Research Article

The Theoretical Research on Grey Catastrophe Model of Traffic System

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As we all know, it is difficult for traditional traffic models to deal with the phenomenon of "catastrophes" in traffic state and the traffic problem with incomplete information at the same time. Aiming at the traffic congestion, this paper firstly takes traffic volume as a state variable, speed, and density as control variables to establish the system potential function and theoretically clarifies that the structure of the traffic system has catastrophe characteristics. Secondly, combined with the grey characteristics of traffic system, we construct the grey catastrophe model for traffic system and obtain the traffic potential function with the system time as the state variable by substituting the variables of the model. Finally, the traffic stability is analyzed theoretically based on the potential function. In this paper, the rationality of the grey catastrophe model of traffic system is expounded theoretically, and the grey model theory is well expanded.

1. Introduction

With the acceleration of China’s economic and social development and urbanization, the transportation system is developing rapidly, and traffic congestion and traffic accidents are increasing. Traffic problems are the bottleneck restricting the development of smart cities. The intelligent transportation system [1] is a prerequisite for the development of a smart city, and short-term traffic state prediction is an important part of the intelligent transportation system to realize traffic information service, traffic control, and guidance. Therefore, it is very necessary and practical for the development of the intelligent transportation system to study scientific and effective traffic flow prediction theories and methods, accurately evaluate the traffic conditions from the accumulated information, and then guide the operation of the transportation system. This has motivated many researchers to focus on constructing the right traffic flow models to supply effective information for the intelligent transportation system. Currently, many published studies have been applied to construct the prediction models for traffic system, which can be categorized into three types: statistical models, machine learning methods, and grey models.

Some scholars use statistical models and intelligent models to analyze the traffic system, and Queen and Albers developed a multivariate Bayesian dynamic model to predict traffic flow [2]. Ghosh and Basu [3] applied the Bayesian method to estimate the parameters of the SARIMA model and predict the traffic flow. Xie and Zhao proposed a Gaussian processes model, which has received attention for its outstanding generalization capabilities and superior nonlinear approximation [4]. Chaos in traffic flow was demonstrated by Xue and Shi, who sampled traffic flow every five minutes and overall provided better prediction than traditional statistical methods [5]. Huang and Sadek developed a novel forecasting approach inspired by human memory for short-term traffic volume forecasting [6]. The algorithms commonly used in machine learning methods include support vector machine algorithm, random forest algorithm, artificial neural network algorithm, and decision tree algorithm. Some scholars improve on the basic model, and others combine different algorithms to get a more accurate combination model. By focusing on a special
challenge of how to develop robust responsive algorithms and prediction models for short-term traffic forecasting in different traffic conditions, Chen and Zhang [7] present an ensemble learning algorithm via the integration of GBRT and Lasso. Wang et al. [8] proposed a machine learning-assisted intelligent traffic monitoring system for improving transportation protection and reliability.

Although machine learning methods are applied in various cases, they also have disadvantages, such as many machine learning methods are difficult to explain the meaning of the models, and some methods are expensive to calculate. Some methods require a large amount of data for training and are not suitable for cases with small sample data and limited information. The traffic system has a large number of uncertain factors such as environment (visibility, wind direction, and temperature) and man-made (traffic accidents and incidents, and the driver’s psychological state). These factors all determine the nonlinear and uncertainty characteristics of traffic flow. For small sample data and uncertainty systems, the grey system model proposed by Deng [9] can avoid the above problems well, so as to better fit and predict. Some scholars have improved on the classical model and proposed more perfect univariate and multivariate models for various areas [10–15]. Especially for traffic system, Guo et al. [16] forecast the short-term traffic flow based on the grey GM(1, 1) model by concerning the delay and nonlinear properties of traffic flow in urban road systems. Xiao and Duan [17] proposed a grey model of traffic flow in a road section, and this model obtained traffic flow information about traffic flow inflow and congestion via matrix least-squares technology and obtained the time response function and modeling steps of the model using a mathematical analysis method. Duan et al. [18] used the tucker tensor decomposition least-squares algorithm to establish the tensor alternating least-squares GM (1, 1) model by combining the modeling mechanism of the grey classical model GM (1, 1) with the algorithm. Kang et al. [19] established the fractional viscoelastic traffic flow model in combination with the modeling principle of the Bass model and successful application of fractional calculus in viscoelastic fluid.

In summary, these existing models and methods suffer from several limitations in the following aspects:

1. The main limitation of statistical models is that sufficient input data are generally needed for parameter estimation to achieve accurate prediction.

2. Some machine learning methods are difficult to explain the system mechanism and usually have the problem of overfitting, and some methods are expensive to calculate.

3. Some improved grey models cannot comprehensively describe the traffic system, especially the catastrophe phenomenon.

Therefore, a novel grey model is proposed to address the above deficiencies. This paper established a grey catastrophe model of traffic system based on the system evolution mechanism for traffic congestion. System structure evolution-driven modeling method establishes the system potential function through the relationship between system control variables and state variables and then analyses the evolution trend of system state and establishes the model. Compared with the previous researches, the main contributions of this paper are summarized as follows:

1. Considering the mechanism of traffic system, a grey catastrophe model is proposed by analyzing the three basic variables of traffic flow.

2. It is proved theoretically the character of the traffic system by catastrophe model.

3. The stability of traffic system can be analyzed based on the potential function, and the traffic trend can be predicted.

The remaining chapters of this paper are arranged as follows: Section 2 introduces the basis of the catastrophe model and the character of traffic system. In Section 3, the system potential function is established by the interaction and relative relationship of various components of the system and the influence of the system and the environment, while the system potential can be expressed and described by internal state variables and external control variables. In Section 4, based on the grey characteristics and catastrophe characteristics of the transportation system, the grey catastrophe model is constructed by using the modeling method of grey system theory to explain the catastrophe characteristics of the transportation system, and the stability of the system is analyzed, and the trend is analyzed according to the potential function. Finally, the main conclusions and theoretical significance are drawn, and the future work has prospected in Section 5.

2. Catastrophe Model Foundations

This section introduces the principle and characteristics of catastrophe theory, and how to apply catastrophe theory to explain the characteristics of traffic flow.

2.1. Catastrophe Theory Basis. Structural stability and morphogenesis were published by French mathematician Thom in 1972. The foundation of catastrophe theory is laid in the outline of the general theory of modeling. Catastrophe theory attempts to discuss the phenomenon of jumping change of state in the differential dynamic system from the perspective of mathematics and mainly studies the bifurcation of stable equilibrium state in the smooth dynamic system. A system has mutation characteristics, which are mainly reflected in the following five aspects.

Catastrophe: when the ideal delay is adopted, the system jumps from a vanishing minimum to a global minimum or local minimum, and the value of potential has a discontinuous change; when Maxwell convention is adopted, the value of potential changes continuously, but its derivative is discontinuous.

Bifurcation: the limited change of the control variable will lead to the change of the value of the state variable.
2.2. The Catastrophe Characteristic of Traffic Flow.

Whether a system has mutation characteristics mainly depends on whether the system has mutation features. When speed, flow, and density (or lane occupancy) are studied as a system, it is a system with abrupt characteristics. Referring to relevant materials, we can know that a flow value corresponds to two speed values: one is in the noncongested state, and the other is in the congested state, which is consistent with the bimodal characteristics of cusp catastrophe theory. Moreover, in most cases, the state of traffic flow is either in the state of noncongested flow or in the state of congested flow. The transition from noncongested state to congested state or from congested state to noncongested state is not a gradual process, but a leap and a sudden change. Therefore, the traffic flow system is abrupt. Referring to relevant materials, we can also know that there are unreachable domains in some areas studied; that is, the system is unreachable. In addition, when the traffic flow system reaches the capacity, the system is in a critical equilibrium state, but this equilibrium state is ideal and exists only instantaneously. It belongs to an unstable equilibrium. Once it encounters the disturbance of external factors, it may destroy this equilibrium. Therefore, the system may be biased to the congested state or the uncrowded state, but this bias is not fixed. Sometimes, it may be biased to the congested state, and sometimes, it may be biased to the uncrowded state, which is consistent with the bifurcation characteristics of catastrophe theory; that is, the system has bifurcation. The lag of the system exists only when the ideal delay convention is followed but not when the system follows the Maxwell convention. To sum up, the traffic flow has the characteristics of catastrophe, and the catastrophe theory used to explain the traffic flow is feasible.

3. Structure Analysis of Traffic System

Traffic congestion is a discontinuous phenomenon with the change of traffic system operation state. However, the transformation of the system from one state to another cannot be completed instantaneously. The structure, state, and behavior of the system will change with time and evolve according to a certain rule in terms of speed, degree, and form. Potential widely exists in the system, which is determined by the interaction of various components of the system, the relative relationship, and the influence of the system and the environment. System potential, which describes the behavior of the system, is expressed by internal state variables and external control variables. To analyze the stability of the traffic system, the potential function of the traffic system is established as follows.

**Theorem 1.** In a traffic system, the system potential function \( V(k, v_w, q) \), which is composed of traffic density \((k)\), traffic return wave \((v_w)\), and traffic volume \((q)\), is as follows:

\[
V(k, v_w, q) = \frac{1}{4}k^4 + \frac{1}{2}Av_wk^2 - Aq. \tag{1}
\]

**Proof.** According to the general assumption of the traffic system, the three basic variables of traffic flow, traffic volume \(q\) (veh/h), average speed of traffic flow \(v\) (km/h), and traffic density \(k\) (veh/km), are related as follows:

\[
q = k \cdot v. \tag{2}
\]

When the traffic flow encounters some interference and the traffic density suddenly increases, the number of cars passing through a certain point will decrease inevitably. This deceleration trend is called the return wave, which propagates to the rear of the traffic flow at the speed of \(v_w\). We have

\[
v_w = \frac{dq}{dk} = v + k \frac{dv}{dk}. \tag{3}
\]

According to the pipe speed-density model, we have

\[
v = v_f \left[1 - \frac{k}{(k_j)^{s+1}}\right], \tag{4}
\]

where \(v_f\) is the free-flow speed, \(k_j\) is the jam density, \(s\) is an integer coefficient, and \(s \neq -1\). According to equation (4),

\[
\frac{dv}{dk} = -\frac{(s + 1)v_f}{k_j^{s+1}}k^s. \tag{5}
\]

Let \(A' = (k_j^{s+1}/(s + 1)v_f)\), and \(A'\) is the road characteristic coefficient, reflecting the specific characteristics of the road. According to formulas (2), (3), and (5), we have

\[
k^{s+2} + A'v_wk - Aq = 0. \tag{6}
\]

According to the range \(s\) of pipe structure equation, we take \(s = 1\), and the formula (6) is translated to

\[
k^3 + Av_wk - Aq = 0, \tag{7}
\]

where \(A = (k_j^2/2v_f)\). Let \(V(k, v_w, q)\) is the system potential function, then

\[
\frac{\partial V}{\partial k} = k^3 + Av_wk - Aq = 0. \tag{8}
\]
Therefore, formula (1) is established, and the conclusion is proved.

According to Theorem 1, the structure of the potential function of traffic system is consistent with that of the cusp model in the catastrophe theory, which indicates that the structure of traffic system has catastrophic characteristics. Catastrophe is a widespread phenomenon in nonlinear systems. Its representation is characterized by the micro-change of external conditions leading to the drastic change of the system's macroscopic state. That is, the continuous change of external conditions leads to the discontinuous catastrophe of the system state. For example, road traffic is affected by the shockwave, and traffic congestion will occur in a certain critical state, that is, a catastrophe phenomenon. Traffic system is affected by the "human-vehicle-road-environment" together. Although some parameters and their correlations can be studied qualitatively and quantitatively, at present, there are still many factors whose relationships and influence are not clear enough and have obvious gray characteristics. Based on the gray character and catastrophe character of the traffic system, this paper establishes the grey catastrophe model of the traffic system.

4. Grey Catastrophe Model of Traffic System

Traffic congestion is a phenomenon of the sudden change of traffic state caused by the influence of the natural environment (visibility, wind direction, and temperature) and the continuous change of internal control variables of the system such as human factors (traffic accidents, emergencies, and personal psychological state of drivers). This phenomenon is caused by the evolution of the system structure according to a certain law with time and space, which is not completed in an instant. How to theoretically analyze the evolution law and structural characteristics of road traffic systems and establish the grey catastrophe model are the research focus of this paper. In Section 4.1, we propose the grey catastrophe model by the grey characteristics and catastrophe characteristics of the transportation system. Then, the stability of the system is analyzed, and the trend is analyzed according to the potential function in Section 4.2.

4.1. Grey Catastrophe Model. In the traffic system, traffic volume is the primary state variable of the system operation, and its prediction results are directly related to the service level of traffic information and the effect of traffic control induction. Taking traffic volume as the research object, the grey catastrophe model is established as follows.

**Definition 1.** Assume that \( X^{(0)} = (x^{(0)}(t)), t = 1, 2, \ldots, n \) is an original sequence, \( X^{(1)} = (x^{(1)}(t)) \) is the 1-AGO sequence of \( X^{(0)} \), and the grey catastrophe model is

\[
x^{(1)}(t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4 + A_5 t^5,
\]

where \( t \) is the cumulative observation time, and \( A_0, A_1, A_2, A_3, A_4, A_5 \) are obtained by regression analysis.

The essence of grey modeling theory is to weaken the randomness of the original random sequence by using the information processing method of "generation." The time series of the traffic volume in the traffic system is \( x^{(0)}(t) \), whose 1-AGO is \( x^{(1)}(t) \). And any univariate continuous function can always be expanded into a power series by Taylor, so

\[
x^{(1)}(t) = A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n.
\]

According to the catastrophe theory, the correctness of the cusp position can be guaranteed only by inspecting the Taylor series in the fourth power term, and the derivative of equation (9) can be translated into

\[
x^{(0)}(t) = \frac{d x^{(1)}(t)}{dt} = A_1 + 2A_2 t + 3A_3 t^2 + 4A_4 t^3 + 5A_5 t^4.
\]

Therefore, the established model conforms to the characteristics of the catastrophe model.

**Theorem 2.** Taking the traffic volume as the research object, the potential function \((X(T, a, b))\) of the grey catastrophe model \((GCM(1, 1))\) is constructed, and we have

\[
X(T, a, b) = \frac{1}{4} T^4 + b T + C,
\]

where \( T \) is the state variable (indirect time state variable), \( a, b \) are the control variable, and \( C \) is the parameter.

**Proof.** Let \( X = x^{(0)}(t) \), \( a_0 = A_1, a_1 = 2A_2, a_2 = 3A_3, a_3 = 4A_4, a_4 = 5A_5 \), and then, the formula (11) is translated into

\[
X = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4.
\]

Variable substitution is done for formula (13), and let \( t = T_t - d, d = a_4/4a_5 \), so that

\[
X = b_1 T_t^4 + b_2 T_t^3 + b_3 T_t^2 + b_0,
\]

where

\[
b_1 = a_4 d^4 - a_2 d^2 + a_3 d - a_1 d + a_0,
b_2 = -4a_4 d^3 + 3a_3 d^2 - 2a_2 d + a_1,
b_3 = -6a_4 d^2 + 3a_3 d + a_2,
b_4 = a_4.
\]

Furthermore, variable substitution is done for formula (14). Let

\[
T_t = T - \sqrt[\frac{1}{4b_4}]{b_4} \begin{cases} \{ & b_4 > 0 \\ \{ & b_4 < 0 
\end{cases}
\]

when \( b_4 > 0 \), and by substituting formula (15) into formula (14), we have

\[
X = \frac{1}{4} T^4 + \frac{1}{2} a T^2 + b T + C,
\]

where \( T \) is the indirect time state variable and \( T = (t + d) \sqrt[\frac{1}{4b_4}]{C} \), where \( b_4 \) is the parameter with little significance to the system structure. \( a, b \) are control variable of the system potential function, which can be expressed as
system potential function, which can be expressed as
\[ a = \frac{b_2}{\sqrt{B_4}}, \]  \hspace{1cm} (17)
\[ b = \frac{b_1}{\sqrt{4B_4}}. \]  \hspace{1cm} (18)

when \( b_1 < 0 \), and by substituting formula (15) into formula (14), we have
\[ X = \frac{1}{4}T^4 + \frac{1}{2}aT^2 + bT + C, \]  \hspace{1cm} \hspace{1cm} (19)
\[ \text{where } T \text{ is the indirect time state variable, and } T = (t + d) \cdot \sqrt{-4b_4}. C \text{ is the parameter with little significance to the system structure.} \]
\[ a, b \text{ are control variable of the system potential function, which can be expressed as} \]
\[ a = \frac{b_2}{\sqrt{b_4}}, \]  \hspace{1cm} (20)
\[ b = \frac{b_1}{\sqrt{4b_4}}. \]  \hspace{1cm} (21)
\[ \text{To sum up, the potential function (12) can be obtained through variable substitution according to model (9).} \]
\[ \text{In addition to conforming to the characteristics of the abrupt cusp of the system, it is important to determine the important parameters controlling the state change of the system and ignore the minor ones.} \]

4.2. Stability Analysis of Traffic System. The control variable in the potential function is similar to the development coefficient in the classical grey model. They are used to describe and control the development trend of the system. It can be found that the grey time series of the traffic system has a typical topological structure of the cusp catastrophe potential function, which conforms to the predictability condition. It also indicates that the formula of traffic jam can be quantitatively predicted under certain conditions. Next, the stability of the system is analyzed based on the potential function, and the trend is predicted.

**Theorem 3.** Taking the potential function of the traffic system in Theorem 2 as the research object, the following conclusions can be drawn from the analysis of its structural stability:

(1) \( D = 0 \)

(2) \( D < 0 \), and \( 3T^2 + a < 0 \)

where \( D = 4a^3 + 27b^2 \).

**Proof.** According to the catastrophe theory, the equilibrium surface equation of the system can be obtained from the potential function (12),
\[ \frac{\partial X}{\partial T} = T^3 + aT + b = 0. \]  \hspace{1cm} (22)
\[ \text{Equation (20) shows the relationship between state variable } T \text{ and control variables } a, b, \text{ of which the folded or}
\]
sharp inflection point set of the space surface represented is called the singularity set. Its projection on the plane of \( a - b \) is called the bifurcation set, and the equation of the bifurcation set is
\[ 4a^3 + 27b^2 = 0. \]  \hspace{1cm} (23)
\[ \text{Let } D = 4a^3 + 27b^2, \text{ we know that the bifurcation set is a half-cube parabola, and it has a sharp point at } (0, 0). \]
\[ \text{The point } (a, b) \text{ on the bifurcation set corresponds to the unstable state (critical state) of the system, which may change from one equilibrium state to another.} \]
\[ \text{At the same time, the bifurcation set divides the control variable plane into two regions. When } D < 0, \text{ the system may also be unstable under } aob. \]
\[ \text{The criterion for determining the unstable point is} \]
\[ \frac{\partial^2 X}{\partial T^2} = 3T^2 + a < 0. \]  \hspace{1cm} (24)
\[ \text{In summary, the conclusion of the system becoming catastrophic can be proved.} \]

**Theorem 4.** Taking traffic flow as the research object, the time difference before and after the system instability in the critical state is
\[ \Delta t = \sqrt{3}(-a)^{1/2} (4b_4)^{-1/4}. \]  \hspace{1cm} (25)

**Proof.** According to the analysis of Theorem 3, when \( b > 0 \), the catastrophe of the right branch of the bifurcation set refers to the catastrophe of the mathematical structure of the system, and the value of the state variable \( T \) does not jump.
\[ \text{For the early warning of traffic system instability, it is concerned about the unstable state of the corresponding point when spanning the left branch } (b < 0) \text{ of the bifurcation set, and the value of state variable } T \text{ jumps. If the formula (21) is true, when } a = 0, \text{ formula (17) has triple zero roots, and } T_1 = T_2 = T_3 = 0; \text{ when } a < 0, \text{ formula (20) has three real roots; that is,} \]
\[ T_1 = 2\left(\frac{a}{3}\right)^{1/2}, \]  \hspace{1cm} (26)
\[ T_2 = T_3 = -\left(\frac{a}{3}\right)^{1/2}. \]  \hspace{1cm} (27)
\[ \text{When crossing the bifurcation set, the value of the state variable } T \text{ changes abruptly,} \]
\[ \Delta T = T_1 - T_2 = 3\left(\frac{a}{3}\right)^{1/2}. \]  \hspace{1cm} (28)
\[ \Delta t = \Delta T \sqrt{\frac{1}{4b_4}} = \sqrt{3}(-a)^{1/2} (4b_4)^{-1/4}. \]  \hspace{1cm} (29)
\[ \text{The conclusion is clear. By using Theorem 4, the time difference between the critical state } (D = 0) \text{ and the} \]
congestion time $\Delta t$ in the traffic system can be obtained, while the sum of $\Delta t$ and the duration of the critical state are the time when congestion occurs in the traffic system.

5. Conclusion

In this paper, a grey catastrophe model is established by analyzing the potential function of the traffic system. We take traffic volume as state variable, speed, and density as control variables to establish the system potential function and theoretically clarifies that the structure of the traffic system has catastrophe characteristic. Combined with the grey characteristics of traffic system, we construct the grey catastrophe model for traffic system and obtain the traffic potential function with the system time as the state variable by substituting the variables of the model. Also, we analyze the traffic stability theoretically based on the potential function. The model can be used for stability analysis, time series prediction, and system evolution. The grey catastrophe model of traffic, which integrates the grey theory and catastrophe theory, is of certain theoretical and practical significance in solving traffic problems.

Up to this point, the rationality of the grey catastrophe model of traffic system is expounded theoretically, and the grey model theory is well expanded. However, how to solve the actual traffic problems is a problem to be our future research. How to apply the model to practice and better explain the relationship among traffic flow parameters by using the combination of qualitative and quantitative methods implements the optimal control of the traffic system.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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