Hindawi
Journal of Mathematics
Volume 2022, Article ID 4478452, 5 pages
https://doi.org/10.1155/2022/4478452

Research Article

Beurling-Type Theorem for a Class of Reproducing Kernel Hilbert Spaces

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Received 3 October 2021; Accepted 10 December 2021; Published 29 January 2022

Academic Editor: Xiaolong Qin

In this paper, it is proved that the Beurling-type theorem holds for the shift operator on a class of reproducing analytic Hilbert spaces.

1. Introduction

Let T be a bounded linear operator on a Hilbert space H. A closed subspace M of H is said to be invariant for T if $TM \subset M$. In the present paper, we denote by $Lat(T)$ the lattice of invariant subspaces of T on H. A basic problem in functional analysis is to describe the invariant subspaces of T. We say that T is analytic (pure) on H if $H_\infty = \cap_{n\geq 1} T^n H = \{0\}$. If T is an isometric ($\|Tx\| = \|x\|$, $x \in H$) and pure operator on H, the Wold–Kolmogorov decomposition theorem implies that

$$M = [M \ominus TM], \quad \forall M \in Lat(T), \tag{1}$$

where $M \ominus TM$ is the orthogonal complement of TM in M (it is also called the wandering subspace of T on M), and

$$[M \ominus TM] = (M \ominus TM) \oplus \left(TM \ominus T^2 M\right) \oplus \cdots, \tag{2}$$

is the smallest invariant subspace of T containing $M \ominus TM$. The most famous example for the isometric and pure operator on a Hilbert space is the following:

Let ℓ^2 be the space which consists of the collection of square-summable sequences of complex numbers. That is,

$$\ell^2 = \left\{ \{a_n\}_{n=0}^{\infty} : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}. \tag{3}$$

The norm of the vector $\{a_n\}_{n=0}^{\infty}$ is

$$\left\| \{a_n\}_{n=0}^{\infty} \right\|_{\ell^2} = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}. \tag{4}$$

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. The Hardy space H^2 consists of all analytic functions on $\mathbb{D}$ having power series representations with ℓ^2-complex coefficients sequence. That is

$$H^2 = \left\{ f \in H(\mathbb{D}) : f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \{a_n\}_{n=0}^{\infty} \in \ell^2 \right\}, \tag{5}$$

where $H(\mathbb{D})$ is the space of analytic functions in $\mathbb{D}$. The norm of the vector $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of H^2 is defined as the ℓ^2-norm of $\{a_n\}_{n=0}^{\infty}$. The following mapping

$$\{a_n\}_{n=0}^{\infty} \mapsto \sum_{n=0}^{\infty} a_n z^n, \tag{6}$$

is clearly an isomorphism from ℓ^2 onto H^2, then H^2 is a Hilbert space. Let $\mathbb{T}$ be the boundary of the unit disk $\mathbb{D}$. Let $L^2 = L^2(\mathbb{T})$ be the Hilbert space of square-integrable functions on $\mathbb{T}$ with respect to Lebesgue measure, normalized so that the measure of the entire circle is 1. It is well known that there is an isomorphism between H^2 and the closed subspace $\widetilde{H^2}$ of L^2 which consists of the L^2-functions with negative Fourier coefficients vanishing.

It is clear that the unilateral shift T_z on H^2 is an isometric and pure operator, then (1) holds for T_z on H^2. On the other hand, the famous Beurling theorem [1] gives a complete characterization of the invariant subspaces of T_z on H^2, that is, every invariant subspace of T_z on H^2 other than $\{0\}$ has the form ϕH^2, where $\phi(z)$ is an inner function. The Beurling theorem also implies that (1) holds for T_z on H^2, so in this case, the Beurling theorem is a special case of the Wold-Kolmogorov decomposition theorem. To generalize the Beurling theorem, for a bounded linear operator T on a Hilbert space H, we say that the Beurling-type theorem for T holds on H if (1) holds. Hence, the Beurling-type theorem holds for all isometric and pure operators. Another basic problem in functional analysis is to find whether the Beurling-type theorem holds for a nonisometric operator on a Hilbert space. The most famous example for the nonisometric operator is the shift operator (it is also called the Bergman shift) on the Bergman space $A^2(\mathbb{D})$ which is the Hilbert space consisting of the square-integrable analytic functions on $\mathbb{D}$ with Lebesgue area measure.

The study of the wandering subspaces of invariant subspaces for the Bergman shift tells that the dimension of wandering subspace ranges from 1 to ∞ (see [2]). Nevertheless, Aleman, Richter, and Sundberg (see [3]) discovered that all invariant subspaces of the Bergman shift are also generated by their wandering subspaces. This reveals the internal structure of invariant subspaces of the Bergman space and becomes a fundamental theorem on the Bergman space (see [4]). Later, the Beurling-type theorem was studied by many mathematicians (see [5–9]). In [8], Shimorin studied the Beurling-type theorem for a nonisometric operator T which is close to an isometry in some sense (in particular, we can assume T is left invertible), and proved the following theorem.

Theorem 1 (Shimorin's theorem). *Let T be a linear operator on a Hilbert space H with the properties.*

(i) $\|Tx+y\|^2 \le 2(\|x\|^2+\|Ty\|^2)$ for all $x, y \in H$, and

(ii) $\cap_{n=1}^{\infty} T^n H = \{0\}$,

Then $H = [H \ominus TH]$.

Definition 1. We say that Shimorin's condition for T holds on H if T satisfies the above conditions (i) and (ii) (see [10]).

Shimorin's theorem gives a simpler proof for the Beurling type theorem of Bergman shift obtained by Aleman, Richter, and Sundberg. The first step to finding whether the Beurling-type theorem holds for a nonisometric operator T on a Hilbert space is to verify whether Shimorin's condition for T holds on the invariant subspaces. It is always difficult to verify directly according to Definition 1, for example, the reproducing kernel Hilbert spaces with the complicated kernel functions (see [11]). Therefore, we hope to give a convenient and feasible criterion for judging whether the Beurling-type theorem holds for the shift operator on a class of reproducing kernel Hilbert spaces.

In [12], the authors proved the Beurling-type theorem for the shift operator on $H^2(\gamma)$ which is the Hilbert space over the bidisk $\mathbb{D}^2$ generated by a positive sequence $\{\gamma_{nm}\}_{n,m\ge 0}$. In this paper, we mainly study the Beurling-type theorem for reproducing kernel Hilbert spaces. Let $H^2(\beta)$ be the space generated by a positive sequence $\beta = \{\beta_n\}_{n\ge 0}$; that is, the space consists of all formal power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfying

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} \beta_n^2 |a_n|^2 < \infty. \tag{7}$$

We define the map on $H^2(\beta) \otimes H^2(\beta)$ as

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \beta_n^2 a_n \overline{b_n}, \tag{8}$$

where $f, g \in H^2(\beta)$, and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n. \tag{9}$$

Note that $f = g$ in $H^2(\beta)$ if and only $a_n = b_n$ for any $n \ge 0$, then it is easy to check that is an inner product on $H^2(\beta)$, so $H^2(\beta)$ is a Hilbert space. Let

$$K_w(z) = K_\beta(z,w) = \sum_{n=0}^{\infty} \frac{\overline{w}^n z^n}{\beta_n^2} \in H^2(\beta), \tag{10}$$

and it is easy to see that $f(w) = \langle f, K_w\rangle$ for any $f \in H^2(\beta)$. Hence, we call $H^2(\beta)$ the reproducing kernel Hilbert spaces generated by the positive sequence $\beta = \{\beta_n\}_{n\ge 0}$.

A power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $H^2(\beta)$ will satisfy

$$\lim_{n\longrightarrow\infty} \beta_n |a_n| = 0. \tag{11}$$

Hence, for n sufficiently large, we have that $|a_n| \le \beta_n^{-1}$. Thus, the radius of convergence R_f of f satisfying

$$R_f^{-1} = \limsup_{n\longrightarrow\infty} |a_n|^{1/n} \le \limsup_{n\longrightarrow\infty} \beta_n^{-(1/n)} = \left(\liminf_{n\longrightarrow\infty} \beta_n^{1/n}\right)^{-1}. \tag{12}$$

Therefore, $f \in H^2(\beta)$ will have a radius of convergence greater than

$$R := \liminf_{n\longrightarrow\infty} \beta_n^{1/n}. \tag{13}$$

Provided $R > 0$, then every function in $H^2(\beta)$ defines an analytic function on the disk of radius R and thus $H^2(\beta)$ can be viewed as a space of analytic functions on this disk.

The classical examples of $H^2(\beta)$ is as follows.

Example 1. If

(1) If $\beta_n \equiv 1, n = 0, 1, \ldots,$ then $R = 1$, $K_w(z) = 1/(1-\overline{w}z)$. In this case, $H^2(\beta) = H^2(\mathbb{D})$, the Hardy space.

(2) If $\beta_n = \sqrt{n+1}, n = 0, 1, \ldots,$ then $R = 1$, $K_w(z) = (1/\overline{w}z)\log 1/(1-\overline{w}z)$. In this case, $H^2(\beta) = \mathscr{D}$, the Dirichlet space.

(3) If $\beta_n = \sqrt{(n!\Gamma(\gamma)/\Gamma(n+\gamma))}, n = 0, 1, \cdots,$ where $\gamma > 1$, then $R = 1$, $K_w(z) = 1/(1-\overline{w}z)^\gamma$. In this case,

$H^2(\beta) = A^2_{\gamma-2}(\mathbb{D})$, the weighted Bergman space. In particular, when $\gamma = 2$, $A^2_0(\mathbb{D}) = A^2(\mathbb{D})$ is the classical Bergman space.

(4) If $\beta_n = \sqrt{n!}, n = 0, 1, \ldots$, then $R = +\infty$, $K_w(z) = e^{\overline{w}z}$. In this case, $H^2(\beta) = F^2(\mathbb{C})$, the Fock space.

Let T_z be the shift operator on $H^2(\beta)$ defined by $(T_z f)(z) = zf(z)$. Note that T_z is a compression operator on $H^2(\beta)$ when

$$\beta_n \leq \beta_{n-1}, \quad n \geq 1, \tag{14}$$

hence, in the present paper, we assume that T_z is bounded on $H^2(\beta)$. We give a criterion for judging the Beurling-type theorem holds for T_z on $H^2(\beta)$ (Theorem 2), and as an application, we find that the Beurling-type theorem holds for the shift operator on a family of classical reproducing kernel Hilbert spaces.

In what follows, a nontrivial subspace M of $H^2(\beta)$ is said to be invariant if $zM \subset M$, that is, M is invariant for the shift operator T_z. The following is our main result of the paper.

Theorem 2. *Let $H^2(\beta)$ be the reproducing kernel Hilbert spaces generated by the sequence $\beta = \{\beta_n\}_{n\geq 0}$ with $\beta_n > 0$ which satisfies*

$$\begin{aligned} &\beta_0^2 \leq 2\beta_1^2, \\ &\frac{\beta_{n+1}^2 + \beta_{n-1}^2}{2} \leq \frac{\beta_{n+1}^2\beta_{n-1}^2}{\beta_n^2}, \quad n = 1, 2, \ldots, \end{aligned} \tag{15}$$

then for every nontrivial invariant subspace M of $H^2(\beta)$, $M = [M \ominus zM]$.

We note that the above condition is equivalent to the following condition:

$$\frac{1}{\beta_{n-1}^2} + \frac{1}{\beta_{n+1}^2} \leq 2\frac{1}{\beta_n^2}. \tag{16}$$

Namely, the sequence $\{1/\beta_n^2\}_{n\geq 0}$ satisfies some property of "convex up," and it is much convenient to verify the Beurling type theorem for the reproducing kernel Hilbert spaces $H^2(\beta)$ satisfying the above condition.

2. Proof of Main Result and Its Applications

In this section, we first prove our main result and then give several useful applications to a class of classical reproducing kernel Hilbert spaces.

Proof. (Proof of Theorem 2) Let M be a nontrivial invariant subspace of the reproducing kernel Hilbert space $H^2(\beta)$. Since it is closed, M itself is a Hilbert space under the inner product of $H^2(\beta)$. It is easy to see that $\cap_{n=1}^{\infty} T_z^n M = \{0\}$, so the condition (ii) of Theorem 1 is satisfied. Given $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $H^2(\beta)$. Since

$$0 \leq \left| \frac{a_{n-1}}{\beta_{n+1}^2} - \frac{b_n}{\beta_{n-1}^2} \right|^2 = \frac{|a_{n-1}|^2}{\beta_{n+1}^4} + \frac{|b_n|^2}{\beta_{n-1}^4} - \frac{2}{\beta_{n+1}^2\beta_{n-1}^2} \mathrm{Re}\left(b_n \overline{a_{n-1}}\right), \tag{17}$$

we obtain

$$2\mathrm{Re}\left(b_n\overline{a_{n-1}}\right) \leq \left|a_{n-1}\right|^2 \frac{\beta_{n-1}^2}{\beta_{n+1}^2} + \left|b_n\right|^2 \frac{\beta_{n+1}^2}{\beta_{n-1}^2}. \tag{18}$$

So, we have

$$\begin{aligned} \left|a_{n-1} + b_n\right|^2 \beta_n^2 &= \left(\left|a_{n-1}\right|^2 + \left|b_n\right|^2 + 2\mathrm{Re}\left(b_n\overline{a_{n-1}}\right)\right)\beta_n^2 \\ &\leq \beta_n^2 \left[\left(1 + \frac{\beta_{n-1}^2}{\beta_{n+1}^2}\right)\left|a_{n-1}\right|^2 + \left(1 + \frac{\beta_{n+1}^2}{\beta_{n-1}^2}\right)\left|b_n\right|^2\right] \\ &= \left(\frac{\beta_n^2}{\beta_{n-1}^2} + \frac{\beta_n^2}{\beta_{n+1}^2}\right)\left|a_{n-1}\right|^2 \beta_{n-1}^2 \\ &\quad + \left(\frac{\beta_n^2}{\beta_{n+1}^2} + \frac{\beta_n^2}{\beta_{n-1}^2}\right)\left|b_n\right|^2 \beta_{n+1}^2. \end{aligned} \tag{19}$$

By the assumption of the theorem, we have

$$\frac{\beta_{n+1}^2 + \beta_{n-1}^2}{2} \leq \frac{\beta_{n+1}^2\beta_{n-1}^2}{\beta_n^2}, \tag{20}$$

for every $n = 1, 2, 3, \cdots$, which yields

$$\frac{\beta_n^2}{\beta_{n-1}^2} + \frac{\beta_n^2}{\beta_{n+1}^2} \leq 2. \tag{21}$$

It follows from the above that (19) becomes

$$\left|a_{n-1} + b_n\right|^2 \beta_n^2 \leq 2\left|a_{n-1}\right|^2 \beta_{n-1}^2 + 2\left|b_n\right|^2 \beta_{n+1}^2. \tag{22}$$

Hence, for any $f, g \in H^2(\beta)$, we have

$$\begin{aligned} \|zf + g\|_{H^2(\beta)}^2 &= \left\| b_0 + \sum_{n=1}^{\infty} \left(a_{n-1} + b_n\right) z^n \right\|^2 \\ &= \left|b_0\right|^2 \beta_0^2 + \sum_{n=1}^{\infty} \left|a_{n-1} + b_n\right|^2 \beta_n^2 \\ &\leq \left|b_0\right|^2 \beta_0^2 + 2\sum_{n=1}^{\infty} \left(\left|a_{n-1}\right|^2 \beta_{n-1}^2 + \left|b_n\right|^2 \beta_{n+1}^2\right). \end{aligned} \tag{23}$$

On the other hand, we have

$$\begin{aligned} &2\left(\|f\|_{H^2(\beta)}^2 + \|zg\|_{H^2(\beta)}^2\right) \\ &= 2\left(\sum_{n=0}^{\infty} \left|a_n\right|^2 \beta_n^2 + \sum_{n=1}^{\infty} \left|b_{n-1}\right|^2 \beta_n^2\right) \\ &= 2\left(\sum_{n=1}^{\infty} \left|a_{n-1}\right|^2 \beta_{n-1}^2 + \sum_{n=0}^{\infty} \left|b_n\right|^2 \beta_{n+1}^2\right) \\ &= 2\left|b_0\right|^2 \beta_1^2 + 2\sum_{n=1}^{\infty} \left(\left|a_{n-1}\right|^2 \beta_{n-1}^2 + \left|b_n\right|^2 \beta_{n+1}^2\right). \end{aligned} \tag{24}$$

Combining the above with (23), and the assumption $\beta_0^2 \leq 2\beta_1^2$ we get

$$\|zf+g\|^2_{H^2(\beta)} \le \left(\|f\|^2_{H^2(\beta)} + \|zg\|^2_{H^2(\beta)}\right), \quad (25)$$

which implies that condition (i) of Theorem 1 is also satisfied. Hence, applying Shimorin's theorem we finish the proof of the main result.

We point out that the condition of Theorem 2 is only sufficient for the Beurling-type theorem to be true. For example, the classical Dirichlet space $\mathscr{D}$ is a reproducing kernel Hilbert space $H^2(\beta)$ generated by the sequence $\beta = \{\sqrt{n+1}\}_{n\ge 0}$. It is well known that if M is a nontrivial invariant subspace of $\mathscr{D}$, then $M = [M\ominus zM]$ and $\dim[M\ominus zM] = 1$ (see [13]). However, we easily see that, for $n = 1, 2, \cdots$,

$$\frac{\beta^2_{n+1}+\beta^2_{n-1}}{2} = n+1 > n + \left(1 - \frac{1}{n+1}\right) = \frac{n(n+2)}{n+1} = \frac{\beta^2_{n+1}\beta^2_{n-1}}{\beta^2_n}. \quad (26)$$

However, for the Hardy space or some weighted Bergman space case, we have the following two corollaries which are direct applications of the main result.

Corollary 1. *Any nontrivial invariant subspace M of the Hardy space $H^2(\mathbb{D})$ has the property $M = [M\ominus zM]$.*

Proof. Since the classical Hardy space $H^2(\mathbb{D})$ is the space $H^2(\beta)$ generated by sequence $\beta^2_n \equiv 1$ for $n = 0, 1, \cdots$, we have

$$\frac{\beta^2_{n+1}+\beta^2_{n-1}}{2} = \frac{\beta^2_{n+1}\beta^2_{n-1}}{\beta^2_n}, \quad (27)$$

and $\beta^2_0 = 1 < 2 = 2\beta^2_1$. Thus, the corollary follows from Theorem 2.

Corollary 2. *Any nontrivial invariant subspace M of the weighted Bergman space $A^2_{\gamma-2}(\mathbb{D})$ $(1<\gamma\le 2)$ has the property $M = [M\ominus zM]$.*

Proof. When $\gamma > 1$, the weighted Bergman space $A^2_{\gamma-2}(\mathbb{D}) = H^2(\beta)$ generated by the sequence $\beta = \{\beta_n\colon n = 0, 1, 2, \cdots\}$, where $\beta_n = \sqrt{n!\Gamma(\gamma)/\Gamma(n+\gamma)}$. Thus, we have that for $n = 1, 2, \cdots$,

$$\begin{aligned}\frac{\beta^2_{n+1}+\beta^2_{n-1}}{2} &= \frac{1}{2}\left[\frac{(n+1)!\Gamma(\gamma)}{\Gamma(n+1+\gamma)} + \frac{(n-1)!\Gamma(\gamma)}{\Gamma(n-1+\gamma)}\right] \\ &= \frac{1}{2}\left[\frac{n(n+1)}{(n+\gamma)(n-1+\gamma)} + 1\right]\frac{(n-1)!\Gamma(\gamma)}{\Gamma(n-1+\gamma)},\end{aligned} \quad (28)$$

$$\begin{aligned}\frac{\beta^2_{n+1}\beta^2_{n-1}}{\beta^2_n} &= \frac{(n+1)!\Gamma(\gamma)}{\Gamma(n+1+\gamma)}\cdot\frac{(n-1)!\Gamma(\gamma)}{\Gamma(n-1+\gamma)}\cdot\frac{\Gamma(n+\gamma)}{n!\Gamma(\gamma)} \\ &= \frac{n+1}{n+\gamma}\cdot\frac{(n-1)!\Gamma(\gamma)}{\Gamma(n-1+\gamma)}.\end{aligned} \quad (29)$$

Since $1<\gamma\le 2$, then $\gamma^2 - 3\gamma + 2 \le 0$, it is equivalent to

$$(n+\gamma)(n-1+\gamma) + n(n+1) \le 2(n+1)(n-1+\gamma), \quad (30)$$

$$\text{or } \frac{n(n+1)}{(n+\gamma)(n-1+\gamma)} + 1 \le 2\frac{n+1}{n+\gamma}, \quad (31)$$

and hence we have

$$\begin{aligned}&\frac{1}{2}\left[\frac{n(n+1)}{(n+\gamma)(n-1+\gamma)} + 1\right]\frac{(n-1)!\Gamma(\gamma)}{\Gamma(n-1+\gamma)} \\ &\le \frac{n+1}{n+\gamma}\cdot\frac{(n-1)!\Gamma(\gamma)}{\Gamma(n-1+\gamma)},\end{aligned} \quad (32)$$

which combining with (28) and (29) we obtain

$$\frac{\beta^2_{n+1}+\beta^2_{n-1}}{2} \le \frac{\beta^2_{n+1}\beta^2_{n-1}}{\beta^2_n}, \quad n = 1, 2, \cdots. \quad (33)$$

In addition, it is easy to see that $\beta^2_0 = 1$ and $\beta^2_1 = 1/\gamma$, thus we have $\beta^2_0 \le 2\beta^2_1$ since $1<\gamma\le 2$. Applying Theorem 2 we then get the desired conclusion. The proof is complete.

We remark here that Hedenmalm and Zhu (see [14]) showed that the Beurling-type theorem can fail in certain weighted Bergman spaces $A^2_{\gamma-2}(\mathbb{D})$ for some $\gamma > 2$.

Let $g(z) = \sum_{n=0}^{\infty} c_n z^n$, where $c_n > 0$ for all integer $n \ge 0$. Then $K_w(z) = K(z,w) = g(\overline{w}z) = \sum_{n=0}^{\infty} c_n \overline{w}^n z^n$ is a kernel function. Accordingly, we denote the corresponding reproducing kernel Hilbert space generated by g as H^2_g. Without loss of the generality, we may assume the generating function g has convergence radius 1. Thus, H^2_g is a functional Hilbert space consisting of analytic functions on the unit disk $\mathbb{D}$. It is clear that $\{\sqrt{c_n}\,z^n\}_{n\ge 0}$ is an orthonormal basis and $\|z^n\|^2 = (1/c_n)$. If $f \in H^2_g$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then its norm is defined by $\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2/c_n$.

In the framework H^2_g of reproducing kernel Hilbert spaces, if we put $c_n = 1/\beta^2_n$, then $H^2_g = H^2(\beta)$ where $\beta = \{\beta_n\colon n = 0, 1, 2, \cdots\}$. Then Theorem 2 can be restated as follows.

Theorem 3. *Let H^2_g be the reproducing kernel Hilbert spaces generated by $g(z) = \sum_{n=0}^{\infty} c_n z^n$. If $\{c_n\}$ satisfies*

$$\begin{aligned} c_1 &\le 2c_0, \\ c_{n+1} + c_{n-1} &\le 2c_n, \quad n = 1, 2, \cdots,\end{aligned} \quad (34)$$

then for each nontrivial invariant subspace M of H^2_g, $M = [M\ominus zM]$.

If we restrict ourselves to the $H^2(\beta)$ generated by $\beta = \{\beta_n\}$ where $\beta_n = (n+1)^{\nu}$ for some real number ν. These spaces are also known in the literature as weighted Dirichlet spaces or $\mathscr{D}_\nu$ spaces. For $f \in \mathscr{D}_\nu$, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then the norm is simply written as

$$\|f\|^2 = \sum_{n=0}^{\infty} (n+1)^{2\nu} |a_n|^2. \quad (35)$$

Let us consider $\varphi(x) = (1+x)^{\alpha}$, where α is a real number and $x > 0$. It is easy to see that φ is a convex-up function on $(-1,+\infty)$ when $0 \leq \alpha \leq 1$. Setting $c_n = (n+1)^{\alpha} = (1/\beta_n^2)$, we have

$$\frac{1}{\beta_{n+1}^2} + \frac{1}{\beta_{n-1}^2} = c_{n+1} + c_{n-1} \leq 2c_n = 2\frac{1}{\beta_n^2}, \tag{36}$$

for $n = 1, 2, \cdots$, and since $\beta_0^2 = (1/c_0) = 1, \beta_1^2 = (1/c_1) = (1/2^{\alpha})$, we have

$$\beta_0^2 = 1 \leq 2^{1-\alpha} = 2\beta_1^2, \tag{37}$$

when $0 \leq \alpha \leq 1$. From the discussion above, we obtain the following result.

Corollary 3. *If M is a nontrivial invariant subspace of weighted Dirichlet space $\mathscr{D}_{\nu}(-(1/2) \leq \nu \leq 0)$, then $M = [M \ominus zM]$.*

Using Theorem 3, it is convenient to construct a more reproducing kernel Hilbert space for which the nontrivial invariant subspace has a Beurling-type theorem. As an example, we consider a positive function

$$\varphi(x) = \frac{x + ae^2}{\ln\left(x + ae^2\right)}, \tag{38}$$

for $x \geq 0$, where $a > 1$ is a constant. Note that

$$\varphi''(x) = \frac{2 - \ln\left(x + ae^2\right)}{\left(x + ae^2\right)\ln^3\left(x + ae^2\right)} < 0, \tag{39}$$

since $a > 1$, then $\varphi(x)$ is a convex-up function on $[0, +\infty)$. If we write

$$\begin{aligned} c_n &= \frac{n + ae^2}{\ln\left(n + ae^2\right)}, \\ \beta_n^2 &= \frac{1}{c_n}, \end{aligned} \tag{40}$$

then for $n = 1, 2, \cdots$, we have

$$\frac{1}{\beta_{n+1}^2} + \frac{1}{\beta_{n-1}^2} = c_{n+1} + c_{n-1} \leq 2c_n = 2\frac{1}{\beta_n^2}, \tag{41}$$

and $\beta_0^2 \leq 2\beta_1^2$. The application of our main result to this case, we can get the following result.

Corollary 4. *Let $H^2(\beta)$ be reproducing kernel Hilbert space generated by the sequence $\beta = \{\beta_n\}$ where*

$$\beta_n = \sqrt{\frac{\ln\left(n + ae^2\right)}{n + ae^2}}, \quad n = 0, 1, 2, \cdots, \tag{42}$$

and M a nontrivial invariant subspace of $H^2(\beta)$. If $a > 1$, then we have $M = [M \ominus zM]$.

Data Availability

Some or all data, models, or code generated or used during the study are available from the corresponding author by request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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