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Research Article

Exponential Dichotomy of the Two-Dimensional Linear Differential System

Ni Hua, Wang Jing, and Bai Min

School of Mathematical Sciences, Jiangsu University, Zhenjiang 212013, Jiangsu, China

Correspondence should be addressed to Ni Hua; nihua979@126.com

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This paper deals with a class of two-dimensional linear differential systems with periodic coefficients. A sufficient condition for the exponential dichotomy of the linear system is given by the fixed point theorem and variable substitution, and some new results are obtained.

1. Introduction

Consider the second-order linear differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+p(t)\frac{\mathrm{d}x}{\mathrm{d}t}+q(t)x=0. \tag{1}$$

Equation (1) is often encountered in engineering, elasticity, electricity, and various oscillation problems.

Because of its importance, scholars have never stopped studying Equation (1) [1–4], and some scholars used transformation as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=y. \tag{2}$$

Then Equation (1) becomes

$$\begin{pmatrix}\dfrac{\mathrm{d}x}{\mathrm{d}t}\\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t}\end{pmatrix}=\begin{pmatrix}0 & 1\\ -q(t) & -p(t)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}. \tag{3}$$

Huang [5] used Lyapunov's direct method to study the stability of the ordinary solution of Equation (3) and thereby obtained some sufficient conditions for the stability of the ordinary solution of Equation (1).

Exponential dichotomy is an important property of linear differential equation. The theory of exponential dichotomy is a generalization of the concept of hyperbolic rate of linear autonomous equation in linear nonautonomous equation and plays an important role in the analysis of nonautonomous equation. The theory of exponential dichotomy of linear differential equations can be traced back to Perron's study of the stability of linear differential equations and the existence of bounded solutions of nonlinear differential equations. In 1934, Li established the theory of exponential dichotomy on the linear difference equation. The theory of exponential dichotomy has been playing an important role in the qualitative and stability research of differential and difference equations and has been applied to various research fields of mathematics with its rich theoretical ideas and complex mathematical skills [6].

Consider the homogeneous linear differential system

$$\frac{\mathrm{d}x}{\mathrm{d}t}=A(t)x, \tag{4}$$

where $A(t)$ is a square matrix of order n, and $A(t)$ is continuous on $\mathbb{R}$; if there is a projection P and positive constants $K\geq 1, \alpha>0$, then

$$\begin{aligned}\left\|X(t)PX^{-1}(s)\right\|&\leq K\exp(-\alpha(t-s)),\ (t\geq s),\\ \left\|X(t)(I-P)X^{-1}(s)\right\|&\leq K\exp(\alpha(t-s)),\ (s\geq t),\end{aligned} \tag{5}$$

where $X(t)$ is a fundamental solution matrix of (4), I is the n-order unit matrix, then (4) is said to have exponential dichotomy on $\mathbb{R}$. The exponential dichotomy of linear system on the whole axis is a powerful tool in the stability theory and also a very useful tool in the study of (almost) periodic differential systems [7].

Consider the following two-dimensional linear differential system

$$\begin{pmatrix} \dfrac{dx}{dt} \\ \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{6}$$

Remark 1. If $a(t) \equiv 0, b(t) \equiv 1$, then Equation (6) becomes Equation (3). So, without losing generality, we just need to discuss Equation (6).

Shi [8] established some conclusions on the sign of characteristic exponents directly in terms of the coefficients. Theorem 1 in Ref. [8] is as follows:

Theorem 1 (see [8]). *Consider Equation (6), $a(t), b(t), c(t)$, and $d(t)$ are ω-periodic continuous functions on R, $c(t)$ and $d(t)$ have continuous derivatives on $\mathbb{R}$, and assume that the following conditions hold true:*

$$\begin{aligned} &(H_1) \int_0^{\omega} (a(t) + d(t))dt \neq 0, \\ &(H_2) c(t) \neq 0, \\ &(H_3) \frac{d(t)c'(t)}{c(t)} + a(t)d(t) - b(t)c(t) - d'(t) < 0. \end{aligned} \tag{7}$$

Then the characteristic exponents of system (6) must be positive and negative.

In this paper, we are devoted to obtaining the sufficient conditions for the exponential dichotomy of system (6) and generalized Theorem 1 in [8].

2. Some Lemmas and Abbreviations

Consider the following equation

$$\frac{dx}{dt} = A(t)x + f(t). \tag{8}$$

Here, $A(t)$ is an ω-periodic continuous n-order square matrix function on $\mathbb{R}$, $f(t)$ is an ω-periodic continuous n-dimensional vector function on $\mathbb{R}$, and the corresponding homogeneous linear system of Equation (8) is as follows:

$$\frac{dx}{dt} = A(t)x. \tag{9}$$

Lemma 1 (see [7]). *If $A(t)$ is an ω-periodic continuous n-order square matrix function on $\mathbb{R}$, $f(t)$ is an ω-periodic n-dimensional continuous vector function on $\mathbb{R}$, and (9) has exponential dichotomy (5), then the nonhomogeneous linear periodic system (8) has a unique ω-periodic continuous solution, which can be expressed as follows:*

$$\begin{aligned} x(t) = &\int_{-\infty}^{t} X(t)PX^{-1}(s)f(s)ds \\ &- \int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)f(s)ds, \end{aligned} \tag{10}$$

where $\operatorname{mod}((x(t)) \subseteq \operatorname{mod}(A(t), f(t))$, $\|x(t)\| \leq (2K/\alpha)\|f(t)\|$, *and $X(t)$ is a fundamental solution matrix of (9).*

Lemma 2 (see [9]). *Consider the following equation*

$$\frac{dx}{dt} = a(t)x + b(t), \tag{11}$$

where $a(t), b(t)$ are ω-periodic continuous functions on $\mathbb{R}$; if $\int_0^{\omega} a(t)dt \neq 0$, then Equation (11) has a unique ω-periodic continuous solution $\eta(t)$, then $\operatorname{mod}(\eta) \subseteq \operatorname{mod}(a(t), b(t))$, *and $\eta(t)$ can be written as follows:*

$$\eta(t) = \begin{cases} \displaystyle\int_{-\infty}^{t} e^{\int_s^t a(\theta)d\theta} b(s)ds, & \displaystyle\int_0^{\omega} a(t)dt < 0, \\ \displaystyle -\int_{t}^{+\infty} e^{\int_s^t a(\theta)d\theta} b(s)ds. & \displaystyle\int_0^{\omega} a(t)dt > 0. \end{cases} \tag{12}$$

Lemma 3 (see [10]). *Assume that ω-periodic sequence $\{f_n(t)\}$ is convergent uniformly on any compact set of $\mathbb{R}$, $f(t)$is an ω-periodic function, and* $\operatorname{mod}(f_n) \subseteq \operatorname{mod}(f)$ $(n = 1, 2, \ldots)$, *then $\{f_n(t)\}$ is convergent uniformly on*$\mathbb{R}$.

Lemma 4 (see [11]). *Assume that $\mathbb{V}$ is a metric space, $\mathbb{C}$ is a convex closed set of $\mathbb{V}$, and its boundary is $\partial\mathbb{C}$, and if $T: \mathbb{C} \longrightarrow \mathbb{C}$ is a continuous compact mapping, such that $T(\partial\mathbb{C}) \subseteq \mathbb{C}$, then T has at least a fixed point on $\mathbb{C}$.*

Consider one-dimensional periodic differential equation as follows:

$$\frac{dx}{dt} = f(t, x). \tag{13}$$

Here, $f: \mathbb{R} \times I \longrightarrow \mathbb{R}$ is a continuous function, and $f(t + \omega, x) = f(t, x), \omega > 0, I \subseteq \mathbb{R}$.

Lemma 5 (see [12]).*If $f(t, x)$ has second-order continuous partial derivatives on x, and $f''_{xx}(t, x) \neq 0$, then Equation (13) has at most two periodic continuous solutions.*

For the sake of convenience, assume that $f(t)$ is an ω-periodic continuous function on $\mathbb{R}$, then we denote

$$f_M = \sup_{t \in [0,\omega]} f(t), f_L = \inf_{t \in [0,\omega]} f(t). \tag{14}$$

The rest of the paper is arranged as follows. In section 3, we discuss the existence of two periodic solutions of Riccati's equation. In section 4, we discuss the exponential dichotomy of two-dimensional linear differential system. In section 5, we consider the nonhomogeneous second-order linear differential equation and get some results about the existence of the unique periodic solution on the equation. Finally, we draw a conclusion of this paper.

3. Two Periodic Solutions of Riccati's Equation

Topological degree theory and fixed point theorem are often used by scholars to find (almost) periodic solutions of differential equations. Stimulated by the works of [13–15], in this section, we consider the existence of two periodic solutions of the following Riccati's equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(t)x^2 + b(t)x + c(t). \tag{15}$$

The existence of two periodic solutions of Riccati equation is obtained by using the fixed point theorem.

Theorem 2. *Consider Equation (15), $a(t), b(t)$, and $c(t)$ are all ω-periodic continuous functions on $\mathbb{R}$, and assume that the following condition holds true:*

$$(H_1) a(t)c(t) < 0. \tag{16}$$

Remark 2. Without loss of generality, assume $a(t) < 0, c(t) > 0$.

Then Equation (15) has exactly one positive, one negative, and two ω-periodic continuous solutions as follows:

(1) One negative ω-periodic continuous solution is $\gamma_1(t)$, and then

$$\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)_L \le \gamma_1(t) \le \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)_M. \tag{17}$$

(2) Another positive ω-periodic continuous solution is $\gamma_2(t)$, and then

$$\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)_L \le \gamma_2(t) \le \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)_M. \tag{18}$$

Remark 3. In Ref. [16], the author got a proposition as follows:

Proposition 1 (see [16]). *Consider Equation (15), $a(t), b(t)$, and $c(t)$ are all ω-periodic continuous functions on $\mathbb{R}$, and assume that the following conditions hold true:*

$$(H_1) a(t)c(t) < 0, \tag{19}$$

then Equation (15) has exactly one positive, one negative, and two ω-periodic continuous solutions $\gamma_1(t), \gamma_2(t)$, and then

$$\gamma_1(t) < 0 < \gamma_2(t). \tag{20}$$

The proof of Theorem 1 in this paper is different from that of Theorem 1 in [16], and from Equation (17), (18), and (20), the ranges of periodic solutions obtained by us is more accurate. It can be seen that Theorem 1 in this paper is an extension of Theorem 1 in paper [16].

Proof. By (H_1), Equation (15) can be turned into

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= a(t)\left(x + \frac{b(t) - \sqrt{b^2(t) - 4a(t)c(t)}}{2a(t)}\right)\\ &\quad \cdot \left(x + \frac{b(t) + \sqrt{b^2(t) - 4a(t)c(t)}}{2a(t)}\right).\end{aligned} \tag{21}$$

Denote

$$\begin{aligned}\lambda_1(t) &= \frac{-b(t) + \sqrt{b^2(t) - 4a(t)c(t)}}{2a(t)},\\ \lambda_2(t) &= \frac{-b(t) - \sqrt{b^2(t) - 4a(t)c(t)}}{2a(t)}.\end{aligned} \tag{22}$$

It follows from (H_1) that

$$(\lambda_1)_L \le \lambda_1(t) \le (\lambda_1)_M < 0 < (\lambda_2)_L \le \lambda_2(t) \le (\lambda_2)_M. \tag{23}$$

It follows from (22) and Equation (21) that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(t)(x - \lambda_1(t))(x - \lambda_2(t)). \tag{24}$$

Now, we divide the proof into three steps.

(1) We prove the existence of the periodic solution $\gamma_1(t)$ of Equation (15).

Assume that

$$\mathbb{S} = \{\phi(t) \in C(\mathbb{R}, \mathbb{R}) | \phi(t + \omega) = \phi(t)\}. \tag{25}$$

$\forall \phi(t), \varphi(t) \in \mathbb{S}$, then the distance is defined as follows:

$$\rho(\phi, \varphi) = \sup_{t \in [0,\omega]} |\phi(t) - \varphi(t)|. \tag{26}$$

Thus, $(\mathbb{S}, \rho)$ is a complete metric space.

Take a convex closed set of $\mathbb{S}$ as follows:

$$\begin{aligned}\mathbb{B}_1 = \{&\phi(t) \in \mathbb{S} | (\lambda_1)_L \le \phi(t) \le (\lambda_1)_M,\\ &\mathrm{mod}(\phi) \subseteq \mathrm{mod}(a, b, c)\}.\end{aligned} \tag{27}$$

$\forall \phi(t) \in \mathbb{B}_1$, then consider the following equation:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= a(t)(x - \lambda_1(t))(\phi(t) - \lambda_2(t))\\ &= a(t)(\phi(t) - \lambda_2(t))x - a(t)(\phi(t) - \lambda_2(t))\lambda_1(t).\end{aligned} \tag{28}$$

It follows from $a(t) < 0$ (23) and (27) that

$$\begin{aligned}0 &< a_M((\lambda_1)_M - (\lambda_2)_L) \le a(t)(\phi(t) - \lambda_2(t))\\ &\le a_L((\lambda_1)_L - (\lambda_2)_M).\end{aligned} \tag{29}$$

Hence, we have

$$\int_0^{\omega} a(t)(\phi(t)-\lambda_2(t))\mathrm{d}t>0. \quad (30)$$

Since $a(t), \phi(t), \lambda_1(t)$, and $\lambda_2(t)$ are ω-periodic continuous functions, it follows that

$$a(t)(\phi(t)-\lambda_2(t)), a(t)(\phi(t)-\lambda_2(t))\lambda_1(t). \quad (31)$$

are ω-periodic continuous functions, by (30), according to Lemma 2, (24) has a unique ω-periodic continuous solution as follows:

$$\eta(t)=\int_t^{+\infty} e^{\int_s^t a(\theta)(\phi(\theta)-\lambda_2(\theta))d\theta} a(s)(\phi(s)-\lambda_2(s))\lambda_1(s)\mathrm{d}s. \quad (32)$$

and

$$\operatorname{mod}(\eta)\subseteq\operatorname{mod}(a(t)(\phi(t)-\lambda_2(t)), a(t)(\phi(t)-\lambda_2(t))\lambda_1(t)). \quad (33)$$

It follows from (22) and (27) that

$$\begin{aligned}&\operatorname{mod}(a(t)(\phi(t)-\lambda_2(t)))\subseteq\operatorname{mod}(a,b,c),\\ &\operatorname{mod}(a(t)(\phi(t)-\lambda_2(t))\lambda_1(t))\subseteq\operatorname{mod}(a,b,c).\end{aligned} \quad (34)$$

Hence, we have

$$\operatorname{mod}(\eta)\subseteq\operatorname{mod}(a,b,c). \quad (35)$$

It follows from (27), (29), and (32) that

$$\begin{aligned}
\eta(t) &\geq (\lambda_1)_L \int_t^{+\infty} e^{\int_s^t a(\theta)(\phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta} a(s)(\phi(s)-\lambda_2(s))\mathrm{d}s\\
&= -(\lambda_1)_L \int_t^{+\infty} e^{\int_s^t a(\theta)(\phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta} d\left(\int_s^t a(\theta)(\phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta\right)\\
&= -(\lambda_1)_L \left[e^{\int_s^t a(\theta)(\phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta}\right]_t^{+\infty}\\
&= -(\lambda_1)_L \left[e^{\int_{+\infty}^t a(\theta)(\phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta}-1\right]\\
&\geq -(\lambda_1)_L \left[e^{\int_{+\infty}^t a_L((\lambda_1)_L-(\lambda_2)_M)\mathrm{d}\theta}-1\right]\\
&= (\lambda_1)_L,\\
\eta(t) &\leq (\lambda_1)_M \int_t^{+\infty} e^{\int_s^t a(\theta)(\phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta} a(s)(\phi(s)-\lambda_2(s))\mathrm{d}s\\
&= -(\lambda_1)_M \int_t^{+\infty} e^{\int_s^t a(\theta)(\phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta} \mathrm{d}\left(\int_s^t a(\theta)(\phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta\right)\\
&= -(\lambda_1)_M \left[e^{\int_s^t a(\theta)(\phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta}\right]_t^{+\infty}\\
&= -(\lambda_1)_M \left[e^{\int_{+\infty}^t a(\theta)(\phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta}-1\right]\\
&\leq -(\lambda_1)_M \left[e^{\int_{+\infty}^t a_M((\lambda_1)_M-(\lambda_2)_L)\mathrm{d}\theta}-1\right]\\
&= (\lambda_1)_M.
\end{aligned} \quad (36)$$

Hence, $\eta(t) \in \mathbb{B}_1$.

We define a mapping as follows:

$$(T\phi)(t) = \int_t^{+\infty} e^{\int_s^t a(\theta)(\phi(\theta)-\lambda_2(\theta))d\theta} \cdot a(s)(\phi(s) - \lambda_2(s))\lambda_1(s)\mathrm{d}s. \quad (37)$$

Thus, if $\forall \phi(t) \in \mathbb{B}_1$, then $(T\phi)(t) \in \mathbb{B}_1$; hence, $T\colon \mathbb{B}_1 \longrightarrow \mathbb{B}_1$.

Now, we prove that the mapping T is a compact mapping.

Consider any sequence $\{\phi_n(t)\} \subseteq \mathbb{B}_1\ (n = 1, 2, \ldots)$, then it is given as

$$(\lambda_1)_L \le \phi_n(t) \le (\lambda_1)_M, \mathrm{mod}(\phi_n) \subseteq \mathrm{mod}(a, b, c).\ (n = 1, 2, \ldots). \quad (38)$$

On the other hand, $(T\phi_n)(t) = x_{\phi_n}(t)$ satisfies

$$\frac{\mathrm{d}x_{\phi_n}(t)}{\mathrm{d}t} = a(t)(\phi_n(t) - \lambda_2(t))x_{\phi_n}(t) - a(t)(\phi_n(t) - \lambda_2(t))\lambda_1(t). \quad (39)$$

Thus, we have

$$\left|\frac{\mathrm{d}x_{\phi_n}(t)}{\mathrm{d}t}\right| \le 2a_L((\lambda_2)_M - (\lambda_1)_L)(\lambda_1)_L, \mathrm{mod}(x_{\phi_n}(t)) \subseteq \mathrm{mod}(a, b, c). \quad (40)$$

Hence, $\{\mathrm{d}x_{\phi_n}(t)/\mathrm{d}t\}$ is uniformly bounded; therefore, $\{x_{\phi_n}(t)\}$ is uniformly bounded and equicontinuous on $\mathbb{R}$. By the theorem of Ascoli–Arzela, for any sequence $\{x_{\phi_n}(t)\} \subseteq \mathbb{B}_1$, there exists a subsequence (also denoted by $\{x_{\phi_n}(t)\}$) such that $\{x_{\phi_n}(t)\}$ which is convergent uniformly on any compact set of $\mathbb{R}$. From (40), combined with Lemma 3, $\{x_{\phi_n}(t)\}$ is convergent uniformly on $\mathbb{R}$; that is, T is relatively compact on $\mathbb{B}_1$.

Next, we prove that T is a continuous mapping.

Assume $\{\phi_n(t)\} \subseteq \mathbb{B}_1, \phi(t) \in \mathbb{B}_1$, and then

$$\phi_n(t) \longrightarrow \phi(t).\ (n \longrightarrow \infty). \quad (41)$$

It follows from (37) that

$$\begin{aligned}
&|(T\phi_n)(t) - (T\phi)(t)| \\
&= \left|\int_t^{+\infty} e^{\int_s^t a(\theta)(\phi_n(\theta) - \lambda_2(\theta))d\theta} a(s)(\phi_n(s) - \lambda_2(s))\lambda_1(s)\mathrm{d}s - \int_t^{+\infty} e^{\int_s^t a(\theta)(\phi(\theta) - \lambda_2(\theta))\mathrm{d}\theta} a(s)(\phi(s) - \lambda_2(s))\lambda_1(s)\mathrm{d}s\right| \\
&= \left|\int_t^{+\infty} e^{\int_s^t a(\theta)(\phi_n(\theta) - \lambda_2(\theta))\mathrm{d}\theta} a(s)(\phi_n(s) - \phi(s))\lambda_1(s)\mathrm{d}s + \int_t^{+\infty}\left(e^{\int_s^t a(\theta)(\phi_n(\theta) - \lambda_2(\theta))\mathrm{d}\theta} - e^{\int_s^t a(\theta)(\phi(\theta) - \lambda_2(\theta))\mathrm{d}\theta}\right) a(s)(\phi(s) - \lambda_2(s))\lambda_1(s)\mathrm{d}s\right| \\
&= \left|\int_t^{+\infty} e^{\int_s^t a(\theta)(\phi_n(\theta) - \lambda_2(\theta))\mathrm{d}\theta} a(s)(\phi_n(s) - \phi(s))\lambda_1(s)\mathrm{d}s + \int_t^{+\infty} e^{\xi}\left(\int_s^t a(\theta)(\phi_n(\theta) - \phi(\theta))\mathrm{d}\theta\right) a(s)(\phi(s) - \lambda_2(s))\lambda_1(s)\mathrm{d}s\right|.
\end{aligned} \quad (42)$$

Here,ξ is between $\int_s^t a(\theta)(\phi_n(\theta) - \lambda_2(\theta))d\theta$ and $\int_s^t a(\theta)(\phi(\theta) - \lambda_2(\theta))d\theta$; thus, ξ is between $a_L((\lambda_1)_L - (\lambda_2)_M)(t - s)$ and $a_M((\lambda_1)_M - (\lambda_2)_L)(t - s)$; hence, we have&ecmath;

$$\begin{aligned}
&\left|(T\phi_n)(t)-(T\phi)(t)\right| \\
&\leq\left(\int_t^{+\infty} e^{a_M\left((\lambda_1)_M-(\lambda_2)_L\right)(t-s)}|a|_M|\lambda_1|_M\mathrm{d}s+\int_t^{+\infty} e^{a_M\left((\lambda_1)_M-(\lambda_2)_L\right)(t-s)}(s-t)|a|_M^2\left((\lambda_2)_M-(\lambda_1)_L\right)|\lambda_1|_M\mathrm{d}s\right)\rho(\phi_n,\phi) \\
&=\left(\frac{|a|_M|\lambda_1|_M}{a_M\left((\lambda_1)_M-(\lambda_2)_L\right)}+\frac{|a|_M^2\left((\lambda_2)_M-(\lambda_1)_L\right)|\lambda_1|_M}{\left(a_M\left((\lambda_1)_M-(\lambda_2)_L\right)\right)^2}\right)\rho(\phi_n,\phi) \\
&=\left(\frac{a_L(\lambda_1)_L}{a_M\left((\lambda_1)_M-(\lambda_2)_L\right)}-\frac{a_L^2\left((\lambda_2)_M-(\lambda_1)_L\right)(\lambda_1)_L}{\left(a_M\left((\lambda_1)_M-(\lambda_2)_L\right)\right)^2}\right)\rho(\phi_n,\phi).
\end{aligned} \tag{43}$$

By (41) and the above inequality, we get

$$(T\phi_n)(t)\longrightarrow(T\phi)(t),\ (n\longrightarrow\infty). \tag{44}$$

Therefore, T is continuous. From equation (37), $T(\partial\mathbb{B}_1)\subseteq\mathbb{B}_1$. According to Lemma 4, T has at least a fixed point on $\mathbb{B}_1$, and the fixed point is the ω-periodic continuous solution $\gamma_1(t)$ of Equation (15), and then we get

$$(\lambda_1)_L\leq\gamma_1(t)\leq(\lambda_1)_M<0. \tag{45}$$

(2) We prove the existence of the periodic solution $\gamma_2(t)$ of Equation (15).

Assume that

$$\mathbb{S}=\{\phi(t)\in C(\mathbb{R},\mathbb{R})\,|\,\phi(t+\omega)=\phi(t)\}. \tag{46}$$

When $\forall\phi(t),\varphi(t)\in\mathbb{S}$, the distance is defined as follows:

$$\rho(\phi,\varphi)=\sup_{t\in[0,\omega]}|\phi(t)-\varphi(t)| \tag{47}$$

Thus, $(\mathbb{S},\rho)$ is a complete metric space.

Take a convex closed set of $\mathbb{S}$ as follows:

$$\begin{aligned}\mathbb{B}_2=\{\phi(t)\in\mathbb{S}|\,(\lambda_2)_L\leq\phi(t)\leq(\lambda_2)_M,\operatorname{mod}(\phi)\\ \subseteq\operatorname{mod}(a,b,c)\}.\end{aligned} \tag{48}$$

$\forall\phi(t)\in\mathbb{B}_2$, then consider the following equation:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=a(t)\left(\phi(t)-\lambda_1(t)\right)\left(x-\lambda_2(t)\right)\\ &=a(t)\left(\phi(t)-\lambda_1(t)\right)x-a(t)\left(\phi(t)-\lambda_1(t)\right)\lambda_2(t).\end{aligned} \tag{49}$$

It follows from $a(t)<0$ and (23) and (51) that

$$\begin{aligned}a_L\left((\lambda_2)_M-(\lambda_1)_L\right)&\leq a(t)\left(\phi(t)-\lambda_1(t)\right)\\ \&9;&\leq a_M\left((\lambda_2)_L-(\lambda_1)_M\right)<0.\end{aligned} \tag{50}$$

Hence, we have

$$\int_0^\omega a(t)\left(\phi(t)-\lambda_1(t)\right)\mathrm{d}t<0. \tag{51}$$

Since $a(t),\phi(t),\lambda_1(t)$, and $\lambda_2(t)$ are ω-periodic continuous functions, we get

$$a(t)\left(\phi(t)-\lambda_1(t)\right),a(t)\left(\phi(t)-\lambda_1(t)\right)\lambda_2(t). \tag{52}$$

From (54), according to Lemma 2, we get unique ω-periodic continuous solution as follows:

$$\begin{aligned}\eta(t)=-\int_{-\infty}^t e^{\int_s^t a(\theta)\left(\phi(\theta)-\lambda_1(\theta)\right)\mathrm{d}\theta}\\ \cdot a(s)\left(\phi(s)-\lambda_1(s)\right)\lambda_2(s)\mathrm{d}s.\end{aligned} \tag{53}$$

and

$$\begin{aligned}\operatorname{mod}(\eta)\subseteq\operatorname{mod}\big(a(t)\left(\phi(t)-\lambda_1(t)\right),\\ a(t)\left(\phi(t)-\lambda_1(t)\right)\lambda_2(t)\big).\end{aligned} \tag{54}$$

It follows from (22) and (51) that

$$\begin{aligned}\operatorname{mod}\left(a(t)\left(\phi(t)-\lambda_1(t)\right)\right)\subseteq\operatorname{mod}(a,b,c),\\ \operatorname{mod}\left(a(t)\left(\phi(t)-\lambda_1(t)\right)\lambda_2(t)\right)\subseteq\operatorname{mod}(a,b,c).\end{aligned} \tag{55}$$

Hence, we have

$$\operatorname{mod}(\eta)\subseteq\operatorname{mod}(a,b,c). \tag{56}$$

It follows from (51), (53), and (56) that&ecmath;

$$\begin{aligned}
\eta(t) &\geq -(\lambda_2)_L \int_{-\infty}^{t} e^{\int_s^t a(\theta)(\phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta} a(s)(\phi(s)-\lambda_1(s))\mathrm{d}s \\
&= (\lambda_2)_L \int_{-\infty}^{t} e^{\int_s^t a(\theta)(\phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta} d\int_s^t a(\theta)t(\phi(\theta)-\lambda_1(\theta))n\mathrm{d}q\theta) \\
&= (\lambda_2)_L \left[e^{\int_s^t a(\theta)(\phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta} \right]_{-\infty}^{t} \\
&= (\lambda_2)_L \left[1 - e^{\int_{-\infty}^t a(\theta)(\phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta} \right] \\
&\geq (\lambda_2)_L \left[1 - e^{\int_{-\infty}^t a_M((\lambda_2)_L-(\lambda_1)_M)\mathrm{d}\theta} \right] \\
&= (\lambda_2)_L, \\
\eta(t) &\leq -(\lambda_2)_M \int_{-\infty}^{t} e^{\int_s^t a(\theta)(\phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta} a(s)(\phi(s)-\lambda_1(s))\mathrm{d}s, \\
&= (\lambda_2)_M \int_{-\infty}^{t} e^{\int_s^t a(\theta)(\phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta} \mathrm{d}\int_s^t a(\theta)t(\phi(\theta)-\lambda_1(\theta))n\mathrm{d}q\theta) \\
&= (\lambda_2)_M \left[e^{\int_s^t a(\theta)(\phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta} \right]_{-\infty}^{t} \\
&= (\lambda_2)_M \left[1 - e^{\int_{-\infty}^t a(\theta)(\phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta} \right] \\
&\leq (\lambda_2)_M \left[1 - e^{\int_{-\infty}^t a_L((\lambda_2)_M-(\lambda_1)_L)\mathrm{d}\theta} \right] \\
&= (\lambda_2)_M.
\end{aligned} \tag{57}$$

Hence, $\eta(t) \in \mathbb{B}_2$.

We define a mapping as follows:

$$(T\phi)(t) = -\int_{-\infty}^{t} e^{\int_s^t a(\theta)(\phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta} \cdot a(s)(\phi(s)-\lambda_1(s))\lambda_2(s)\mathrm{d}s. \tag{58}$$

Thus, if $\forall \phi(t) \in \mathbb{B}_2$, then $(T\phi)(t) \in \mathbb{B}_2$; hence, $T\colon \mathbb{B}_2 \longrightarrow \mathbb{B}_2$.

Now, we prove that the mapping T is a compact mapping.

Consider any sequence $\{\phi_n(t)\} \subseteq \mathbb{B}_2\ (n = 1, 2, \ldots)$, then it follows that

$$(\lambda_2)_L \leq \phi_n(t) \leq (\lambda_2)_M, \operatorname{mod}(\phi_n) \subseteq \operatorname{mod}(a, b, c).\ (n = 1, 2, \ldots). \tag{59}$$

On the other hand, $(T\phi_n)(t) = x_{\phi_n}(t)$ satisfies

$$\frac{\mathrm{d}x_{\phi_n}(t)}{\mathrm{d}t} = a(t)(\phi_n(t)-\lambda_1(t))x_{\phi_n}(t) - a(t)(\phi_n(t)-\lambda_1(t))\lambda_2(t). \tag{60}$$

Thus, we have

$$\left|\frac{\mathrm{d}x_{\phi_n}(t)}{\mathrm{d}t}\right| \leq -2a_L((\lambda_2)_M-(\lambda_1)_L)(\lambda_2)_M, \operatorname{mod}(x_{\phi_n}(t)) \subseteq \operatorname{mod}(a, b, c). \tag{61}$$

Hence, ℝ e $\{\mathrm{d}x_{\phi_n}(t)/\mathrm{d}t\}$ is uniformly bounded; therefore, $\{x_{\phi_n}(t)\}$ is uniformly bounded and equicontinuous on ℝ. By the theorem of Ascoli–Arzela, for any sequence, $\{x_{\phi_n}(t)\} \subseteq \mathbb{B}_2$, then there exists a subsequence (also denoted by $\{x_{\phi_n}(t)\}$) such that $\{x_{\phi_n}(t)\}$ is convergent uniformly on any compact set . From (40), combined with Lemma 3, $\{x_{\phi_n}(t)\}$ is convergent uniformly on $\mathbb{R}$; that is, T is relatively compact on $\mathbb{B}_2$.

Next, we prove that T is a continuous mapping.

Assume that $\{\phi_n(t)\}\subseteq\mathbb{B}_2, \phi(t)\in\mathbb{B}_2$, and then

$$\phi_n(t) \longrightarrow \phi(t). (n \longrightarrow \infty). \tag{62}$$

It follows from (61) that

$$\begin{aligned}
&\left|(T\phi_n)(t) - (T\phi)(t)\right| \\
&= \left|\int_{-\infty}^{t} e^{\int_s^t a(\theta)(\phi_n(\theta) - \lambda_1(\theta))\mathrm{d}\theta} a(s)(\phi_n(s) - \lambda_1(s))\lambda_2(s)\mathrm{d}s \right.\\
&\quad \left. - \int_{-\infty}^{t} e^{\int_s^t a(\theta)(\phi(\theta) - \lambda_1(\theta))\mathrm{d}\theta} a(s)(\phi(s) - \lambda_1(s))\lambda_2(s)\mathrm{d}s\right| \\
&= \left|\int_{-\infty}^{t} e^{\int_s^t a(\theta)(\phi_n(\theta) - \lambda_1(\theta))d\theta} a(s)(\phi_n(s) - \phi(s))\lambda_2(s)\mathrm{d}s \right.\\
&\quad \left. + \int_{-\infty}^{t}\left(e^{\int_s^t a(\theta)(\phi_n(\theta) - \lambda_1(\theta))\mathrm{d}\theta} - e^{\int_s^t a(\theta)(\phi(\theta) - \lambda_1(\theta))\mathrm{d}\theta}\right) a(s)(\phi(s) - \lambda_1(s))\lambda_2(s)\mathrm{d}s\right| \\
&= \left|\int_{-\infty}^{t} e^{\int_s^t a(\theta)(\phi_n(\theta) - \lambda_1(\theta))\mathrm{d}\theta} a(s)(\phi_n(s) - \phi(s))\lambda_2(s)\mathrm{d}s \right.\\
&\quad \left. + \int_{-\infty}^{t} e^{\xi}\left(\int_s^t a(\theta)(\phi_n(\theta) - \phi(\theta))\mathrm{d}\theta\right) a(s)(\phi(s) - \lambda_1(s))\lambda_2(s)\mathrm{d}s\right|,
\end{aligned} \tag{63}$$

Here,ξ is between $\int_s^t a(\theta)(\phi_n(\theta) - \lambda_1(\theta))\mathrm{d}\theta$ and $\int_s^t a(\theta)(\phi(\theta) - \lambda_1(\theta))\mathrm{d}\theta$; thus, ξ is between $a_L((\lambda_2)_M - (\lambda_1)_L)(t-s)$ and $a_M((\lambda_2)_L - (\lambda_1)_M)(t-s)$; hence, we have

$$\begin{aligned}
&\left|(T\phi_n)(t) - (T\phi_n)(t)\right| \\
&\leq \left(\int_{-\infty}^{t} e^{a_M((\lambda_2)_L - (\lambda_1)_M)(t-s)}|a|_M|\lambda_2|_M\mathrm{d}s + \int_{-\infty}^{t} e^{a_M((\lambda_2)_L - (\lambda_1)_M)(t-s)}(t-s)|a|_M^2((\lambda_2)_M - (\lambda_1)_L)|\lambda_2|_M\mathrm{d}s\right)\rho(\phi_n,\phi) \\
&= \left(\frac{|a|_M|\lambda_2|_M}{-a_M((\lambda_2)_L - (\lambda_1)_M)} + \frac{|a|_M^2((\lambda_2)_M - (\lambda_1)_L)|\lambda_2|_M}{(a_M((\lambda_2)_L - (\lambda_1)_M))^2}\right)\rho(\phi_n,\phi) \\
&= \left(\frac{a_L(\lambda_2)_M}{a_M((\lambda_2)_L - (\lambda_1)_M)} + \frac{a_L^2((\lambda_2)_M - (\lambda_1)_L)(\lambda_2)_M}{(a_M((\lambda_2)_L - (\lambda_1)_M))^2}\right)\rho(\phi_n,\phi).
\end{aligned} \tag{64}$$

From (65) and the above inequality, we get

$$(T\phi_n)(t) \longrightarrow (T\phi)(t), (n \longrightarrow \infty). \tag{65}$$

Therefore, T is continuous. From Equation (61), $T(\partial\mathbb{B}_2)\subseteq\mathbb{B}_2$. According to Lemma 4, T has at least a fixed point on $\mathbb{B}_2$, then the fixed point is the ω-periodic continuous solution $\gamma_2(t)$ of Equation (15), and then

$$0 < (\lambda_2)_L \leq \gamma_2(t) \leq (\lambda_2)_M. \tag{66}$$

(3) We prove Equation (15) has exactly two periodic solutions $\gamma_1(t)$ and $\gamma_2(t)$.

Let

$$f(t,x) = a(t)x^2 + b(t)x + c(t), \tag{67}$$

then

$$f_{xx}''(t,x) = 2a(t) < 0. \quad (68)$$

From Equation (68), according to Lemma 5, Equation (15) has at most two periodic continuous solutions, and we have known that Equation (15) has two periodic continuous solutions: $\gamma_1(t), \gamma_2(t)$; thus, it follows that Equation (15) has exactly two periodic solutions $\gamma_1(t), \gamma_2(t)$, and then

$$\begin{aligned}&(\lambda_1)_L \le \gamma_1(t) \le (\lambda_1)_M < 0,\\ &0 < (\lambda_2)_L \le \gamma_2(t) \le (\lambda_2)_M.\end{aligned} \quad (69)$$

This is the end of the proof of Theorem 2. □

4. Exponential Dichotomy

In this section, we consider Equation (6) and get the sufficient condition for the exponential dichotomy of Equation (6).

Theorem 3. *Consider Equation (6),* $a(t), b(t), c(t)$, *and* $d(t)$ *are* ω*-periodic continuous functions on* $\mathbb{R}$, $a(t)$ *and* $b(t)$ *have continuous derivatives on* $\mathbb{R}$, *then assume that the following conditions hold true:*

$$\begin{aligned}&(H_1) b(t) \ne 0,\\ &(H_2) \frac{a(t)b'(t)}{b(t)} + a(t)d(t) - b(t)c(t) - a'(t) < 0.\end{aligned} \quad (70)$$

Then, Equation (6) has exponential dichotomy.

Proof. From Equation (6), the following equation can be obtained:

$$\begin{aligned}&\frac{d^2x}{dt^2} - \left(a(t) + d(t) + \frac{b'(t)}{b(t)}\right)\frac{dx}{dt}\\ &+\left(\frac{a(t)b'(t)}{b(t)} + a(t)d(t) - b(t)c(t) - a'(t)\right)(t)x = 0,\end{aligned} \quad (71)$$

where $(dx/dt) = a(t)x + b(t)y$.

Let

$$x = e^u, \quad (72)$$

then Equation (72) becomes

$$\begin{aligned}&\frac{d^2u}{dt^2} + \left(\frac{du}{dt}\right)^2 - \left(a(t) + d(t) + \frac{b'(t)}{b(t)}\right)\frac{du}{dt}\\ &+\left(\frac{a(t)b'(t)}{b(t)} + a(t)d(t) - b(t)c(t) - a'(t)\right) = 0.\end{aligned} \quad (73)$$

Let

$$z = \frac{du}{dt}, \quad (74)$$

then Equation (73) becomes

$$\begin{aligned}\frac{dz}{dt} = &-z^2 + \left(a(t) + d(t) + \frac{b'(t)}{b(t)}\right)z\\ &-\left(\frac{a(t)b(t)}{b(t)} + a(t)d(t) - b(t)c(t) - a'(t)\right).\end{aligned} \quad (75)$$

This is Riccati's equation. From (H_1), (H_2), Equation (75) satisfies all the conditions of Theorem 2. According to Theorem 2, Equation (75) has exactly one negative, one positive, and two ω-periodic continuous solutions $\gamma_1(t) < 0$ and $\gamma_2(t) > 0$.

It follows from Equation (74) that Equation (73) has two continuous solutions:

$$\begin{aligned}&u_1(t) = u_1(0) + \int_0^t \gamma_1(s)ds,\\ &u_2(t) = u_2(0) + \int_0^t \gamma_2(s)ds.\end{aligned} \quad (76)$$

Here, $u_1(0) \ne u_2(0)$.

It follows from Equation (72) that Equation (71) has two continuous solutions:

$$\begin{aligned}\Phi_1(t) &= e^{u_1(t)} = e^{u_1(0)+\int_0^t \gamma_1(s)ds} = e^{u_1(0)}e^{\int_0^t \gamma_1(s)ds_1}\\ &= k_1 e^{\int_0^t \gamma_1(s)ds},\\ \Phi_2(t) &= e^{u_2(t)} = e^{u_2(0)+\int_0^t \gamma_2(s)ds} = e^{u_2(0)}e^{\int_0^t \gamma_2(s)ds}\\ &= k_2 e^{\int_0^t \gamma_2(s)ds}.\end{aligned} \quad (77)$$

Here, $k_1 = e^{u_1(0)}, k_2 = e^{u_2(0)}$.

Morveover, we get

$$\begin{aligned}\frac{\Phi_2(t)}{\Phi_1(t)} &= \frac{e^{u_2(0)}e^{\int_0^t \gamma_2(s)ds}}{e^{u_1(0)}e^{\int_0^t \gamma_1(s)ds}}\\ &= e^{(u_2(0)-u_1(0))}e^{\int_0^t (\gamma_2(s)-\gamma_1(s))ds} \ne C.\end{aligned} \quad (78)$$

Thus, $\Phi_1(t), \Phi_2(t)$ are linearly independent.

Since

$$\frac{dx}{dt} = a(t)x + b(t)y, \quad (79)$$

then Equation (6) has two sets of linearly independent solutions as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k_1 e^{\int_0^t \gamma_1(\theta)d\theta} \\ \dfrac{k_1(\gamma_1(t) - a(t))e^{\int_0^t \gamma_1(\theta)d\theta}}{b(t)} \end{pmatrix} \begin{pmatrix} k_2 e^{\int_0^t \gamma_2(\theta)d\theta} \\ \dfrac{k_2(\gamma_2(t) - a(t))e^{\int_0^t \gamma_2(\theta)d\theta}}{b(t)} \end{pmatrix}. \quad (80)$$

Taking the fundamental solution matrix $X(t)$ of Equation (6), we get as follows:

$$X(t)=\begin{pmatrix} k_1 e^{\int_0^t \gamma_1(\theta)d\theta} & k_2 e^{\int_0^t \gamma_2(\theta)d\theta} \\ \dfrac{k_1(\gamma_1(t)-a(t))e^{\int_0^t \gamma_1(\theta)d\theta}}{b(t)} & \dfrac{k_2(\gamma_2(t)-a(t))e^{\int_0^t \gamma_2(\theta)d\theta}}{b(t)} \end{pmatrix}. \tag{81}$$

Thus, we have

$$X^{-1}(s)=\frac{\begin{pmatrix} (\gamma_2(s)-a(s))e^{-\int_0^s \gamma_1(\theta)\mathrm{d}\theta}/k_1 & -b(s)e^{-\int_0^s \gamma_1(\theta)\mathrm{d}\theta}/k_1 \\ (a(s)-\gamma_1(s))e^{-\int_0^s \gamma_2(\theta)\mathrm{d}\theta}/k_2 & b(s)e^{-\int_0^s \gamma_2(\theta)\mathrm{d}\theta}/k_2 \end{pmatrix}}{\gamma_2(s)-\gamma_1(s)}. \tag{82}$$

Take projection P, we get as follows:

$$P=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{83}$$

and then it follows

$$\begin{aligned} X(t)PX^{-1}(s) &= \frac{\begin{pmatrix} \gamma_2(s)-a(s) & -b(s) \\ (\gamma_1(t)-a(t))(\gamma_2(s)-a(s))/b(t) & (a(t)-\gamma_1(t))b(s)/b(t) \end{pmatrix} e^{\int_s^t \gamma_1(\theta)d\theta}}{\gamma_2(s)-\gamma_1(s)}, \\ X(t)(I-P)X^{-1}(s) &= \frac{\begin{pmatrix} a(s)-\gamma_1(s) & b(s) \\ (\gamma_2(t)-a(t))(a(s)-\gamma_1(s))/b(t) & (\gamma_2(t)-a(t))b(s)/b(t) \end{pmatrix} e^{\int_s^t \gamma_2(\theta)d\theta}}{\gamma_2(s)-\gamma_1(s)}. \end{aligned} \tag{84}$$

Since $a(t), b(t), \gamma_1(t), \gamma_2(t)$ are ω-periodic continuous functions on $\mathbb{R}$, it follows that there exist $K \geq 1, \alpha > 0$ such that the following inequalities hold true:

$$\begin{aligned} \|X(t)PX^{-1}(s)\| &\leq Ke^{-\alpha(t-s)}, (t \geq s), \\ \|X(t)(I-P)X^{-1}(s)\| &\leq Ke^{\alpha(t-s)}, (s \geq t). \end{aligned} \tag{85}$$

Thus, Equation (6) has exponential dichotomy, and the dichotomy constants are (K, α).

This is the end of the proof of Theorem 3.

If we turn Equation (6) into a second-order linear differential equation about y, we can get the following: □

Theorem 4. *Consider Equation (6), and $a(t), b(t), c(t)$, and $d(t)$ are ω-periodic continuous functions on $\mathbb{R}$, and $c(t)$ and $d(t)$ have continuous derivatives on $\mathbb{R}$, then assume that the following conditions hold true:*

$$\begin{aligned} &(H_1) c(t) \neq 0, \\ &(H_2) \frac{\mathrm{d}(t)c'(t)}{c(t)} + a(t)\mathrm{d}(t) - b(t)c(t) - \mathrm{d}'(t) < 0, \end{aligned} \tag{86}$$

then Equation (6) has exponential dichotomy.

Proof. From Equation (6), the following equation can be obtained:

$$\begin{aligned} &\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \left(a(t) + d(t) + \frac{c'(t)}{c(t)}\right)\frac{\mathrm{d}y}{\mathrm{d}t} \\ &+ \left(\frac{\mathrm{d}(t)c'(t)}{c(t)} + a(t)d(t) - b(t)c(t) - d'(t)\right)y = 0, \end{aligned} \tag{87}$$

where

$$\frac{\mathrm{d}y}{\mathrm{d}t} = c(t)x + d(t)y, \tag{88}$$

Let

$$y = e^u, \tag{89}$$

then Equation (89) becomes

$$\begin{aligned} &\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 - \left(a(t) + d(t) + \frac{c'(t)}{c(t)}\right)\frac{\mathrm{d}u}{\mathrm{d}t} \\ &+ \left(\frac{\mathrm{d}(t)c'(t)}{c(t)} + a(t)d(t) - b(t)c(t) - d'(t)\right) = 0. \end{aligned} \tag{90}$$

Let

$$z = \frac{\mathrm{d}u}{\mathrm{d}t}, \tag{91}$$

then Equation (87) becomes

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -z^2 + \left(a(t) + d(t) + \frac{c'(t)}{c(t)}\right)z - \left(\frac{\mathrm{d}(t)c(t)}{c(t)} + a(t)d(t) - b(t)c(t) - d'(t)\right), \tag{92}$$

and this is Riccati's equation. From (H_1), (H_2), Equation (92) satisfies all the conditions of Theorem 2. According to Theorem 2, Equation (92) has exactly one negative, one positive, and two ω-periodic continuous solutions $\gamma_1(t) < 0$ and $\gamma_2(t) > 0$.

It follows from Equation (92) two continuous solutions as follows:

$$u_1(t) = u_1(0) + \int_0^t \gamma_1(s)ds, \tag{93}$$

and

$$u_2(t) = u_2(0) + \int_0^t \gamma_2(s)ds. \tag{94}$$

Here, $u_1(0) \neq u_2(0)$.

It follows from Equation (89) that Equation (87) has two continuous solutions as follows:

$$\Phi_1(t) = e^{u_1(t)} = e^{u_1(0)}e^{\int_0^t \gamma_1(s)ds} = k_1 e^{\int_0^t \gamma_1(s)ds}, \tag{95}$$

and

$$\Phi_2(t) = e^{u_2(t)} = e^{u_2(0)}e^{\int_0^t \gamma_2(s)ds} = k_2 e^{\int_0^t \gamma_2(s)ds}. \tag{96}$$

Here, $k_1 = e^{u_1(0)}, k_2 = e^{u_2(0)}$.

Moreover,

$$\frac{\Phi_2(t)}{\Phi_1(t)} = \frac{k_2 e^{\int_0^t \gamma_2(s)ds}}{k_1 e^{\int_0^t \gamma_1(s)ds}} = \frac{k_2}{k_1}e^{\int_0^t (\gamma_2(s)-\gamma_1(s))ds} \neq C. \tag{97}$$

Thus, $\Phi_1(t), \Phi_2(t)$ are linearly independent.

Since

$$\frac{\mathrm{d}y}{\mathrm{d}t} = c(t)x + d(t)y, \tag{98}$$

Then Equation (6) has two sets of linearly independent solutions given as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{k_1(\gamma_1(t) - \mathrm{d}(t))e^{\int_0^t \gamma_1(\theta)d\theta}}{c(t)} \\ k_1 e^{\int_0^t \gamma_1(\theta)d\theta} \end{pmatrix} \begin{pmatrix} \dfrac{k_2(\gamma_2(t) - \mathrm{d}(t))e^{\int_0^t \gamma_2(\theta)d\theta}}{c(t)} \\ k_2 e^{\int_0^t \gamma_2(\theta)d\theta} \end{pmatrix}. \tag{99}$$

Take the fundamental solution matrix $Y(t)$ of Equation (6) as follows:

$$Y(t) = \begin{pmatrix} \dfrac{k_1(\gamma_1(t) - \mathrm{d}(t))e^{\int_0^t \gamma_1(\theta)d\theta}}{c(t)} & \dfrac{k_2(\gamma_2(t) - \mathrm{d}(t))e^{\int_0^t \gamma_2(\theta)d\theta}}{c(t)} \\ k_1 e^{\int_0^t \gamma_1(\theta)d\theta} & k_2 e^{\int_0^t \gamma_2(\theta)d\theta} \end{pmatrix}. \tag{100}$$

Thus, we have

$$Y^{-1}(s) = \frac{\begin{pmatrix} c(s)e^{-\int_0^s \gamma_1(\theta)d\theta}/k_1 & (\mathrm{d}(s) - \gamma_2(s))e^{-\int_0^s \gamma_1(\theta)d\theta}/k_1 \\ -c(s)e^{-\int_0^s \gamma_2(\theta)d\theta}/k_2 & (\gamma_1(s) - \mathrm{d}(s))e^{-\int_0^s \gamma_2(\theta)d\theta}/k_2 \end{pmatrix}}{\gamma_1(s) - \gamma_2(s)}. \tag{101}$$

Take projection P as

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (102)$$

Then it follows

$$Y(t)PY^{-1}(s) = \frac{\begin{pmatrix} (\gamma_1(t) - d(t))c(s)/c(t) & (\gamma_1(t) - d(t))(d(s) - \gamma_2(s))/c(t) \\ c(s) & d(s) - \gamma_2(s) \end{pmatrix} e^{\int_s^t \gamma_1(\theta)d\theta}}{\gamma_1(s) - \gamma_2(s)},$$
$$Y(t)(I - P)Y^{-1}(s) = \frac{\begin{pmatrix} (d(t) - \gamma_2(t))c(s)/c(t) & (\gamma_2(t) - d(t))(\gamma_1(s) - d(s))/c(t) \\ -c(s) & \gamma_1(s) - d(s) \end{pmatrix} e^{\int_s^t \gamma_2(\theta)d\theta}}{\gamma_1(s) - \gamma_2(s)}. \quad (103)$$

Since $a(t), b(t), \gamma_1(t), \gamma_2(t)$ are ω-periodic continuous functions on $\mathbb{R}$, it follows that there exist $L \geq 1, \beta > 0$ such that the following inequalities hold true:

$$\left\|Y(t)PY^{-1}(s)\right\| \leq Le^{-\beta(t-s)}, \ (t \geq s),$$
$$\left\|Y(t)(I - P)Y^{-1}(s)\right\| \leq Le^{\beta(t-s)}, \ (s \geq t). \quad (104)$$

Thus, Equation (6) has exponential dichotomy, and the dichotomy constants are (L, β).

This is the end of the proof of Theorem 4.

Next, we will calculate the characteristic exponents of system (6). We first give the definition of the characteristic exponent of the linear system. □

Definition 1 (see[17]). *Consider the following linear differential system:*

$$\frac{dx}{dt} = A(t)x, \quad (105)$$

where $A(t)$ is a square matrix of order n, and $A(t)$ continuous on $[t_0, +\infty)$, $x = x(t)$ is a nonzero solution of (105). Define

$$\chi(x(t)) = \overline{\lim_{t \longrightarrow +\infty}} \frac{1}{t} \ln\|x(t)\|, \quad (106)$$

where $\chi(x(t))$ is called the characteristic exponent of $x = x(t)$.

According the proof of Theorem 4, we get as follows.

Theorem 5. *Consider Equation (6), and $a(t), b(t), c(t)$ and $d(t)$ are ω-periodic continuous functions on $\mathbb{R}$, and $c(t)$ and $d(t)$ have continuous derivatives on $\mathbb{R}$, assume that the following conditions hold true:*

$$(H_1) c(t) \neq 0,$$
$$(H_2) \frac{d(t)c'(t)}{c(t)} + a(t)d(t) - b(t)c(t) - d'(t) < 0. \quad (107)$$

Then the characteristic exponents of system Equation (6) must be positive and negative.

Proof. According to the proof of Theorem 4, by (95)–(97), and (99), Equation (6) has two linearly independent solutions as follows:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \dfrac{k_1(\gamma_1(t) - d(t))e^{\int_0^t \gamma_1(\theta)d\theta}}{c(t)} \\ k_1 e^{\int_0^t \gamma_1(\theta)d\theta} \end{pmatrix},$$
$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \dfrac{k_2(\gamma_2(t) - d(t))e^{\int_0^t \gamma_2(\theta)d\theta}}{c(t)} \\ k_2 e^{\int_0^t \gamma_2(\theta)d\theta} \end{pmatrix}, \quad (108)$$

where $\gamma_1(t) < 0, \gamma_2(t) > 0$. Thus, we have

$$\chi\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \overline{\lim_{t\longrightarrow+\infty}}\frac{1}{t}\ln\left\|\begin{pmatrix} \dfrac{k_1(\gamma_1(t)-\mathrm{d}(t))e^{\int_0^t\gamma_1(\theta)d\theta}}{c(t)} \\ k_1e^{\int_0^t\gamma_1(\theta)d\theta} \end{pmatrix}\right\|$$

$$= \overline{\lim_{t\longrightarrow+\infty}}\frac{1}{t}\ln\left(\left(\frac{k_1(\gamma_1(t)-\mathrm{d}(t))e^{\int_0^t\gamma_1(\theta)d\theta}}{c(t)}\right)^2 + \left(k_1e^{\int_0^t\gamma_1(\theta)d\theta}\right)^2\right)^{\frac{1}{2}} \tag{109}$$

$$= \frac{1}{2}\overline{\lim_{t\longrightarrow+\infty}}\frac{1}{t}\ln\left(\frac{k_1^2\left(\gamma_1(t)-\mathrm{d}(t)^2+c^2(t)\right)}{c^2(t)}e^{2\int_0^t\gamma_1(\theta)d\theta}\right)$$

$$= \frac{1}{2}\overline{\lim_{t\longrightarrow+\infty}}\frac{1}{t}\left(\ln\frac{k_1^2\left(\gamma_1(t)-\mathrm{d}(t)^2+c^2(t)\right)}{c^2(t)} + 2\int_0^t\gamma_1(\theta)d\theta\right) = (\gamma_1)_M < 0.$$

Similarly, we have

$$\chi\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lim_{t\longrightarrow+\infty}\frac{1}{t}\ln\left\|\begin{pmatrix} \dfrac{k_2(\gamma_2(t)-\mathrm{d}(t))e^{\int_0^t\gamma_2(\theta)d\theta}}{c(t)} \\ k_2e^{\int_0^t\gamma_2(\theta)d\theta} \end{pmatrix}\right\| = (\gamma_2)_M > 0. \tag{110}$$

Thus, the characteristic exponents of system (6) must be positive and negative.

This is the end of the proof of Theorem 5.

Similar as Theorem 5, we get as follows. □

Theorem 6. *Consider Equation (6), and $a(t), b(t), c(t)$, and $d(t)$ are ω-periodic continuous functions on ℝ, and $c(t)$ and $d(t)$ have continuous derivatives on ℝ, assume that the following conditions hold true:*

$$(H_1) b(t) \neq 0,$$
$$(H_2) \frac{a(t)b'(t)}{b(t)} + a(t)d(t) - b(t)c(t) - a'(t) < 0, \tag{111}$$

then the characteristic exponents of system (1.5) must be positive and negative.

Remark 4. Comparing Theorem 5 of this paper with Theorem 1 of (88), it can be found that the conditions of Theorem 5 of this paper is weaker than those of Theorem 1 of (88). It can be seen that Theorem 5 of this paper is the generalization of Theorem 1 of (88).

5. Existence of a Unique Periodic Solution

Consider the following equation

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e(t) \\ f(t) \end{pmatrix}. \tag{112}$$

By Theorem 3 and Theorem 4, according to Lemma 1, we can easily get as follows.

Theorem 7. *Consider Equation (114), and $a(t), b(t), c(t), d(t), e(t)$ and $f(t)$ are ω-periodic continuous functions on ℝ, and $a(t)$ and $b(t)$ have continuous derivatives on ℝ, assume that the following conditions hold true:*

$$(H_1) b(t) \neq 0,$$
$$(H_2) \frac{a(t)b'(t)}{b(t)} + a(t)d(t) - b(t)c(t) - a'(t) < 0, \tag{113}$$

then Equation (114) has a unique ω-periodic continuous solution.

Theorem 8. *Consider Equation (114), and $a(t), b(t), c(t), d(t), e(t)$ and $f(t)$ are ω-periodic continuous functions on $\mathbb{R}$, and $c(t)$ and $d(t)$ have continuous derivatives on $\mathbb{R}$, assume that the following conditions hold true:*

$$\begin{aligned} &(H_1) c(t) \neq 0, \\ &(H_2) \frac{d(t)c'(t)}{c(t)} + a(t)d(t) - b(t)c(t) - d'(t) < 0, \end{aligned} \tag{114}$$

Then Equation (114) has a unique ω-periodic continuous solution.

In Equation (114), if $a(t) \equiv 0, b(t) \equiv 1, e(t) \equiv 0$, $c(t) = -q(t), d(t) = -p(t)$, then (112) becomes the following equation:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}. \tag{115}$$

Set $x' = y$, then (115) is equivalent to the following equation:

$$x'' + p(t)x' + q(t)x = f(t). \tag{116}$$

So we can easily get what as follows.

Theorem 9. *Consider Equation (116), and $p(t), q(t)$, and $f(t)$ are ω-periodic continuous functions on $\mathbb{R}$, and assume that the following condition holds true:*

$$(H_1) q(t) < 0, \tag{117}$$

Then Equation (116) has a unique ω-periodic continuous solution.

Further, if $p(t) \equiv 0$, then we can obtain what as follows.

Theorem 10. *Consider the following Hill's equation*

$$\frac{d^2 x}{dt^2} + q(t)x = r(t). \tag{118}$$

$q(t), r(t)$ are both ω-periodic continuous functions on $\mathbb{R}$, and assume that the following condition holds true:

$$(H_1) q(t) < 0, \tag{119}$$

Then equation (117) has a unique ω-periodic continuous solution.

6. Conclusions

In this paper, the exponential dichotomy of two-dimensional linear system is studied. To verify whether the system has exponential dichotomy, the key is to find out the fundamental solution matrix of the system, but it is not easy to find the fundamental solution matrix of the system. In this paper, a new method for judging the exponential dichotomy of two-dimensional linear system is provided; that is, the two-dimensional linear system is transformed into a second-order linear system by using variable substitution; Then, the second-order linear system is transformed into Riccati's equation, and the existence of two linearly independent periodic solutions of Riccati's equation is obtained by using the fixed point theorem; thus, the fundamental solution matrix of the two-dimensional linear system is obtained. Under certain conditions, the two-dimensional linear system has exponential dichotomy.

This method to verify the exponential dichotomy of two-dimensional linear system in this paper is new and feasible. It will have certain application value.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed to each part of this paper equally. The authors read and approved the final manuscript.

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