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Research Article

Stability Analysis of a Ratio-Dependent Predator-Prey Model

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In this study, a ratio-dependent predator-prey model is investigated. The local stability and global stability of the nonnegative boundary equilibrium and positive equilibrium of the model are discussed, respectively. Sufficient condition is obtained for the existence of Hopf bifurcation at the positive equilibrium.

1. Introduction

Recently, the predator-prey models have been studied by many authors [1–8]. In general, a predator-prey model has the following forms:

$$\begin{cases} \dot{x} = xf(x) - p(x)y, \\ \dot{y} = kp(x)y - yg(y), \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ are the densities of the prey and predator population at time t, respectively. The function $f(x)$ represents the growth of the prey population rate, $g(y)$ represents the growth rate of predator population, and $p(x)$ represents the functional response function of predator population to prey population. In [1], Xu et al. used the function $p(x) = x^2/(x^2 + my^2)$ as the functional response function of predator population to prey population. The time delay due to the gestation of the predator is discussed in [1].

It is noted that in model (1), each individual's prey admits the same risk to be attacked by predators and each individual predator admits the same ability to feed on prey. This assumption seems not to be realistic for many animals. In natural world, there are many species whose individuals pass through an immature stage. Stage structure is a natural phenomenon and represents, for example, the division of a population into immature and mature individuals. In the last two decades, stage-structured models have received great attention [3–7, 9].

Based on above discussion, we study the following predator-prey model:

$$\begin{cases} \dot{x}_1(t) = rx_2(t) - (d_1 + r_1)x_1(t) - \dfrac{a_1 x_1^2(t) y_2(t)}{x_1^2(t) + my_2^2(t)}, \\ \dot{x}_2(t) = r_1 x_1(t) - \mathrm{d}_2 x_2(t) - ax_2^2(t), \\ \dot{y}_1(t) = \dfrac{a_2 x_1^2(t-\tau) y_2(t-\tau)}{x_1^2(t-\tau) + my_2^2(t-\tau)} - (r_2 + d_3)y_1(t), \\ \dot{y}_2(t) = r_2 y_1(t) - d_4 y_2(t), \end{cases} \tag{2}$$

where $x_1(t)$ and $x_2(t)$ are the densities of the immature and mature prey at time t and $y_1(t)$ and $y_2(t)$ are the densities of the immature and mature predators at time t. In model (2), all parameters are positive constants. $\tau \geq 0$ is the time delay due to the gestation of the predator. $x^2/(x^2 + my^2)$ is the ratio-dependent functional response.

Model (2) is of the following initial conditions:

$$\begin{aligned}
&x_1(\theta) = \phi_1(\theta) \geq 0,\\
&x_2(\theta) = \phi_2(\theta) \geq 0,\\
&y_1(\theta) = \varphi_1(\theta) \geq 0,\\
&y_2(\theta) = \varphi_2(\theta) \geq 0, \quad \theta \in [-\tau, 0),\\
&\phi_1(0) > 0,\\
&\phi_2(0) > 0,\\
&\varphi_1(0) > 0,\\
&\varphi_2(0) > 0, \phi_1(\theta), \phi_2(\theta), \varphi_1(\theta), \varphi_2(\theta)) \in C\left([-\tau, 0], R^4_{+0}\right).
\end{aligned} \tag{3}$$

The organization of this study is as follows. In Section 2, we discuss the local stability of the nonnegative boundary equilibrium and the positive equilibrium of models (2) and (3). The existence of a Hopf bifurcation for models (2) and (3) at the positive equilibrium is also established. Sufficient conditions are derived for the global stability of the nonnegative boundary equilibrium and positive equilibrium of models (2) and (3) in Section 3, respectively.

2. Local Stability and Hopf Bifurcation

In this section, by analyzing the corresponding characteristic equations, we study the local stability of each of nonnegative equilibria and the existence of a Hopf bifurcation at the positive equilibrium of models (2) and (3).

If $rr_1 > d_2(r_1 + d_1)$, model (2) has a nonnegative boundary equilibrium $E_1(x_1', x_2', 0, 0)$, where

$$\begin{aligned}
x_1' &= \frac{r[rr_1 - d_2(r_1 + d_1)]}{a(r_1 + d_1)^2},\\
x_2' &= \frac{rr_1 - d_2(r_1 + d_1)}{a(r_1 + d_1)}.
\end{aligned} \tag{4}$$

If $(H_1) a_2r_2 > d_4(r_2 + d_3), rr_1 - d_2(r_1 + d_1)/a_1d_2 > d_4(r_2 + d_3)/a_2r_2h$, model (2) has a positive equilibrium $E^+(x_1^+, x_2^+, y_1^+, y_2^+)$, where

$$\begin{aligned}
x_1^+ &= \frac{r}{r_1 + d_1 + a_1h/1 + mh^2} x_2^+,\\
x_2^+ &= \frac{1}{a}\left(\frac{rr_1}{r_1 + d_1 + a_1h/1 + mh^2} - d_2\right),\\
y_1^+ &= \frac{d_4}{r_2} h x_1^* +\\
y_2^+ &= h x_1^+,\\
h &= \sqrt{\frac{a_2r_2 - d_4(r_2 + d_3)}{md_4(r_2 + d_3)}}.
\end{aligned} \tag{5}$$

The characteristic equation of model (2) at $E_1(x_1', x_2', 0, 0)$ takes the following form:

$$\begin{aligned}
&\left[\lambda^2 + (r_1 + d_1 + d_2 + 2ax_2')\lambda + rr_1 - d_2(r_1 + d_1)\right]\\
&\quad \cdot \left[\lambda^2 + g_1\lambda + g_0 + h_0e^{-\lambda\tau}\right] = 0,
\end{aligned} \tag{6}$$

where $g_1 = r_2 + d_3 + d_4, g_0 = d_4(r_2 + d_3), h_0 = -a_2r_2$. When $rr_1 > d_2(r_1 + d_1)$, all roots of equation,

$$\lambda^2 + (r_1 + d_1 + d_2 + 2ax_2^+)\lambda + rr_1 - d_2(r_1 + d_1) = 0, \tag{7}$$

are negative. Now, we consider the roots of the following equation. $\lambda^2 + g_1\lambda + g_0 + h_0e^{-\lambda\tau} = 0$. By calculating, we obtain

$$\begin{aligned}
g_1^2 - 2g_0 &= d_4^2 + (r_2 + d_3)^2 > 0, g_0^2 - h_0^2\\
&= d_4^2(r_2 + d_3)^2 - (a_2r_2)^2.
\end{aligned} \tag{8}$$

When $d_4(r_2 + d_3) > a_2r_2$, we get $g_0^2 - h_0^2 > 0$. Therefore, E_1 is locally stable for all $\tau > 0$. When $d_4(r_2 + d_3) < a_2r_2$, we get $g_0^2 - h_0^2 < 0$. Thus, E_1 is unstable.

The characteristic equation of model (2) at E^+ is of the form

$$\begin{aligned}
&\lambda^4 + P_3\lambda^3 + P_2\lambda^2 + P_1\lambda + P_0\\
&\quad + \left(Q_2\lambda^2 + Q_1\lambda + Q_0\right)e^{-\lambda\tau} = 0,
\end{aligned} \tag{9}$$

where $P_3 = r_1 + d_1 + a_1\alpha + d_2 + 2ax_2^+ + r_2 + d_3 + d_4, P_2 = d_4(r_2 + d_3) + (r_2 + d_3 + d_4) \quad (r_1 + d_1 + a_1\alpha + d_2 + 2ax_2^+) + (r_1 + d_1 + a_1\alpha)(d_2 + 2ax_2^+) - rr_1, \quad P_1 = d_4(r_2 + d_3)(r_1 + d_1 + a_1\alpha + d_2 + 2ax_2^+) + (r_2 + d_3 + d_4)[(r_1 + d_1 + a_1\alpha)(d_2 + 2ax_2^+) - rr_1], P_0 = d_4(r_2 + d_3)[(r_1 + d_1 + a_1\alpha)(d_2 + 2ax_2^+) - rr_1], Q_2 = -a_2r_2\beta, Q_1 = -a_2r_2\beta(r_1 + d_1 + d_2 + 2ax_2^+), \quad Q_0 = -a_2r_2\beta[(r_1 + d_1)(d_2 + 2ax_2^+) - rr_1], \ \alpha = 2mx_1^+(y_2^+)^3/[(x_1^+)^2 m(y_2^+)^2]^2, \beta = (x_1^+)^4/[(x_1^+)^2 + m(y_2^+)^2]^2$.

Let $\tau = 0$; then, (9) has the following form:

$$\lambda^4 + P_3\lambda^3 + (P_2 + Q_2)\lambda^2 + (P_1 + Q_1)\lambda + P_0 + Q_0 = 0. \tag{10}$$

Note that $P_3 > 0$. When

$$\begin{aligned}
&[(H_2)] P_3(P_2 + Q_2) > (P_1 + Q_1), (P_1 + Q_1)\\
&\quad \cdot [P_3(P_2 + Q_2) - (P_1 + Q_1)] > P_3^2(P_0 + Q_0) > 0,
\end{aligned} \tag{11}$$

then positive equilibrium E^+ is locally asymptotically stable.

Let (H_1) and (H_2) hold. If $i\omega (\omega > 0)$ is a solution of (9), by calculation, we can obtain

$$\omega^8 + f_3\omega^6 + f_2\omega^4 + f_1\omega^2 + f_0 = 0, \tag{12}$$

where

$$
\begin{aligned}
f_3 &= P_3^2 - 2P_2 = d_4^2 + (r_2 + d_3)^2 + (r_1 + d_1 + a_1\alpha)^2 + (d_2 + 2ax_2^+)^2 + 2rr_1 > 0,\\
f_2 &= P_2^2 + 2P_0 - 2P_1P_3 - Q_2^2 = \left[d_4^2(r_2 + d_3)^2 - (a_2r_2\beta)^2\right] + \left[(r_1 + d_1 + a_1\alpha)(d_2 + 2ax_2^+) - rr_1\right]^2\\
&\quad + \left[d_4^2 + (r_2 + d_3)^2\right]\left[(r_1 + d_1 + a_1\alpha)^2 + (d_2 + 2ax_2^+)^2 + 2rr_1\right] > 0,\\
f_1 &= P_1^2 - 2P_0P_2 + 2Q_0Q_2 - Q_1^2 = \left[d_4^2(r_2 + d_3)^2 - (a_2r_2\beta)^2\right]\left[(r_1 + d_1)^2 + (d_2 + 2ax_2^+)^2 + 2rr_1\right]\\
&\quad + \left[d_4^2 + (r_2 + d_3)^2\right]\left[(r_1 + d_1 + a_1\alpha)(d_2 + 2ax_2^+) - rr_1\right]^2\\
&\quad + d_4^2(r_2 + d_3)^2\left[2a_1\alpha(r_1 + d_1) + (a_1\alpha)^2\right] > 0,\\
f_0 &= P_0^2 - Q_0^2 = (P_0 + Q_0)(P_0 - Q_0),
\end{aligned}
\tag{13}
$$

when $P_0 > Q_0$, E^+ is locally asymptotically stable for all $\tau > 0$. When $P_0 < Q_0$, ω_0 is the positive root of (12); in this case, (9) has a pair of roots $\pm i\omega_0$. By (12), we obtain

$$
\tau_k = \frac{2k\pi}{\omega_0} + \frac{1}{\omega_0}\arccos\frac{\left(Q_2\omega_0^2 - Q_0\right)\left(\omega_0^4 - P_2\omega_0^2 + P_0\right) + Q_1\omega_0\left(P_3\omega_0^3 - P_1\omega_0\right)}{\left(Q_1\omega_0\right)^2 + \left(Q_2\omega_0^2 - Q_0\right)^2}, \quad k = 0, 1, 2, \ldots,. \tag{14}
$$

Therefore, E^+ remains stable for $\tau < \tau_0$.

Differentiating (9) with respect to τ, we obtain that

$$
\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{4\lambda^3 + 3P_3\lambda^2 + 2P_2\lambda + P_1}{-\lambda\left(\lambda^4 + P_3\lambda^3 + P_2\lambda^2 + P_1\lambda + P_0\right)} + \frac{2Q_2\lambda + Q_1}{\lambda\left(Q_2\lambda^2 + Q_1\lambda + Q_0\right)} - \frac{\tau}{\lambda}. \tag{15}
$$

Hence, we get

$$
\begin{aligned}
\operatorname{sgn}\left\{\frac{d(\mathrm{Re}\lambda)}{d\tau}\right\}_{\lambda=i\omega_0} &= \operatorname{sgn}\left\{\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega_0}\\
&= \operatorname{sgn}\left\{\frac{\left(3P_3\omega_0^2 - P_1\right)\left(P_3\omega_0^2 - P_1\right) + 2\left(2\omega_0^2 - P_2\right)\left(\omega_0^4 - P_2\omega_0^2 + P_0\right)}{\omega_0^2\left(P_1 - P_3\omega_0^2\right)^2 + \left(\omega_0^4 - P_2\omega_0^2 + P_0\right)^2}\right.\\
&\quad \left. + \frac{2Q_2\left(Q_0 - Q_2\omega_0^2\right) - Q_1^2}{\left(Q_3\omega_0^3 - Q_1\omega_0\right)^2 + \left(Q_2\omega_0^2 - q_0\right)^2}\right\}\\
&= \operatorname{sgn}\left\{\frac{4\omega_0^6 + 3f_3\omega_0^4 + 2f_2\omega_0^2 + f_1}{\left(Q_1\omega_0\right)^2 + \left(Q_2\omega_0^2 - Q_0\right)^2}\right\} > 0.
\end{aligned}
\tag{16}
$$

Therefore, as $\tau = \tau_0$, $\omega = \omega_0$, there is Hopf bifurcation. From above discussion, we have the following results.

Theorem 1. *For model (2) with (3), we have the following:*

(i) Let $rr_1 > d_2(r_1 + d_1)$; if $a_2r_2 < d_4(r_2 + d_3)$, then E_1 is locally asymptotically stable; if $a_2r_2 > d_4(r_2 + d_3)$, then E_1 is unstable.

(ii) Assume (H_1) and (H_2) hold; if $P_0 > Q_0$, then E^+ is locally asymptotically stable for all $\tau \geq 0$; if $P_0 < Q_0$,

then there exists a $\tau_0 > 0$, s.t., E^+ is locally asymptotically stable if $0 < \tau < \tau_0$ and unstable if $\tau > \tau_0$. When $\tau = \tau_0$, models (2) and (3) undergo Hopf bifurcation at E^+.

3. Global Stability

In this section, by using an iteration technique, we discuss the global stability of the nonnegative equilibria E_1 and E^+ of models (2) and (3), respectively.

Theorem 2. *Let*

$$[(H_3)] r r_1 > d_2(r_1 + d_1) + \frac{a_1 d_2}{2\sqrt{m}}, \quad a_2 r_2 < d_4(r_2 + d_3), \tag{17}$$

hold; then, the nonnegative boundary equilibrium E_1 of model (2) is globally stable.

Proof. It follows from the positive solution of model (2), and we can obtain

$$\begin{aligned} &\dot{x}_1(t) \leqslant r x_2(t) - (d_1 + r_1) x_1(t), \\ &\dot{x}_2(t) = r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \end{aligned} \tag{18}$$

By Lemma 2.2 of [5] and comparison, we have

$$\begin{aligned} &\limsup_{t\longrightarrow+\infty} x_1(t) \leq \frac{r[r r_1 - d_2(r_1 + d_1)]}{a(r_1 + d_1)^2}, \\ &\limsup_{t\longrightarrow+\infty} x_2(t) \leq \frac{r r_1 - d_2(r_1 + d_1)}{a(r_1 + d_1)}. \end{aligned} \tag{19}$$

Therefore, there is a positive number t_1, for sufficiently small positive number ε, such that as $t > t_1$, $x_1(t) \leq x_1' + \varepsilon$. Hence, for $t > t_1 + \tau$, we derive that

$$\begin{aligned} &\dot{y}_1(t) \leq \frac{a_2(x_1' + \varepsilon)^2 y_2(t-\tau)}{(x_1' + \varepsilon)^2 + m y_2^2(t-\tau)} - (r_2 + d_3) y_1(t), \\ &u\dot{y}_2(t) = r_2 y_1(t) - d_4 y_2(t). \end{aligned} \tag{20}$$

By Lemma 2.2 of [5] and comparison, we can obtain

$$\begin{aligned} &\lim_{t\longrightarrow+\infty} y_1(t) = 0, \\ &\lim_{t\longrightarrow+\infty} y_2(t) = 0. \end{aligned} \tag{21}$$

Therefore, there is a positive number $t_2.t_1$, such that if $t > t_2$, $y_2(t) < \varepsilon$.

For $t > t_2$, we derive from model (2) that

$$\begin{aligned} &\dot{x}_1(t) \geq r x_2(t) - (r_1 + d_1) x_1(t) - \frac{a_1}{2\sqrt{m}} x_1(t) \\ &\dot{x}_2(t) = r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \end{aligned} \tag{22}$$

By Lemma 2.2 of [5] and comparison, we have

$$\begin{aligned} &\lim\inf_{t\longrightarrow+\infty} x_1(t) \geq \frac{r}{a(r_1 + d_1 + a_1/2\sqrt{m})}\left[\frac{r r_1}{r_1 + d_1 + a_1/2\sqrt{m}} - d_2\right] := \underline{x}_1, \\ &\lim\inf_{t\longrightarrow+\infty} x_1(t) \geq \frac{1}{a}\left[\frac{r r_1}{r_1 + d_1 + a_1/2\sqrt{m}} - d_2\right]. \end{aligned} \tag{23}$$

By model (2), it follows that

$$\begin{aligned} &\dot{x}_1(t) \geq r x_2(t) - (r_1 + d_1) x_1(t) - \frac{a_1 \varepsilon}{\underline{x}_1} x_1(t), \\ &\dot{x}_2(t) = r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \end{aligned} \tag{24}$$

By Lemma 2.4 of [3] and comparison, we obtain that

$$\begin{aligned} &\liminf_{t\longrightarrow+\infty} x_1(t) \geq \frac{r[r r_1 - d_2(r_1 + d_1)]}{a(r_1 + d_1)^2}, \\ &\liminf_{t\longrightarrow+\infty} x_2(t) \geq \frac{r r_1 - d_2(r_1 + d_1)}{a(r_1 + d_1)}, \end{aligned} \tag{25}$$

which together with (19) and (21) yields

$$\lim_{t\longrightarrow+\infty} (x_1(t), x_2(t), y_1(t), y_2(t)) = (x_1', x_2', 0, 0). \tag{26}$$

Hence, the equilibrium $E_1(x_1', x_2', 0, 0)$ of model (2) is globally stable. □

Theorem 3. *Assume (H_1), (H_2), and $P_0 > Q_0$ hold; if*

$$\begin{aligned} &[(H_4)] \frac{r r_1 - d_2(r_1 + d_1)}{a_1 d_2} > \frac{1}{2\sqrt{m}}, a_2 r_2 (r_1 + d_1) \\ &< a_1 d_4 (r_2 + d_3) h, \end{aligned} \tag{27}$$

then the positive equilibrium $E^+(x_1^+, x_2^+, y_1^+, y_2^+)$ of model (2) is global stability.

Proof. Let

$$\begin{aligned} U_{x_i} &= \limsup_{t\longrightarrow+\infty} x_i(t),\\ L_{x_i} &= \liminf_{t\longrightarrow+\infty} x_i(t),\\ U_{y_i} &= \limsup_{t\longrightarrow+\infty} y_i(t),\\ L_{y_i} &= \liminf_{t\longrightarrow+\infty} y_i(t), \quad (i=1,2). \end{aligned} \tag{28}$$

By the first two equations of model (2), we can obtain that

$$\begin{aligned} \dot{x}_1(t) &\le r x_2(t) - (d_1 + r_1)x_1(t),\\ \dot{x}_2(t) &= r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \end{aligned} \tag{29}$$

By Lemma 2.2 of [5] and comparison, we have

$$\begin{aligned} U_{x_1} &= \limsup_{t\longrightarrow+\infty} x_1(t) \le \frac{r[rr_1 - d_2(r_1+d_1)]}{a(r_1+d_1)^2} := M_1^{x_1},\\ U_{x_2} &= \limsup_{t\longrightarrow+\infty} x_2(t) \le \frac{rr_1 - d_2(r_1+d_1)}{a(r_1+d_1)} := M_1^{x_2}. \end{aligned} \tag{30}$$

So, for sufficiently small positive number ε, there exists a positive number t_1, such that if $t > t_1$, then $x_1(t) \le M_1^{x_1} + \varepsilon$.

For $t > t_1 + \tau$, by the last two equations of model (2), we get

$$\begin{aligned} \dot{y}_1(t) &\le \frac{a_2(M_1^{x_1}+\varepsilon)^2 y_2(t-\tau)}{(M_1^{x_1}+\varepsilon)^2 + m y_2^2(t-\tau)}\\ &\quad - (r_2+d_3)y_1(t).\dot{x}_2(t) = r_2 x_1(t) - d_4 x_2(t). \end{aligned} \tag{31}$$

By Lemma 2.2 of [5] and comparison, we obtain

$$\begin{aligned} U_{y_1} &= \limsup_{t\longrightarrow+\infty} y_1(t) \le \frac{d_4}{r_2} h M_1^{x_1} := M_1^{y_1},\\ U_{y_2} &= \limsup_{t\longrightarrow+\infty} y_2(t) = h M_1^{x_1} := M_1^{y_2}. \end{aligned} \tag{32}$$

Hence, $U_{y_1} \le M_1^{y_1}$, $U_{y_2} \le M_1^{y_2}$, in which

$$\begin{aligned} M_1^{y_1} &= \frac{a_2 r_2 - d_4(r_2+d_3)}{m r_2 (r_2+d_3)} M_1^{x_1},\\ M_1^{y_2} &= \frac{a_2 r_2 - d_4(r_2+d_3)}{m d_4 (r_2+d_3)} M_1^{x_1}. \end{aligned} \tag{33}$$

Therefore, for sufficiently small positive number ε, there is $t_2 \ge t_1 + \tau$, such that if $t > t_2$, $y_2(t) \le M_1^{y_2} + \varepsilon$.

For $t > t_2$, by the first two equations of model (2), we have

$$\begin{aligned} \dot{x}_1(t) &\ge r x_2(t) - (r_1+d_1)x_1(t) - \frac{a_1}{2\sqrt{m}} x_1(t),\\ \dot{x}_2(t) &= r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \end{aligned} \tag{34}$$

By Lemma 2.4 of [3] and comparison, we derive that

$$\begin{aligned} L_{x_1} &= \liminf_{t\longrightarrow\infty} x_1(t) \ge \frac{r[rr_1 - d_2(r_1+d_1+a_1/2\sqrt{m})]}{a(r_1+d_1+a_1/2\sqrt{m})^2} := N_1^{x_1},\\ L_{x_2} &= \liminf_{t\longrightarrow\infty} x_2(t) \ge \frac{rr_1 - d_2(r_1+d_1+a_1/2\sqrt{m})}{a(r_1+d_1+a_1/2\sqrt{m})} := N_1^{x_2}. \end{aligned} \tag{35}$$

Hence, for sufficiently small positive number ε, there is $t_3 \ge t_2$, such that if $t > t_3$, $x_1(t) \ge N_1^{x_1} - \varepsilon$.

For $t > t_3 + \tau$, it follows from the last two equations of model (2) that

$$\begin{aligned} \dot{y}_1(t) &\ge \frac{a_2(N_1^{x_1}-\varepsilon)^2 y_2(t-\tau)}{(N_1^{x_1}-\varepsilon)^2 + m y_2^2(t-\tau)}\\ &\quad - (d_3+r_2)y_1(t).\dot{x}_2(t) = r_2 x_1(t) - d_4 x_2(t). \end{aligned} \tag{36}$$

By Lemma 2.4 of [3] and comparison, we can obtain

$$\begin{aligned} L_{y_1} &= \liminf_{t\longrightarrow+\infty} y_1(t) \le \frac{d_4}{r_2} h N_1^{x_1} := N_1^{y_1},\\ L_{y_2} &= \limsup_{t\longrightarrow+\infty} y_2(t) = h N_1^{x_1} := N_1^{y_2}. \end{aligned} \tag{37}$$

Therefore, for sufficiently small positive number ε, there is a positive number $t_4 \ge t_3 + \tau$, such that if $t > t_4$, $y_2(t) \ge N_1^{y_2} - \varepsilon$. In this case, by the first two equations of model (2), we have

$$\begin{aligned} \dot{x}_1(t) &\le r x_2(t) - (d_1+r_1)x_1(t)\\ &\quad - \frac{a_1(N_1^{x_1}-\varepsilon)(N_1^{y_2}-\varepsilon)}{(M_1^{x_1}+\varepsilon)^2 + m(M_1^{y_2}+\varepsilon)^2} x_1(t),\\ \dot{x}_2(t) &= r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \end{aligned} \tag{38}$$

For sufficiently small positive number ε, if (H_4) holds, by Lemma 2.2 of [5] and a comparison argument, we can obtain

$$\begin{aligned} U_{x_1} &= \limsup_{t\longrightarrow+\infty} x_1(t) \le \frac{r\left[rr_1 - d_2\left(r_1+d_1+a_1 N_1^{x_1} N_1^{y_2}/(M_1^{x_1})^2 + m(M_1^{y_2})^2\right)\right]}{a\left(r_1+d_1+a_1 N_1^{x_1} N_1^{y_2}/(M_1^{x_1})^2 + m(M_1^{y_2})^2\right)^2} := M_2^{x_1},\\ U_{x_2} &= \limsup_{t\longrightarrow+\infty} x_2(t) \le \frac{rr_1 - d_2\left(r_1+d_1+a_1 N_1^{x_1} N_1^{y_2}/(M_1^{x_1})^2 + m(M_1^{y_2})^2\right)}{a\left(r_1+d_1+a_1 N_1^{x_1} N_1^{y_2}/(M_1^{x_1})^2 + m(M_1^{y_2})^2\right)} := M_2^{x_2}. \end{aligned} \tag{39}$$

Therefore, for sufficiently small positive number ε, there is $t_5 \geq t_4$, such that if $t > t_5$, $x_1(t) \leq M_2^{x_1} + \varepsilon$.

From the last two equations of model (2), we obtain that for $t > t_5 + \tau$,

$$\begin{aligned} \dot{y}_1(t) &\leq \frac{a_2 (M_2^{x_1} + \varepsilon)^2 y_2(t-\tau)}{(M_2^{x_1} + \varepsilon)^2 + m y_2^2(t-\tau)} - (d_3 + r_2) y_1(t), \\ \dot{x}_2(t) &= r_2 x_1(t) - d_4 x_2(t). \end{aligned} \tag{40}$$

By Lemma 2.2 of [5] and comparison, if $a_2 r_2 > d_4(r_2 + d_3)$ holds, we have

$$\begin{aligned} U_{y_1} &= \limsup_{t \longrightarrow +\infty} y_1(t) \leq \frac{d_4}{r_2} M_2^{x_1} := M_2^{y_1}, \\ U_{y_2} &= \limsup_{t \longrightarrow +\infty} y_2(t) \leq h M_2^{x_1} := M_2^{y_2}. \end{aligned} \tag{41}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_6 \geq T_5 + \tau$, such that if $t > T_6$, $y_2(t) \leq M_2^{y_2} + \varepsilon$.

Again, for sufficiently small positive number ε and $t > t_6$, by the first two equations of model (2), we have

$$\begin{aligned} \dot{x}_1(t) &\geq r x_2(t) - (d_1 + r_1) x_1(t) \\ &\quad - \frac{a_1 (M_2^{x_1} + \varepsilon)(M_2^{y_2} + \varepsilon)}{(N_1^{x_1} - \varepsilon)^2 + m (N_1^{y_2} - \varepsilon)^2} x_1(t), \\ \dot{x}_2(t) &= r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \end{aligned} \tag{42}$$

By Lemma 2.4 of [3] and comparison, if (H_4) holds, we can obtain

$$\begin{aligned} L_{x_1} &= \liminf_{t \longrightarrow +\infty} x_1(t) \geq \frac{r\left[r r_1 - d_2\left(r_1 + d_1 + a_1 M_2^{x_1} M_2^{y_2} / (N_1^{x_1})^2 + m (N_1^{y_2})^2\right)\right]}{a\left(r_1 + d_1 + a_1 M_2^{x_1} M_2^{y_2} / (N_1^{x_1})^2 + m (N_1^{y_2})^2\right)^2} := N_2^{x_1}, \\ L_{x_2} &= \liminf_{t \longrightarrow +\infty} x_2(t) \geq \frac{r r_1 - d_2\left(r_1 + d_1 + a_1 M_2^{x_1} M_2^{y_2} / (N_1^{x_1})^2 + m (N_1^{y_2})^2\right)}{a\left(r_1 + d_1 + a_1 M_2^{x_1} M_2^{y_2} / (N_1^{x_1})^2 + m (N_1^{y_2})^2\right)} := N_2^{x_2}. \end{aligned} \tag{43}$$

So, there is a positive number $t_7 \geq t_6$, for $t > t_7$, $x_1(t) \geq N_2^{x_1} - \varepsilon$.

For sufficiently small positive number ε and $t >_7 + \tau$, from the last two equations of model (2), we can derive

$$\begin{aligned} \dot{y}_1(t) &\geq \frac{a_2 (N_2^{x_1} - \varepsilon)^2 y_2(t-\tau)}{(N_2^{x_1} - \varepsilon)^2 + m y_2^2(t-\tau)} - (d_3 + r_2) y_1(t), \\ \dot{x}_2(t) &= r_2 x_1(t) - d_4 x_2(t). \end{aligned} \tag{44}$$

By Lemma 2.4 of [3] and comparison, if $a_2 r_2 > d_4(d_3 + r_2)$, we have

$$\begin{aligned} U_{y_1} &= \limsup_{t \longrightarrow +\infty} y_1(t) \geq \frac{d_4}{r_2} N_2^{x_1} := N_2^{y_1}, \\ U_{y_2} &= \limsup_{t \longrightarrow +\infty} y_2(t) \geq h N_2^{x_1} := N_2^{y_2}. \end{aligned} \tag{45}$$

Repeat the above process; for $n \geq 2$, we can obtain eight sequences:

$$M_n^{x_1}, M_n^{x_2}, M_n^{y_1}, M_n^{y_2}, N_n^{x_1}, N_n^{x_2}, N_n^{y_1}, N_n^{y_2} \ (n = 1, 2,), \tag{46}$$

in which

$$\begin{aligned} M_n^{x_1} &= \frac{r}{r_1 + d_1 + a_1 N_{n-1}^{x_1} N_{n-1}^{y_2} / (M_{n-1}^{x_1})^2 + m (M_{n-1}^{y_2})^2} M_n^{x_2}, \\ M_n^{x_2} &= \frac{r r_1 - d_2\left(r_1 + d_1 + a_1 N_{n-1}^{x_1} N_{n-1}^{y_2} / (M_{n-1}^{x_1})^2 + m (M_{n-1}^{y_2})^2\right)}{a\left(r_1 + d_1 + a_1 N_{n-1}^{x_1} N_{n-1}^{y_2} / (M_{n-1}^{x_1})^2 + m (M_{n-1}^{y_2})^2\right)}, \\ M_n^{y_1} &= \frac{d_4}{r_2} h M_n^{x_1}, \\ M_n^{y_2} &= h M_n^{x_1}, \\ N_n^{x_1} &= \frac{r}{r_1 + d_1 + a_1 N_{n-1}^{x_1} N_{n-1}^{y_2} / (M_{n-1}^{x_1})^2 + m (M_{n-1}^{y_2})^2} N_n^{x_2}, \\ N_n^{x_2} &= \frac{r r_1 - d_2\left(r_1 + d_1 + a_1 M_n^{x_1} M_n^{y_2} / (N_{n-1}^{x_1})^2 + m (N_{n-1}^{y_2})^2\right)}{a\left(r_1 + d_1 + a_1 M_n^{x_1} M_n^{y_2} / (N_{n-1}^{x_1})^2 + m (N_{n-1}^{y_2})^2\right)}, \\ N_n^{y_1} &= \frac{d_4}{r_2} h N_n^{x_1}, \\ N_n^{y_2} &= h N_n^{x_1}. \end{aligned} \tag{47}$$

It is noted that

$$N_n^{x_i} \leq L_{x_i} \leq U_{x_i} \leq M_n^{x_i}, N_n^{y_i} \leq L_{y_i} \leq U_{y_i} \leq M_n^{y_i}, \ (i = 1, 2). \tag{48}$$

Direct calculation, we have $M_n^{x_i}$ and $M_n^{y_i}$ as nonincreasing, and $N_n^{x_i}$ and $N_n^{y_i}$ as nondecreasing. Therefore, the limits of sequences in $M_n^{x_i}$, $M_n^{y_i}$, $N_n^{x_i}$, and $N_n^{y_i}$ exist. Let

$$\begin{aligned} &\lim t \longrightarrow +\infty M_n^{x_i} = \overline{x}_i, \\ &\lim t \longrightarrow +\infty N_n^{x_i} = \underline{x}_i, \\ &\lim t \longrightarrow +\infty M_n^{y_i} = \overline{y}_i, \\ &\lim t \longrightarrow +\infty N_n^{y_i} = \underline{y}_i, \ (i = 1, 2). \end{aligned} \tag{49}$$

We have

$$\overline{x}_1 = \frac{r}{r_1 + d_1 + a_1\underline{x}_1\underline{y}_2/(\overline{x}_1)^2 + m(\overline{y}_2)^2}\overline{x}_2,$$

$$\overline{x}_2 = \frac{rr_1 - d_2\left(r_1 + d_1 + a_1\underline{x}_1\underline{y}_2/(\overline{x}_1)^2 + m(\overline{y}_2)^2\right)}{a\left(r_1 + d_1 + a_1\underline{x}_1\underline{y}_2/(\overline{x}_1)^2 + m(\overline{y}_2)^2\right)},$$

$$\overline{y}_1 = \frac{d_4}{r_2}h\overline{x}_1,$$

$$\overline{y}_2 = h\overline{x}_1,$$

$$\underline{x}_1 = \frac{r}{r_1 + d_1 + a_1\overline{x}_1\overline{y}_2/(\underline{x}_1)^2 + m(\underline{y}_2)^2}\underline{x}_2,$$

$$\underline{x}_2 = \frac{rr_1 - d_2\left(r_1 + d_1 + a_1\overline{x}_1\overline{y}_2/(\underline{x}_1)^2 + m(\underline{y}_2)^2\right)}{a\left(r_1 + d_1 + a_1\overline{x}_1\overline{y}_2/(\underline{x}_1)^2 + m(\underline{y}_2)^2\right)}, \tag{50}$$

$$\underline{y}_1 = \frac{d_4}{r_2}h\underline{x}_1,$$

$$\underline{y}_2 = h\underline{x}_1.$$

Now, we prove that $\overline{x}_i = \underline{x}_i, \overline{y}_i = \underline{y}_i$, $(i = 1, 2)$. By (50), we can obtain

$$\begin{aligned} a\left[(r_1 + d_1)\left(1 + mh^2\right)(\overline{x}_1)^2 + a_1h(\underline{x}_1)^2\right] &= r\left(1 + mh^2\right)\left[rr_1 - d_2(r_1 + d_1)\right](\overline{x}_1)^3 \\ &\quad - rd_2a_1h\left(1 + mh^2\right)(\overline{x}_1)(\underline{x}_1)^2, \\ a\left[(r_1 + d_1)\left(1 + mh^2\right)(\underline{x}_1)^2 + a_1h(\overline{x}_1)^2\right] &= r\left(1 + mh^2\right)\left[rr_1 - d_2(r_1 + d_1)\right](\underline{x}_1)^3 \\ &\quad - rd_2a_1h\left(1 + mh^2\right)(\underline{x}_1)(\overline{x}_1)^2. \end{aligned} \tag{51}$$

From above two equations, we have

$$\begin{aligned} &a\left[(r_1 + d_1)^2\left(1 + mh^2\right)^2 - (a_1h)^2\right]\left[(\underline{x}_1)^2 + (\overline{x}_1)^2\right](\overline{x}_1 + \underline{x}_1)(\overline{x}_1 - \underline{x}_1) \\ &= \left[\left(1 + mh^2\right)(rr_1 - d_2(r_1 + d_1))\left((\overline{x}_1)^2 + \overline{x}_1\underline{x}_1 + (\underline{x}_1)^2\right) + rd_2a_1h\left(1 + mh^2\right)\overline{x}_1\underline{x}_1\right](\overline{x}_1 - \underline{x}_1). \end{aligned} \tag{52}$$

If $\overline{x}_1 \neq \underline{x}_1$, then we obtain

$$\begin{aligned} &a\left[(r_1 + d_1)^2\left(1 + mh^2\right)^2 - (a_1h)^2\right]\left[(\underline{x}_1)^2 + (\overline{x}_1)^2\right](\overline{x}_1 + \underline{x}_1) \\ &= \left(1 + mh^2\right)\left[rr_1 - d_2(r_1 + d_1)\right]\left[(\overline{x}_1)^2 + \overline{x}_1\underline{x}_1 + (\underline{x}_1)^2\right] + rd_2a_1h\left(1 + mh^2\right)\overline{x}_1\underline{x}_1 \end{aligned}. \tag{53}$$

Since $rr_1 > d_2(r_1 + d_1), \overline{x}_1 > 0, \underline{x}_1 > 0$, therefore, $(r_1 + d_1)(1 + mh^2) > a_1h$. This is a contradiction. So, $\overline{x}_1 = \underline{x}_1$. By (50), we have $\overline{x}_2 = \underline{x}_2\overline{y}_1 = \underline{y}_1$ and $\overline{y}_2 = \underline{y}_2$. Therefore, the positive equilibrium E^+ is globally stable.

4. Discussion

In this study, we have studied a ratio-dependent predator-prey model with stage structure for the prey and predator. A time delay due to the gestation of the predator is considered. By using the eigenvalue theory, we have obtained the sufficient conditions for the local stability of the nonnegative equilibria of model (2). The existence of Hopf bifurcation is given. By the iteration technique and comparison arguments, sufficient conditions have been established for the global stability of the nonnegative equilibria. From Theorem 2, we know that if (H_3) holds, the predator population will go to extinction. By Theorem 3, we learn that if (H_1) and (H_4) hold, then both the predator and prey species of model (2) are permanent [10, 11].

Data Availability

The [DATA TYPE] data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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