Research Article

Stability Analysis of a Ratio-Dependent Predator-Prey Model

Pei Yao,1 Zuocheng Wang,2 and Lingshu Wang3

1Shijiazhuang Information Engineering Vocational College, Hebei, Shijiazhuang, China
2The Architecture of Hebei University, Hebei, China
3Hebei University of Economics and Business, Hebei, China

Correspondence should be addressed to Lingshu Wang; wanglingshu@126.com

Received 13 January 2022; Accepted 5 February 2022; Published 17 March 2022

Academic Editor: Sun Young Cho

Copyright © 2022 Pei Yao et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this study, a ratio-dependent predator-prey model is investigated. The local stability and global stability of the nonnegative boundary equilibrium and positive equilibrium of the model are discussed, respectively. Sufficient condition is obtained for the existence of Hopf bifurcation at the positive equilibrium.

1. Introduction

Recently, the predator-prey models have been studied by many authors [1–8]. In general, a predator-prey model has the following forms:

\[
\begin{align*}
\dot{x} &= xf(x) - p(x)y, \\
\dot{y} &= kp(x)y - yg(y),
\end{align*}
\]

where \(x(t)\) and \(y(t)\) are the densities of the prey and predator population at time \(t\), respectively. The function \(f(x)\) represents the growth rate of predator population, and \(g(y)\) represents the functional response function of predator population to prey population. In [1], Xu et al. used the function \(p(x) = x^2/(x^2 + my^2)\) as the functional response function of predator population to prey population. The time delay due to the gestation of the predator is discussed in [1].

It is noted that in model (1), each individual’s prey admits the same risk to be attacked by predators and each individual predator admits the same ability to feed on prey. This assumption seems not to be realistic for many animals. In natural world, there are many species whose individuals pass through an immature stage. Stage structure is a natural phenomenon and represents, for example, the division of a population into immature and mature individuals. In the last two decades, stage-structured models have received great attention [3–7, 9].

Based on above discussion, we study the following predator-prey model:

\[
\begin{align*}
\dot{x}_1(t) &= rx_2(t) - (d_1 + r_1)x_1(t) - \frac{a_1x_1^2(t)y_2(t)}{x_1^2(t) + my_2^2(t)}, \\
\dot{x}_2(t) &= r_1x_1(t) - d_2x_2(t) - ax_2^2(t), \\
\dot{y}_1(t) &= \frac{a_2x_1^2(t - \tau)y_2(t - \tau)}{x_1^2(t - \tau) + my_2^2(t - \tau)} - (r_2 + d_3)y_1(t), \\
\dot{y}_2(t) &= r_2y_1(t) - d_4y_2(t),
\end{align*}
\]

where \(x_1(t)\) and \(x_2(t)\) are the densities of the immature and mature prey at time \(t\) and \(y_1(t)\) and \(y_2(t)\) are the densities of the immature and mature predators at time \(t\). In model (2), all parameters are positive constants. \(\tau \geq 0\) is the time delay due to the gestation of the predator. \(x^2/(x^2 + my^2)\) is the ratio-dependent functional response.

Model (2) is of the following initial conditions:
The characteristic equation of model (2) at $E_1(x'_1, x'_2, 0, 0)$ takes the following form:

$$[\lambda^2 + (r_1 + d_1 + d_2 + 2ax_1^2)\lambda + rr_1 - d_2(r_1 + d_1)]$$

$$\cdot [\lambda^2 + g_1\lambda + g_0 + h_0e^{-\lambda\tau}] = 0,$$

where $g_1 = r_2 + d_3 + d_4, g_0 = d_4(r_2 + d_3), h_0 = -a_2r_2$. When $rr_1 > d_2(r_1 + d_1)$, all roots of equation,

$$\lambda^2 + (r_1 + d_1 + d_2 + 2ax_1^2)\lambda + rr_1 - d_2(r_1 + d_1) = 0,$$

are negative. Now, consider the roots of the following equation. $\lambda^2 + g_1\lambda + g_0 + h_0e^{-\lambda\tau} = 0$. By calculating, we obtain

$$g_1^2 - 2g_0 = d_4^2 + (r_2 + d_3)^2 > 0, g_0^2 - h_0^2$$

$$= d_4^2(r_2 + d_3)^2 - (a_2r_2)^2.$$
\( f_3 = P_3^2 - 2P_2 = d_4^2 + (r_1 + d_1)^2 + (r_4 + d_4 + a_1a)^2 + (d_2 + 2ax^*_2)^2 + 2rr_1 > 0, \)
\( f_2 = P_2^2 + 2P_0 - 2P_1P_3 - Q_2^2 = [d_4^2 (r_2 + d_3)^2 - (a_2r_3^2\beta)^2] + [(r_1 + d_1 + a_1a)(d_2 + 2ax^*_3) - rr_1]^2 \)
\( + [d_4^2 + (r_2 + d_3)^2] [(r_1 + d_1 + a_1a)^2 + (d_2 + 2ax^*_3)^2 + 2rr_1] > 0, \)
\( f_1 = P_1^2 - 2P_0P_2 + 2Q_0Q_2 - Q_1^2 = [d_4^2 (r_2 + d_3)^2 - (a_2r_3^2\beta)^2] [(r_1 + d_1)^2 + (d_2 + 2ax^*_2)^2 + 2rr_1] \)
\( + [d_4^2 + (r_2 + d_3)^2] [(r_1 + d_1 + a_1a)(d_2 + 2ax^*_2) - rr_1]^2 \)
\( + d_4^2 (r_2 + d_3)^2 [2a_1a(r_1 + d_1) + (a_1a)^2] > 0, \)
\( f_0 = P_0^2 - Q_0^2 = (P_0 + Q_0)(P_0 - Q_0), \)

\[ \text{when } P_0 > Q_0, \text{ } E^+ \text{ is locally asymptotically stable for all } \tau > 0. \text{ When } P_0 < Q_0, \omega_0 \text{ is the positive root of } (12); \text{ in this case, } (9) \text{ has a pair of roots } \pm i\omega_0. \text{ By } (12), \text{ we obtain} \]
\[
\tau_k = \frac{2k\pi}{\omega_0} + \frac{1}{\omega_0} \arccos \frac{(Q_2\omega_0^2 - Q_0)(\omega_0^4 - P_3\omega_0^2 + P_0) + Q_1\omega_0(Q_3\omega_0^3 - P_1\omega_0)}{(Q_4\omega_0^2 - Q_0)^2 + (Q_4\omega_0^2 - Q_0)^2}, \quad k = 0, 1, 2, \ldots.
\]

Therefore, \( E^+ \) remains stable for \( \tau < \tau_0. \) Differentiating (9) with respect to \( \tau, \) we obtain that
\[ \left( \frac{d\lambda}{d\tau} \right)^{-1} = -\frac{4\lambda^3 + 3P_3\lambda^2 + 2P_2\lambda + P_1}{-\lambda(\lambda^4 + P_3\lambda^3 + P_2\lambda^2 + P_1\lambda + P_0)} + \frac{2Q_2\lambda + Q_1}{\lambda Q_2\lambda^3 + Q_2\lambda + Q_0} - \frac{\tau}{\lambda} \]

Hence, we get
\[
\text{sgn} \left\{ \frac{d \text{ (Re}\lambda)}{d\tau} \right\}_{\lambda=\pm i\omega_0} = \text{sgn} \left\{ \text{ Re} \left( \frac{d\lambda}{d\tau} \right)^{-1} \right\}_{\lambda=\pm i\omega_0} = \text{sgn} \left\{ \frac{(3P_3\omega_0^2 - P_1)(P_3\omega_0^2 - P_0) + 2(2\omega_0^2 - P_3)(\omega_0^4 - P_2\omega_0^2 + P_0)}{\omega_0^2(P_1 - P_3\omega_0^2)^2 + (\omega_0^4 - P_2\omega_0^2 + P_0)^2} \right. \]
\[
+ \left. \frac{2Q_2(Q_3\omega_0^3 - Q_2\omega_0^3)}{(Q_2\omega_0^3 - Q_1\omega_0)^2 + (Q_2\omega_0^3 - Q_0)^2} \right\} > 0.
\]

Therefore, as \( \tau = \tau_0, \omega = \omega_0, \) there is Hopf bifurcation. From above discussion, we have the following results.

**Theorem 1.** For model (2) with (3), we have the following:

(i) Let \( rr_1 > d_1(r_1 + d_1); \text{ if } a_2r_2 < d_4(r_2 + d_4), \text{ then } E_1 \text{ is locally asymptotically stable}; \text{ if } a_2r_2 > d_4(r_2 + d_4), \text{ then } E_1 \text{ is unstable.} \)

(ii) Assume (H1) and (H2) hold; if \( P_0 > Q_0, \text{ then } E^+ \text{ is locally asymptotically stable for all } \tau \geq 0; \text{ if } P_0 < Q_0, \text{ then } E^+ \text{ is
then there exists a $\tau_0 > 0$, s.t., $E^+$ is locally asymptotically stable if $0 < \tau < \tau_0$ and unstable if $\tau > \tau_0$. When $\tau = \tau_0$, models (2) and (3) undergo Hopf bifurcation at $E^+$.

**3. Global Stability**

In this section, by using an iteration technique, we discuss the global stability of the nonnegative equilibria $E_1$ and $E^+$ of models (2) and (3), respectively.

**Theorem 2.** Let

$$[(H_2)] \frac{r r_1 + d_2 (r_1 + d_1)}{a r_2} < \frac{a_2 d_4}{2 \sqrt{m}},$$

$$a_2 r_2 < d_4 (r_2 + d_4), \quad (17)$$

hold, then, the nonnegative boundary equilibrium $E_1$ of model (2) is globally stable.

**Proof.** It follows from the positive solution of model (2), and we can obtain

$$\begin{align*}
\dot{x}_1(t) &\leq r x_2(t) - (d_1 + r_1) x_1(t), \\
\dot{x}_2(t) &= r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \\
\end{align*} \quad (18)$$

By Lemma 2.2 of [5] and comparison, we have

$$\begin{align*}
l \limsup_{t \to \infty} x_1(t) &\leq \frac{r (r r_1 - d_2 (r_1 + d_1))}{a (r_1 + d_1)^2}, \\
l \limsup_{t \to \infty} x_2(t) &\leq \frac{r r_1 - d_2 (r_1 + d_1)}{a (r_1 + d_1)}, \quad (19)
\end{align*}$$

Therefore, there is a positive number $t_1$, for sufficiently small positive number $\varepsilon$, such that as $t > t_1$, $x_1(t) \leq x_1^* + \varepsilon$.

Hence, for $t > t_1 + \tau$, we derive that

$$\begin{align*}
y_1(t) &\leq \frac{a_2 (x_1^* + \varepsilon)^2 y_2(t - \tau)}{(x_1^* + \varepsilon)^2 + m y_1^2(t - \tau)} - (r_2 + d_3) y_1(t), \\
y_2(t) &= r_2 y_1(t) - d_4 y_2(t). \quad (20)
\end{align*}$$

By Lemma 2.2 of [5] and comparison, we can obtain

$$\begin{align*}
l \lim_{t \to \infty} y_1(t) &= 0, \\
l \lim_{t \to \infty} y_2(t) &= 0. \quad (21)
\end{align*}$$

Therefore, there is a positive number $t_2$, such that if $t > t_2$, $y(t) < \varepsilon$.

For $t > t_2$, we derive from model (2) that

$$\dot{x}_1(t) \geq r x_2(t) - (r_1 + d_1) x_1(t) - \frac{a_2}{2 \sqrt{m}} x_1(t)$$

$$\dot{x}_2(t) = r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \quad (22)$$

By Lemma 2.2 of [5] and comparison, we have

$$\begin{align*}
l \liminf_{t \to \infty} x_1(t) &\geq \frac{r}{a (r_1 + d_1 + 2 \sqrt{m})} \left[ \frac{r r_1}{r_1 + d_1 + a_2/2 \sqrt{m}} - d_2 \right], \\
l \liminf_{t \to \infty} x_2(t) &\geq \frac{r r_1}{a (r_1 + d_1 + 2 \sqrt{m})} - d_2. \quad (23)
\end{align*}$$

By model (2), it follows that

$$\begin{align*}
\dot{x}_1(t) &\geq r x_2(t) - (r_1 + d_1) x_1(t) - \frac{a_4 \varepsilon}{x_1^*} x_1(t), \\
\dot{x}_2(t) &= r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \quad (24)
\end{align*}$$

By Lemma 2.4 of [3] and comparison, we obtain that

$$\begin{align*}
l \liminf_{t \to \infty} x_1(t) &\geq \frac{r (r r_1 - d_2 (r_1 + d_1))}{a (r_1 + d_1)^2}, \\
l \liminf_{t \to \infty} x_2(t) &\geq \frac{r r_1 - d_2 (r_1 + d_1)}{a (r_1 + d_1)}, \\
\end{align*} \quad (25)$$

which together with (19) and (21) yields

$$\begin{align*}
l \liminf_{t \to \infty} (x_1(t), x_2(t), y_1(t), y_2(t)) = (x_1^*, x_2^*, 0, 0). \quad (26)
\end{align*}$$

Hence, the equilibrium $E_1(x_1^*, x_2^*, 0, 0)$ of model (2) is globally stable.

**Theorem 3.** Assume $(H_1)$, $(H_2)$, and $P_0 > Q_0$ hold; if

$$[(H_4)] \frac{r r_1 - d_2 (r_1 + d_1)}{a_4 d_2} > \frac{1}{2 \sqrt{m}} a_2 r_2 (r_1 + d_1)$$

$$< a_4 d_4 (r_2 + d_4) h, \quad (27)$$

then the positive equilibrium $E^+(x_1^*, x_2^*, y_1^*, y_2^*)$ of model (2) is global stability.

**Proof.** Let
By the first two equations of model (2), we can obtain that
\[
\dot{x}_1(t) \leq r x_2(t) - (d_1 + r_1) x_1(t), \\
\dot{x}_2(t) = r_1 x_1(t) - d_2 x_2(t) - a x^2(t). 
\] (29)

By Lemma 2.2 of [5] and comparison, we have
\[
U_{x_1} = \limsup_{t \to \infty} x_1(t) \leq \limsup_{t \to \infty} \frac{r [r_1 - d_1 (r_1 + d_1)]}{a (r_1 + d_1)} := M_{x_1}, \\
U_{x_2} = \limsup_{t \to \infty} x_2(t) \leq \limsup_{t \to \infty} \frac{r r_1 - d_2 (r_1 + d_1)}{a (r_1 + d_1)} := M_{x_2}. 
\] (30)

So, for sufficiently small positive number \( \varepsilon \), there exists a positive number \( t_1 \), such that if \( t > t_1 \), then \( x_1(t) \leq M_{x_1} + \varepsilon \).

For \( t > t_1 + \tau \), by the last two equations of model (2), we get
\[
\dot{y}_1(t) \leq \frac{a_1 (M_{x_1} + \varepsilon)^2 + a_1 (M_{x_2} + \varepsilon)^2 + a_3 (N_{1}^{x_1} + \varepsilon)^2 + a_3 (N_{1}^{x_2} + \varepsilon)^2}{(N_{1}^{x_1} + \varepsilon)^2 + (N_{1}^{x_2} + \varepsilon)^2} y_2(t - \tau) \\
- (r_1 + r_2) y_1(t) \cdot \dot{x}_2(t) = r_2 x_1(t) - d_4 x_2(t). 
\] (31)

By Lemma 2.2 of [5] and comparison, we obtain
\[
U_{y_1} = \limsup_{t \to \infty} y_1(t) \leq \frac{d_4}{r_2} h M_{x_1} := M_{y_1}, \\
U_{y_2} = \limsup_{t \to \infty} y_2(t) = h M_{x_2} := M_{y_2}. 
\] (32)

Hence, \( U_{y_1} \leq M_{y_1}, U_{y_2} \leq M_{y_2}, \) in which
\[
M_{y_1} = \frac{a_1 r_2 - d_4 (r_1 + d_3) M_{x_2}}{m r_2 (r_1 + d_3)}, \\
M_{y_2} = \frac{a_1 r_2 - d_4 (r_1 + d_3) M_{x_2}}{m d_4 (r_2 + d_3)}. 
\] (33)

Therefore, for sufficiently small positive number \( \varepsilon \), there is \( t_2 \geq t_1 + \tau \), such that if \( t > t_2 \), \( y_2(t) \leq M_{y_2} + \varepsilon \).

For \( t > t_2 \), by the first two equations of model (2), we have
\[
\dot{x}_1(t) \geq r x_2(t) - (r_1 + d_1) x_1(t) - \frac{a_1}{2 \sqrt{m}} x_{1}(t), \\
\dot{x}_2(t) = r_1 x_1(t) - d_2 x_2(t) - a x^2(t). 
\] (34)

By Lemma 2.4 of [3] and comparison, we derive that
\[
L_{x_1} = \liminf_{t \to \infty} x_1(t) \geq \frac{r r_1 - d_2 (r_1 + d_1 + a/2 \sqrt{m})}{a (r_1 + d_1 + a/2 \sqrt{m})^2} = N_{1}^{x_1}, \\
L_{x_2} = \liminf_{t \to \infty} x_2(t) \geq \frac{r r_1 - d_2 (r_1 + d_1 + a/2 \sqrt{m})}{a (r_1 + d_1 + a/2 \sqrt{m})} = N_{1}^{x_2}. 
\] (35)

Hence, for sufficiently small positive number \( \varepsilon \), there is \( t_3 \geq t_2 + \tau \), such that if \( t > t_3 \), \( x_1(t) \geq N_{1}^{x_1} - \varepsilon \).

For \( t > t_3 + \tau \), it follows from the last two equations of model (2) that
\[
\dot{y}_1(t) \geq \frac{a_3 (N_{1}^{x_1} - \varepsilon)^2 y_2(t - \tau)}{(N_{1}^{x_1} - \varepsilon)^2 + (N_{1}^{x_2} - \varepsilon)^2} \\
- (r_1 + r_2) y_1(t) \cdot \dot{x}_2(t) = r_2 x_1(t) - d_4 x_2(t). 
\] (36)

By Lemma 2.4 of [3] and comparison, we can obtain
\[
L_{y_1} = \liminf_{t \to \infty} y_1(t) \leq \frac{d_4}{r_2} h N_{1}^{x_1} := N_{1}^{y_1}, \\
L_{y_2} = \limsup_{t \to \infty} y_2(t) = h N_{1}^{x_2} := N_{1}^{y_2}. 
\] (37)

Therefore, for sufficiently small positive number \( \varepsilon \), there is a positive number \( t_4 \geq t_3 + \tau \), such that if \( t > t_4 \), \( y_2(t) \geq N_{1}^{y_2} - \varepsilon \). In this case, by the first two equations of model (2), we have
\[
\dot{x}_1(t) \leq r x_2(t) - (d_1 + r_1) x_1(t) \\
- \frac{a_1 (N_{1}^{x_1} - \varepsilon) (N_{1}^{y_2} - \varepsilon)}{(N_{1}^{x_1} - \varepsilon)^2 + (N_{1}^{y_2} - \varepsilon)^2} x_1(t), \\
\dot{x}_2(t) = r_1 x_1(t) - d_2 x_2(t) - a x^2(t). 
\] (38)

For sufficiently small positive number \( \varepsilon \), if \( (H_i) \) holds, by Lemma 2.2 of [5] and a comparison argument, we can obtain
\[
U_{x_1} = \limsup_{t \to \infty} x_1(t) \leq \limsup_{t \to \infty} \frac{r r_1 - d_2 (r_1 + d_1 + a_1 N_{1}^{x_1} N_{1}^{y_2}/(M_{x_1}^{y_2})^2 + m (M_{x_2}^{y_2})^2)}{a (r_1 + d_1 + a_1 N_{1}^{x_1} N_{1}^{y_2}/(M_{x_1}^{y_2})^2 + m (M_{x_2}^{y_2})^2)} := M_{x_1}^{y_2}, \\
U_{x_2} = \limsup_{t \to \infty} x_2(t) \leq \limsup_{t \to \infty} \frac{r r_1 - d_2 (r_1 + d_1 + a_1 N_{1}^{x_1} N_{1}^{y_2}/(M_{x_1}^{y_2})^2 + m (M_{x_2}^{y_2})^2)}{a (r_1 + d_1 + a_1 N_{1}^{x_1} N_{1}^{y_2}/(M_{x_1}^{y_2})^2 + m (M_{x_2}^{y_2})^2)} := M_{x_2}^{y_2}. 
\] (39)
Therefore, for sufficiently small positive number $\varepsilon$, there is $t_5 \geq t_4$, such that if $t > t_5$, $x_1(t) \leq M_2^2 + \varepsilon$.

From the last two equations of model (2), we obtain that for $t > t_5 + \tau$,

$$y_1(t) \leq c \left( \frac{a_2 (M_2^2 + \varepsilon)^2}{(M_2^2 + \varepsilon)^2 + m y_2^2(t - \tau)} - (d_3 + r_2) y_1(t) \right),$$

$$\hat{x}_1(t) = r_2 x_1(t) - d_4 x_2(t).$$

By Lemma 2.2 of [5] and comparison, if $a_2 r_2 > d_4 (r_2 + d_3)$ holds, we have

$$U_{y_1} = \limsup_{t \to +\infty} y_1(t) \leq \frac{c}{r_2} M_2^2 = M_2^2,$$

$$U_{y_2} = \limsup_{t \to +\infty} y_2(t) \leq h M_2^2 = M_2^2.$$  \hfill (40)

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_6 \geq T_5 + \tau$, such that if $t > T_6$, $y_2(t) \leq M_2^2 + \varepsilon$.

Again, for sufficiently small positive number $\varepsilon$ and $t > t_6$, by the first two equations of model (2), we have

$$\hat{x}_1(t) \geq r x_1(t) - (d_1 + r_1) x_1(t)$$

$$- \frac{a_1 (M_1^2 + \varepsilon)^2 (M_1^2 + \varepsilon)}{(N_1^2 - \varepsilon)^2 + m (N_1^2 - \varepsilon)^2} x_1(t),$$

$$\hat{x}_2(t) = r_1 x_1(t) - d_2 x_2(t) - ax_2^2(t).$$

By Lemma 2.4 of [3] and comparison, if $(H_4)$ holds, we can obtain

$$L_{x_1} = \liminf_{t \to +\infty} x_1(t) \geq \frac{r r_1 - d_4 (r_1 + d_1 + a_1 M_2^2 M_2^2 / (N_1^2)^2 + m N_1^2)}{a (r_1 + d_1 + a_1 M_2^2 M_2^2 / (N_1^2)^2 + m N_1^2)} = N_2^2,$$

$$L_{x_2} = \liminf_{t \to +\infty} x_2(t) \geq \frac{r r_1 - d_2 (r_1 + d_1 + a_1 M_2^2 M_2^2 / (N_1^2)^2 + m N_1^2)}{a (r_1 + d_1 + a_1 M_2^2 M_2^2 / (N_1^2)^2 + m N_1^2)} = N_2^2.$$  \hfill (41)

So, there is a positive number $t_7 \geq t_6$, for $t > t_7$, $x_1(t) \geq N_2^2 - \varepsilon$.

For sufficiently small positive number $\varepsilon$ and $t > t_7$, from the last two equations of model (2), we can derive

$$y_1(t) \geq \frac{c}{(N_2^2 - \varepsilon)^2 + m y_2^2(t - \tau)} \frac{r r_1 - d_4 (r_1 + d_1 + a_1 M_2^2 M_2^2 / (N_1^2)^2 + m N_1^2)}{a (r_1 + d_1 + a_1 M_2^2 M_2^2 / (N_1^2)^2 + m N_1^2)} - (d_3 + r_2) y_1(t),$$

$$\hat{x}_1(t) = r_2 x_1(t) - d_4 x_2(t).$$

By Lemma 2.4 of [3] and comparison, if $a_2 r_2 > d_4 (d_3 + r_2)$, we have

$$U_{y_1} = \limsup_{t \to +\infty} y_1(t) \geq \frac{c}{r_2} N_2^2 = N_2^2,$$

$$U_{y_2} = \limsup_{t \to +\infty} y_2(t) \geq h N_2^2 = N_2^2.$$  \hfill (42)

Repeat the above process; for $n \geq 2$, we can obtain eight sequences:

$$M_n^{x_i}, M_n^{x_2}, M_n^{x_3}, M_n^{x_4}, N_n^{x_1}, N_n^{x_2}, N_n^{x_3}, N_n^{x_4} (n = 1, 2, \ldots),$$

in which

$$M_n^{x_1} = \frac{r}{r_1 + d_1 + a_1 N_2^2 N_2^2 / (N_1^2 n)^2 + m N_2^2} M_n^{x_1},$$

$$M_n^{x_2} = \frac{r r_1 - d_2 (r_1 + d_1 + a_1 N_2^2 N_2^2 / (N_1^2 n)^2 + m N_2^2)}{a (r_1 + d_1 + a_1 N_2^2 N_2^2 / (N_1^2 n)^2 + m N_2^2)} M_n^{x_2},$$

$$M_n^{x_3} = \frac{r r_1 - d_3 (r_1 + d_1 + a_1 M_2^2 / (N_1^2 n)^2 + m N_1^2)}{a (r_1 + d_1 + a_1 M_2^2 / (N_1^2 n)^2 + m N_1^2)} M_n^{x_3},$$

$$M_n^{x_4} = \frac{r}{r_2} h N_n^{x_3}. \hfill (43)$$

$$M_n^{x_5} = \frac{d_2}{r_2} M_n^{x_4},$$

$$M_n^{x_6} = h N_n^{x_5},$$

$$N_n^{x_1} = \frac{r}{r_1 + d_1 + a_1 N_2^2 N_2^2 / (M_2^2 n)^2 + m (M_2^2 n)^2} N_n^{x_1},$$

$$N_n^{x_2} = \frac{r r_1 - d_2 (r_1 + d_1 + a_1 M_2^2 M_2^2 / (M_2^2 n)^2 + m (M_2^2 n)^2)}{a (r_1 + d_1 + a_1 M_2^2 M_2^2 / (M_2^2 n)^2 + m (M_2^2 n)^2)} N_n^{x_2},$$

$$N_n^{x_3} = \frac{d_2}{r_2} h N_n^{x_5},$$

$$N_n^{x_4} = h N_n^{x_3}. \hfill (44)$$

It is noted that

$$N_2^1 \leq L_{x_i} \leq U_{x_i} \leq M_2^1, N_2^i \leq L_{y_i} \leq U_{y_i} \leq M_2^i, (i = 1, 2).$$

Direct calculation, we have $M_n^{x_i}$ and $M_n^{y_i}$ as nonincreasing, and $N_n^{x_i}$ and $N_n^{y_i}$ as nondecreasing. Therefore, the limits of sequences in $M_n^{x_i}, M_n^{y_i}, N_n^{x_i}$, and $N_n^{y_i}$ exist. Let

$$\lim \longrightarrow + \infty M_n^{x_i} = \bar{x}_i,$$

$$\lim \longrightarrow + \infty N_n^{x_i} = \bar{y}_i,$$

$$\lim \longrightarrow + \infty M_n^{y_i} = \bar{y}_i,$$

$$\lim \longrightarrow + \infty N_n^{y_i} = \bar{y}_i, (i = 1, 2).$$  \hfill (45)
We have
\[ \bar{x}_1 = \frac{r}{r_1 + d_1 + \alpha_1 x_1 y_1/(\bar{x}_1)^2 + m(y_2)^2} \bar{x}_2, \]
\[ \bar{x}_2 = \frac{r r_1 - d_2 (r_1 + d_1 + \alpha_1 x_1 y_1/(\bar{x}_1)^2 + m(y_2)^2)}{a (r_1 + d_1 + \alpha_1 x_1 y_1/(\bar{x}_1)^2 + m(y_2)^2)} \bar{x}_2. \]
\[ y_1 = \frac{d_1}{r_2 h \bar{x}_1}, \]
\[ y_2 = h \bar{x}_1, \]
\[ \bar{x}_1 = \frac{r}{r_1 + d_1 + \alpha_1 \bar{x}_1 y_1/(\bar{x}_1)^2 + m(y_2)^2} \bar{x}_2, \]
\[ \bar{x}_2 = \frac{r r_1 - d_2 (r_1 + d_1 + \alpha_1 \bar{x}_1 y_2/(\bar{x}_1)^2 + m(y_2)^2)}{a (r_1 + d_1 + \alpha_1 \bar{x}_1 y_2/(\bar{x}_1)^2 + m(y_2)^2)} \bar{x}_2. \] (50)

Now, we prove that \( \bar{x}_1, \bar{x}_2, y_1, y_2 \) (i = 1, 2). By (50), we can obtain
\[ a \left[ (r_1 + d_1) (1 + mh^2) (\bar{x}_1)^2 + a_1 h (\bar{x}_1) \right] = r \left( 1 + mh^2 \right) \left( r r_1 - d_2 (r_1 + d_1) \right) (\bar{x}_1)^3 \]
\[ - r d_2 a_1 h (1 + mh^2) (\bar{x}_1)^2, \]
\[ a \left[ (r_1 + d_1) (1 + mh^2) (\bar{x}_1)^2 + a_1 h (\bar{x}_1) \right] = r \left( 1 + mh^2 \right) \left( r r_1 - d_2 (r_1 + d_1) \right) (\bar{x}_1)^3 \]
\[ - r d_2 a_1 h (1 + mh^2) (\bar{x}_1)^2. \] (51)

From above two equations, we have
\[ a \left[ (r_1 + d_1)^2 (1 + mh^2)^2 - (a_1 h)^2 \right] \left[ (\bar{x}_1)^2 + (\bar{x}_2)^2 \right] (\bar{x}_1 + \bar{x}_2) (\bar{x}_1 - \bar{x}_2) \]
\[ = \left[ (1 + mh^2) (r r_1 - d_2 (r_1 + d_1)) (\bar{x}_1)^2 + a_1 h (1 + mh^2) \bar{x}_1 \bar{x}_2 \right] (\bar{x}_1 - \bar{x}_2). \] (52)

If \( \bar{x}_1 \neq \bar{x}_2 \), then we obtain
\[ a \left[ (r_1 + d_1)^2 (1 + mh^2)^2 - (a_1 h)^2 \right] \left[ (\bar{x}_1)^2 + (\bar{x}_2)^2 \right] (\bar{x}_1 + \bar{x}_2)
\[ = \left[ (1 + mh^2) (r r_1 - d_2 (r_1 + d_1)) (\bar{x}_1)^2 + a_1 h (1 + mh^2) \bar{x}_1 \bar{x}_2 \right] + r d_2 a_1 h (1 + mh^2) \bar{x}_1 \bar{x}_2. \] (53)

Since \( r r_1 > d_2 (r_1 + d_1), x_1 > 0, x_2 > 0 \), therefore, \( r_1 + d_1 (1 + mh^2) > a_1 h \). This is a contradiction. So, \( \bar{x}_1 = \bar{x}_2 \). By (50), we have \( \bar{x}_1 = \bar{x}_2, y_1 = y_2, \) and \( y_2 = y_2 \). Therefore, the positive equilibrium \( E^* \) is globally stable.

4. Discussion

In this study, we have studied a ratio-dependent predator-prey model with stage structure for the prey and predator. A time delay due to the gestation of the predator is considered. By using the eigenvalue theory, we have obtained the sufficient conditions for the local stability of the nonnegative equilibria of model (2). The existence of Hopf bifurcation is given. By the iteration technique and comparison arguments, sufficient conditions have been established for the global stability of the nonnegative equilibria. From Theorem 2, we know that if \( H_3 \) holds, the predator population will go to extinction. By Theorem 3, we learn that if \( H_1 \) and \( H_4 \) hold, then both the predator and prey species of model (2) are permanent [10, 11].

Data Availability

The [DATA TYPE] data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.
Acknowledgments

This work was supported by the National Natural Science Foundation of China (12105073), the Natural Science Foundation of Hebei Province (A2019207070), and the Scientific Research Foundation of Hebei University of Economics and Business(2021ZD07).

References


