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Research Article

Relational Quasicontractions and Related Fixed Point Theorems

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In this article, some results on fixed points under quasicontractions in the framework of metric space endowed with binary relation are proved. Our newly proved results improve and extend several noted fixed point theorems of the existing literature besides their relation-theoretic analogues. We conclude this article by constructing an example to affirm the efficacy of our results.

1. Introduction

Given a metric space $(\mathbb{M}, \varrho)$, a mapping $\mathscr{F}\colon \mathbb{M} \longrightarrow \mathbb{M}$ is said to be contraction if it satisfies

$$\varrho(\mathscr{F}r, \mathscr{F}t) \leq k\varrho(r,t), \quad \text{for some } 0 \leq k < 1. \tag{1}$$

Banach contraction principle [1] plays a key role in the area of metrical fixed point theory. This core result guarantees of the existence and uniqueness of fixed point under the hypotheses that the ambient space remains a complete metric space, whereas the underlying mapping should be a contraction mapping. Many authors extended this classical result employing relatively more general contractive conditions. One of the interesting extensions of contraction mapping was given by Ćirić [2], often referred as quasicontraction. We say that a self-mapping $\mathscr{F}$ defined on a metric space $(\mathbb{M}, \varrho)$ is a quasicontraction if for some $0 \leq q < 1$ and for all $r, t \in \mathbb{M}$, we have

$$\varrho(\mathscr{F}r, \mathscr{F}t) \leq q \cdot \max\{\varrho(r,t), \varrho(x, \mathscr{F}r), \varrho(t, \mathscr{F}t), \varrho(r, \mathscr{F}t), \varrho(t, \mathscr{F}r)\}. \tag{2}$$

Indeed quasicontraction mappings subsume several noted generalized contractions due to Kannan [3], Chatterjea [4], Reich [5], Hardy and Rogers [6], Bianchini [7], Rhoades [8], and Zamfirescu [9]. For further details regarding quasicontractions, we refer some recent works due to Aydi et al. [10], Karapınar et al. [11], Bachar and Khamsi [12], Alfuraidan [13], Fallahi and Aghanians [14], Darko et al. [15], and Karapınar et al. [16].

A variant of Banach contraction principle under binary relation is proved by Alam and Imdad [17], wherein the ambient metric space and other involved notions are compatible with an amorphous relation. Under universal relation, the result reduces to classical Banach contraction principle, while under partially ordered relation, the result is transformed into classical order-theoretic metrical fixed point theorems of Ran and Reurings [18] and Nieto and Rodríguez-López [19]. In recent years, various metrical fixed theorems are proved under different types of contractive conditions employing certain binary relations (e.g., [20–34]).

The main intent of this article is to prove a fixed point theorem in the setting of metric space endowed with a binary relation under relation-preserving quasicontraction condition. We also deduce some consequences from our newly proved results. An example is also furnished to demonstrate our main results.

2. Preliminaries

This section deals with certain relevant notions and basic results which are utilized in our subsequent discussions. For any set $\mathbb{M} \neq \varnothing$, a subset $\mathfrak{R}$ of $\mathbb{M}^2$ is termed as a binary relation on $\mathbb{M}$. We sometimes write $r\mathfrak{R}t$ instead of $(r,t) \in \mathfrak{R}$, e.g., in case of the relations of "less than equals to" ($\mathfrak{R} \coloneqq \leq$) and "greater than equals to" ($\mathfrak{R} \coloneqq \geq$) on $\mathbb{R}$, the set of real numbers is expressed, respectively, as $r \leq t$ and $r \geq t$.

Definition 1 (see [35]). If $\mathfrak{R}$ is any binary relation on a set $\mathbb{M}$, then the set

$$\mathfrak{R}^{-1}\colon = \left\{(r,t) \in \mathbb{M}^2\colon\ (t,r) \in \mathfrak{R}\right\} \tag{3}$$

remains again a relation on $\mathbb{M}$, which is termed as an inverse relation of $\mathfrak{R}$. Also, the set

$$\mathfrak{R}^s \coloneqq \mathfrak{R} \cup \mathfrak{R}^{-1} \tag{4}$$

forms again a relation on $\mathbb{M}$, which is called the symmetric-closure of $\mathfrak{R}$. Clearly, $\mathfrak{R}^s$ remains the least symmetric relation on $\mathbb{M}$ among those binary relations which contain $\mathfrak{R}$.

Definition 2 (see [17]). Any two elements r and t of a set $\mathbb{M}$ are called $\mathfrak{R}$-comparative, whereas $\mathfrak{R}$ remains a relation on $\mathbb{M}$ if either $(r,t) \in \mathfrak{R}$ or $(t,r) \in \mathfrak{R}$. Usually $[r,t] \in \mathfrak{R}$ means that "r and t are $\mathfrak{R}$-comparative."

Proposition 1 (see [17]). *If $\mathfrak{R}$ remains a relation on $\mathbb{M}$, then*

$$(r,t) \in \mathfrak{R}^s \Leftrightarrow [r,t] \in \mathfrak{R}. \tag{5}$$

Definition 3 (see [35]). If $\mathfrak{R}$ remains a relation on a set $\mathbb{M}$, then $\mathfrak{R}$ is said to be complete if each pair of elements of $\mathbb{M}$ is $\mathfrak{R}$-comparative, i.e.,

$$[r,t] \in \mathfrak{R}, \quad \forall r,t \in \mathbb{M}. \tag{6}$$

Definition 4 (see [35]). If $\mathbb{E} \subseteq \mathbb{M}$ and $\mathfrak{R}$ stands for a relation on $\mathbb{M}$, then the set

$$\mathfrak{R}|_{\mathbb{E}} \coloneqq \mathfrak{R} \cap \mathbb{E}^2 \tag{7}$$

is termed as the restriction of $\mathfrak{R}$ to $\mathbb{E}$. It is clear that $\mathfrak{R}|_{\mathbb{E}}$ remains a relation on $\mathbb{E}$ induced by $\mathfrak{R}$.

Definition 5 (see [17]). By an $\mathfrak{R}$-preserving sequence, whereas $\mathfrak{R}$ remains a relation on a set $\mathbb{M}$, we meant that the sequence $\{r_n\} \subset \mathbb{M}$, which satisfy

$$\left(r_n, r_{n+1}\right) \in \mathfrak{R}, \quad \forall n \in \mathbb{N}_0. \tag{8}$$

Definition 6 (see [17]). Given any set $\mathbb{M}$ and a mapping $\mathcal{F}$ from $\mathbb{M}$ into itself, a relation $\mathfrak{R}$ on $\mathbb{M}$ is termed as $\mathcal{F}$-closed if for any $r,t \in \mathbb{M}$,

$$(r,t) \in \mathfrak{R} \Rightarrow (\mathcal{F}r, \mathcal{F}t) \in \mathfrak{R}. \tag{9}$$

Proposition 2 (see [20]). *Given any set $\mathbb{M}$, suppose that $\mathfrak{R}$ remains a relation on $\mathbb{M}$ while $\mathcal{F}$ remains a function from $\mathbb{M}$ into itself. If $\mathfrak{R}$ is $\mathcal{F}$-closed, then $\mathfrak{R}^s$ must be $\mathcal{F}$-closed.*

Proposition 3 (see [21]). *Given any set $\mathbb{M}$, suppose that $\mathfrak{R}$ remains a relation on $\mathbb{M}$ while $\mathcal{F}$ remains a function from $\mathbb{M}$ into itself. If $\mathfrak{R}$ is $\mathcal{F}$-closed, then $\mathfrak{R}$ must be $\mathcal{F}^n$-closed (where $n \in \mathbb{N}_0$).*

Definition 7 (see [36, 37]). Given any set $\mathbb{M}$, suppose that $\mathcal{F}$ remains a function from $\mathbb{M}$ into itself and $\alpha\colon \mathbb{M} \times \mathbb{M} \longrightarrow [0,\infty)$ is a mapping. We say that $\mathcal{F}$ is α-admissible if for all $r,t \in \mathbb{M}$,

$$\alpha(r,t) \geq 1 \Rightarrow \alpha(\mathcal{F}r, \mathcal{F}t) \geq 1. \tag{10}$$

If we define the function $\alpha(r,t) = \begin{cases} 1, & \text{if } (r,t) \in \mathfrak{R} \\ 0, & \text{if } (r,t) \notin \mathfrak{R} \end{cases}$, then the α-admissibility of $\mathcal{F}$ is equivalent to $\mathcal{F}$-closedness of $\mathfrak{R}$.

Definition 8 (see [20]). A metric space $(\mathbb{M}, \varrho)$ is termed as $\mathfrak{R}$-complete (whereas $\mathfrak{R}$ remains a relation on $\mathbb{M}$) if each $\mathfrak{R}$-preserving Cauchy sequence in $\mathbb{M}$ converges.

Obviously, given any metric space, completeness implies $\mathfrak{R}$-completeness whatever the relation $\mathfrak{R}$. In particular, if $\mathfrak{R}$ remains the universal relation, then the concepts of $\mathfrak{R}$-completeness and usual completeness are equivalent.

Definition 9 (see [20]). A mapping $\mathcal{F}\colon \mathbb{M} \longrightarrow \mathbb{M}$, whereas $(\mathbb{M}, \varrho)$ remains a metric space while $\mathfrak{R}$ remains a relation on $\mathbb{M}$, is termed $\mathfrak{R}$-continuous at a point $r \in \mathbb{M}$ if for any $\mathfrak{R}$-preserving sequence $\{r_n\}$ such that $r_n \xrightarrow{\varrho} r$, we have $\mathcal{F}(r_n) \xrightarrow{\varrho} \mathcal{F}(r)$. Moreover, $\mathcal{F}$ is termed as $\mathfrak{R}$-continuous if it is $\mathfrak{R}$-continuous at every point of $\mathbb{M}$.

Obviously, for any mapping on a metric space endowed with the relation $\mathfrak{R}$, continuity implies $\mathfrak{R}$-continuity. In particular, if $\mathfrak{R}$ remains the universal relation, then the concepts of $\mathfrak{R}$-continuity and usual continuity are equivalent.

Definition 10 (see [17]). A relation $\mathfrak{R}$ on a metric space $(\mathbb{M}, \varrho)$ is termed as ϱ-self-closed if each $\mathfrak{R}$-preserving sequence $\{r_n\}$ with $r_n \xrightarrow{\varrho} r$ has a subsequence $\left\{r_{n_k}\right\}$ whose terms are $\mathfrak{R}$-comparative with r, i.e.,

$$\left[r_{n_k}, r\right] \in \mathfrak{R}, \quad \forall k \in \mathbb{N}_0. \tag{11}$$

In the sequel, for a metric space $(\mathbb{M}, \varrho)$, a relation $\mathfrak{R}$ and a function $\mathcal{F}$ from $\mathbb{M}$ into itself, the following notations will be adopted:

(i) $F(\mathcal{F})$:= the collection of the fixed points of $\mathcal{F}$

(ii) $\mathbb{M}(\mathcal{F}, \mathfrak{R})\colon = \{r \in \mathbb{M}\colon\ (r, \mathcal{F}r) \in \mathfrak{R}\}$

(iii) $\delta(r) \coloneqq \sup\{\varrho(\mathcal{F}^n r, \mathcal{F}^m r)\colon\ n, m \in \mathbb{N}\}$

3. Main Results

In the following lines, we prove a fixed point theorem for quasicontractions employing a binary relation.

Theorem 1. *Let $(\mathbb{M}, \varrho)$ be a metric space while $\mathfrak{R}$ is a relation on $\mathbb{M}$ and $\mathcal{F}$ is a function from $\mathbb{M}$ into itself. Also, suppose that*

(1) *[(a)]* $(\mathbb{M}, \varrho)$ *is* $\mathfrak{R}$*-complete.*

(2) *[(b)]*$\mathfrak{R}$ *is* $\mathcal{F}$*-closed.*

(3) *[(c)]*$\mathcal{F}$ *is* $\mathfrak{R}$*-continuous or* $\mathfrak{R}$ *is* ϱ*-self-closed.*

(4) *[(d)]*$\mathbb{M}(\mathcal{F}, \mathfrak{R})$ *is nonempty.*

(5) *[(e)] for some* $q \in [0, (1/2))$ *and for all* $r, t \in \mathbb{M}$ *with* $(r, t) \in \mathfrak{R}$*, the following assumption is satisfied:*

$$\begin{aligned}\varrho(\mathcal{F}r, \mathcal{F}t) \leq q \cdot \max\{&\varrho(r, t), \varrho(r, \mathcal{F}r),\\ &\varrho(t, \mathcal{F}t), \varrho(r, \mathcal{F}t), \varrho(t, \mathcal{F}r)\}.\end{aligned} \tag{12}$$

Then, $\mathcal{F}$ admits a fixed point. Further, the completeness of $\mathfrak{R}_{F(\mathcal{F})}$ implies the uniqueness of fixed point.

Proof. In view of assumption (d), we can choose $r_0 \in \mathbb{M}(\mathcal{F}, \mathfrak{R})$. Define a sequence $\{r_n\}$ of Picard iteration based at the initial point r_0 so that

$$r_n = \mathcal{F}^n(r_0), \quad \forall n \in \mathbb{N}_0. \tag{13}$$

As $(r_0, \mathcal{F}r_0) \in \mathfrak{R}$, using $\mathcal{F}$-closedness of $\mathfrak{R}$ and Proposition 3, we get

$$\left(\mathcal{F}^n r_0, \mathcal{F}^{n+1} r_0\right) \in \mathfrak{R}, \tag{14}$$

so that

$$\left(r_n, r_{n+1}\right) \in \mathfrak{R}, \quad \forall n \in \mathbb{N}_0. \tag{15}$$

It follows that the sequence $\{r_n\}$ is $\mathfrak{R}$-preserving.

Denote $\varrho_n \coloneqq \varrho(r_{n+1}, r_n)$. Applying the contractive condition (e) and using (13) and (15), we obtain for all $n \in \mathbb{N}_0$ that

$$\begin{aligned}\varrho_n = \varrho(r_n, r_{n+1}) &= \varrho(\mathcal{F}r_{n-1}, \mathcal{F}r_n)\\ &\leq q \cdot \max\{\varrho(r_{n-1}, r_n), \varrho(r_{n-1}, \mathcal{F}r_{n-1}),\\ &\quad \varrho(r_n, \mathcal{F}r_n), \varrho(r_{n-1}, \mathcal{F}r_n), \varrho(r_n, \mathcal{F}r_{n-1})\}\\ &= q \cdot \max\{\varrho_{n-1}, \varrho_n, \varrho(r_{n-1}, r_{n+1}), 0\},\end{aligned} \tag{16}$$

which gives rise to

$$\varrho_n \leq q \cdot (\varrho_{n-1} + \varrho_n), \tag{17}$$

so that

$$\varrho_n \leq \frac{q}{1-q} \cdot \varrho_{n-1} = p \cdot \varrho_{n-1}, \quad \forall n \in \mathbb{N}. \tag{18}$$

As $q \in [0, (1/2))$, we have $p = (q/(1-q)) \in [0, 1)$. By induction, equation (18) reduces to

$$\varrho_n \leq p\varrho_{n-1} \leq p^2\varrho_{n-2} \leq \cdots \leq p^n\varrho_0, \tag{19}$$

so that

$$\varrho_n \leq p^n\varrho_0, \quad \forall n \in \mathbb{N}. \tag{20}$$

For $n < m$, using triangular inequality and (20), we obtain

$$\begin{aligned}\varrho(r_n, r_m) &\leq \varrho(r_n, r_{n+1}) + \varrho(r_{n+1}, r_{n+2}) + \cdots + \varrho(r_{m-1}, r_m)\\ &\leq \left(p^n + p^{n+1} + \cdots + p^{m-1}\right)\varrho_0\\ &= p^n\left(1 + p + p^2 + \cdots + p^{n-m+1}\right)\varrho_0\\ &< \frac{p^n}{1-p}\varrho_0 \longrightarrow 0, \quad \text{as } m, n \longrightarrow \infty,\end{aligned} \tag{21}$$

yielding thereby $\{r_n\}$ as Cauchy. Hence, $\{r_n\}$ is an $\mathfrak{R}$-preserving Cauchy sequence. By $\mathfrak{R}$-completeness of $(\mathbb{M}, \varrho)$, one can find $r \in \mathbb{M}$ satisfying

$$r_n \xrightarrow{\varrho} r. \tag{22}$$

Now, we use assumption (c) to show that r is a fixed point of $\mathcal{F}$. Suppose that $\mathcal{F}$ is $\mathfrak{R}$-continuous. Since $\{r_n\}$ is $\mathfrak{R}$-preserving with $r_n \xrightarrow{\varrho} r$, therefore $\mathfrak{R}$-continuity of $\mathcal{F}$ asserts that $r_{n+1} = \mathcal{F}(r_n) \xrightarrow{\varrho} \mathcal{F}(r)$. The uniqueness of limit ensures that $\mathcal{F}(r) = r$ so that r is a fixed point of $\mathcal{F}$. Next, we assume that $\mathfrak{R}$ is ϱ-self-closed, then since $\{r_n\}$ remains an $\mathfrak{R}$-preserving sequence converging to r, therefore one can a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ satisfying $[r_{n_k}, r] \in \mathfrak{R}$ for all $k \in \mathbb{N}_0$. Obviously, we have

$$r_{n_k} \xrightarrow{\varrho} r. \tag{23}$$

On using $[r_{n_k}, r] \in \mathfrak{R}$, symmetry of ϱ and assumption (e), we have

$$\begin{aligned}\varrho\left(r_{n_k+1}, \mathcal{F}r\right) &= \varrho\left(\mathcal{F}r_{n_k}, \mathcal{F}r\right)\\ &\leq q \cdot \max\left\{\varrho\left(r_{n_k}, r\right), \varrho\left(r_{n_k}, \mathcal{F}r_{n_k}\right),\right.\\ &\quad \left.\varrho(r, \mathcal{F}r), \varrho\left(r_{n_k}, \mathcal{F}r\right), \varrho\left(r, \mathcal{F}r_{n_k}\right)\right\}.\end{aligned} \tag{24}$$

Making use of the triangular inequality $\varrho(r_{n_k}, \mathcal{F}r) \leq \varrho(r_{n_k}, r) + \varrho(r, \mathcal{F}r)$, equation (24) reduces to

$$\varrho\left(r_{n_k+1}, \mathcal{F}r\right) \leq q \cdot \left[\varrho\left(r_{n_k}, r\right) + \varrho\left(r_{n_k}, r_{n_k+1}\right) + \varrho(r, \mathcal{F}r) + \varrho\left(r, r_{n_k+1}\right)\right]. \tag{25}$$

Using (25) and triangular inequality, we get

$$\begin{aligned}\varrho(r, \mathcal{F}r) &\leq \varrho\left(r, r_{n_k+1}\right) + \varrho\left(r_{n_k+1}, \mathcal{F}r\right)\\ &\leq (1+q)\varrho\left(r, r_{n_k+1}\right)\\ &\quad + q \cdot \left[\varrho\left(r_{n_k}, r\right) + \varrho\left(r_{n_k}, r_{n_k+1}\right) + \varrho(r, \mathcal{F}r)\right],\end{aligned} \tag{26}$$

thereby yielding

$$\begin{aligned}\varrho(r, \mathcal{F}r) &\leq \frac{1+q}{1-q}\varrho\left(r, r_{n_k+1}\right)\\ &\quad + \frac{q}{1-q}\left[\varrho\left(r_{n_k}, r\right) + \varrho_{n_k}\right] \longrightarrow 0, \quad \text{as } k \longrightarrow 0,\end{aligned} \tag{27}$$

(owing to (20) and (23)) so that $\mathcal{F}(r) = r$. Thus, r is a fixed point.

Finally, to prove the uniqueness of fixed point, we take $r, t \in F(\mathcal{F})$, then we have

$$\begin{aligned} \mathcal{F}(r) &= r, \\ \mathcal{F}(t) &= t. \end{aligned} \tag{28}$$

As $\mathfrak{R}_{F(\mathcal{F})}$ is complete, we have $[r, t] \in \mathfrak{R}$. Applying contractive condition (e) on these points, we get

$$\begin{aligned} \varrho(r, t) &= \varrho(\mathcal{F}r, \mathcal{F}t) \\ &\leq q \cdot \max\{\varrho(r, t), \varrho(r, r), \varrho(t, t), \varrho(r, t), \varrho(t, r)\}, \end{aligned} \tag{29}$$

so that

$$\varrho(r, t) \leq q \cdot \varrho(r, t), \tag{30}$$

which gives rise to

$$r = t. \tag{31}$$

It follows that $\mathcal{F}$ has a unique fixed point. This completes the proof.

Now, we present the following results which ensure that Theorem 1 is true for $q \in [0, 1)$ under some restricted hypotheses. □

Theorem 2. *Let $(\mathbb{M}, \varrho)$ be a metric space while $\mathfrak{R}$ is a relation on $\mathbb{M}$ and $\mathcal{F}$ is a function from $\mathbb{M}$ into itself. Also, suppose that*

(1) [(a)] $(\mathbb{M}, \varrho)$ is $\mathfrak{R}$-complete.
(2) [(b)] $\mathfrak{R}$ is $\mathcal{F}$-closed and transitive.
(3) [(c)] $\mathcal{F}$ is $\mathfrak{R}$-continuous or $\mathfrak{R}$ is ϱ-self-closed.
(4) [(d)] there exists $r_0 \in \mathbb{M}(\mathcal{F}, \mathfrak{R})$ such that $\delta(r_0) < \infty$.
(5) [(e)] for some $q \in [0, 1)$ and for all $r, t \in \mathbb{M}$ with $(r, t) \in \mathfrak{R}$, the following assumption is satisfied:

$$\begin{aligned} \varrho(\mathcal{F}r, \mathcal{F}t) \leq q \cdot \max\{&\varrho(r, t), \varrho(r, \mathcal{F}r), \\ &\varrho(t, \mathcal{F}t), \varrho(r, \mathcal{F}t), \varrho(t, \mathcal{F}r)\}. \end{aligned} \tag{32}$$

Then, $\mathcal{F}$ admits a fixed point. Further, the completeness of $\mathfrak{R}_{F(\mathcal{F})}$ implies the uniqueness of fixed point.

Proof. Following the lines of the proof of Theorem 1, one can show that the sequence r_n defined by $r_n = \mathcal{F}^n(r_0)$ is $\mathfrak{R}$-preserving. Using (15) and transitivity of $\mathfrak{R}$, we obtain

$$(r_n, r_m) \in \mathfrak{R}, \quad \forall n, m \in \mathbb{N}_0 \text{ with } n < m. \tag{33}$$

Using (33) and assumption (e), we get

$$\begin{aligned} \varrho(r_n, r_m) \leq q \cdot \max\{&\varrho(r_{n-1}, r_{m-1}), \varrho(r_{n-1}, r_n), \\ &\varrho(r_{m-1}, r_m), \varrho(r_{n-1}, r_m), \varrho(r_n, r_{m-1})\}, \end{aligned} \tag{34}$$

which implies that

$$\delta(r_n) \leq q\delta(r_{n-1}), \quad \forall n \in \mathbb{N}. \tag{35}$$

Hence, we have

$$\delta(r_n) \leq q^n \delta(r_0), \quad \forall n \in \mathbb{N}, \tag{36}$$

which yields that

$$\varrho(r_n, r_{n+m}) \leq \delta(r_n) \leq q^n \delta(r_0), \quad \forall n, m \in \mathbb{N}. \tag{37}$$

As $\delta(r_0) < \infty$, the sequence $\{r_n\}$ is Cauchy. Rest of the proof can be completed by proceeding the lines of the proof of Theorem 1. □

4. Applications

As applications, we deduce the following consequences of Theorem 1, which are indeed relation-theoretic versions of some well-known theorems existing in the literature.

Corollary 1 (Hardy-Rogers type). *The conclusion of Theorem 1 holds under the hypotheses (a)–(d) along with the following:*

(1) [(e_1)] for some $\alpha, \eta, \delta, \lambda, \mu \geq 0$ satisfying $\alpha + \eta + \delta + \lambda + \mu < 1$ and for all $r, t \in \mathbb{M}$ with $(r, t) \in \mathfrak{R}$, the following assumption is satisfied:

$$\begin{aligned} \varrho(\mathcal{F}r, \mathcal{F}t) \leq\ &\alpha\varrho(r, t) + \eta\varrho(r, \mathcal{F}r) + \delta\varrho(t, \mathcal{F}t) \\ &+ \lambda\varrho(t, \mathcal{F}r) + \mu\varrho(r, \mathcal{F}t). \end{aligned} \tag{38}$$

Proof. Take $r, t \in \mathbb{M}$ with $(r, t) \in \mathfrak{R}$ and write $q = \alpha + \eta + \delta + \lambda + \mu$, and then, we have

$$\begin{aligned} \varrho(\mathcal{F}r, \mathcal{F}t) &\leq \alpha\varrho(r, t) + \eta\varrho(r, \mathcal{F}r) + \delta\varrho(t, \mathcal{F}t) \\ &\quad + \lambda\varrho(t, \mathcal{F}r) + \mu\varrho(r, \mathcal{F}t) \\ &= q \cdot \max\{\varrho(r, t), \varrho(r, \mathcal{F}r), \\ &\quad \varrho(t, \mathcal{F}t), \varrho(r, \mathcal{F}t), \varrho(t, \mathcal{F}r)\}. \end{aligned} \tag{39}$$

Thus, the result follows from Theorem 1. □

Corollary 2 (Reich type). *The conclusion of Theorem 1 holds under the hypotheses (a)–(d) along with the following:*

(1) [(e_2)] for some $\alpha, \eta, \delta \geq 0$ satisfying $\alpha + \eta + \delta < 1$ and for all $r, t \in \mathbb{M}$ with $(r, t) \in \mathfrak{R}$, the following assumption is satisfied:

$$\varrho(\mathcal{F}r, \mathcal{F}t) \leq \alpha\varrho(r, t) + \eta\varrho(r, \mathcal{F}r) + \delta\varrho(t, \mathcal{F}t). \tag{40}$$

Proof. Choosing $\lambda = \mu = 0$ in Corollary 1, the assumption (e_1) reduces to (e_2). Consequently, the result follows from Corollary 1. □

Corollary 3 (Kannan type). *The conclusion of Theorem 1 holds under the hypotheses (a)–(d) along with the following:*

(1) [(e_3)] for some $\eta, \delta \geq 0$ satisfying $\eta + \delta < 1$ and for all $r, t \in \mathbb{M}$ with $(r, t) \in \mathfrak{R}$, the following assumption is satisfied:

$$\varrho(\mathcal{F}r, \mathcal{F}t) \leq \eta\varrho(r, \mathcal{F}r) + \delta\varrho(t, \mathcal{F}t). \tag{41}$$

Proof. Choosing $\alpha = \lambda = \mu = 0$ in Corollary 1, the assumption (e_1) reduces to (e_3). Consequently, the result follows from Corollary 1. □

Corollary 4 (Chatterjea type). *The conclusion of Theorem 1 holds under the hypotheses (a)–(d) along with the following:*

(1) $[(e_4)]$ *for some* $\lambda, \mu \geq 0$ *satisfying* $\lambda + \mu < 1$ *and for all* $r, t \in \mathbb{M}$ *with* $(r, t) \in \mathfrak{R}$, *the following assumption is satisfied:*

$$\varrho(\mathcal{F}r, \mathcal{F}t) \leq \lambda\varrho(t, \mathcal{F}r) + \mu\varrho(r, \mathcal{F}t). \tag{42}$$

Proof. Choosing $\alpha = \eta = \delta = 0$ in Corollary 1, the assumption (e_1) reduces to (e_4). Consequently, the result follows from Corollary 1. □

Corollary 5 (Bianchini type). *The conclusion of Theorem 1 holds under the hypotheses (a)–(d) along with the following:*

(1) $[(e_5)]$ *for some* $q \in [0, 1)$ *and for all* $r, t \in \mathbb{M}$ *with* $(r, t) \in \mathfrak{R}$, *the following assumption is satisfied:*

$$\varrho(\mathcal{F}r, \mathcal{F}t) \leq q \cdot \max\{\varrho(r, \mathcal{F}r), \varrho(t, \mathcal{F}t)\}, \quad \forall r, t \in \mathbb{M} \text{ with } (r, t) \in \mathfrak{R}. \tag{43}$$

Proof. Take $r, t \in \mathbb{M}$ with $(r, t) \in \mathfrak{R}$ and $q \in [0, 1)$, then we have

$$\begin{aligned}\varrho(\mathcal{F}r, \mathcal{F}t) &\leq q \cdot \max\{\varrho(r, \mathcal{F}r), \varrho(t, \mathcal{F}t)\}\\ &= q \cdot \max\{\varrho(r, t), \varrho(r, \mathcal{F}r),\\ &\quad \varrho(t, \mathcal{F}t), \varrho(r, \mathcal{F}t), \varrho(t, \mathcal{F}r)\}.\end{aligned} \tag{44}$$

Thus, the result follows from Theorem 1. □

Corollary 6 (Zamfirescu type). *The conclusion of Theorem 1 holds under the hypotheses (a)–(d) along with the following:*

(1) $[(e_6)]$ *for some* $0 \leq a < 1$, $0 \leq b < (1/2)$, $0 \leq c < (1/2)$ *and for all* $r, t \in \mathbb{M}$ *with* $(r, t) \in \mathfrak{R}$, *at least one of the following is true:*

(i) $\varrho(\mathcal{F}r, \mathcal{F}t) \leq a\varrho(r, t)$
(ii) $\varrho(\mathcal{F}r, \mathcal{F}t) \leq b[\varrho(r, \mathcal{F}r) + \varrho(t, \mathcal{F}t)]$
(iii) $\varrho(\mathcal{F}r, \mathcal{F}t) \leq c[\varrho(t, \mathcal{F}r) + \varrho(r, \mathcal{F}t)]$

5. An Example

In this section, we furnish the following example which demonstrates the importance of Theorem 1.

Example 1. Consider $\mathbb{M} := M_1 \cup M_2$ with the standard (usual) metric ϱ, where

$$\begin{aligned}M_1 &= \left\{\frac{p}{q}: p = 0, 1, 3, 9, \ldots; q = 1, 4, 7, \ldots, 3k-2, \ldots\right\},\\ M_2 &= \left\{\frac{p}{q}: p = 1, 3, 9, 27, \ldots; q = 2, 5, 8, \ldots, 3k-1, \ldots\right\}.\end{aligned} \tag{45}$$

On $\mathbb{M}$, define a relation $\mathfrak{R}$ as follows:

$$\mathfrak{R} = \{(r, t) \in \mathbb{M} \times \mathbb{M}: r - t \geq 0\}. \tag{46}$$

Define the mapping $\mathcal{F}: \mathbb{M} \longrightarrow \mathbb{M}$ by

$$\mathcal{F}(r) = \begin{cases}\dfrac{3r}{5}, & \text{if } r \in M_1,\\[2ex] \dfrac{r}{8}, & \text{if } r \in M_2.\end{cases} \tag{47}$$

Take $r, t \in \mathbb{M}$ with $(r, t) \in \mathfrak{R}$. Then, $\mathfrak{R}$ is $\mathcal{F}$-closed. Also, by routine calculation, it can easily verify that $\mathcal{F}$ satisfies the contractive condition (e) with $q = (2/5)$. Consequently, $\mathcal{F}$ has a unique fixed point. Indeed, here $\mathfrak{R}$ is complete and so $\mathfrak{R}_{F(\mathcal{F})}$. Indeed, (unique) fixed point of $\mathcal{F}$ is $r = 0$.

6. Conclusion

In this paper, we have proved the fixed point theorems in the setting of relational metric space under relation-preserving quasicontractions. For possible works, one can prove the variants of the results in the context of relational quasimetric space, relational symmetric space, relational partial metric space, relational G-metric space, relational generalized metric space, relational JS-metric space, relational cone metric space, relational complex-valued metric space, relational rectangular metric space, relational b-metric space.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

All the authors declare that there are no conflicts of interest.

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