \(s_1, s_2, s_3 \in S\), we have

\[\delta((r_1, s_1), (r_2, s_2), (r_3, s_3)) = (\delta_1(r_1, r_2, r_3), \delta_2(s_1, s_2, s_3)),\]

(35)
(ii) there exists an element $m^* \in M$ such that, for every $r_1, r_2, r_3 \in R$ and $s_1, s_2, s_3 \in S$, we have
\[
\gamma((r_1, s_1), (r_2, s_2), (r_3, s_3)) = r_1 r_2 r_3 m^* + r_1 m^* s_3 s_2 \\
+ r_2 m^* s_1 s_3 + r_3 m^* s_1 s_2 \\
- r_1 r_2 m^* s_3 - r_1 r_3 m^* s_2 \\
- r_2 r_3 m^* s_1 - m^* s_1 s_2 s_3,
\]
(36)

\[
\beta_1((r_1, s_1), m_2, m_3) = r_1 h_1(m_2, m_3) - h_1(m_2, m_3) s_1, \\
\beta_4(m_1, m_2, (r_3, s_1)) = r_3 h_4(m_1, m_2) - h_4(m_1, m_2) s_1, \\
\beta_6(m_1, (r_2, s_2), m_3) = r_2 h_6(m_1, m_3) - h_6(m_1, m_3) s_2, \\
\beta_2((r_1, s_1), (r_2, s_2), m_3) = r_1 r_2 h_2(m_3) - r_1 h_2(m_3) s_2 - r_2 h_2(m_3) s_1 + h_2(m_3) s_1 s_2, \\
\beta_3((r_1, s_1), m_2, (r_3, s_3)) = r_1 r_3 h_3(m_2) - r_1 h_3(m_2) s_3 - r_3 h_3(m_2) s_1 + h_3(m_2) s_1 s_3, \\
\beta_5(m_1, (r_2, s_2), (r_3, s_3)) = r_2 r_3 h_5(m_1) - r_2 h_5(m_1) s_3 - r_3 h_5(m_1) s_2 + h_5(m_1) s_2 s_3,
\]
where $h_1, h_4, h_6: M \times M \to M, h_2, h_3, h_5: M \to M$ are mappings

\[
\delta((1, 0), (r, 0), (0, s)) = \delta((1, 0), (1, 0), (r, 0), (0, s)) \\
= (1, 0) \delta((1, 0), (r, 0), (0, s)) + \delta((1, 0), (r, 0), (0, s))(1, 0),
\]
(38)

**Proof.** (i) Since $\delta$ is a triple derivation on $R \times S$, we have
\[
\delta((1, 0), (r, 0), (0, s)) = \delta((1, 0), (1, 0), (r, 0), (0, s)) \\
= (1, 0) \delta((1, 0), (r, 0), (0, s)) \\
+ \delta((1, 0), (r, 0), (0, s))(1, 0),
\]
(39)
for all $r \in R, s \in S$. It follows that
\[
\delta((1, 0), (r, 0), (0, s)) = 0. 
\]
(40)
Similarly, we get that
\[
\delta((1, 0), (r_2, 0), (r_3, 0)) = \delta((r_1, 0), (r_2, 0), (0, 1)) = 0. 
\]
(41)

Set $\delta((r_1, 0), (r_2, 0), (0, s)) = (t, z) \in R \times S$. In view of (40), we have
\[
(t, z) = \delta((r_1, 0), (1, 0), (r_2, 0), (0, s)) \\
= \delta((r_1, 0), (r_2, 0), (0, s))(1, 0) \\
+ (r_1, 0) \delta((1, 0), (r_2, 0), (0, s))
\]
(42)
This implies that $z = 0$. On the other hand, we have
\[ (t, z) = \delta((r_1, 0), (r_2, 0), (0, s_3)) (0, 1) \]
\[ = (0, s_2)\delta((r_1, 0), (r_2, 0), (0, 1)) \]
\[ + \delta((r_1, 0), (r_2, 0), (0, s_3)) (0, 1) \]
\[ = (t, z) (0, 1) = (0, z). \]

This yields that \( t = 0. \) Hence, \( \delta((r_1, 0), (r_2, 0), (0, s_3)) = 0. \) Likewise,
\[ \delta((r_1, 0), (0, s_2), (0, s_3)) \]
\[ = \delta((r_1, 0), (0, s_2), (r_3, 0)) \]
\[ = \delta((0, s_1), (r_2, 0), (0, s_3)) \]
\[ = \delta((0, s_1), (r_2, 0), (0, s_3)) = 0. \]

Assume that \( \delta((r_1, 0), (r_2, 0), (r_3, 0)) = (u, v). \) By (41), we obtain
\[ \delta((r_1, 0), (r_2, 0), (r_3, 0)) = \delta((r_1, 0)(1, 0), (r_2, 0), (r_3, 0)) \]
\[ = (r_1, 0, 0, 0, 0, 0) \]
\[ + \delta((r_1, 0), (r_2, 0), (r_3, 0))(1, 0) = (u, 0). \]

Therefore, we define \( \delta((r_1, 0), (r_2, 0), (r_3, 0)) = (\delta_1, (r_1, r_2, r_3), 0). \) It follows that
\[ \delta((r_1, r_2, r_3), 0) = \delta((r_1, 0)(r_2, 0), (r_3, 0)) \]
\[ = (r_1, 0, 0, 0, 0, 0) \]
\[ + \delta((r_1, 0), (r_2, 0), (r_3, 0)) (r_1, 0) \]
\[ = (r_1, 0, 0, 0, 0, 0) = (0, 0, 0). \]

Thus, \( \delta_1 \) is a derivation in the first coordinate. Similarly, \( \delta_2 \) is a derivation in the second and third coordinates, so that \( \delta_3 \) is a triple derivation of \( R. \) By an analogue computation, one can show that there exists a triple derivation \( \delta_2 \) of \( S \) such that
\[ \delta((0, s_1), (0, s_2), (0, s_3)) = (0, 0, 0). \]

Hence, \( \delta((r_1, s_1), (r_2, s_2), (r_3, s_3)) = (\delta_1, (r_1, r_2, r_3), \delta_2, (s_1, s_2, s_3)). \)

(iii) Assume that \( \gamma((1, 0), (1, 0), (1, 0)) = m^{*} \in M, \) we have
\[ \gamma((1, 0), (1, 0), (0, 1)) = -m^{*}, \gamma((1, 0), (0, 1), (1, 0)) = -m^{*}, \]
\[ \gamma((0, 1), (1, 0), (1, 0)) = m^{*}, \gamma((0, 1), (0, 1), (1, 0)) = m^{*}, \]
\[ \gamma((1, 0), (1, 0), (1, 0)) = -m^{*}. \]

Since \( \gamma \) is a triple derivation, it follows that
\[ \gamma((1, 0), (r, 0), (1, 0)) = \gamma((1, 0), (1, 0), (1, 0)) \]
\[ = (r, 0) \gamma((1, 0), (1, 0), (1, 0)) \]
\[ + \gamma((1, 0), (r, 0), (1, 0))(1, 0) \]
\[ = rm^{*}, \]
for all \( r \in R. \) According to the above relation, we have
\[ \gamma((r_1, 0), (r_2, 0), (1, 0)) = \gamma((r_1, 0)(1, 0), (r_2, 0), (1, 0)) \]
\[ = (r_1, 0) \gamma((1, 0), (1, 0), (1, 0)) \]
\[ + \gamma((r_1, 0), (r_2, 0), (1, 0))(1, 0) \]
\[ = r_1r_2m^{*}, \]
for all \( r_1, r_2 \in R. \) This yields that
\[ \gamma((r_1, 0), (r_2, 0), (r_3, 0)) = \gamma((r_1, 0), (r_2, 0), (r_3, 0))(1, 0) \]
\[ = (r_3, 0) \gamma((1, 0), (1, 0), (1, 0)) \]
\[ + \gamma((r_1, 0), (r_2, 0), (r_3, 0))(1, 0) \]
\[ = r_3r_1r_2m^{*}, \]
for all \( r_1, r_2, r_3 \in R. \) Similarly, we get that
\[ \gamma((r_1, 0), (r_2, 0), (r_3, 0)) = r_1r_2r_3m^{*} = r_1r_2r_3m^{*}, \]
\[ = r_2r_1r_3m^{*} = r_2r_1r_3m^{*} \]
\[ = r_3r_2r_1m^{*}, \]
for all \( r_1, r_2, r_3 \in R. \) Analogue computation shows that, for every \( r_1, r_2, r_3 \in R \) and \( s_1, s_2, s_3 \in S, \) we have
\[ y((r_1, 0), (r_2, 0), (0, s_3)) = -r_1 r_2 m^* s_3 = -r_2 r_1 m^* s_3, \]
\[ y((r_1, 0), (0, s_2), (0, s_3)) = r_1 m^* s_2 s_3 = r_1 m^* s_2 s_3, \]
\[ y((r_1, 0), (0, s_2), (r_3, 0)) = -r_1 r_3 m^* s_2 = -r_3 r_1 m^* s_2, \]
\[ y((0, s_1), (r_2, 0), (r_3, 0)) = -r_2 r_3 m^* s_1 = -r_3 r_2 m^* s_1, \]
\[ y((0, s_1), (0, s_2), (r_3, 0)) = r_m^* s_1 s_3 = r_m^* s_3 s_1, \]
\[ y((0, s_1), (0, s_2), (0, s_3)) = m^* s_1 s_2 s_3 = -m^* s_2 s_1 s_2 \]
\[ = -m^* s_2 s_1 s_3 \]
\[ = -m^* s_2 s_1 s_2 \]
\[ = -m^* s_2 s_1. \]

This proves (ii).

(iii) Since \( f: M \times M \times M \rightarrow R \times S \) is a bimodule homomorphism in each coordinate, we have
\[
\begin{align*}
f(m_1, m_2, m_3) &= f((0, 0)m_1 (0, 1), m_2, m_3) \\
&= (1, 0) f(m_1, m_2, m_3) (0, 1) = (0, 0). \quad (54)
\end{align*}
\]

Since \( a_i \) is a bimodule homomorphism in the coordinate of the element in \( M \), we get \( a_i = 0 \), \( i \in \{1, 2, 3, 4, 5, 6\}. \)

(iv) Recalling that \( \beta_i: (R \times S) \times M \rightarrow M \) is a derivation in the first coordinate, for every \( r_1, r_2 \in R, s_1 \in S, m_2, m_3 \in M \), we have
\[
\begin{align*}
\beta_1((r_1, 0), m_2, m_3) &= \beta_1((r_1, 0), (1, 0), m_2, m_3) \\
&= \beta_1((r_1, 0), m_2, m_3)((1, 0) + (r_1, 0)) \\
&= r_1 \beta_1((1, 0), m_2, m_3). \quad (55)
\end{align*}
\]
\[
\begin{align*}
\beta_1((0, s_1), m_2, m_3) &= \beta_1((0, 1), (0, s_1), m_2, m_3) \\
&= (0, 1) \beta_1((0, s_1), m_2, m_3) + \beta_1((0, 1), m_2, m_3)) (0, s_1)) \\
&= \beta_1((0, 1), m_2, m_3) s_1. \quad (56)
\end{align*}
\]

Set \( \beta_1((1, 0), m_2, m_3) = h_1(m_2, m_3). \) It follows that \( \beta_1((1, 0), m_2, m_3) = -h_1(m_2, m_3). \) Using (55) and (56), we have
\[
\begin{align*}
\beta_1((r_1, s_1), m_2, m_3) &= r_1 h_1(m_2, m_3) - h_1(m_2, m_3)s_1. \quad (57)
\end{align*}
\]

Similarly, there exist two mappings \( h_4, h_6: M \times M \rightarrow M \) such that
\[
\begin{align*}
h_4(m_1, m_2, (r_3, s_1)) &= r_3 h_4(m_1, m_2) - h_4(m_1, m_2)s_1, \\
h_6(m_1, (r_2, s_2), m_3) &= r_2 h_6(m_1, m_3) - h_6(m_1, m_3)s_2. \quad (58)
\end{align*}
\]

for all \( r_2, r_3 \in R, s_2, s_1 \in S, m_1, m_2, m_3 \in M. \)

Set \( \beta_2((1, 0), (1, 0), m_3) = h_2(m_3). \) Since \( \beta_2: (R \times S) \times (R \times S) \times M \rightarrow M \) is a derivation in the first and second coordinates, it follows that
\[
\begin{align*}
\beta_2((r_1, 0), (r_2, 0), m_3) &= \beta_2((r_1, 0), (1, 0), (r_2, 0), m_3) \\
&= (r_1, 0)h_2((1, 0), (r_2, 0), m_3)) (1, 0) \\
&= (r_1, 0)h_2((1, 0), (r_2, 0), m_3) + r_1 r_2 h_2(m_3) \quad (59)
\end{align*}
\]
\[
\begin{align*}
\beta_2((0, s_1), (0, s_2), m_3) &= \beta_2((0, s_1), (0, 1)) (0, s_2), m_3) \\
&= (0, 1) h_2((0, s_1), (0, s_2), m_3) + r_1 r_2 h_2(m_3) s_2 s_1 \quad (59)
\end{align*}
\]
\[
\begin{align*}
\beta_2((0, s_1), (0, s_2), m_3) &= \beta_2((0, 1), (0, s_2), m_3) s_2 s_1 \\
&= h_2(m_3) s_2 s_1. \quad (60)
\end{align*}
\]

for all \( r_1, r_2 \in R, s_1, s_2 \in S, m_3 \in M. \) Similarly, we have
\[
\begin{align*}
\beta_2((r_1, 0), (r_2, 0), m_3) &= r_2 r_3 h_2(m_3), \\
\beta_2((0, s_1), (0, s_2), m_3) &= h_2(m_3) s_2 s_1, \quad (61)
\end{align*}
\]

for all \( r_1, r_2 \in R, s_1, s_2 \in S, m_3 \in M. \) Therefore,
\[
\begin{align*}
\beta_2((r_1, 0), (r_2, 0), m_3) &= r_1 r_2 h_2(m_3) - r_1 r_2 h_2(m_3) \quad (59)
\end{align*}
\]
\[
\begin{align*}
\beta_2((r_1, 0), (r_2, 0), m_3) &= r_1 r_2 h_2(m_3) - r_1 r_2 h_2(m_3) s_1 + h_2(m_3) s_2 s_1. \quad (61)
\end{align*}
\]

for all \( r_1, r_2 \in R, s_1, s_2 \in S, m_3 \in M. \) Likewise, there exist two mappings \( h_3, h_5: M \rightarrow M \) such that
\[
\begin{align*}
\beta_3((r_1, s_1), (m_2, (r_3, s_2)) &= r_3 h_3(m_2) - r_1 h_3(m_2) s_3 \quad (58)
\end{align*}
\]
\[
\begin{align*}
\beta_3(m_1, (r_2, s_2), (r_3, s_3)) &= r_3 h_5(m_1) - r_2 h_5(m_1) s_3 \quad (58)
\end{align*}
\]

for all \( r_1, r_2, r_3 \in R, s_1, s_2, s_3 \in S, m_1, m_2 \in M. \) In addition, we have
\[
\begin{align*}
r_1 r_2 h_3(m_2) &= r_1 r_2 h_3(m_2), \quad h_3(m_2) s_1 s_3, \\
r_1 r_2 h_5(m_1) &= r_3 h_5(m_1) + h_5(m_1) s_2 s_1, \quad (60)
\end{align*}
\]

By (11), we have
Similarly, according to (12)–(28), we obtain that $h_i \in \text{extensions can be decomposed into the sum of several}$

$$h_1(r m_2, m_3) = \beta_1 ((1,0), r m_2, m_3) = \beta_1 ((1,0), (r,0)m_2, m_3)$$

$$= (r,0)\beta_1 ((1,0), m_2, m_3) = r\beta_1 ((1,0), m_2, m_3)$$

$$= rh_1 (m_2, m_3),$$

(64)

for all $r \in R, m_2, m_3 \in M$. Hence, $h_i$ is a left $R$-module on $M$. Similarly, according to (12)–(28), we obtain that $h_i$ is a bimodule homomorphism in each coordinate on $M$, $i \in \{1,2,3,4,5,6\}$. Since

$$r'' r'' h_i (m_2, m_3) = h_i (r'' m_2, r'' m_3) = r'' r h_i (m_2, m_3),$$

(65)

we get $[r', r''] h_i (m_2, m_3) = 0$ for all $r', r'' \in R, m_2, m_3 \in M$. Likewise,

$$[r', r''] h_2 (m_1, m_2) = [r', r''] h_6 (m_1, m_3) = h_1 (m_2, m_3) [s', s'']$$

$$= h_4 (m_1, m_2) [s', s''] = h_6 (m_1, m_3) [s', s''] = 0,$$

(66)

for all $r', r'' \in R, s', s'' \in S, m_1, m_2, m_3 \in M$. 

(v) Using (i), (15) and (61), we arrive at

$$r_1 r_2 r_3 h_2 (m_3) = \beta_2 ((r_1, 0), (r_2, 0), (r_3, 0)m_3)$$

$$= \beta_2 ((r_1, 0), (r_2, 0), (r_3, 0), m_3)$$

$$+ \delta ((r_1, 0), (r_2, 0), (r_3, 0)) m_3$$

$$= r_3 r_1 r_2 h_2 (m_3) + (\delta_1 (r_1, r_2, r_3), \delta_2 (0, 0, 0)) m_3$$

$$= r_3 r_1 r_2 h_2 (m_3) + \delta_1 (r_1, r_2, r_3) m_3,$$

(67)

for all $r_1, r_2, r_3 \in R, m_1, m_2, m_3 \in M$. According to (59) and (60), we have $\delta_1 (r_1, r_2, r_3)m_3 = 0$. Similarly,

$$m_3 \delta_2 (s_1, s_2, s_3) = \delta_1 (r_1, r_2, r_3) m_1 = \delta_1 (r_1, r_2, r_3) m_2$$

$$= m_1 \delta_2 (s_1, s_2, s_3) = m_2 \delta_2 (s_1, s_2, s_3) = 0,$$

(68)

for all $r_1, r_2, r_3 \in R, s_1, s_2, s_3 \in S$. \hfill $\Box$

Corollary 2. Let $d$ and $T (R \times S, M)$ be as above. Then, the mappings

$$D, F, G, H: T (R \times S, M) \times T (R \times S, M) \times T (R \times S, M)$$

$$\longrightarrow T (R \times S, M)$$

are derivations of $T (R \times S, M)$, and $d = D + F + G + H$, for all $r_1, r_2, r_3 \in R, s_1, s_2, s_3 \in S, m_1, m_2, m_3 \in M$, where $m^* \in M$.

In conclusion, we conjecture that $n$-derivations on trivial extensions can be decomposed into the sum of several $n$-derivations.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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