Unification of Two-Variable Family of Apostol-Type Polynomials with Applications

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In this paper, the two-variable unified family of generalized Apostol-type polynomials is introduced, and some implicit forms and general symmetry identities are derived. Also, we obtain new degenerate Apostol-type numbers and polynomials constructed from the new 2-variable unified family. We derive explicit formulae of polynomials and identities that include some special numbers and polynomials. In addition, a probabilistic representation of the new family and some statistical properties are obtained.

1. Introduction

Khan and Raza [1] defined the 2-variable general polynomials (2VGP) \( p_n(x, y) \) by means of the following generating function:

\[
e^{xt+yt} = \sum_{n=0}^{\infty} p_n(x, y) \frac{t^n}{n!}, \quad p_0(x, y) = 1,
\]

where \( \varphi(y, t) \) has series expansion

\[
\varphi(y, t) = \sum_{n=0}^{\infty} \varphi_n(y) \frac{t^n}{n!} \quad \varphi_0(y) \neq 0.
\]

The 2-variable general polynomials \( p_n(x, y) \) contain a number of important special polynomials of two variables. Generating functions for certain members that belong to the 2VGP are given as follows:

The higher-order Hermite polynomials, sometimes called the Gould–Hopper polynomials \( H_n^{(m)}(x, y) \) are defined by [2].

\[
e^{xt+yt} = \sum_{n=0}^{\infty} H_n^{(m)}(x, y) \frac{t^n}{n!}
\]

The 2-variable Hermite polynomials \( H_n(x, y) \) are defined by [3].

\[
e^{xt+yt} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}
\]

The 2-variable generalized Laguerre polynomials \( mL_n(y, x) \) are defined by the following generating function [4]:

\[
e^{xt} C_0(\gamma t) = \sum_{n=0}^{\infty} mL_n(y, x) \frac{t^n}{n!}
\]

where \( C_0(y) \) is the 0-th order Tricomi function [5].

\[
C_0(y) = \sum_{r=0}^{\infty} \frac{(-1)^r y^r}{(r!)^2}
\]

The 2-variable Laguerre polynomials \( L_n(y, x) \) are defined by [6]

\[
e^{xt} C_0(\gamma t) = \sum_{n=0}^{\infty} L_n(y, x) \frac{t^n}{n!}
\]

The 2-variable truncated exponential polynomials of order \( r ) \) \( (x, y) \) are defined [7]
\[
e^{-\alpha y^2} = \sum_{n=0}^{\infty} e^{(r)}_n(x, y) \frac{t^n}{n!}.
\]

(8)

In particular, we note that
\[
e^{(2)}_n(x, y) = n! [2] e_n(x, y),
\]

(9)

where \([2] e_n(x, y)\) denotes the 2-variable truncated exponential polynomials [8].

The 2-variable truncated exponential polynomials \([2] e^{(r)}_n(x, y)\) are defined by [8].
\[
e^{(r)}_n(x, y) = \sum_{n=0}^{\infty} [2] e_n(x, y) \frac{t^n}{n!}
\]

(10)

Khan et al. [9] introduced the 2-variable Apostol type polynomials of order \(\alpha\), by
\[
\sum_{n=0}^{\infty} p^{(\alpha)}_n(x, y; \lambda; \mu, y) \frac{t^n}{n!} = \left(\frac{2\mu y^r}{\lambda e^t + 1}\right)^\alpha e^{xt} \phi(y, t), \quad |t| < |\log(-\lambda)|.
\]

(11)

Araci et al. [10] introduced and investigate a new unified class of generalized Apostol type polynomials by the following generating function:
\[
\sum_{n=0}^{\infty} \mathcal{H}^{(n)}_{\alpha}(x, y; a, b, c; \mu, y; \lambda) \frac{t^n}{n!} = \left(\frac{2\mu y^r}{\lambda b^2 + \alpha^2}\right)^\alpha c^{xt+\lambda y}, \quad c > 1; j > 2; |t| < \left|\frac{\log - \lambda}{\log(c/b)}\right|.
\]

(12)

In [11], Cartilz introduced a new study of degenerate versions of Bernoulli and Euler numbers and polynomials. Lately, many researchers have begun to study the degenerate Bell numbers and polynomials, degenerate complete Bell polynomials and numbers, degenerate Hermite polynomials and numbers, and so on (see [12–14]).

Furthermore, mathematicians have done recently a few interesting works on Apostol-type polynomials in the field of approximation theory (for more details, see [15–18]).

This manuscript is a modified and extended version of [19].

The paper is organized as follows: in Section 2, we construct a new version of the 2-variable Apostol-type polynomials and numbers and related special polynomials and numbers. In Section 3, by using generating functions, we investigate some summation formulae, explicit expression, and some symmetric identities. In Section 4, we construct a new version of 2-variable degenerate Apostol-type polynomials and numbers and related special polynomials. In Section 5, we obtain the generating functions for new special polynomials that belong to the new version of 2-variable unified Apostol-type polynomials. In Section 6, the probabilistic representation of the new family and its statistical properties are presented.

2. Unification of Two-Variable Apostol-Type Polynomials

The two-variable unified family of generalized Apostol-type polynomials of order \(r\), denoted by \(p^{(r)}(x, y; a, b, c, v, \mu; \overline{\alpha})\) is defined as the Apostol type convolution of the 2-variable general polynomials \(p_n(x, y)\).

**Definition 1.** Let \(a, b \in \mathbb{R}^+\) and \(a \neq b\). A new generalization of the Apostol Hermite–Genocchi polynomials \(p^{(r)}(x, y; a, b, c, v, \mu; \overline{\alpha})\) for nonnegative integer \(n\) is defined by the generating function
\[
\sum_{n=0}^{\infty} p^{(r)}(x, y; a, b, c, v, \mu; \overline{\alpha}) \frac{t^n}{n!} = \frac{(-1)^r t^{rv} 2^{\mu t}}{\prod_{i=0}^{r-1}(a_i b^i - a^i)} e^{xt} \phi(y, t),
\]

(13)

where \(r \in \mathbb{C}; \overline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1})\) is a sequence of complex numbers.

Setting \(c = c\) and \(\phi(y, t) = 1\) in (13), we get the following definition.

**Definition 2.** A unified family \(\mathcal{H}^{(r)}(x; a, b, v, \mu; \overline{\alpha})\) of generalized Apostol-type polynomials is given by
\[
\sum_{n=0}^{\infty} \mathcal{H}^{(r)}(x; a, b, v, \mu; \overline{\alpha}) \frac{t^n}{n!} = \frac{(-1)^r t^{rv} 2^{\mu t}}{\prod_{i=0}^{r-1}(a_i b^i - a^i)} e^{xt}.
\]

(14)

**Remark 1.** Setting \(x = 0\) in (14), then we obtain the new unified family of generalized Apostol-type numbers, which is defined as
\[
\sum_{n=0}^{\infty} \mathcal{U}_n^{(r)}(a, b, \nu, \mu; \overline{\kappa}) t^n = \frac{(-1)^r t^r 2^{\nu m}}{\prod_{i=0}^{r-1} (a_i b^t - a^t)} t^n.
\]

Also, setting \( c = e \) in (13), we get the following definition.

**Definition 3.** The two-variable unified family of generalized Apostol-type polynomials of order \( r \)
\( p \mathcal{U}_n^{(r)}(x, y; a, b; \nu, \mu; \overline{\kappa}) \) is defined by the following generating function
\[
\sum_{n=0}^{\infty} p \mathcal{U}_n^{(r)}(x, y; a, b; \nu, \mu; \overline{\kappa}) t^n = \frac{(-1)^r t^r 2^{\nu m}}{\prod_{i=0}^{r-1} (a_i b^t - a^t)} e^{xt} \psi(y, t).
\]

\( \psi(y, t) \)

We obtain the series definition of \( p \mathcal{U}_n^{(r)}(x, y; a, b, \nu, \mu; \overline{\kappa}) \) by the following theorem.

**Theorem 1.** The two-variable unified family of generalized Apostol-type of order \( r \) \( p \mathcal{U}_n^{(r)}(x, y; a, b, \nu, \mu; \overline{\kappa}) \) is defined by the following series:
\[
p \mathcal{U}_n^{(r)}(x, y; a, b, \nu, \mu; \overline{\kappa}) = 2^{\nu m} \sum_{m_i, m_2, \ldots, m_r \geq 0} \binom{n}{m} \prod_{i=1}^{r} (\alpha_{i-1})^{m_i} \left( \log \left( \frac{b}{a} \sum_{i=1}^{r} \frac{c^i}{a^i} \right) \right)^{n-m} \varphi_{m}(y).
\]

**Proof.** The left-hand side of (13) is equal to
\[
\frac{(-1)^r 2^{\nu m} e^{xt} \psi(y, t)}{\prod_{i=0}^{r-1} (a_i b^t - a^t)} = \frac{2^{\nu m} e^{xt} / (\log a^t)}{\prod_{i=0}^{r-1} (1 - \alpha_i e^{t \log b/a^t})} \psi(y, t) = \sum_{m_i, m_2, \ldots, m_r \geq 0} \prod_{i=1}^{r-1} (\alpha_{i-1})^{m_i} e^{xt} / (\log b/a + \sum_{i=1}^{r} \log c^i / a^i) \psi(y, t).
\]

By (13), we get
\[
\sum_{n=0}^{\infty} p \mathcal{U}_n^{(r)}(x, y; a, b, \nu, \mu; \overline{\kappa}) t^n = \sum_{n=0}^{\infty} 2^{\nu m} \sum_{m_i, m_2, \ldots, m_r \geq 0} \binom{n}{m} \prod_{i=1}^{r-1} (\alpha_{i-1})^{m_i} \left( \log \left( \frac{b}{a} \sum_{i=1}^{r} \frac{c^i}{a^i} \right) \right)^{n-m} \varphi_{m}(y) t^n.
\]

By comparing the coefficients on both sides in the last equation, we obtain (18).
Table 1: Special cases of the $p\mathcal{U}_n^{(r)}(x, y; a, b, v, \mu; \overline{\nu})$.

<table>
<thead>
<tr>
<th>No.</th>
<th>Values of the parameters</th>
<th>Relation between $p\mathcal{U}_n^{(r)}(x, y; a, b, v, \mu; \overline{\nu})$ and its special cases</th>
<th>Name of the results special polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\alpha_i = -\lambda$ and $\varphi(y, t) = 1$ in (13)</td>
<td>$p\mathcal{U}_n^{(r)}(x; a, b, v, \mu; -\lambda) = E_n^{(r)}(x; a, b, c, \mu, v; \lambda)$</td>
<td>The generalized Apostol-type Gould-Hopper polynomials, [10]</td>
</tr>
<tr>
<td>2</td>
<td>$\alpha_i = -\lambda$ and $\varphi(y, t) = e^{ct}$, $j &gt; 2$ in (13)</td>
<td>$p\mathcal{U}_n^{(r)}(x; y; a, b, v, \mu; -\lambda) = HF_n^{(r)}(x, y; a, b, c, \mu, v; \lambda)$</td>
<td>The generalized Apostol type polynomials of order $r$, [10]</td>
</tr>
<tr>
<td>3</td>
<td>$\alpha_i = -\lambda, b = e$ and $a = 1$ in (16)</td>
<td>$p\mathcal{U}_n^{(r)}(x, y; a, b, v, \mu; -\lambda) = p\mathcal{P}_n^{(r)}(x, y; \lambda, \mu, v)$</td>
<td>The 2-variable Apostol type polynomials of order $r$, [9]</td>
</tr>
<tr>
<td>4</td>
<td>$\alpha_i = \lambda, b = e, c = 1$ and $a = 1$ in (16)</td>
<td>$p\mathcal{U}_n^{(r)}(x, y; 1, e, 1, 0; \lambda) = \varphi(y, t)$</td>
<td>The 2-variable Apostol-Bernoulli polynomials of order $r$, [9]</td>
</tr>
<tr>
<td>5</td>
<td>$\alpha_i = -\lambda, b = \mu$ and $a = 1$ and $v = 0$ in (16)</td>
<td>$p\mathcal{U}_n^{(r)}(x, y; 1, 0, 1; \lambda) = p\mathcal{E}_n^{(r)}(x, y; \lambda)$</td>
<td>The 2-variable Apostol-Euler polynomials of order $r$, [9]</td>
</tr>
<tr>
<td>6</td>
<td>$\alpha_i = -\lambda, b = e, c = 1$ and $a = 1$ and $\mu = v = 1$ in (16)</td>
<td>$p\mathcal{U}_n^{(r)}(x, y; 1, 1, 1, -\lambda) = p\mathcal{C}_n^{(r)}(x, y; \lambda)$</td>
<td>The 2-variable Apostol-Gegenbauer polynomials of order $r$, [10]</td>
</tr>
<tr>
<td>7</td>
<td>$\alpha_i = -\lambda, b = e, c = 1$ and $\varphi(y, t) = 1$ in (16)</td>
<td>$p\mathcal{U}_n^{(r)}(x, y; 1, e, \mu; -\lambda) = F_n^{(r)}(x, y; \lambda, \mu, v)$</td>
<td>The generalized Apostol type polynomials of order $r$, [20]</td>
</tr>
<tr>
<td>8</td>
<td>$\alpha_i = 1, b = e, \mu = (1 - k), v = k$ in (14)</td>
<td>$p\mathcal{U}_n^{(r)}(x, y; 1, e, \mu; \overline{\nu}) = (-1)^k M_n^{(r)}(x, \overline{\nu})$</td>
<td>The unified family of generalized Apostol-Euler, Bernoulli and Gegenbauer polynomials, [21]</td>
</tr>
<tr>
<td>9</td>
<td>$\alpha_i = 1, b = e, \mu = (1 - k), v = k$ in (14)</td>
<td>$p\mathcal{U}_n^{(r)}(x, y; 1, e, \mu; \overline{\nu}) = (-1)^k M_n^{(r)}(x, \overline{\nu})$</td>
<td>The generalization of Apostol-Hermite polynomials, [22]</td>
</tr>
</tbody>
</table>

3. Implicit Summation Formulas for the Two-Variable Unified Family of Generalized Apostol-Type Polynomials

Theorem 3. Let $a, b > 0$ and $a \neq b$. Then for $x, y, z \in \mathbb{R}$ and $n \geq 0$. The following implicit summation formula for $p\mathcal{U}_n^{(r)}(x, y; a, b; v, \mu; \overline{\nu})$ holds true as follows:

$$
p\mathcal{U}_n^{(r)}(x + z, y; a, b, v, \mu; \overline{\nu}) = \sum_{n=0}^{\infty} p\mathcal{U}_n^{(r)}(x, y; a, b, v, \mu; \overline{\nu}) z^n. \tag{21}
$$

Proof. Replacement of $x$ by $x + z$ in (16) gives

$$
\begin{align*}
\sum_{n=0}^{\infty} p\mathcal{U}_n^{(r)}(x + z, y; a, b, v, \mu; \overline{\nu}) t^n &= (-1)^r t^r x^{2n} \prod_{i=0}^{r-1} \left( a_i b_i^r - a_i^r \right) e^{(x+z)t} \varphi(y, t) \\
&= \sum_{n=0}^{\infty} p\mathcal{U}_n^{(r)}(x, y; a, b, v, \mu; \overline{\nu}) z^n t^{n+m} \frac{1}{n!m!}.
\end{align*}
$$

We get

$$
\sum_{n=0}^{\infty} p\mathcal{U}_n^{(r)}(x + z, y; a, b, v, \mu; \overline{\nu}) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} p\mathcal{U}_n^{(r)}(x, y; a, b, v, \mu; \overline{\nu}) t^{n-k}.
$$

Equating the coefficients of $t^n$ on both sides, yields (23). \qed
Theorem 5. Let $a, b > 0$ and $a \neq b$. Then for $x, y, z \in \mathbb{R}$ and $n \geq 0$, the following implicit summation formula for $p^{(r)}_{n+m}(x; y, a, b; \mu; \overline{\alpha})$ holds true as follows:

\[
p^{(r)}_{n+m}(z; y, a, b, v, \mu; \overline{\alpha}) = \sum_{n,m=0}^{\infty} \binom{n}{p} \binom{m}{q} (z-x)^{p+q} p^{(r)}_{n+m-p-q}(x; y, a, b, \mu; \overline{\alpha}).
\]  

(26)

Proof. Replacing $t$ by $t + u$ in the generating function (16) and using the following rule [23]:

\[
\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(m+n) \frac{x^n}{n!} \frac{y^m}{m!}
\]  

(27)

Replacing $x$ by $z$ in the previous equation and then equal both sides, we get

\[
e^{-x(t+u)} \sum_{n,m=0}^{\infty} p^{(r)}_{n+m}(x, y; a, b, \mu, \nu; \overline{\alpha}) \frac{t^n u^m}{n! m!} = \sum_{n,m=0}^{\infty} p^{(r)}_{n+m}(z, y; a, b, \mu, \nu; \overline{\alpha}) \frac{t^n u^m}{n! m!}
\]  

(29)

On expanding exponential function (29) gives

\[
\sum_{N=0}^{\infty} [(z-x)(t+u)]^N \frac{N!}{N!} \sum_{n,m=0}^{\infty} p^{(r)}_{n+m}(x, y; a, b, \mu, \nu; \overline{\alpha}) \frac{t^n u^m}{n! m!} = \sum_{n,m=0}^{\infty} p^{(r)}_{n+m}(z, y; a, b, \mu, \nu; \overline{\alpha}) \frac{t^n u^m}{n! m!}
\]  

(30)

Using equation (27) in the left-hand side of equation (30), we find

\[
\sum_{p,q=0}^{\infty} \frac{(z-x)^{p+q} t^p u^q}{p! q!} \sum_{n,m=0}^{\infty} p^{(r)}_{n+m}(x, y; a, b, \mu, \nu; \overline{\alpha}) \frac{t^n u^m}{n! m!} = \sum_{n,m=0}^{\infty} p^{(r)}_{n+m}(z, y; a, b, \mu, \nu; \overline{\alpha}) \frac{t^n u^m}{n! m!}
\]  

(31)

Now replacing $n$ by $n-p$, $m$ by $m-q$, and using the Cauchy-product rule in the left-hand side of (31), we get

\[
\sum_{n,m=0}^{\infty} \frac{(z-x)^{p+q}}{p! q!} p^{(r)}_{n+m-p-q}(x, y; a, b, \mu, \nu; \overline{\alpha}) \frac{t^n}{(n-p)!} \frac{u^m}{(m-q)!} = \sum_{n,m=0}^{\infty} p^{(r)}_{n+m}(z, y; a, b, \mu, \nu; \overline{\alpha}) \frac{t^n u^m}{n! m!}
\]  

(32)

Finally, on equating the coefficients of the like powers of $t$ and $u$ in the above equation, yields (26).

\[\square\]

Theorem 6. The following implicit summation formula for $p^{(r)}_{n}(x; y, a, b; \mu; \overline{\alpha})$ holds true as follows:
Proof. Using equation (16), we can easily obtain (33) and (34).

(i) 
\[
p^r \mathcal{U}^{(r)}_n(x + 1, y; a, b, \nu; \overline{\alpha}) = \sum_{m=0}^{n} \binom{n}{m} p^r \mathcal{U}^{(r)}_{n-m}(x, y; a, b, \nu; \overline{\alpha}).
\]  

(ii) 
\[
p^r \mathcal{U}^{(r)}_n(x + z; a, b, \nu; \overline{\alpha}) = \sum_{m=0}^{n} \binom{n}{m} p^r \mathcal{U}^{(r)}_{n-m}(z, a, b, \nu; \overline{\alpha}) p^r_m(x, y).
\]  

Proof. Let

\[
G(t) = \frac{(-1)^r \gamma^m(t^r)^r}{(\prod_{i=0}^{r-1}(\alpha_i b_i^a - a_i^a))(\prod_{i=0}^{r-1}(\alpha_i b_i^a - a_i^a))} e^{\gamma x dt} \varphi(y, dt) \varphi(y, ct).
\]

Then the expression for \( G(t) \) is symmetric in \( \gamma \) and \( d \) and we can expand \( G(t) \) into series in two ways.

Firstly

\[
G(t) = \frac{1}{(\alpha b)^r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \binom{m}{m} \sum_{m=0}^{n} \frac{(\text{dx} y; a, b, \nu; \overline{\alpha})}{m!} \mathcal{U}^{(r)}_{n-m}(\text{dx}, y; a, b, \nu; \overline{\alpha}) (t^m)^n.
\]

Secondly

\[
G(t) = \frac{1}{(cd)^r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \binom{m}{m} \sum_{m=0}^{n} \frac{(\text{dx} y; a, b, \nu; \overline{\alpha})}{m!} \mathcal{U}^{(r)}_{n-m}(\text{dx}, y; a, b, \nu; \overline{\alpha}) (t^m)^n.
\]

Form equations (37) and (38), by comparing the coefficients of \( t^n \) on both sides, yields (35).
4. Applications

The $2\text{VGP}$ family $p_n(x, y)$ contains a number of important special polynomials of two variables. Some members belonging to the $2\text{VGP}$ family are considered in Section 1. We notice that for every member belonging to the $2\text{VGP}$, there is a new special polynomial that belongs to the $pU_n^{\{\alpha\}}(x, y, a, b, v, \mu; \tau_\alpha)$ family. Thus, by selecting a suitable choice for the function $\varphi(y, t)$ in equation (16), the generating function for the corresponding member belongs to $pU_n^{\{\alpha\}}(x, y, a, b, v, \mu; \tau_\alpha)$ is a family that can be obtained.

Example 1. Setting $\varphi(y, t) = e^{yt}$ in the left-hand side of generating function (16), gives Gould–Hopper type polynomials (GHATP), denoted by $H^{(m)}U_n^{(r)}(x, y, a, b, v, \mu; \tau_\alpha)$ are defined by

$$
\sum_{n=0}^{\infty} H^{(m)}U_n^{(r)}(x, y, a, b, v, \mu; \tau_\alpha) \frac{t^n}{n!} = \frac{(-1)^r y^r 2^\mu}{\prod_{i=0}^{r-1} (a_i b^i - a^i)} e^{xt+yt}.
$$

(39)

Setting suitable values of the parameters in the results of the GHATP $H^{(m)}U_n^{(r)}(x, y, a, b, v, \mu; \tau_\alpha)$, we obtain the following results:

1. If $\alpha_i = \lambda, b = e, a = 1, \mu = 0$, then $H^{(m)}U_n^{(r)}(x, y; 1, e, 0; \lambda) = (-1)^r H^{(m)}U_n^{(r)}(x, y; \lambda)$ (Gould–Hopper–Apostol–Bernoulli polynomials of order $r$) (see [9]).
2. If $\alpha_i = -\lambda, b = e, a = 1, \mu = 0$, then $H^{(m)}U_n^{(r)}(x, y; 1, e, 0; -\lambda) = H^{(m)}C_n^{(r)}(x, y; \lambda)$ (Gould–Hopper–Apostol–Euler polynomials of order $r$), see [9].
3. If $\alpha_i = -\lambda, b = e, a = 1, \mu = 1$, then $H^{(m)}U_n^{(r)}(x, y; 1, e, 1; -\lambda) = H^{(m)}U_n^{(r)}(x, y; \lambda)$ (Gould–Hopper–Apostol–Genocchi polynomials of order $r$), [9].

The series definitions and other results for the GHATP $H^{(m)}U_n^{(r)}(x, y, a, b, v, \mu; \tau_\alpha)$ are given in Table 2.

Remak 2. For $m = 2$, the $H^{(m)}U_n^{(r)}(x, y)$ reduce to $H_n(x, y)$. Therefore, setting $m = 2$ in equation (39), we obtain the following generating function for the 2-variable Hermite Apostol type polynomials, denoted by $H_u^{\beta}(x, y, a, b, v, \mu; \tau_\alpha)$ as follows:

$$
\sum_{n=0}^{\infty} H_u^{\beta}(x, y, a, b, v, \mu; \tau_\alpha) \frac{t^n}{n!} = \frac{(-1)^r y^r 2^\mu}{\prod_{i=0}^{r-1} (a_i b^i - a^i)} e^{xt+yt}.
$$

(40)

The series definitions and other results for the 2-variable Hermite Apostol type polynomials $H_u^{\beta}(x, y, a, b, v, \mu; \tau_\alpha)$ can be deduced by setting $m = 2$ in the results given in Table 2.

Example 2. Setting $\varphi(y, t) = C_\alpha(\gamma^y)$ (for which the $p_n(x, y)$ reduce to the $mL_n(y, x)$ in the left-hand side of generating function (16), we find that the resultant 2-variable generalized Laguerre Apostol type polynomials (2VGLATP), denoted by $mL_u^{\beta}(y, x, a, b, v, \mu; \tau_\alpha)$ are defined by the following generating function:

$$
\sum_{n=0}^{\infty} mL_u^{\beta}(y, x, a, b, v, \mu; \tau_\alpha) \frac{t^n}{n!} = \frac{(-1)^r y^r 2^\mu}{\prod_{i=0}^{r-1} (a_i b^i - a^i)} e^{xt}.
$$

(41)

The series definitions and other results for the 2VGLATP $mL_u^{\beta}(y, x, a, b, v, \mu; \tau_\alpha)$ are given in Table 3.

Remark 3. Since for $m = 1$ and $y \rightarrow -y$, then $mL_u(y, x)$ reduce to the $L_n(x, y)$. Therefore, setting $m = 1$ and $y \rightarrow -y$ in equation (41), we obtain the generating function for the 2-variable Laguerre Apostol type polynomials, denoted by $L_u^{\beta}(y, x, a, b, v, \mu; \tau_\alpha)$ as

$$
\sum_{n=0}^{\infty} L_u^{\beta}(y, x, a, b, v, \mu; \tau_\alpha) \frac{t^n}{n!} = \frac{(-1)^r y^r 2^\mu}{\prod_{i=0}^{r-1} (a_i b^i - a^i)} e^{xt}.
$$

(42)

The series definitions and other results for the $L_u^{\beta}(y, x, a, b, v, \mu; \tau_\alpha)$ can be obtained by setting $m = 1$ and $y \rightarrow -y$ in the results given in Table 3.

Remark 4. Since for $x = 1$, the $L_n(x, y)$ reduce to the classical Laguerre polynomials $L_n(y)$. Therefore, setting $x = 1$ in equation (42), we obtain the following generating function for the Laguerre Apostol type polynomials, denoted by $L_u^{(r)}(y, a, b, v, \mu; \tau_\alpha)$:

$$
\sum_{n=0}^{\infty} L_u^{(r)}(y, a, b, v, \mu; \tau_\alpha) \frac{t^n}{n!} = (-1)^r y^r 2^\mu \prod_{i=0}^{r-1} (a_i b^i - a^i) e^{xt}.
$$

(43)

The series definitions and other results for the $L_u^{(r)}(y, a, b, v, \mu; \tau_\alpha)$ can be obtained by setting $m = 1$, $y \rightarrow -y$, and $x = 1$ in the results given in Table 3.

Setting suitable values of the parameters in the results of the $mL_u^{\beta}(y, x, a, b, v, \mu; \tau_\alpha)$, we obtain results for 2-variable generalized Laguerre Apostol polynomials related to the $mL_u^{\beta}(y, x, a, b, v, \mu; \tau_\alpha)$ (for more details see [9]).

Example 3. Setting $\varphi(y, t) = 1 + yt^\beta$ (for which the $p_n(x, y)$ reduce to the $c_{\gamma^y}(x, y)$ in the left-hand side of generating function (16), we obtain the 2-variable truncated exponential Apostol type polynomials of order $\beta$ (2VTEATP), denoted by $e(\beta)U_n^{(r)}(x, y, a, b, v, \mu; \tau_\alpha)$ are defined by

$$
\sum_{n=0}^{\infty} e(\beta)U_n^{(r)}(x, y, a, b, v, \mu; \tau_\alpha) \frac{t^n}{n!} = \frac{(-1)^r y^r 2^\mu}{\prod_{i=0}^{r-1} (a_i b^i - a^i)} \left( e^{xt} \right).
$$

(44)

The series definitions and other results for the 2VTEATP $e(\beta)U_n^{(r)}(x, y, a, b, v, \mu; \tau_\alpha)$ are given in Table 4.

Setting suitable values of the parameters in the results of the $U_n^{(r)}(y, x, a, b, v, \mu; \tau_\alpha)$, we obtain results for 2-variable
Table 2: Results for $H^{(m)} \mathcal{Y}_n^{(r)}(x, y, a, b, \nu; \mu; \pi)$.

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Results</th>
<th>Expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Series definition</td>
<td>$H^{(m)} \mathcal{Y}<em>n^{(r)}(x, y, a, b, \nu; \mu; \pi) = \sum</em>{z=0}^{n} \left( \frac{n}{z} \right)^{m} \mathcal{Y}_n^{(r)}(a, b, \nu; \mu; \pi) H_z^{(m)}(x, y)$</td>
</tr>
<tr>
<td>2</td>
<td>Summation formulae</td>
<td>$H^{(m)} \mathcal{Y}<em>n^{(r)}(x + w, y, a, b, \nu; \mu; \pi) = \sum</em>{z=0}^{n} \left( \frac{n}{z} \right)^{m} H^{(m)} \mathcal{Y}_n^{(r)}(x, y, a, b, \nu; \mu; \pi) w^{n-z}$</td>
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</tbody>
</table>

Table 3: Results for mL $\mathcal{Y}_n^{(r)}(x, y, a, b, \nu; \mu; \pi)$.

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<tbody>
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<td>Series definition</td>
<td>mL $\mathcal{Y}<em>n^{(r)}(y, x, a, b, \nu; \mu; \pi) = \sum</em>{z=0}^{n} \left( \frac{n}{z} \right)^{m} \mathcal{Y}_n^{(r)}(a, b, \nu; \mu; \pi) mL_z^{(r)}(y, x)$</td>
</tr>
<tr>
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<td>Summation formulae</td>
<td>mL $\mathcal{Y}<em>n^{(r)}(y + w, x, a, b, \nu; \mu; \pi) = \sum</em>{z=0}^{n} \left( \frac{n}{z} \right)^{m} \mathcal{Y}_n^{(r)}(y, x, a, b, \nu; \mu; \pi) w^{n-z}$</td>
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</table>

Table 4: Results for $e(\beta) \mathcal{Y}_n^{(r)}(x, y, a, b, \nu; \mu; \pi)$.

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<td>Summation formulae</td>
<td>$e(\beta) \mathcal{Y}<em>n^{(r)}(x + w, y, a, b, \nu; \mu; \pi) = \sum</em>{z=0}^{n} \left( \frac{n}{z} \right)^{m} e(\beta) \mathcal{Y}_n^{(r)}(x, y, a, b, \nu; \mu; \pi) w^{n-z}$</td>
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Remark 5. Since for $\beta = 2$, the $e^{(\beta)}(x, y)$ of order $\beta$ reduce to the [2]e$_p$(x, y). Therefore, taking $\beta = 2$ in equation (44), we obtain the following generating function for the 2-variable truncated exponential Apostol type polynomials, denoted by [2]e$^{(2)}$($\mathcal{Y}_n^{(r)}(x, y, a, b, \nu; \mu; \pi)$) as

$$
\sum_{n=0}^{\infty} [2]e^{(2)}(x, y, a, b, \nu; \mu; \pi) \frac{t^n}{n!} \frac{e^{xt}}{\prod_{i=0}^{\nu}(a_i b_i - a_i)} (1-t)^\nu.
$$

The series definitions and other results for the 2VTEATP [2]e$^{(2)}$($\mathcal{Y}_n^{(r)}(x, y, a, b, \nu; \mu; \pi)$) can be obtained by taking $\beta = 2$ in the results given in Table 4.

Remark 6. Since for $\gamma = 1$, the [2]e$_p$(x, y) of order $\beta$ reduce to the truncated exponential polynomials [2]e$_p$(x). Therefore, taking $\gamma = 1$ in equation (45), we get the following generating function for the truncated exponential polynomials, denoted by [2]e$^{(2)}$($\mathcal{Y}_n^{(r)}(x, a, b, \nu; \mu; \pi)$):

$$
\sum_{n=0}^{\infty} [2]e^{(2)}(x, a, b, \nu; \mu; \pi) \frac{t^n}{n!} \frac{(-1)^\nu t^{\nu/2}}{\prod_{i=0}^{\nu}(a_i b_i - a_i)} (e^x - 1 - t)^\nu.
$$

5. Two-Variable Degenerate Apostol-type Polynomials

In this section, replacing $e^{(2)} = (1 + \lambda)^{\nu/2}$, and $b = 1, a = e = e$ in (13), a new class of two-variable degenerate Apostol...
type polynomials $\mathcal{U}_n^{(r)}(x, y; \mu; \lambda, \lambda)$ is given. Some identities and properties are obtained.

\[ \sum_{n=0}^{\infty} pD \mathcal{U}_n^{(r)}(x, y; \nu; \lambda, \lambda) \frac{t^n}{n!} = \frac{t^{r+2r^n}}{\prod_{i=0}^{r-1} \left( (1 + \lambda t)^{1/\lambda} - \alpha \right)} (1 + \lambda t)^{x/\lambda} \varphi(y, t). \] (47)

5.1. Special Cases

1. If $\alpha_i = 1, r = 1, \nu = 1, \mu = 0$ and $\varphi(y, t) = 1$ in (47), we obtain

\[ pD \mathcal{U}_n^{(1)}(x, y; 1, 0; \lambda, 1) = B_{n, \lambda}(x). \] (48)

(Degenerate Bernoulli polynomials, see [13]).

2. If $\alpha_i = -1, r = 1, \nu = 0, \mu = 1$ and $\varphi(y, t) = 1$ in (47), we obtain

\[ pD \mathcal{U}_n^{(1)}(x; -1; 1; \lambda, 0) = E_{n, \lambda}(x). \] (49)

(Degenerate Euler polynomials, see [13]).

3. If $\alpha_i = 1, i = 0, 1, \ldots, r - 1, \nu = \mu = 1$ and $\varphi(y, t) = 1$ in (47), we obtain

\[ pD \mathcal{U}_n^{(r)}(\alpha, \beta; \lambda; \mu) = \sum_{x_1, \ldots, x_n=0}^{\infty} \frac{1}{\lambda^n} (\alpha_{i-1})^x \sum_{k=0}^{[n/2]} \frac{\beta^k}{k!} \left( n - 2k \right)! \prod_{i=0}^{n-2k-1} \left( \alpha - \sum_{i=1}^{r} x_i - r - i \lambda \right). \] (52)

Proof. Putting $x = \alpha, y = \beta$, and $\varphi(\beta, t) = e^{\beta t}$ in (47), we have

\[ \sum_{n=0}^{\infty} pD \mathcal{U}_n^{(r)}(\alpha, \beta; \lambda; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{x_1, \ldots, x_n=0}^{\infty} \frac{1}{\lambda^n} \left( \sum_{r=1}^{\infty} x_r \right)^{n-2} \left( \sum_{i=1}^{r} x_i - r - i \lambda \right)^2 \varphi(t). \] (53)

The generating function of degenerate generalized Hermite polynomials is given by [12].

\[ (1 + \lambda t)^{\alpha/\lambda} e^{\beta t} = \sum_{n=0}^{\infty} H_n(\alpha, \beta; \lambda) \frac{t^n}{n!}. \] (54)

Then,

\[ \sum_{n=0}^{\infty} pD \mathcal{U}_n^{(r)}(\alpha, \beta; \lambda; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{x_1, \ldots, x_n=0}^{\infty} \frac{1}{\lambda^n} \left( \sum_{r=1}^{\infty} x_r \right)^{n-2} H_n \left( \alpha - \sum_{i=1}^{r} x_i - r - i \lambda \right) \frac{t^n}{n!}. \] (55)

Definition 4. For non-negative $n$, $pD \mathcal{U}_n^{(r)}(x, y; \nu; \lambda, \lambda)$ is given by means of the generating function

\[ pD \mathcal{U}_n^{(r)}(x; 1, 1; \lambda, 1) = G_n^{(r)}(x). \] (50)

(Degenerate Genocchi polynomials of order $r$, see [13]).

4. If $\alpha_i = 0, r = 1, \nu = \mu = 0$ and $\varphi(y, t) = e^{\mu t}$ in (47), we obtain

\[ pD \mathcal{U}_n^{(1)}(x, y; 0, 0; \lambda, 0) = H_n(x - 1, y, \lambda). \] (51)

(Two-variable partially degenerate Hermite polynomials, see [12]).

Theorem 8. The explicit formula of degenerate generalized Apostol Hermite polynomials is given by

\[ pD \mathcal{U}_n^{(r)}(x, y; \nu; \lambda, \lambda) = \sum_{n=0}^{\infty} \frac{1}{\lambda^n} \left( \sum_{r=1}^{\infty} x_r \right)^{n-2} H_n \left( \alpha - \sum_{i=1}^{r} x_i - r - i \lambda \right) \frac{t^n}{n!}. \] (55)
Equating the coefficients of $t^n$ on both sides in the last equation, yields (52).

5.2. Some Identities of $pD\mathcal{U}_n^{(r)}(x, y; \nu, \mu; \lambda, \overline{\alpha},)$

\[
\sum_{m=0}^{n} pD\mathcal{U}_n^{(r)}(x, y; \nu, \mu; \lambda, \overline{\alpha})S(n, m) = \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} D\mathcal{U}_k^{(r)}(x; \nu, \mu; \lambda, \overline{\alpha}) S(k, m) \text{Bel}_{n-k, \lambda}(y),
\]

where $\text{Bel}_{n-k, \lambda}(x)$ are degenerate Bell polynomials, see ([14]) and $S(n, m)$ are Stirling numbers of the second kind, see ([24]).

\[
\frac{(e^\epsilon - 1)^{\nu} 2^{\mu}}{\prod_{i=0}^{r-1} \left( (1 + \lambda(e^\epsilon - 1))^{1/\lambda} - \alpha_i \right)} \left( 1 + \lambda(e^\epsilon - 1) \right)^{\chi/\lambda} \phi(y, (e^\epsilon - 1)) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} pD\mathcal{U}_n^{(r)}(x, y; \nu, \mu; \lambda, \overline{\alpha}) S(n, m) \frac{(e^\epsilon - 1)^m}{m!}.
\]

Proof. Replacing $t$ by $(e^\epsilon - 1)$ in (47), we get

\[
\frac{(e^\epsilon - 1)^{\nu} 2^{\mu}}{\prod_{i=0}^{r-1} \left( (1 + \lambda(e^\epsilon - 1))^{1/\lambda} - \alpha_i \right)} \left( 1 + \lambda(e^\epsilon - 1) \right)^{\chi/\lambda} \phi(y, (e^\epsilon - 1)) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} pD\mathcal{U}_n^{(r)}(x, y; \nu, \mu; \lambda, \overline{\alpha}) S(n, m) \frac{(e^\epsilon - 1)^m}{m!}.
\]

It is well known that the Stirling numbers of the second kind are defined by (see [24])

\[
\frac{(e^\epsilon - 1)^m}{m!} = \sum_{n=0}^{\infty} S(n, m) \frac{t^n}{n!}.
\]

Let us put, [14].

\[
\phi(y, (e^\epsilon - 1)) = (1 + \lambda(e^\epsilon - 1))^{\chi/\lambda} = \sum_{k=0}^{\infty} \text{Bel}_{k, \lambda}(y) \frac{t^k}{k!}.
\]

\[
\frac{(e^\epsilon - 1)^{\nu} 2^{\mu}}{\prod_{i=0}^{r-1} \left( (1 + \lambda(e^\epsilon - 1))^{1/\lambda} - \alpha_i \right)} \left( 1 + \lambda(e^\epsilon - 1) \right)^{\chi/\lambda} \left( 1 + \lambda(e^\epsilon - 1) \right)^{\gamma/\lambda} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} pD\mathcal{U}_n^{(r)}(x; \nu, \mu; \lambda, \overline{\alpha}) S(k, m) \text{Bel}_{n-k, \lambda}(y) \frac{t^n}{n!}.
\]

Therefore, by comparing the coefficients of $t^n$ on both sides of (59) and (61), we obtain (56).

\[
pD\mathcal{U}_n^{(r)}(x, y; \nu, \mu; \lambda, \overline{\alpha}) = \sum_{m=0}^{n} \binom{n}{m} D\mathcal{U}_{n-m}^{(r)}(y; \nu, \mu; \lambda, \overline{\alpha}) \sum_{\ell=0}^{m} s(m, \ell) x^\ell \lambda^{m-\ell},
\]

Theorem 9. For $n \geq 0$, we have

\[
Theorem 10. For $n \geq 0$, we have
where $s(n, m)$ are the Stirling numbers of the first kind, see [24].

$$
\frac{t^n 2^m}{\prod_{i=0}^{n-1} (1 + \lambda t)} \left( 1 + \lambda t \right)^{\frac{1}{2}} \phi(y, t) = \sum_{n=0}^{\infty} pD \mathcal{H}_m^{(r)}(y; v, \mu, \lambda, \pi_r) \frac{t^n}{n!} \sum_{m=0}^{\infty} \left( \frac{x}{\lambda} \right)_m \left( \lambda t \right)^m
\]

$$
= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} D \mathcal{H}_m^{(r)}(y; v, \mu, \lambda, \pi_r) \sum_{\ell=0}^{m} s(m, \ell)x^{\ell} \lambda^{m-\ell} t^n / n!.
\]

Therefore, by equating the right-hand side of (47) and the last equation, we obtain (62). □

6. Probabilistic Application

**Definition 11.** Let $X_1, X_2, \ldots, X_r$ be a nonnegative random variable. Then $\mathbb{X} = (X_1, X_2, \ldots, X_r)$ is said to be generalized

$$
D(\alpha, \gamma, \beta, \lambda) = \sum_{n=0}^{\infty} \prod_{i=0}^{r} (\alpha_{i-1})^{\kappa} H_n \left( \gamma - \sum_{i=1}^{r} \ell_i - r, \beta, \lambda \right),
\]

and

$$
M_\mathbb{X}(t) = E(e^{tX})
\]

$$
= D(\sum_{i=1}^{r} (\alpha_{i-1})^{\kappa}, \gamma, \beta, \lambda)
\]

6.1. Statistical Properties of GDHD Model

6.1.1. Cumulative Distribution Function. The cumulative distribution function (cdf) of GDHD is given by

$$
P(X \leq x_i) = 1 - \frac{\alpha_i x_i + 1}{D(\alpha, \gamma, \beta, \lambda)}.
\]

6.1.2. Moments and Related Measures. The moment generating function is given by

$$
E(X) = \sum_{x_1, x_2, \ldots, x_r = 0}^{\infty} x_1 \prod_{i=1}^{r} (\alpha_{i-1})^{\kappa} H_n \left( \gamma - \sum_{i=1}^{r} x_i - r, \beta, \lambda \right) / D(\alpha, \gamma, \beta, \lambda).
\]

$$
\begin{align*}
\text{Var}(X) &= \left[ \sum_{x_1, x_2, \ldots, x_r = 0}^{\infty} x_i^2 \left( \prod_{i=1}^{r} (\alpha_{i-1})^{\kappa} H_n \left( \gamma - \sum_{i=1}^{r} x_i - r, \beta, \lambda \right) \right) / D(\alpha, \gamma, \beta, \lambda) \right] - \left[ \sum_{x_1, x_2, \ldots, x_r = 0}^{\infty} x_1 \prod_{i=1}^{r} (\alpha_{i-1})^{\kappa} H_n \left( \gamma - \sum_{i=1}^{r} x_i - r, \beta, \lambda \right) / D(\alpha, \gamma, \beta, \lambda) \right]^2.
\end{align*}
\]

Proof. Form equation (47), we have degenerate Hermite distribution (GDHD), if its probability mass function is

$$
P(X) = \frac{\prod_{i=1}^{r} (\alpha_{i-1})^{\kappa} H_n \left( \gamma - \sum_{i=1}^{r} x_i - r, \beta, \lambda \right)}{D(\alpha, \gamma, \beta, \lambda)}
\]

where

$$
\mathbb{X} = (x_1, x_2, \ldots, x_r) \quad \text{and} \quad t = (t_1, t_2, \ldots, t_r).
\]

The $r$-th moments $\mu_r' \text{ of GDHD}$ is

$$
\mu_r' = \left( \sum_{x_1, x_2, \ldots, x_r = 0}^{\infty} x_i^r \prod_{i=1}^{r} (\alpha_{i-1})^{\kappa} H_n \left( \gamma - \sum_{i=1}^{r} x_i - r, \beta, \lambda \right) / D(\alpha, \gamma, \beta, \lambda) \right).
\]

The mean and the variance of GDHD are
Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest with this study.

References