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Research Article

Bicomplex Landau and Ikehara Theorems for the Dirichlet Series

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The aim of this paper is to generalize the Landau-type Tauberian theorem for the bicomplex variables. Our findings extend and improve on previous versions of the Ikehara theorem. Also boundedness result for the bicomplex version of Ikehara–Korevaar theorem is derived. The purpose of this article is to substantially extend the various complex Tauberian theorems for the Dirichlet series to the bicomplex domain.

1. Introduction

For a long time, bicomplex numbers have been investigated, and a lot of work has been carried out in this area. Bicomplex numbers are introduced by Segre [1] in 1882. Different algebraic and geometric features of bicomplex numbers, as well as their applications, have been the focus of recent research. Many properties and applications of bicomplex numbers have been discovered (see, [2–8]). In recent developments, efforts have been made to extend the integral transforms [9–14], and a number of special functions like [5, 15–19] to the bicomplex variable from their complex counterparts.

The aim of this paper is to extend the various complex Tauberian theorems for the Dirichlet series to the bicomplex domain. Generalization of Landau-type theorem and Ikehara theorem is introduced. Boundedness condition for the bicomplex Tauberian theorem has been included. In the proof of these results, the decomposition theorem of Ringleb plays a vital role.

1.1. Bicomplex Numbers. The set of bicomplex numbers was defined by Segre [1] in the following way:

Definition 1 (Bicomplex number). The set of bicomplex numbers is defined in terms of real components as

$$\mathbb{T} = \{\xi\colon \xi = x_0 + i_1 x_1 + i_2 x_2 + j x_3 | x_0, x_1, x_2, x_3 \in \mathbb{R}\}, \quad (1)$$

and it can be represented as in terms of complex numbers as

$$\mathbb{T} = \{\xi\colon \xi = z_1 + i_2 z_2 | z_1, z_2 \in \mathbb{C}\}, \quad (2)$$

where $i_1^2 = i_2^2 = -1, i_1 i_2 = i_2 i_1 = j, j^2 = 1$.

The notations we will use are as follows:
$x_0 = \mathrm{Re}(\xi), x_1 = \mathrm{Im}_{i_1}(\xi), x_2 = \mathrm{Im}_{i_2}(\xi), x_3 = \mathrm{Im}_j(\xi)$.

The set of all zero divisor elements of $\mathbb{T}$ is called null cone, and it is denoted by $\mathbb{NC}$ and is defined as follows:

$$\mathbb{NC} = \left\{z_1 + z_2 i_2 | z_1^2 + z_2^2 = 0\right\}. \quad (3)$$

Segre [1] noticed that the two zero divisor elements $(1 + i_1 i_2)/2$ and $(1 - i_1 i_2)/2$ are idempotent elements and play a vital role in the theory of the bicomplex numbers. e_1 and e_2, the two nontrivial idempotent elements of $\mathbb{T}$, are defined as follows:

$$
\begin{aligned}
e_1 &= \frac{1 + i_1 i_2}{2} = \frac{1 + j}{2},\\
e_2 &= \frac{1 - i_1 i_2}{2} = \frac{1 - j}{2}.
\end{aligned}
\tag{4}
$$

Also,

$$
\begin{aligned}
e_1 + e_2 &= 1,\\
e_1 \cdot e_2 &= 0,\\
e_1^2 &= e_1,\\
e_2^2 &= e_2.
\end{aligned}
\tag{5}
$$

Definition 2 (idempotent representation). $\mathbb{T}$ has a unique idempotent representation for each element [4, 20–22] defined by

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2. \tag{6}$$

So, if $\xi_1 = (z_1 - i_1 z_2)$ and $\xi_2 = (z_1 + i_1 z_2)$, then $\xi = \xi_1 e_1 + \xi_2 e_2$.

Writing ξ in real components and idempotent components as

$$\xi = x_0 + i_1 x_1 + i_2 x_2 + i_1 i_2 x_3 = (x_0 + i_1 x_1) + i_2 (x_2 + i_1 x)_3 = \xi_1 e_1 + \xi_2 e_2, \tag{7}$$

and comparing them, we get $\xi_1 = (x_0 + x_3) + i_1 (x_1 - x_2)$ and $\xi_2 = (x_0 - x_3) + i_1 (x_1 + x_2)$.

The set of hyperbolic numbers $\mathbb{D} = \{x_1 + x_3 j | x_1, x_3 \in \mathbb{R}, j^2 = 1 \text{ and } j \notin \mathbb{R}\}$ and the set of complex numbers $\mathbb{C}$ are two important proper subsets which are unified by the set of bicomplex numbers $\mathbb{T}$ (see, [[6], p.19]). The sets $\mathbb{T}, \mathbb{D}$ are connected to the theory of Clifford algebras. The set of bicomplex number is a two-dimensional complex Clifford algebra which has a set of hyperbolic numbers as its real (Clifford) subalgebra (see [[6], p.24]), or $\mathbb{T} \cong CI_{\mathbb{C}}(1, 0) \cong CI_{\mathbb{C}}(0, 1)$ and $\mathbb{D} \cong CI_{\mathbb{R}}(0, 1)$ (see [[7], p.1]).

Definition 3 (bicomplex moduli). Let $\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2 = x_0 + x_1 i_1 + x_2 i_2 + x_3 j \in \mathbb{T}$ (see [6, 22, 23]).

The norm of ξ is defined as

$$\|\xi\| = \sqrt{|z_1|^2 + |z_2|^2} = \frac{1}{\sqrt{2}} \sqrt{|\xi_1|^2 + |\xi_2|^2} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}. \tag{8}$$

The i_1 modulus of ξ is given by

$$|\xi|_{i_1} = \sqrt{z_1^2 + z_2^2}. \tag{9}$$

The i_2 modulus of ξ is given by

$$|\xi|_{i_2} = \sqrt{\left(|z_1|^2 - |z_2|^2\right) + 2\mathrm{Re}\left(z_1 \overline{z_1}\right) i_2}. \tag{10}$$

The j modulus of ξ is given by

$$|\xi|_j = |z_1 - i_1 z_2| e_1 + |z_1 + i_1 z_2| e_2. \tag{11}$$

The absolute value of ξ is given by

$$|\xi|_{abs} = \sqrt{|z_1^2 + z_2^2|} = \sqrt{|(z_1 - i_1 z_2)(z_1 + i_1 z_2)|} = \sqrt{|\xi_1 \xi_2|} = \sqrt{|\xi_1||\xi_2|}. \tag{12}$$

Ringleb [24] (see also [22]), investigated the analyticity of a bicomplex function with respect to its idempotent complex component functions in the following theorem. When studying the convergence of bicomplex functions, this theorem is crucial.

Theorem 1 (decomposition theorem of Ringleb [24]). *Let $f(\xi)$ be analytic in a region $U \subseteq \mathbb{T}$, and let $T_1 \subseteq \mathbb{C}$ and $T_2 \subseteq \mathbb{C}$ be the component regions of $\mathbb{T}$, in the ξ_1 and ξ_2 planes, respectively. Then, there exists a unique pair of complex-valued analytic functions, $f_1(\xi_1)$ and $f_2(\xi_2)$, defined in $U_1 \subseteq T_1$ and $U_2 \subseteq T_2$, respectively, such that*

$$f(\xi) = f_1(\xi_1) e_1 + f_2(\xi_2) e_2, \quad \xi \in U. \tag{13}$$

Conversely, if $f_1(\xi_1)$ is any complex-valued analytic function in a region T_1 and $f_2(\xi_2)$ any complex-valued analytic function in a region T_2, then the bicomplex-valued function $f(\xi)$ defined by equation (13) is an analytic function of the bicomplex variable ξ in the product region $U = U_1 \times_e U_2$.

In 1826, Abel proved the following result for the real power series (see [25–27]).

Theorem 2 (Abel's theorem). *Let*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{14}$$

be a power series with coefficients $a_n \in \mathbb{R}$ that converges on $(-1, 1)$. We assume that $\sum_{n=0}^{\infty} a_n$ converges. Then,

$$\lim_{x \longrightarrow 1} f(x) = \sum_{n=0}^{\infty} a_n. \tag{15}$$

In general, the converse is not true, i.e., if $\lim_{x \longrightarrow 1} f(x)$ exists, one cannot conclude that $\sum_{n=0}^{\infty} a_n$ converges. In 1897, Tauber [28] proved the converse to Abel's theorem but under an additional hypothesis.

Theorem 3 (Tauberian theorem). *Let*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{16}$$

be a power series with coefficients $a_n \in \mathbb{R}$ that converges on the real interval $(-1, 1)$. We assume that

$$\lim_{x \longrightarrow 1} f(x) = A, \tag{17}$$

exists, and moreover,

$$\lim_{n \longrightarrow \infty} n a_n = 0. \tag{18}$$

Then, $\sum_{n=0}^{\infty} a_n$ converges and is equal to A.

Detailed proof of the above theorem may be found in [[27], p.435].

Tauber's result directed to many other Tauberian theorems. Later, various other converse theorems have been proved by Hardy and Littlewood and they named them the "Tauberian theorems" (see [26, 29]).

Tauberian theory provides many techniques for resolving difficult problems in analysis. Tauberian type theorems have numerous applications in mathematics, including rapidly decaying distributions and their applications to stable laws [30], generalized functions [31], Dirichlet series [32], and the solution of the prime number theorem [26]. In the bicomplex variable [10], the Tauberian theorem for the Laplace–Stieltjes transform is proved. Tauberian theory provides novel answers to complex situations. It has a variety of applications in number theory [26, 33]. In the area of mathematical physics, applications are studied in the quantum field theory [31, 34].

Landau [35] (see also [[32], p.4]) studied the following Tauberian result for complex power series.

Theorem 4 (Landau's theorem). *Let G be given for $\mathrm{Re}(w) > 1, w \in \mathbb{C}$ by a convergent Dirichlet series*

$$G(w) = \sum_{n=1}^{\infty} \frac{a_n}{n^w}, \tag{19}$$

with $a_n \geq 0, \forall n \in \mathbb{N}$. We suppose that for some constant α, the analytic function

$$H(w) = G(w) - \frac{\alpha}{w - 1}, \quad \mathrm{Re}(w) > 1, \tag{20}$$

has an analytic or just continuous extension (also called H) to the closed half-plane $\mathrm{Re}(w) \geq 1$. Finally, we suppose that there is a constant K such that

$$H(w) = O\left(|w|^K\right), \quad \mathrm{Re}(w) \geq 1,\ K > 0. \tag{21}$$

Then,

$$\frac{1}{n} \sum_{k=1}^{n} a_k \longrightarrow \alpha, \quad \text{as } n \longrightarrow \infty. \tag{22}$$

Ikehara's theorem [25] extends the result of Landau (see [29]).

Theorem 5 (Ikehara's theorem). *Let G be given by the Dirichlet series*

$$G(w) = \sum_{n=1}^{\infty} \frac{a_n}{n^w}, \tag{23}$$

convergent for $\mathrm{Re}(w) > 1$, where the coefficients satisfy the Tauberian condition $a_n \geq 0, \forall n \in \mathbb{N}$. If there exists a constant α such that

$$G(w) - \frac{\alpha}{w - 1}, \tag{24}$$

admits a continuous extension to the line $\mathrm{Re}(w) = 1$, then

$$\sum_{k=1}^{n} a_k \backsim \alpha n, \quad \text{as } n \longrightarrow \infty. \tag{25}$$

In [36, 37], the authors defined the bicomplex Dirichlet series as $f(\xi) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n \xi}, \xi \in \mathbb{T}$, where $\{a_n\}$, $a_n = \alpha_{1_n} e_1 + \alpha_{2_n} e_2$ is a bicomplex number sequence. Substituting $\lambda_n = \log n$, the following form of the bicomplex Dirichlet series is obtained:

$$f(\xi) = \sum_{n=1}^{\infty} a_n n^{-\xi}. \tag{26}$$

In terms of idempotent components, $f(\xi)$ can be written as

$$\begin{aligned} f(\xi) &= \sum_{n=1}^{\infty} a_n n^{-\xi} = \sum_{n=1}^{\infty} \alpha_{1n} n^{-\xi_1} e_1 + \sum_{n=1}^{\infty} \alpha_{2n} n^{-\xi_2} e_2 \\ &= f_1(\xi_1) e_1 + f_2(\xi_2) e_2. \end{aligned} \tag{27}$$

The idempotent components of $f(\xi)$, $f_1(\xi_1) = \sum_{n=1}^{\infty} \alpha_{1n} n^{-\xi_1}$ and $f_2(\xi_2) = \sum_{n=1}^{\infty} \alpha_{2n} n^{-\xi_2}$ are the complex Dirichlet Series.

If the abscissae of convergence of the series $f_1(\xi_1) = \sum_{n=1}^{\infty}\alpha_{1n}n^{-\xi_1}$ and $f_2(\xi_2) = \sum_{n=1}^{\infty}\alpha_{2n}n^{-\xi_2}$ are denoted by σ_1 and σ_2, respectively, then the region

$$\mathbb{E} = \{\xi \in \mathbb{T}\colon \operatorname{Re}(\xi_1) > \sigma_1 \text{ and } \operatorname{Re}(\xi_2) > \sigma_2\}, \tag{28}$$

or equivalently

$$\mathbb{E} = \left\{\xi \in \mathbb{T}\colon -\operatorname{Re}(\xi) + \sigma_1 < \operatorname{Im}_j(\xi) < \operatorname{Re}(\xi) - \sigma_2\right\}, \tag{29}$$

is the region of convergence of the bicomplex Dirichlet series $f(\xi)$ defined in equation (26).

Inspired by the work of Agarwal et al. [10] and Srivastava and Kumar [37], here, the bicomplex Landau-type Tauberian theorem is investigated. Also, the bicomplex version of the Ikehara's Tauberian theorem, which is generalization of the Landau-type Tauberian theorem, has been studied.

2. Bicomplex Versions of the Landau and Ikehara Theorems

Motivated by the work of Landau, we have derived the bicomplex version of Theorem 4 as follows:

Theorem 6 (bicomplex Landau theorem). *Let f be given for $\xi = \xi_1 e_1 + \xi_2 e_2$, $|\operatorname{Im}_j(\xi)| < \operatorname{Re}(\xi) - 1$ by a convergent Dirichlet series*

$$f(\xi) = \sum_{n=1}^{\infty}\frac{a_n}{n^{\xi}}, \quad \xi, a_n \in \mathbb{T}, \tag{30}$$

where $a_n = a_{n_1} + j a_{n_4} \in \mathbb{D}$ with $a_{n_1} \geq |a_{n_4}|$, $\forall n \in \mathbb{N}$. We suppose that for some hyperbolic constant $A = A_1 e_1 + A_2 e_2$, the analytic function

$$g(\xi) = f(\xi) - \frac{A}{\xi - 1}, \quad \left|\operatorname{Im}_j(\xi)\right| < \operatorname{Re}(\xi) - 1, \tag{31}$$

has an analytic or just continuous extension (also called g) to the closed half-plane $|\operatorname{Im}_j(\xi)| \leq \operatorname{Re}(\xi) - 1$.

Finally, we suppose that there is a constant M such that

$$g(\xi) = O\left(|\xi|_j^M\right), \tag{32}$$

for $|\operatorname{Im}_j(\xi)| \leq \operatorname{Re}(\xi) - 1$. Then,

$$\frac{1}{n}S_n = \frac{1}{n}\sum_{k=1}^{n} a_k \longrightarrow A, \quad \text{as } n \longrightarrow \infty. \tag{33}$$

Proof. We consider the Dirichlet series

$$f(\xi) = \sum_{n=1}^{\infty}\frac{a_n}{n^{\xi}} = f(\xi_1)e_1 + f(\xi_2)e_2, \quad a_n, \xi \in \mathbb{T}, \tag{34}$$

and let $A_1 e_1 + A_2 e_2 = A \in \mathbb{T}$. Here,

$$\begin{aligned} f(\xi_1) &= \sum_{n=1}^{\infty}\frac{\alpha_{1n}}{n^{\xi_1}}, \quad \operatorname{Re}(\alpha_{1n}) > 0,\\ f(\xi_2) &= \sum_{n=1}^{\infty}\frac{\alpha_{2n}}{n^{\xi_2}}, \quad \operatorname{Re}(\alpha_{2n}) > 0, \end{aligned} \tag{35}$$

are convergent for $\operatorname{Re}(\xi_1) > 1$ and $\operatorname{Re}(\xi_2) > 1$, respectively.

For some constants A_i, $(i = 1, 2)$,

$$g_i(\xi_i) = f(\xi_i) - \frac{A_i}{\xi_i - 1}, \quad \operatorname{Re}(\xi_i) > 1,\ i = 1, 2, \tag{36}$$

are analytic functions in the complex domain. By Theorem 4, function $g_i(\xi_i)$, $(i = 1, 2)$ has an analytic or just continuous extension (also called g_i) to the closed half plane Re $(\xi_i) \geq 1$, $(i = 1, 2)$.

Since $g_1(\xi_1)$ and $g_2(\xi_2)$ are analytic functions, thereby taking the idempotent linear combination of (36) for $i = 1, 2$,

$$\begin{aligned} g(\xi) &= g_1(\xi_1)e_1 + g_2(\xi_2)e_2\\ &= \left(f(\xi_1) - \frac{A_1}{\xi_1 - 1}\right)e_1 + \left(f(\xi_2) - \frac{A_2}{\xi_2 - 1}\right)e_2\\ &= f(\xi_1)e_1 + f(\xi_2)e_2 - \left(\frac{A_1 e_1 + A_2 e_2}{\xi_1 e_1 + \xi_2 e_2 - 1}\right)\\ &= f(\xi) - \frac{A}{\xi - 1}. \end{aligned} \tag{37}$$

With the help of equation (7), the conditions $\operatorname{Re}(\xi_1) > 1, \operatorname{Re}(\xi_2) > 1$ can be rewritten as

$$\begin{aligned} &x_0 + x_3 > 1,\\ &x_0 - x_3 > 1,\\ &\Rightarrow |x_3| < x_0 - 1\\ &\Rightarrow \left|\operatorname{Im}_j(\xi)\right| < \operatorname{Re}(\xi) - 1. \end{aligned} \tag{38}$$

By assumption of the theorem, the j-modulus of $g(\xi)$, in (37), $\xi \in \mathbb{T}$, we have

$$\begin{aligned} &g(\xi) = O\left(|\xi|_j^M\right),\\ &\text{Since, } |\xi|_j^M = |\xi_1|^M e_1 + |\xi_2|^M e_2,\\ &\Rightarrow g(\xi_1) = O\left(|\xi_1|^M\right)\\ &\quad\ g(\xi_2) = O\left(|\xi_2|^M\right). \end{aligned} \tag{39}$$

Thus, by Theorem 4 for complex domain,

$$\begin{aligned} \frac{1}{n}S_{1n} &= \frac{1}{n}\sum_{k=1}^{n}\alpha_{1k} \longrightarrow A_1,\\ \frac{1}{n}S_{2n} &= \frac{1}{n}\sum_{k=1}^{n}\alpha_{2k} \longrightarrow A_2,\\ &\text{as } n \longrightarrow \infty. \end{aligned} \tag{40}$$

By idempotent combination of the above series,

$$\frac{1}{n}S_{1n}e_1 + \frac{1}{n}S_{2n}e_2 = \frac{1}{n}S_n \longrightarrow A_1e_1 + A_2e_2 = A, \qquad \frac{1}{n}S_n \longrightarrow A, \quad \text{as}\, n \longrightarrow \infty. \tag{41}$$

Furthermore, the relation $a_n = \alpha_{1n}e_1 + \alpha_{2n}e_2 = a_{n_1} + i_1a_{n_2} + i_2a_{n_3} + ja_{n_4}$ gives

$$\begin{aligned}\alpha_{1n} &= \left(a_{n_1} + a_{n_4}\right) + i_1\left(a_{n_2} - a_{n_3}\right),\\ \alpha_{2n} &= \left(a_{n_1} - a_{n_4}\right) + i_1\left(a_{n_2} + a_{n_3}\right).\end{aligned} \tag{42}$$

The conditions $\alpha_{1n} \geq 0$, $\alpha_{2n} \geq 0$ imply

$$\begin{aligned}&a_{n_1} + a_{n_4} \geq 0,\\ &a_{n_2} - a_{n_3} = 0;\\ &a_{n_1} - a_{n_4} \geq 0,\\ &a_{n_2} + a_{n_3} = 0.\\ &\Rightarrow a_{n_1} \geq \left|a_{n_4}\right|;\\ &a_{n_2} = a_{n_3} = 0.\\ &\Rightarrow a_n = \left(a_{n_1} + a_{n_4}\right)e_1 + \left(a_{n_1} - a_{n_4}\right)e_2 = a_{n_1} + ja_{n_4}.\end{aligned} \tag{43}$$

Hence, a_n is a hyperbolic number with $a_{n_1} \geq |a_{n_4}|$. □

Remark 1. In the proof of the above theorem, it is observed that the results and conditions focus on the hyperbolic coefficients and not on coefficients of imaginary units i_1 and i_2; hence, it can be called the hyperbolic version of the Landau theorem.

Theorem 7 (bicomplex Ikehara theorem). *Let $\xi, a_n \in \mathbb{T}$ where $\xi = \xi_1e_1 + \xi_2e_2$ and $a_n = a_{n_1} + ja_{n_4}, n \in \mathbb{N}$ is a sequence of hyperbolic numbers [6]. Let f be given by the Dirichlet series*

$$f(\xi) = \sum_{n=1}^{\infty} \frac{a_n}{n^{\xi}}, \quad a_{n_1} \geq \left|a_{n_4}\right|, \tag{44}$$

convergent for $|\mathrm{Im}_j(\xi)| < \mathrm{Re}(\xi) - 1$.

If there exists a hyperbolic constant $\beta \in \mathbb{D}$ such that

$$f(\xi) - \frac{\beta}{\xi - 1}, \tag{45}$$

admits a continuous extension to the plane $\mathrm{Re}(\xi) = 1, \mathrm{Im}_j(\xi) = 0$, then

$$S_n = \sum_{k=1}^{n} a_n \backsim \beta n, \quad \text{as}\, n \longrightarrow \infty. \tag{46}$$

Proof. We consider the Dirichlet Series

$$f(\xi) = \sum_{n=1}^{\infty} \frac{a_n}{n^{\xi}} = f(\xi_1)e_1 + f(\xi_2)e_2, \tag{47}$$

where

$$f(\xi_i) = \sum_{n=1}^{\infty} \frac{\alpha_{in}}{n^{\xi_i}}, \quad \alpha_{in} \geq 0,\ \mathrm{Re}(\xi_i) > 1,\ i = 1, 2, \tag{48}$$

where $f(\xi)$ is convergent for $\alpha_{1n} \geq 0$, $\alpha_{2n} \geq 0$ i.e. $a_{n_1} \geq |a_{n_4}|$ (from equation (43)) and $|\mathrm{Im}_j(\xi)| < \mathrm{Re}(\xi) - 1$.

By Theorem 5, for some constants β_i, $(i = 1, 2)$, the analytic functions

$$f_i(\xi_i) - \frac{\beta_i}{\xi_i - 1}, \quad i = 1, 2, \tag{49}$$

admit a continuous extension to the lines Re $(\xi_i) = 1$, $(i = 1, 2)$. Taking idempotent linear combination of the functions defined in equation (49), we get for $\mathrm{Re}(\xi_i) > 1$, $(i = 1, 2)$ or equivalently $|\mathrm{Im}_j(\xi)| < \mathrm{Re}(\xi) - 1$,

$$\left(f(\xi_1) - \frac{\beta_1}{\xi_1 - 1}\right)e_1 + \left(f(\xi_2) - \frac{\beta_2}{\xi_2 - 1}\right)e_2 = f(\xi) - \frac{\beta}{\xi - 1}, \tag{50}$$

where $\beta = \beta_1e_1 + \beta_2e_2 \in \mathbb{D}$. Hence, $f(\xi) - (\beta/(\xi - 1))$ admits a continuous extension to the plane $\mathrm{Re}(\xi_1) = 1, \mathrm{Re}(\xi_2) = 1$, i.e., $\mathrm{Re}(\xi) = 1, \mathrm{Im}_j(\xi) = 0$ which means $\xi = 1 + x_1i_1 + x_2i_2$, $x_1, x_2 \in \mathbb{R}$.

Furthermore,

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \alpha_{1k}e_1 + \sum_{n=1}^{n} \alpha_{2k}e_2 \backsim \beta n, \quad \text{as}\, n \longrightarrow \infty. \tag{51}$$

□

3. Ikehara's Theorem Involving Boundedness

In this section, we discuss some results about Schwartz functions, tempered distributions, and the Fourier transform (see [38–40]). Schwartz [41] (see also [38]) chooses the class of test function ϕ that is infinitely continuously differentiable and that vanishes outside some bounded set. All functionals defined on this class that are linear and continuous are named distributions by Schwartz.

Space $S(\mathbb{R})$ is the Schwartz space of rapidly decreasing smooth test functions ϕ (see [29]), i.e., those C^{∞} functions over the real field such that

$$\sup_{u\in R}\left|u^p\phi^{(q)}(u)\right| < \infty, \quad p, q \in \mathbb{N}. \tag{52}$$

The space of tempered distributions is represented by $S'(\mathbb{R})$, which is the dual of $S(\mathbb{R})$ (see [29]). The evaluation of $g \in S'(\mathbb{R})$ at $\psi \in S(\mathbb{R})$ is denoted by $\langle g, \psi\rangle = \int_{-\infty}^{\infty} g(u)\psi(u)\mathrm{d}u$. Thus, $g \in S'(\mathbb{R})$ if and only if

$$\begin{aligned}&\langle g, a\psi + \phi\rangle = a\langle g, \psi\rangle + \langle g, \phi\rangle,\\ &\lim_{n\longrightarrow\infty} \langle g, \psi_n\rangle = \langle g, \lim_{n\longrightarrow\infty} \psi_n\rangle,\end{aligned} \tag{53}$$

whenever $\{\psi_n\}_{n=0}^{\infty}$ is convergent in $S(\mathbb{R})$.

If a tempered distribution is the Fourier transform of a bounded (measurable) function, then it is called a pseudomeasure.

Let $\sum_{n=1}^{\infty} a_n/n^w$ be a complex Dirichlet series with coefficients $a_n \geq 0$ that converges to a function $f(w)$ for

$\operatorname{Re}(w) > 1$. In 2008, Korevaar [42] proved following theorem for boundedness of S_N/N in complex space as follows:

Theorem 8 (Ikehara–Korevaar theorem). *Let $\sum_{n=1}^{\infty} a_n/n^w$ be a Dirichlet series with coefficients $a_n \geq 0$ converging to $g(w)$ for $\operatorname{Re}(w) > 1$. Let $S_N = \sum_{n\leq N} a_n$; the sequence $\{S_N/N\}$ will remain bounded if the quotient*

$$f(w) = \frac{g(w)}{w}, \quad w = u + i_1 v \in \mathbb{C},\ \operatorname{Re}(w) = u > 1, \tag{54}$$

converges in the sense of tempered distribution to a pseudomeasure $f(1 + i_1 v)$, as $u \longrightarrow 1$.

Remark 2. The distributional convergence in the above theorem is convergence in the Schwartz space S'. In other words,

$$< f(u + i_1 v), \phi(v) \longrightarrow f(1 + i_1 v), \phi(v) >, \quad \text{as}\, u \longrightarrow 1, \tag{55}$$

for all testing functions $\phi(v) \in S$, that is, all rapidly decreasing C^{∞} functions.

We hereby provide the bicomplex version of Theorem 8.

Theorem 9 (bicomplex Ikehara–Korevaar theorem). *Let $\sum_{n=1}^{\infty} a_n/n^{\xi}$ be a bicomplex Dirichlet series where $\xi = \xi_1 e_1 + \xi_2 e_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2 \in \mathbb{T}$, $z_1 = x_1 + i_1 y_1$, $z_2 = x_2 + i_1 y_2 \in \mathbb{C}$, and $a_n = a_{n_1} + j a_{n_4} \in \mathbb{D}$ with $a_{n_1} \geq |a_{n_4}|$ that converges to $f(\xi)$ for $|\operatorname{Im}_j(\xi)| < \operatorname{Re}(\xi) - 1$.*

Let $S_N = \sum_{n\leq N} a_n$; then, a necessary and sufficient condition for the boundedness of S_N/N is that the quotient $q(\xi) = (f(\xi)/\xi)$, $\xi \notin \mathbb{NC}$ converges in the sense of tempered distribution to a pseudomeasure $q(1 + i_1 y_1 + i_2 x_2)$, as $x_1 \longrightarrow 1, y_2 \longrightarrow 0$.

Proof. Let the Dirichlet series $\sum_{n=1}^{\infty} a_n/n^{\xi}$ converges to $f(\xi) = f_1(\xi_1)e_1 + f_2(\xi_2)e_2$ where

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{\xi}} = \sum_{n=1}^{\infty} \frac{\alpha_{1n}}{n^{\xi_1}} e_1 + \sum_{n=1}^{\infty} \frac{\alpha_{2n}}{n^{\xi_2}} e_2, \quad a_n = \alpha_{1n} e_1 + \alpha_{2n} e_2. \tag{56}$$

For $\operatorname{Re}(\xi_1) > 1$ and $\operatorname{Re}(\xi_2) > 1$, equivalently, $|\operatorname{Im}_j(\xi)| < \operatorname{Re}(\xi) - 1$, the Dirichlet series $\sum_{n=1}^{\infty} \alpha_{1n}/n^{\xi_1}$, $\alpha_{1n} \geq 0$ converges to the function $f_1(\xi_1)$ and the Dirichlet series $\sum_{n=1}^{\infty} \alpha_{2n}/n^{\xi_2}$, $\alpha_{2n} \geq 0$ converges to the function $f_2(\xi_2)$.

Let us denote $\sum_{n\leq N} \alpha_{1n} = S_{1N}$ and $\sum_{n\leq N} \alpha_{2n} = S_{2N}$; then,

$$\begin{aligned} S_N &= \sum_{n\leq N} a_n \\ &= \sum_{n\leq N} \alpha_{1n} e_1 + \sum_{n\leq N} \alpha_{2n} e_2 \\ &= S_{1N} e_1 + S_{2N} e_2. \end{aligned} \tag{57}$$

From Theorem 8, the necessary and sufficient condition for the boundedness of S_{1N}/N is that the quotient

$$q_1(z_1 - i_1 z_2) = q_1((x_1 + y_2) + i_1(y_1 - x_2)) = \frac{f_1(z_1 - i_1 z_2)}{z_1 - i_1 z_2}, \tag{58}$$

converges in the sense of tempered distribution to a pseudomeasure $q_1(1 + i_1(y_1 - x_2))$, as $x_1 + y_2 \longrightarrow 1$.

Similarly, the necessary and sufficient condition for boundedness of S_{2N}/N is that the quotient

$$q_2(z_1 + i_1 z_2) = q_2((x_1 - y_2) + i_1(y_1 + x_2)) = \frac{f_2(z_1 + i_1 z_2)}{z_1 + i_1 z_2}, \tag{59}$$

converges in the sense of tempered distribution to a pseudomeasure $q_2(1 + i_1(y_1 + x_2))$, as $x_1 - y_2 \longrightarrow 1$.

Again, by the application of the Ringleb theorem, the necessary and sufficient condition for the boundedness of $S_N/N = (S_{1N}/N)e_1 + (S_{2N}/N)e_2$ is that the quotient

$$\begin{aligned} q(z_1 + i_2 z_2) &= q_1 e_1 + q_2 e_2 \\ &= \left(\frac{f_1(z_1 - i_1 z_2)}{z_1 - i_1 z_2}\right) e_1 + \left(\frac{f_2(z_1 + i_1 z_2)}{z_1 + i_1 z_2}\right) e_2 \\ &= (f_1(z_1 - i_1 z_2)e_1 + f_2(z_1 + i_1 z_2)e_2)\left(\frac{1}{z_1 + i_2 z_2}\right) \\ &= \frac{f(\xi)}{\xi}, \quad \xi \notin \mathbb{NC}, \end{aligned} \tag{60}$$

converges to $q_1(1 + i_1(y_1 - x_2))e_1 + q_2(1 + i_1(y_1 + x_2))e_2 = q(1 + i_1 y_1 + i_2 x_2)$ in the sense of tempered distribution to a pseudomeasure as $x_1 + y_2 \longrightarrow 1$ and $x_1 - y_2 \longrightarrow 1$, i.e., $x_1 \longrightarrow 1, y_2 \longrightarrow 0$. □

4. Conclusion

In this paper, Landau-type Tauberian theorem in bicomplex space which is the generalization of Landau-type Tauberian theorem has been derived. The necessary and sufficient condition for the boundedness of the partial sum $S_N = \sum_{n\leq N} a_n$ for bicomplex Dirichlet series with hyperbolic coefficients is obtained. The conditions of convergence are affected by the j coefficient of bicomplex numbers, and hence the theorems can be seen as the hyperbolic versions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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