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Retraction

Retracted: Statistical Inference and Mathematical Properties of Burr X Logistic-Exponential Distribution with Applications to Engineering Data

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] M. M. Al Sobhi, "Statistical Inference and Mathematical Properties of Burr X Logistic-Exponential Distribution with Applications to Engineering Data," *Journal of Mathematics*, vol. 2022, Article ID 4688871, 21 pages, 2022.

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Research Article

Statistical Inference and Mathematical Properties of Burr X Logistic-Exponential Distribution with Applications to Engineering Data

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The Burr X logistic-exponential distribution is introduced in this study as a novel logistic-exponential distribution extension that may be utilized to efficiently describe engineering data. There are J-shape, symmetrical, left-skewed, reversed-J shape, and right-skewed densities available, as well as decreasing, rising, bathtub, unimodal, J-shape, and reversed-J shape hazard rates. The fundamental mathematical features of the proposed model were obtained. The new model's parameters were estimated using seven different approaches, including maximum likelihood, Anderson–Darling, maximum product of spacing, least-squares, Cramér–von Mises, percentiles, and weighted least squares. To evaluate the performance of the recommended estimation methods, a full simulation study was carried out. Finally, the adaptability of the provided distribution was tested using two real datasets from engineering science, revealing that the new model can yield a close match when compared to competing models.

1. Introduction

Survival and reliability analysis is an important field of statistics with several applications in engineering, actuarial science, biomedical investigations, demography, and dependability. In these applicable disciplines, several writers have constructed generalized distributions to model various data. The exponential distribution is a popular data modeling model because it is analytically tractable and has a low memory need. However, due to its declining probability density function (PDF) and constant hazard rate function, its use was limited (HRF). As a result, various academics have developed generalized variants of the exponential distribution in order to improve its capacity to represent data in applicable domains and increase its flexibility in terms of PDF and HRF. Some important generlizations of the exponential (E) distribution are proposed as follows: the exponentiated-E [1], beta-exponential [2], logoistic-E [15], beta generalized-E [3], Nadarajah–Haghighi [4], transmuted generalized-E [5], Harris extended-E [6], Marshall–Olkin Nadarajah–Haghighi [7], transmuted Topp–Leone E [8], alpha power-E [9], Marshall–Olkin logistic E [10], extended-E [11], odd inverse power generalized Weibull-E [12, 13] studied the heavy-tailed E and Type I half logistic Burr E [14] distributions.

The logistic-E (LE) distribution [15] is a significant generalization of the E distribution. We suggested a more flexible variant of the LE model termed the Burr X logistic-E (BXLE) distribution in this study. The BXLE allows for greater flexibility and application when modeling engineering data. The BXLE distribution was constructed using the Burr X family (Yousof et al. [16]).

The Burr X family's cumulative distribution function (CDF) looks like this:

$$F(x;\kappa,\varphi)=\left\{1-e^{-[G(x;\varphi)/1-(G(x;\varphi))]^{2}}\right\}^{\kappa},\quad \kappa>0,\ x\in\Re, \tag{1}$$

where $G(x;\varphi)$ is the baseline model CDF and φ is the baseline parameter vector. The Burr X family's PDF shrinks to

$$f(x;\kappa,\varphi)=\frac{2\kappa g(x;\varphi)G(x;\varphi)}{[1-G(x;\varphi)]^{3}}e^{-\left[G(x;\varphi)/1-(G(x;\varphi))^{2}\right]}\left\{1-e^{-\left[G(x;\varphi)^{2\delta}/1-(G(x;\varphi))^{2}\right]}\right\}^{\kappa-1},\quad \kappa>0,\ x\in\Re. \tag{2}$$

Some qualities can inspire the suggested BXLE distribution, for example, the BXLE model includes the Burr X E distribution as a special submodel; the BXLE distribution provides J-shape, symmetrical, left-skewed, reversed-J shape, and right-skewed densities, as well as decreasing, increasing, bathtub, then unimodal, J-shape, and reversed-J shape hazard rates; its PDF and CDF have simple closed forms and can thus be used effectively in analyzing censored data, and it has also been employed to model engineering datasets, where it outperforms other competing distributions in terms of fit.

The key goal of this article is to investigate and deduce some of the basic distributional features of a new extension of the LE model based on the Burr X family. We are also interested in investigating the estimation of BXLE parameters using seven classical estimation methods, including maximum likelihood estimators (MLEs), Anderson–Darling estimators (ADEs), maximum product of spacing estimators (MPSEs), least-squares estimators (LSEs), Cramér–von Mises estimators (CVMEs), percentile estimators (PCEs), and weighted least-squares estimators (WLSEs). Extensive simulations were used to examine and analyze the performance of the suggested estimate approaches.

The CDF of the BXLE model is obtained by substituting the CDF of the LE model in (1), as follows:

$$F(x)=\left[1-e^{-\left(e^{\rho x}-1\right)^{2\delta}}\right]^{\kappa},\quad \rho,\delta,\kappa>0,\ x>0. \tag{3}$$

The corresponding PDF of the BXLE distribution follows, by inserting the PDF and CDF of the LE model in (2), as

$$f(x)=2\delta\kappa\rho e^{\rho x-\left(e^{\rho x}-1\right)^{2\delta}}\left(e^{\rho x}-1\right)^{2\delta-1}\left[1-e^{-\left(e^{\rho x}-1\right)^{2\delta}}\right]^{\kappa-1},\quad \rho,\delta,\kappa>0,\ x>0. \tag{4}$$

The BXLE distribution's survival function (SF) and HRF take the required forms:

$$S(x)=1-\left[1-e^{-\left(e^{\rho x}-1\right)^{2\delta}}\right]^{\kappa},$$
$$h(x)=\frac{2\delta\kappa\rho e^{\rho x}\left(e^{\rho x}-1\right)^{2\delta-1}\left[1-e^{-\left(e^{\rho x}-1\right)^{2\delta}}\right]^{\kappa}}{\left[-e^{\left(e^{\rho x}-1\right)^{2\delta}}+1\right]\left\{\left[1-e^{-\left(e^{\rho x}-1\right)^{2\delta}}\right]^{\kappa}-1\right\}}. \tag{5}$$

Figures 1 and 2 provide plots of the PDF and HRF of the BXLE distribution, respectively. The BXLE model produces J-shape, symmetrical, left-skewed, reversed-J shape, and right-skewed densities, as well as decreasing, increasing, bathtub, then unimodal, J-shape, and reversed-J shape hazard rates.

The remainder of this work is structured as follows. Section 2 determined some basic distributional features of the BXLE distribution. In Section 3, the BXLE parameters were calculated using seven different approaches. Section 4 investigated the performance of these estimators using numerical simulations. In Section 5, two engineering actual datasets were evaluated to demonstrate the relevance and adaptability of the BLLE distribution. Finally, Section 6 gave the conclusions.

2. Mathematical Properties

In this section, we will introduce some important statistical properties such as linear representation, quantile function, moments, and order statistics.

2.1. Linear Representation. An expansion for (3) can be derived using the power series

$$(1-z)^{b}=\sum_{j=0}^{\infty}(-1)^{j}\binom{b}{j}z^{j}, \tag{6}$$

where $|Z|<1$ and $b>0$.

Then, the BXLE CDF could be written as

$$F(x)=\sum_{j=0}^{\infty}(-1)^{j}\binom{\kappa}{j}e^{-j\left(e^{\rho x}-1\right)^{2\delta}}. \tag{7}$$

Applying the E series,

$$F(x)=\sum_{j,k=0}^{\infty}\frac{(-1)^{j+k}j^{k}}{k!}\binom{\kappa}{j}e^{2\delta k\rho x}\left(1-e^{-\rho x}\right)^{2\delta k}, \tag{8}$$

and then

$$F(x)=\sum_{j,k,m=0}^{\infty}\frac{(-1)^{j+k+m}j^{k}}{k!}\binom{\kappa}{j}\binom{2\delta k}{m}e^{-\rho(m-2\delta k)x}. \tag{9}$$

By differentiating the previous equation, we have

$$f(x)=\sum_{k,m=0}^{\infty}\phi_{k,m}g_{\rho(m-2\delta k)}(x), \tag{10}$$

where g_A is the PDF of E model with scale parameter $\rho(m-2\delta k)$ and $\phi_{k,m}=\sum_{j=0}^{\infty}\left((-1)^{j+k+m+1}j^{k}/k!\right)\binom{\kappa}{j}\binom{2\delta k}{m}$.

2.2. Quantile Function. Obtaining the inverse CDF yields the quantile function (QF) of the BXLE distribution (3) as

$$Q(p)=\frac{\log\left\{\left[-\log\left(1-p^{(1/\kappa)}\right)\right]^{(1/2\delta)}+1\right\}}{\rho},\quad 0<p<1. \tag{11}$$

The QF may be used to generate random data from the BXLE distribution:

$$x_i=\frac{\log\left\{\left[-\log\left(1-p_i^{(1/\kappa)}\right)\right]^{1/2\delta}+1\right\}}{\rho},\quad i=1,2,\ldots,n, \tag{12}$$

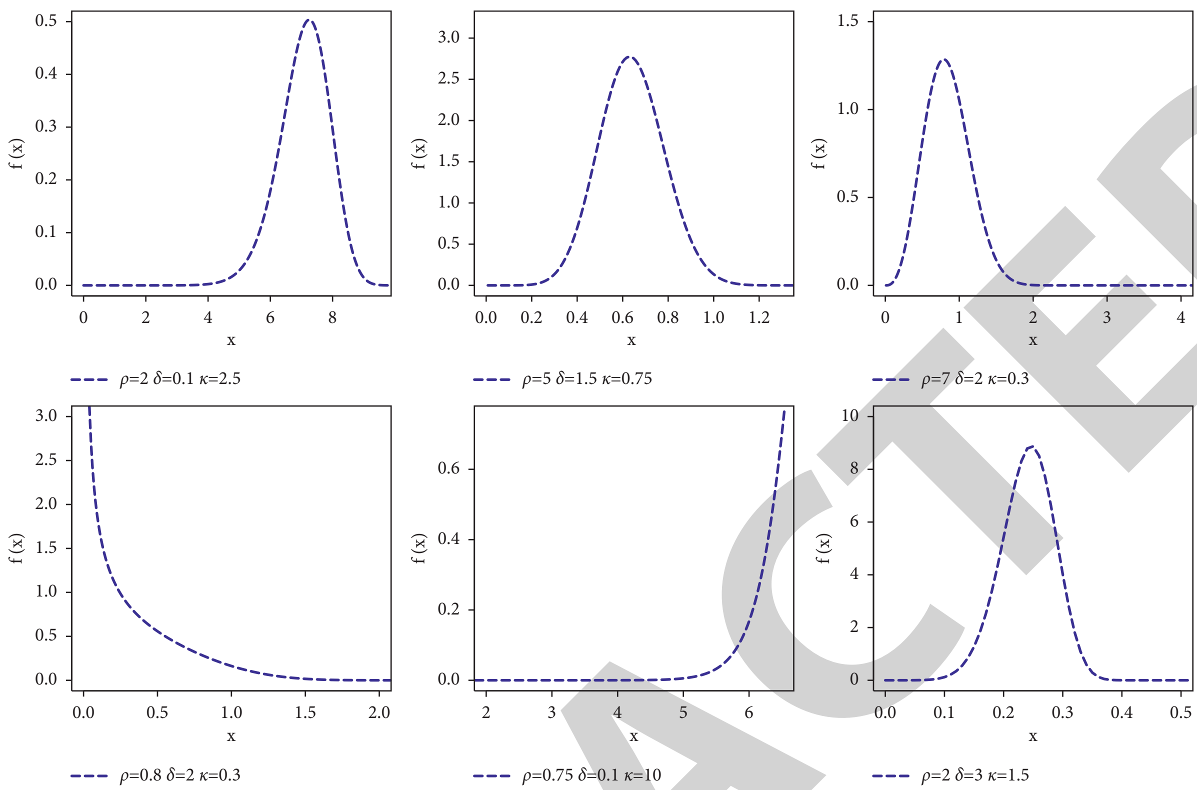

Figure 1: PDF plots of BXLE distribution.

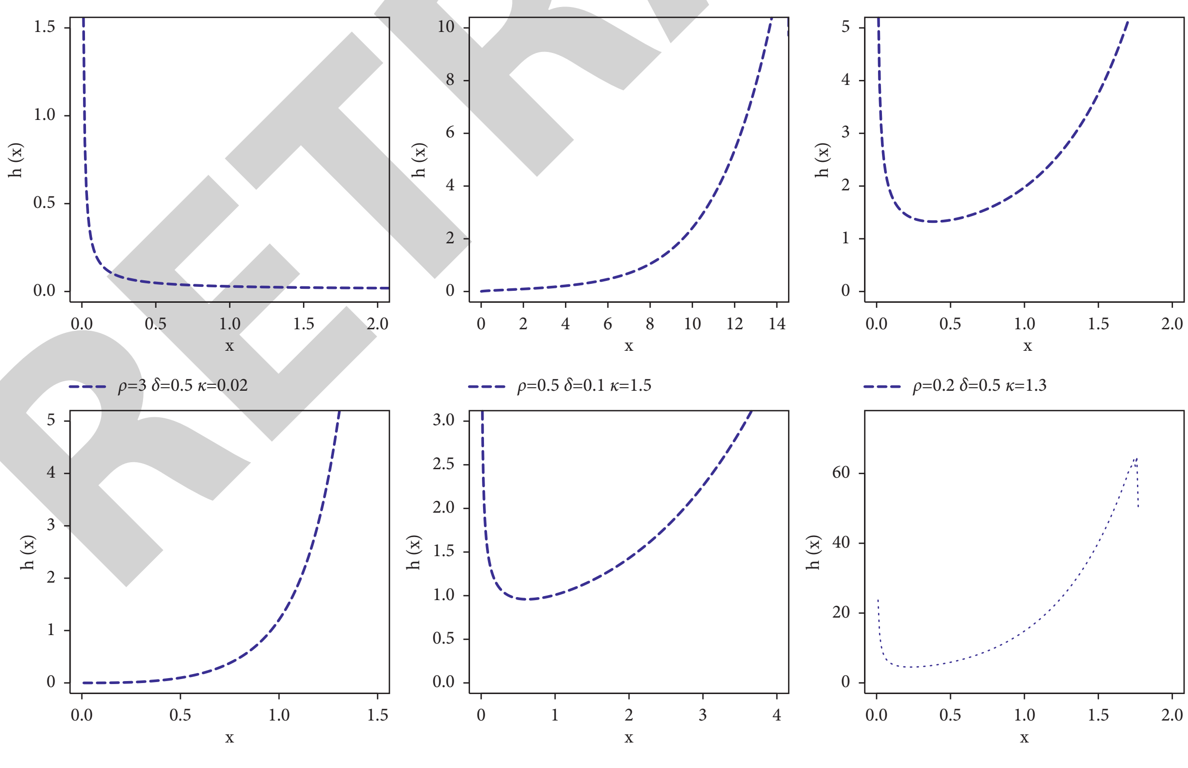

Figure 2: HRF plots of BXLE distribution.

where $p \in (0,1)$ follows the uniform distribution.

2.3. Moments. The rth moment of the BXLE distribution has the form

$$\begin{aligned}\mu_r' = E(X^r) &= \int_0^\infty x^r f(x)\mathrm{d}x \\ &= \sum_{k,m=0}^{\infty} \phi_{k,m}[\rho(m-2\delta k)]^{-r}\Gamma(r+1).\end{aligned} \tag{13}$$

One can obtain the first four original moments of the BXLE distribution by setting $r = 1, 2, 3$, and 4 in the last formula.

The BXLE distribution's moment generating function has the following form:

$$M(t) = \sum_{k,m=0}^{\infty} \phi_{k,m}\frac{\rho(m-2\delta k)}{\rho(m-2\delta k) - t}. \tag{14}$$

Characteristic function of the BXLE distribution follows from the last formula by replacing t with it.

2.4. Order Statistics. The PDF and CDF of the BXLE distribution's ith order statistic (OS) are provided by

$$\begin{aligned}f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x) = \frac{2\delta\kappa\rho n! e^{\rho x}\left(e^{\rho x}-1\right)^{2\delta-1}\left[1-e^{-(e^{\rho x}-1)^{2\delta}}\right]^{i\kappa}\left\{1-\left[1-e^{-(e^{\rho x}-1)^{2\delta}}\right]^{\kappa}\right\}^{n-i}}{\Gamma(i)\Gamma(-i+n+1)\left[e^{(e^{\rho x}-1)^{2\delta}}-1\right]}, \\ F_{i:n}(x) &= \sum_{r=i}^{n}\binom{n}{r}(F(x))^r(1-F(x))^{n-r} = \frac{\Gamma(n+1)\left[1-e^{-(e^{\rho x}-1)^{2\delta}}\right]^{i\kappa}\left\{1-\left[1-e^{-(e^{\rho x}-1)^{2\delta}}\right]^{\kappa}\right\}^{n-i}\Omega}{\Gamma(i+1)\Gamma(-i+n+1)},\end{aligned} \tag{15}$$

where $\Omega = {}_2F_1[1, i-n; i+1; 1 + ((1/(1-e^{-(-(1+e^{x\rho}))^{2\delta}})^{\kappa} - 1))]$ is a hyper-geometric function.

$$\begin{aligned}f_{i:n}(x) &= \frac{f(x)}{B(i,n-i+1)}\sum_{h=0}^{\infty}(-1)^h\binom{n-i}{h}F^{h+i-1}(x) \\ &= \frac{\sum_{k,m=0}^{\infty}\phi_{k,m}g_{\rho(m-2\delta k)}(x)}{B(i,n-i+1)} \\ &\sum_{h=0}^{\infty}(-1)^h\binom{n-i}{h}\left[1-e^{-(e^{\rho x}-1)^{2\delta}}\right]^{\kappa(h+i-1)},\end{aligned} \tag{16}$$

where $B(\cdot,\cdot)$ is beta function.

3. Different Estimation Methods

In this part, we will look at how to estimate the BXLE parameters using seven different approaches, including the MLE, ADE, MPSE, CVME, LSE, WLSE, and PCE.

3.1. Maximum Likelihood Method of Estimation. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the PDF (4); then, the log-likelihood function holds to

$$\begin{aligned}L &= (\kappa-1)\sum_{i=1}^{n}\log\left[1-e^{-(e^{\rho x_i}-1)^{2\delta}}\right] + \sum_{i=1}^{n}\left[\rho x_i - (e^{\rho x_i}-1)^{2\delta}\right] \\ &+ 2(\delta-1)\sum_{i=1}^{n}\log(e^{\rho x_i}-1) + n\log(2\delta\kappa\rho).\end{aligned} \tag{17}$$

By differentiating (17) with respect to ρ, δ, and κ, respectively, and equating to 0, then

$$\begin{aligned}\frac{\partial L}{\partial \rho} &= (\kappa-1)\sum_{i=1}^{n}\frac{2\delta x_i e^{\rho x_i-(e^{\rho x_i}-1)^{2\delta}}(e^{\rho x_i}-1)^{2\delta-1}}{1-e^{-(e^{\rho x_i}-1)^{2\delta}}} \\ &+ (2\delta-1)\sum_{i=1}^{n}\frac{x_i e^{\rho x_i}}{e^{\rho x_i}-1} + \sum_{i=1}^{n}\left[x_i - 2\delta x_i e^{\rho x_i}(e^{\rho x_i}-1)^{2\delta-1}\right] + \frac{n}{\rho}, \\ \frac{\partial L}{\partial \delta} &= (\kappa-1)\sum_{i=1}^{n}\frac{2e^{-(e^{\rho x_i}-1)^{2\delta}}(e^{\rho x_i}-1)^{2\delta}\log(e^{\rho x_i}-1)}{1-e^{-(e^{\rho x_i}-1)^{2\delta}}} \\ &+ \sum_{i=1}^{n}-2(e^{\rho x_i}-1)^{2\delta}\log(e^{\rho x_i}-1) + 2\sum_{i=1}^{n}\log(e^{\rho x_i}-1) + \frac{n}{\delta}, \\ \frac{\partial L}{\partial \kappa} &= \sum_{i=1}^{n}\log\left(1-e^{-(e^{\rho x_i}-1)^{2\delta}}\right) + \frac{n}{\kappa}.\end{aligned} \tag{18}$$

By solving the above equations, we derive MLEs of the BXLE parameters.

3.2. Ordinary and Weighted Least-Squares Methods of Estimations. Let $x_{1:n}, x_{2:n}, \ldots, x_{2:n}$ be the OS of a random sample of size n from the BXLE model. Hence, we have the OLSE of the BXLE parameters by minimizing the next equation:

$$O = \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\left\{\left[1 - e^{-(e^{\rho x_{i:n}}-1)^{2\delta}}\right]^{\kappa} - \frac{i}{n+1}\right\}^2. \tag{19}$$

The OLSE of the BXLE parameters may also be calculated by solving the nonlinear equations:

$$\sum_{i=1}^{n}\left\{\left[1 - e^{-(e^{\rho x_{i:n}}-1)^{2\delta}}\right]^{\kappa} - \frac{i}{n+1}\right\}\Delta_s(x_{i:n}) = 0, \quad s = 1,2,3, \tag{20}$$

where

$$\Delta_1(x_{i:n}) = \frac{\partial}{\partial\rho}F(x_{i:n}) = 2\delta\kappa x_{i:n}e^{\rho x_{i:n}-(e^{\rho x_{i:n}}-1)^{2\delta}}\left(e^{\rho x_{i:n}}-1\right)^{2\delta-1}\left[1 - e^{-(e^{\rho x_{i:n}}-1)^{2\delta}}\right]^{\kappa-1}, \tag{21}$$

$$\Delta_2(x_{i:n}) = \frac{\partial}{\partial\delta}F(x_{i:n}) = \frac{2\kappa\left(e^{\rho x_{i:n}}-1\right)^{2\delta}\log\left(e^{\rho x_{i:n}}-1\right)\left[1 - e^{-(e^{\rho x_{i:n}}-1)^{2\delta}}\right]^{\kappa}}{e^{(e^{\rho x_{i:n}}-1)^{2\delta}}-1}, \tag{22}$$

$$\Delta_3(x_{i:n}) = \frac{\partial}{\partial\kappa}F(x_{i:n}) = \left[1 - e^{-(e^{\rho x_{i:n}}-1)^{2\delta}}\right]^{\kappa}\log\left[1 - e^{-(e^{\rho x_{i:n}}-1)^{2\delta}}\right]. \tag{23}$$

The WLSE of the BXLE parameters can be calculated by minimizing the following equation:

$$W = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left\{\left[1 - e^{-(e^{\rho x_{i:n}}-1)^{2\delta}}\right]^{\kappa} - \frac{i}{n+1}\right\}^2. \tag{24}$$

Furthermore, the WLSE of the BXLE parameters can be obtained by solving the following nonlinear equations:

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}) - \frac{i}{n+1}\right]\Delta_s(x_{i:n}) = 0, \tag{25}$$

where $\Delta_s(x_{i:n})$, $s = 1,2,3$, were defined in (21), (22), and (23), respectively.

3.3. Anderson–Darling Estimation. The ADEs of the BXLE parameters are obtained by minimizing the following equation:

$$A = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(x_{i:n}) + \log S(x_{i:n})\right]. \tag{26}$$

The ADE can also be calculated by solving the following nonlinear equations:

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_s(x_{i:n})}{F(x_{i:n})} - \frac{\Delta_s(x_{n+1-i:n})}{S(x_{n+1-i:n})}\right] = 0, \tag{27}$$

where $\Delta_s(x_{i:n})$, $s = 1,2,3$, were defined in (21), (22), and (23), respectively.

3.4. Cramér–von Mises Estimators. The CVMEs of BXLE parameters are obtained by minimizing the following equation:

$$\begin{aligned}\mathrm{CV}\frac{1}{12n}\sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{2i-1}{2n}\right]^2 \\ = \frac{2}{12n} + \sum_{i=1}^{n}\left\{\left[1 - e^{-(e^{\rho x_{i:n}}-1)^{2\delta}}\right]^{\kappa} - \frac{2i-1}{2n}\right\}^2,\end{aligned} \tag{28}$$

or by solving the following nonlinear equations:

$$\sum_{i=1}^{n}\left\{\left[1 - e^{-(e^{\rho x_{i:n}}-1)^{2\delta}}\right]^{\kappa} - \frac{2i-1}{2n}\right\}\Delta_s(x_{i:n}) = 0, \tag{29}$$

where $\Delta_s(x_{i:n})$, $s = 1,2,3$, were defined in (21), (22), and (23), respectively.

3.5. Maximum Product of Spacing Method of Estimation. As an alternative to the ML approach, the maximum product of spacing (MPS) method is used to estimate the parameters of continuous univariate models. The uniform spacings of a random sample of size n drawn from the BXLE distribution may be defined as follows:

$$D_i = F(x_i) - F(x_{i-1}), \quad (30)$$

where D_i denotes the uniform spacings, $F(x_0) = 0$, $F(x_{n+1} = 1)$, and $\sum_{i=1}^{n+1} D_i = 1$. MPS estimators (MPSEs) of the BXLE parameters can be obtained by maximizing

$$G = \frac{1}{n+1}\sum_{i=1}^{n+1}\log(D_i), \quad (31)$$

with respect to ρ, δ, and κ. Further, the MPSE of the BXLE parameters can also be obtained by solving

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i}\left[\Delta_s(x_{i:n}) - \Delta_s(x_{i-1:n})\right] = 0, \quad (32)$$

where $\Delta_s(x_{i:n})$, $s = 1, 2, 3$, were defined in (21), (22), and (23), respectively.

3.6. Percentile Method of Estimation. If we consider $pi = (i/(n+1))$ to be an estimate of $F(xi\colon n)$, then the PCE of the BXLE parameters is derived by minimizing the following expression:

$$\begin{aligned} \mathrm{PCE} &= \sum_{i=1}^{n}\left[x_{i:n} - Q(p_i)\right]^2 \\ &= \sum_{i=1}^{n}\left(x_{i:n} - \frac{\log\left\{\left[-\log\left(1 - p_i^{1/\kappa}\right)\right]^{1/2\delta} + 1\right\}}{\rho}\right)^2. \end{aligned} \quad (33)$$

Alternatively, solve the associated nonlinear equations:

$$\sum_{i=1}^{n}\left(x_{i:n} - \frac{\log\left\{\left[-\log\left(1 - p_i^{1/\kappa}\right)\right]^{1/2\delta} + 1\right\}}{\rho}\right)\Phi_s(x_{i:n}) = 0, \quad s = 1, 2, 3, \quad (34)$$

where

$$\begin{aligned} \Phi_1(x_{i:n}) &= \frac{\partial}{\partial\rho}Q(p_i) = -\frac{\log\left\{\left[-\log\left(1 - p_i^{1/\kappa}\right)\right]^{1/2\delta} + 1\right\}}{\rho^2}, \\ \Phi_2(x_{i:n}) &= \frac{\partial}{\partial\rho}Q(p_i) = -\frac{\log\left[-\log\left(1 - p_i^{1/\kappa}\right)\right]}{2\delta^2\rho\left\{\left[-\log\left(1 - p_i^{1/\kappa}\right)\right]^{-(1/2\delta)} + 1\right\}}, \\ \Phi_3(x_{i:n}) &= \frac{\partial}{\partial\rho}Q(p_i) = -\frac{p_i^{1/\kappa}\log(p_i)\left[-\log\left(1 - p_i^{1/\kappa}\right)\right]^{(1/2\delta)-1}}{2\delta\kappa^2\rho\left(p_i^{1/\kappa} - 1\right)\left\{\left[-\log\left(1 - p_i^{1/\kappa}\right)\right]^{(1/2\delta)} + 1\right\}}. \end{aligned} \quad (35)$$

4. Numerical Outcomes

Based on comprehensive simulation findings, this section investigates the performance of the seven estimate approaches in estimating the BXLE parameters. We explore several sample sizes, $n = 20, 30, 50, 100, 200, 400$, as well as some parametric values for ρ, δ, and κ, $\rho = \{0.50, 0.75, 1.5, 2.0, 2.5, 3.0\}$, $\delta = \{0.50, 0.75, 1.5, 2.0, 3.0\}$, and $\kappa = \{0.25, 0.5, 0.75, 1.5, 2.0, 2.5, 3.0\}$. We generate $n = 2000$ random samples from the BXLE distribution using its QF and calculate the average values of the estimates (AVEs) with their associated average mean square errors (MSEs), average absolute biases (AVBs), and average mean relative estimates (MREs) for all sample sizes and parameter combinations using the R software.

The MSE, AVB, and MRE were calculated by the following equations:

$$\begin{aligned} \mathrm{MSE} &= \frac{1}{N}\sum_{i=1}^{N}(\hat{\eta} - \eta)^2, \\ \mathrm{AVB} &= \frac{1}{N}\sum_{i=1}^{N}|\hat{\eta} - \eta|, \\ \mathrm{MRE} &= \frac{1}{N}\sum_{i=1}^{\eta}\frac{|\hat{\eta} - \eta|}{\eta}, \end{aligned} \quad (36)$$

where $\eta = (\rho, \delta, \kappa)'$.

Table 1: Simulation results of the AVE, AVB, MSE, and MRE of BXLE distribution for ($\rho = 0.5, \delta = 0.25, \kappa = 0.75$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	WLSE
20	AVE	$\hat{\rho}$	0.75626	0.74294	0.77114	0.63616	0.66827	0.66066	0.68538
		$\hat{\delta}$	0.32870	0.32225	0.33404	0.28857	0.30368	0.29438	0.31013
		$\hat{\kappa}$	1.28915	1.10508	1.29370	1.17167	1.29074	1.29088	1.18169
	AVB	$\hat{\rho}$	0.57509	0.50104	0.56988	0.44217	0.48610	0.44857	0.48385
		$\hat{\delta}$	0.13429	0.12232	0.13450	0.10197	0.11572	0.10502	0.11975
		$\hat{\kappa}$	0.84827	0.68373	0.88523	0.73831	0.86113	0.88817	0.76063
	MSE	$\hat{\rho}$	0.61131	0.43585	0.55599	0.32051	0.37521	0.30429	0.38361
		$\hat{\delta}$	0.03684	0.02765	0.03335	0.01838	0.02384	0.01723	0.02682
		$\hat{\kappa}$	1.34536	0.97456	1.51227	1.16675	1.42122	1.76670	1.13282
	MRE	$\hat{\rho}$	1.15018	1.00208	1.13975	0.88434	0.97220	0.89714	0.96769
		$\hat{\delta}$	0.53715	0.48927	0.53801	0.40789	0.46288	0.42008	0.47901
		$\hat{\kappa}$	1.13103	0.91164	1.18031	0.98441	1.14818	1.18423	1.01418
30	AVE	$\hat{\rho}$	0.74980	0.73553	0.70742	0.64032	0.68392	0.67673	0.70238
		$\hat{\delta}$	0.32590	0.31460	0.31839	0.28646	0.30457	0.29386	0.30745
		$\hat{\kappa}$	1.11324	1.01445	1.23892	1.03511	1.17335	1.14738	1.05079
	AVB	$\hat{\rho}$	0.53103	0.47971	0.50014	0.40962	0.46876	0.44152	0.46056
		$\hat{\delta}$	0.12802	0.11374	0.11864	0.09400	0.10969	0.10188	0.10946
		$\hat{\kappa}$	0.66919	0.59122	0.81142	0.57234	0.74324	0.73838	0.62155
	MSE	$\hat{\rho}$	0.52477	0.38597	0.40381	0.30499	0.35316	0.28967	0.35777
		$\hat{\delta}$	0.03443	0.02367	0.02524	0.01729	0.02085	0.01574	0.02189
		$\hat{\kappa}$	0.79795	0.65169	1.28599	0.65605	1.08587	1.23344	0.75033
	MRE	$\hat{\rho}$	1.06207	0.95941	1.00028	0.81924	0.93752	0.88303	0.92112
		$\hat{\delta}$	0.51207	0.45496	0.47458	0.37600	0.43875	0.40753	0.43785
		$\hat{\kappa}$	0.89226	0.78830	1.08189	0.76313	0.99099	0.98451	0.82873
50	AVE	$\hat{\rho}$	0.69689	0.71560	0.75253	0.67167	0.67426	0.67517	0.67767
		$\hat{\delta}$	0.30795	0.31005	0.32236	0.29655	0.30238	0.29341	0.30187
		$\hat{\kappa}$	0.97495	0.87418	0.99148	0.87106	1.06047	0.97671	0.94581
	AVB	$\hat{\rho}$	0.43983	0.40735	0.47510	0.37968	0.43243	0.40228	0.39965
		$\hat{\delta}$	0.10706	0.10000	0.11235	0.09237	0.10150	0.09259	0.09683
		$\hat{\kappa}$	0.48884	0.42880	0.58480	0.42059	0.61858	0.54587	0.48790
	MSE	$\hat{\rho}$	0.39635	0.29597	0.36214	0.28129	0.29993	0.24958	0.27337
		$\hat{\delta}$	0.02674	0.01899	0.02254	0.01768	0.01836	0.01351	0.01783
		$\hat{\kappa}$	0.41368	0.33758	0.66812	0.33158	0.72395	0.67687	0.46245
	MRE	$\hat{\rho}$	0.87965	0.81470	0.95020	0.75936	0.86486	0.80455	0.79929
		$\hat{\delta}$	0.42826	0.40002	0.44939	0.36947	0.40598	0.37038	0.38731
		$\hat{\kappa}$	0.65178	0.57173	0.77974	0.56079	0.82477	0.72782	0.65053
100	AVE	$\hat{\rho}$	0.64897	0.66463	0.66279	0.61251	0.63576	0.67257	0.66188
		$\hat{\delta}$	0.29199	0.29387	0.29946	0.27945	0.28980	0.29256	0.29370
		$\hat{\kappa}$	0.83676	0.81012	0.93017	0.79547	0.93685	0.80283	0.85312
	AVB	$\hat{\rho}$	0.32170	0.32725	0.37202	0.27099	0.35471	0.33264	0.34039
		$\hat{\delta}$	0.08002	0.07787	0.08794	0.06545	0.08167	0.07821	0.07992
		$\hat{\kappa}$	0.31060	0.31096	0.45045	0.27546	0.44797	0.34072	0.36769
	MSE	$\hat{\rho}$	0.25320	0.22288	0.24784	0.16443	0.22054	0.19068	0.22617
		$\hat{\delta}$	0.01776	0.01372	0.01601	0.01102	0.01306	0.01094	0.01367
		$\hat{\kappa}$	0.16281	0.16200	0.40000	0.13830	0.37298	0.24699	0.25588
	MRE	$\hat{\rho}$	0.64341	0.65451	0.74405	0.54199	0.70942	0.66528	0.68077
		$\hat{\delta}$	0.32006	0.31150	0.35174	0.26179	0.32666	0.31284	0.31969
		$\hat{\kappa}$	0.41413	0.41461	0.60061	0.36728	0.59729	0.45429	0.49026
200	AVE	$\hat{\rho}$	0.62806	0.61441	0.63894	0.59507	0.62384	0.62830	0.58400
		$\hat{\delta}$	0.28473	0.28028	0.28828	0.27427	0.28341	0.28064	0.27245
		$\hat{\kappa}$	0.77515	0.77074	0.81983	0.76058	0.83602	0.76306	0.81231
	AVB	$\hat{\rho}$	0.25696	0.24422	0.29983	0.21929	0.29219	0.26479	0.23074
		$\hat{\delta}$	0.06349	0.05754	0.06868	0.05258	0.06628	0.06067	0.05306
		$\hat{\kappa}$	0.22577	0.22809	0.31003	0.20476	0.31524	0.24611	0.24667
	MSE	$\hat{\rho}$	0.18981	0.13768	0.17886	0.12369	0.16951	0.13519	0.12078
		$\hat{\delta}$	0.01303	0.00851	0.01076	0.00816	0.01001	0.00742	0.00712
		$\hat{\kappa}$	0.08315	0.08413	0.17360	0.07187	0.18961	0.10257	0.11142
	MRE	$\hat{\rho}$	0.51391	0.48845	0.59966	0.43859	0.58438	0.52959	0.46149
		$\hat{\delta}$	0.25396	0.23017	0.27474	0.21033	0.26512	0.24267	0.21222
		$\hat{\kappa}$	0.30102	0.30412	0.41338	0.27302	0.42032	0.32815	0.32889

Table 1: Continued.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	WLSE
400	AVE	$\hat{\rho}$	0.54278	0.56400	0.59055	0.54946	0.59618	0.60443	0.55423
		$\hat{\delta}$	0.26135	0.26722	0.27407	0.26175	0.27455	0.27381	0.26417
		$\hat{\kappa}$	0.76724	0.76195	0.78147	0.74278	0.76910	0.71854	0.76840
	AVB	$\hat{\rho}$	0.13906	0.16768	0.21650	0.13422	0.21968	0.18995	0.16118
		$\hat{\delta}$	0.03348	0.03898	0.04861	0.03165	0.04877	0.04269	0.03623
		$\hat{\kappa}$	0.14364	0.16593	0.22004	0.13548	0.21739	0.16307	0.17109
	MSE	$\hat{\rho}$	0.05849	0.07369	0.10487	0.04410	0.10851	0.07949	0.05898
		$\hat{\delta}$	0.00398	0.00466	0.00608	0.00267	0.00600	0.00419	0.00344
		$\hat{\kappa}$	0.03479	0.04481	0.08409	0.03130	0.07771	0.04229	0.04917
	MRE	$\hat{\rho}$	0.27812	0.33536	0.43299	0.26843	0.43937	0.37990	0.32236
		$\hat{\delta}$	0.13393	0.15593	0.19442	0.12662	0.19509	0.17077	0.14491
		$\hat{\kappa}$	0.19152	0.22124	0.29339	0.18064	0.28986	0.21742	0.22812

Table 2: Simulation results of the AVE, AVB, MSE, and MRE of BXLE distribution for ($\rho = 0.5, \delta = 0.25, \kappa = 2.5$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	WLSE
20	AVE	$\hat{\rho}$	1.79838	1.78970	3.72711	1.26641	2.56295	1.37714	2.13722
		$\hat{\delta}$	0.29334	0.30627	0.34361	0.28445	0.33228	0.28874	0.31289
		$\hat{\kappa}$	4.18376	3.56065	4.14725	3.59192	3.73590	3.67903	3.50675
	AVB	$\hat{\rho}$	1.61879	1.55097	3.51563	1.04734	2.34417	1.18384	1.90137
		$\hat{\delta}$	0.06430	0.07386	0.10995	0.05517	0.10056	0.06101	0.08100
		$\hat{\kappa}$	2.61040	2.17815	3.01901	2.13379	2.61237	2.20280	2.27485
	MSE	$\hat{\rho}$	27.24052	18.29810	203.49892	7.90529	41.28473	9.92185	28.59804
		$\hat{\delta}$	0.02077	0.02546	0.04469	0.01434	0.03817	0.01850	0.02930
		$\hat{\kappa}$	13.02115	9.19147	17.70745	10.29252	13.32810	8.42028	10.35202
	MRE	$\hat{\rho}$	3.23757	3.10194	7.03127	2.09468	4.68835	2.36768	3.80274
		$\hat{\delta}$	0.25720	0.29544	0.43979	0.22068	0.40226	0.24405	0.32402
		$\hat{\kappa}$	1.04416	0.87126	1.20761	0.85352	1.04495	0.88112	0.90994
30	AVE	$\hat{\rho}$	0.99099	1.32921	2.07930	0.95420	1.53393	0.98388	1.35775
		$\hat{\delta}$	0.27125	0.28688	0.31226	0.27055	0.29855	0.27138	0.28670
		$\hat{\kappa}$	3.61831	3.12526	3.73172	3.20562	3.65227	3.29690	3.28674
	AVB	$\hat{\rho}$	0.75889	1.04541	1.84504	0.69777	1.31490	0.74802	1.10170
		$\hat{\delta}$	0.03911	0.05178	0.07787	0.03944	0.06690	0.04132	0.05350
		$\hat{\kappa}$	1.88820	1.59461	2.40894	1.64291	2.25227	1.71862	1.84758
	MSE	$\hat{\rho}$	4.30156	8.59982	24.55273	3.60766	11.38860	4.80813	10.04901
		$\hat{\delta}$	0.00776	0.01501	0.02769	0.00763	0.02158	0.00886	0.01474
		$\hat{\kappa}$	7.31598	5.21073	11.46680	5.87619	10.04647	5.36600	6.74280
	MRE	$\hat{\rho}$	1.51779	2.09083	3.69008	1.39553	2.62980	1.49604	2.20341
		$\hat{\delta}$	0.15645	0.20714	0.31150	0.15778	0.26759	0.16526	0.21402
		$\hat{\kappa}$	0.75528	0.63784	0.96358	0.65716	0.90091	0.68745	0.73903
50	AVE	$\hat{\rho}$	0.72897	0.87809	1.30793	0.67147	1.14520	0.68726	0.94686
		$\hat{\delta}$	0.26094	0.26977	0.28827	0.25766	0.28002	0.25869	0.27062
		$\hat{\kappa}$	3.19562	2.77012	3.22326	2.88887	3.25550	2.99237	2.97697
	AVB	$\hat{\rho}$	0.45319	0.54537	1.02266	0.36814	0.87346	0.40681	0.64862
		$\hat{\delta}$	0.02671	0.03093	0.05103	0.02294	0.04451	0.02554	0.03453
		$\hat{\kappa}$	1.33854	1.16317	1.78693	1.12205	1.75687	1.25366	1.37743
	MSE	$\hat{\rho}$	1.88471	2.09399	7.42802	0.81609	5.56387	0.90985	3.96788
		$\hat{\delta}$	0.00372	0.00481	0.01309	0.00201	0.01061	0.00258	0.00648
		$\hat{\kappa}$	3.69003	2.59408	6.49205	2.69052	6.21149	2.75350	3.75536
	MRE	$\hat{\rho}$	0.90638	1.09073	2.04533	0.73627	1.74691	0.81362	1.29724
		$\hat{\delta}$	0.10683	0.12372	0.20413	0.09177	0.17806	0.10218	0.13811
		$\hat{\kappa}$	0.53542	0.46527	0.71477	0.44882	0.70275	0.50146	0.55097

Table 2: Continued.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	WLSE
100	AVE	$\hat{\rho}$	0.55522	0.59565	0.73694	0.54867	0.65998	0.54406	0.62726
		$\hat{\delta}$	0.25137	0.25550	0.26245	0.25206	0.25804	0.25193	0.25698
		$\hat{\kappa}$	2.73597	2.70173	2.92681	2.62728	2.89390	2.79649	2.70926
	AVB	$\hat{\rho}$	0.20372	0.24386	0.41585	0.19054	0.33922	0.21185	0.27539
		$\hat{\delta}$	0.01554	0.01542	0.02334	0.01334	0.01982	0.01405	0.01699
		$\hat{\kappa}$	0.73542	0.78757	1.21745	0.69786	1.11703	0.83807	0.89749
	MSE	$\hat{\rho}$	0.09625	0.25577	1.05703	0.06916	0.48862	0.10010	0.23843
		$\hat{\delta}$	0.00092	0.00075	0.00217	0.00031	0.00121	0.00039	0.00079
		$\hat{\kappa}$	1.02468	1.12126	3.10865	0.93664	2.68427	1.30829	1.54429
	MRE	$\hat{\rho}$	0.40743	0.48773	0.83169	0.38108	0.67843	0.42371	0.55078
		$\hat{\delta}$	0.06218	0.06168	0.09337	0.05334	0.07928	0.05621	0.06797
		$\hat{\kappa}$	0.29417	0.31503	0.48698	0.27914	0.44681	0.33523	0.35900
200	AVE	$\hat{\rho}$	0.51672	0.54133	0.54552	0.53728	0.56798	0.51174	0.54949
		$\hat{\delta}$	0.24980	0.25229	0.25268	0.25183	0.25355	0.25036	0.25282
		$\hat{\kappa}$	2.63383	2.61698	2.79988	2.49776	2.69918	2.67996	2.59225
	AVB	$\hat{\rho}$	0.13323	0.15520	0.19892	0.13056	0.21263	0.14086	0.16293
		$\hat{\delta}$	0.01036	0.01015	0.01216	0.00900	0.01281	0.00958	0.01067
		$\hat{\kappa}$	0.50072	0.55770	0.77807	0.43559	0.76143	0.56505	0.57875
	MSE	$\hat{\rho}$	0.03135	0.05073	0.09565	0.03116	0.11396	0.03749	0.04950
		$\hat{\delta}$	0.00051	0.00019	0.00033	0.00014	0.00036	0.00016	0.00020
		$\hat{\kappa}$	0.44381	0.55024	1.18116	0.34815	1.03225	0.57667	0.60758
	MRE	$\hat{\rho}$	0.26645	0.31040	0.39784	0.26112	0.42525	0.28173	0.32585
		$\hat{\delta}$	0.04143	0.04062	0.04864	0.03600	0.05125	0.03831	0.04270
		$\hat{\kappa}$	0.20029	0.22308	0.31123	0.17423	0.30457	0.22602	0.23150
400	AVE	$\hat{\rho}$	0.50100	0.52236	0.52823	0.51783	0.52287	0.51285	0.51279
		$\hat{\delta}$	0.24898	0.25119	0.25180	0.25086	0.25118	0.25060	0.25073
		$\hat{\kappa}$	2.57622	2.54449	2.61466	2.49300	2.61337	2.55073	2.57562
	AVB	$\hat{\rho}$	0.08862	0.10704	0.14043	0.08619	0.12979	0.09737	0.10665
		$\hat{\delta}$	0.00729	0.00722	0.00869	0.00606	0.00825	0.00672	0.00705
		$\hat{\kappa}$	0.33083	0.39449	0.53050	0.28869	0.50412	0.37270	0.41309
	MSE	$\hat{\rho}$	0.01298	0.01903	0.03637	0.01352	0.03009	0.01617	0.01763
		$\hat{\delta}$	0.00037	0.00008	0.00014	0.00006	0.00012	0.00007	0.00008
		$\hat{\kappa}$	0.18116	0.25401	0.46145	0.16352	0.44900	0.24111	0.28127
	MRE	$\hat{\rho}$	0.17725	0.21407	0.28086	0.17238	0.25958	0.19474	0.21331
		$\hat{\delta}$	0.02915	0.02889	0.03476	0.02424	0.03300	0.02690	0.02820
		$\hat{\kappa}$	0.13233	0.15779	0.21220	0.11548	0.20165	0.14908	0.16523

Table 3: Simulation results of the AVE, AVB, MSE, and MRE of BXLE distribution for $(\rho = 0.75, \delta = 1.50, \kappa = 0.25)$.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	WLSE
20	AVE	$\hat{\rho}$	0.99360	0.95711	0.96270	0.86926	0.98110	0.86442	0.92370
		$\hat{\delta}$	2.42543	2.42282	2.66289	2.08319	2.56447	3.53535	2.35381
		$\hat{\kappa}$	0.34295	0.30690	0.34230	0.31174	0.31461	0.36856	0.34015
	AVB	$\hat{\rho}$	0.50582	0.48037	0.45163	0.45122	0.48677	0.50786	0.47104
		$\hat{\delta}$	1.29066	1.31734	1.45196	1.18737	1.45493	2.70680	1.30148
		$\hat{\kappa}$	0.19613	0.15841	0.18684	0.15798	0.17649	0.25179	0.19231
	MSE	$\hat{\rho}$	0.45214	0.46317	0.45474	0.38317	0.49133	0.41989	0.43086
		$\hat{\delta}$	4.06209	5.56485	7.76750	4.11920	6.57387	48.76375	4.70223
		$\hat{\kappa}$	0.12030	0.07043	0.14299	0.07264	0.10355	0.39286	0.12948
	MRE	$\hat{\rho}$	0.67443	0.64049	0.60217	0.60163	0.64902	0.67715	0.62806
		$\hat{\delta}$	0.86044	0.87823	0.96797	0.79158	0.96995	1.80453	0.86765
		$\hat{\kappa}$	0.78450	0.63364	0.74735	0.63194	0.70596	1.00714	0.76924

Table 3: Continued.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	WLSE
30	AVE	$\hat{\rho}$	0.95938	0.87601	0.92287	0.84252	0.92927	0.87334	0.89492
		$\hat{\delta}$	2.22733	2.07721	2.38510	1.93928	2.32628	2.82560	2.18518
		$\hat{\kappa}$	0.30217	0.29895	0.31209	0.29869	0.28824	0.32625	0.30640
	AVB	$\hat{\rho}$	0.44834	0.39020	0.40942	0.40810	0.42097	0.46867	0.42713
		$\hat{\delta}$	1.08404	0.96531	1.17402	1.00478	1.17111	1.91451	1.11519
		$\hat{\kappa}$	0.15229	0.13439	0.15337	0.13590	0.13904	0.20183	0.15072
	MSE	$\hat{\rho}$	0.34095	0.28513	0.34354	0.29757	0.35432	0.34798	0.34303
		$\hat{\delta}$	2.66924	2.49728	4.10740	2.68863	4.11698	15.16534	3.64459
		$\hat{\kappa}$	0.06268	0.04899	0.08541	0.04280	0.06511	0.25334	0.06424
	MRE	$\hat{\rho}$	0.59779	0.52026	0.54590	0.54413	0.56130	0.62489	0.56950
		$\hat{\delta}$	0.72269	0.64354	0.78268	0.66985	0.78074	1.27634	0.74346
		$\hat{\kappa}$	0.60915	0.53757	0.61347	0.54358	0.55616	0.80732	0.60289
50	AVE	$\hat{\rho}$	0.91163	0.86778	0.92852	0.76703	0.90027	0.84528	0.87249
		$\hat{\delta}$	2.03322	1.99131	2.22980	1.65948	2.12220	2.31959	2.00310
		$\hat{\kappa}$	0.29140	0.27962	0.28109	0.29744	0.27958	0.27304	0.28505
	AVB	$\hat{\rho}$	0.37058	0.36321	0.38886	0.32528	0.38555	0.41703	0.37056
		$\hat{\delta}$	0.86939	0.87081	1.01492	0.72622	0.98087	1.36792	0.89554
		$\hat{\kappa}$	0.13116	0.11350	0.12246	0.11279	0.12262	0.13345	0.11958
	MSE	$\hat{\rho}$	0.23360	0.23192	0.29373	0.18427	0.27272	0.25789	0.25123
		$\hat{\delta}$	1.56149	1.81361	2.69281	1.25685	2.50704	5.78669	2.05061
		$\hat{\kappa}$	0.04840	0.02696	0.04590	0.02649	0.04063	0.07676	0.03561
	MRE	$\hat{\rho}$	0.49411	0.48428	0.51848	0.43371	0.51407	0.55604	0.49408
		$\hat{\delta}$	0.57959	0.58054	0.67662	0.48415	0.65392	0.91195	0.59703
		$\hat{\kappa}$	0.52466	0.45399	0.48984	0.45115	0.49050	0.53381	0.47833
100	AVE	$\hat{\rho}$	0.88398	0.83183	0.86478	0.75635	0.86641	0.81495	0.88168
		$\hat{\delta}$	1.90133	1.81269	1.95739	1.57353	1.96806	1.94150	1.91637
		$\hat{\kappa}$	0.27396	0.27360	0.26871	0.28069	0.26751	0.25885	0.26220
	AVB	$\hat{\rho}$	0.32223	0.30217	0.31393	0.26218	0.32813	0.35825	0.33039
		$\hat{\delta}$	0.73675	0.66249	0.74721	0.54375	0.78939	0.96608	0.73887
		$\hat{\kappa}$	0.10653	0.09510	0.09814	0.08624	0.09925	0.10375	0.09643
	MSE	$\hat{\rho}$	0.16553	0.15050	0.17177	0.11858	0.19127	0.17936	0.17703
		$\hat{\delta}$	1.03517	0.88563	1.25841	0.64749	1.48242	1.98362	1.04959
		$\hat{\kappa}$	0.03014	0.01913	0.02273	0.01442	0.02174	0.02017	0.01795
	MRE	$\hat{\rho}$	0.42964	0.40289	0.41857	0.34957	0.43751	0.47767	0.44052
		$\hat{\delta}$	0.49117	0.44166	0.49814	0.36250	0.52626	0.64405	0.49258
		$\hat{\kappa}$	0.42614	0.38039	0.39256	0.34498	0.39698	0.41501	0.38571
200	AVE	$\hat{\rho}$	0.80515	0.80691	0.82173	0.73513	0.82507	0.74742	0.82156
		$\hat{\delta}$	1.67192	1.71497	1.78058	1.50274	1.79591	1.66370	1.74915
		$\hat{\kappa}$	0.26800	0.26074	0.26166	0.27287	0.26112	0.27281	0.26201
	AVB	$\hat{\rho}$	0.24761	0.24322	0.25420	0.17948	0.26823	0.29254	0.26068
		$\hat{\delta}$	0.52815	0.52432	0.55804	0.34659	0.60818	0.69585	0.56620
		$\hat{\kappa}$	0.07914	0.07238	0.07655	0.05840	0.07850	0.09053	0.07860
	MSE	$\hat{\rho}$	0.09236	0.09356	0.10633	0.06331	0.11724	0.11168	0.10681
		$\hat{\delta}$	0.45233	0.51497	0.64130	0.30021	0.76518	0.89761	0.60677
		$\hat{\kappa}$	0.01156	0.00903	0.01146	0.00740	0.01110	0.00987	0.01097
	MRE	$\hat{\rho}$	0.33015	0.32430	0.33894	0.23930	0.35763	0.39005	0.34758
		$\hat{\delta}$	0.35210	0.34954	0.37203	0.23106	0.40545	0.46390	0.37747
		$\hat{\kappa}$	0.31656	0.28951	0.30622	0.23360	0.31400	0.36213	0.31439
400	AVE	$\hat{\rho}$	0.78216	0.79246	0.80316	0.72550	0.80397	0.72395	0.79052
		$\hat{\delta}$	1.61856	1.64791	1.70873	1.46216	1.69433	1.53857	1.64473
		$\hat{\kappa}$	0.26564	0.25990	0.26092	0.26924	0.25947	0.27606	0.26031
	AVB	$\hat{\rho}$	0.20920	0.21347	0.22881	0.12526	0.22547	0.24047	0.20776
		$\hat{\delta}$	0.44192	0.44367	0.49432	0.22955	0.48918	0.52659	0.43133
		$\hat{\kappa}$	0.06805	0.06447	0.06871	0.04216	0.06758	0.07965	0.06396
	MSE	$\hat{\rho}$	0.06600	0.06737	0.07930	0.03591	0.07975	0.07370	0.06616
		$\hat{\delta}$	0.31604	0.32293	0.44119	0.15385	0.43808	0.41215	0.31885
		$\hat{\kappa}$	0.00854	0.00694	0.00854	0.00456	0.00798	0.00813	0.00710
	MRE	$\hat{\rho}$	0.27893	0.28462	0.30508	0.16702	0.30062	0.32062	0.27702
		$\hat{\delta}$	0.29461	0.29578	0.32955	0.15304	0.32612	0.35106	0.28755
		$\hat{\kappa}$	0.27218	0.25786	0.27486	0.16864	0.27031	0.31859	0.25582

Table 4: Simulation results of the AVE, AVB, MSE, and MRE of BXLE distribution for $(\rho = 2, \delta = 0.75, \kappa = 0.5)$.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	WLSE
20	AVE	$\hat{\rho}$	2.19972	2.10860	2.18055	1.83387	2.02412	1.89513	2.04918
		$\hat{\delta}$	0.80464	0.82806	0.81834	0.76080	0.84119	0.76588	0.83656
		$\hat{\kappa}$	1.28376	0.95015	1.24156	0.96376	1.19295	0.97828	1.11270
	AVB	$\hat{\rho}$	1.48464	1.27651	1.42915	0.84619	1.22818	1.09457	1.33954
		$\hat{\delta}$	0.32012	0.32444	0.31584	0.22769	0.33701	0.26718	0.34797
		$\hat{\kappa}$	0.92999	0.60028	0.89061	0.60676	0.85783	0.63555	0.76614
	MSE	$\hat{\rho}$	3.08643	2.61610	3.47436	1.56670	2.76591	1.78122	3.07681
		$\hat{\delta}$	0.20631	0.36524	0.20891	0.26242	0.58276	0.13725	0.36637
		$\hat{\kappa}$	3.21804	1.35341	2.97730	1.80724	2.82349	1.49541	2.12508
	MRE	$\hat{\rho}$	0.74232	0.63826	0.71457	0.42310	0.61409	0.54729	0.66977
		$\hat{\delta}$	0.42683	0.43259	0.42111	0.30359	0.44934	0.35624	0.46396
		$\hat{\kappa}$	1.85999	1.20057	1.78123	1.21352	1.71567	1.27109	1.53228
30	AVE	$\hat{\rho}$	2.07208	2.09879	1.93054	1.81643	1.87920	1.80982	1.97688
		$\hat{\delta}$	0.77345	0.79470	0.75483	0.73911	0.76213	0.73608	0.79586
		$\hat{\kappa}$	1.02110	0.82572	1.17015	0.81542	1.10133	0.84551	0.93708
	AVB	$\hat{\rho}$	1.35315	1.18342	1.27698	0.78084	1.20001	1.03024	1.23836
		$\hat{\delta}$	0.29055	0.27508	0.27860	0.20257	0.28535	0.24667	0.30605
		$\hat{\kappa}$	0.65961	0.46780	0.79891	0.44044	0.73575	0.47929	0.57549
	MSE	$\hat{\rho}$	2.35464	2.20850	2.39316	1.25677	2.26625	1.50754	2.29529
		$\hat{\delta}$	0.10427	0.14760	0.10481	0.22213	0.21766	0.10972	0.31649
		$\hat{\kappa}$	1.51082	0.78721	2.20739	0.85698	1.93455	0.76941	1.10169
	MRE	$\hat{\rho}$	0.67657	0.59171	0.63849	0.39042	0.60000	0.51512	0.61918
		$\hat{\delta}$	0.38739	0.36677	0.37147	0.27009	0.38047	0.32889	0.40807
		$\hat{\kappa}$	1.31923	0.93561	1.59783	0.88089	1.47150	0.95859	1.15098
50	AVE	$\hat{\rho}$	2.21638	2.01940	1.98735	1.84738	1.91368	1.98021	1.99521
		$\hat{\delta}$	0.80572	0.76881	0.77159	0.73132	0.75873	0.76822	0.77033
		$\hat{\kappa}$	0.70792	0.69409	0.86623	0.67736	0.83730	0.67283	0.78059
	AVB	$\hat{\rho}$	1.20455	1.04262	1.16157	0.69744	1.07214	0.97911	1.14091
		$\hat{\delta}$	0.26408	0.23746	0.26207	0.17030	0.24731	0.23206	0.26052
		$\hat{\kappa}$	0.35563	0.32068	0.49499	0.28592	0.46832	0.31635	0.41074
	MSE	$\hat{\rho}$	1.82687	1.42665	1.96539	0.91267	1.64635	1.30571	1.78899
		$\hat{\delta}$	0.08468	0.08372	0.10039	0.04297	0.08239	0.06919	0.14433
		$\hat{\kappa}$	0.37512	0.29818	0.78968	0.28940	0.73692	0.26978	0.50205
	MRE	$\hat{\rho}$	0.60228	0.52131	0.58079	0.34872	0.53607	0.48956	0.57045
		$\hat{\delta}$	0.35210	0.31662	0.34943	0.22707	0.32975	0.30941	0.34736
		$\hat{\kappa}$	0.71126	0.64136	0.98999	0.57184	0.93664	0.63270	0.82148
100	AVE	$\hat{\rho}$	2.16724	2.14223	1.97914	1.95534	1.97042	2.11936	2.02067
		$\hat{\delta}$	0.79778	0.79347	0.76016	0.75013	0.76123	0.79596	0.76815
		$\hat{\kappa}$	0.59543	0.58523	0.68696	0.56600	0.67915	0.57161	0.63676
	AVB	$\hat{\rho}$	1.02791	0.93518	1.01950	0.51636	1.00702	0.91638	0.97203
		$\hat{\delta}$	0.23146	0.21315	0.22653	0.13179	0.23056	0.21854	0.21899
		$\hat{\kappa}$	0.23138	0.21866	0.30926	0.15552	0.30441	0.21339	0.25835
	MSE	$\hat{\rho}$	1.32814	1.11433	1.26780	0.60659	1.28261	1.12964	1.17428
		$\hat{\delta}$	0.06693	0.05548	0.05985	0.03010	0.06435	0.06556	0.05745
		$\hat{\kappa}$	0.10069	0.09467	0.24364	0.06331	0.23705	0.08434	0.14838
	MRE	$\hat{\rho}$	0.51395	0.46759	0.50975	0.25818	0.50351	0.45819	0.48601
		$\hat{\delta}$	0.30861	0.28420	0.30204	0.17572	0.30742	0.29139	0.29199
		$\hat{\kappa}$	0.46277	0.43732	0.61852	0.31105	0.60881	0.42678	0.51670
200	AVE	$\hat{\rho}$	2.20581	2.15610	2.13158	1.92166	2.01104	2.22879	2.12827
		$\hat{\delta}$	0.80440	0.79620	0.79485	0.73774	0.76525	0.81614	0.78879
		$\hat{\kappa}$	0.54241	0.54382	0.58959	0.54591	0.61300	0.51987	0.55222
	AVB	$\hat{\rho}$	0.89638	0.83000	0.95399	0.34023	0.91264	0.84794	0.84106
		$\hat{\delta}$	0.20432	0.19064	0.21694	0.08864	0.20523	0.19965	0.19182
		$\hat{\kappa}$	0.17065	0.17054	0.22367	0.10093	0.23182	0.16210	0.17306
	MSE	$\hat{\rho}$	1.06258	0.86982	1.12036	0.36602	1.01362	0.93002	0.89204
		$\hat{\delta}$	0.05631	0.04548	0.05775	0.01784	0.04974	0.05204	0.04582
		$\hat{\kappa}$	0.04495	0.04693	0.09351	0.03008	0.10902	0.04136	0.04966
	MRE	$\hat{\rho}$	0.44819	0.41500	0.47700	0.17012	0.45632	0.42397	0.42053
		$\hat{\delta}$	0.27242	0.25418	0.28926	0.11819	0.27364	0.26620	0.25576
		$\hat{\kappa}$	0.34131	0.34107	0.44733	0.20186	0.46365	0.32421	0.34612

Table 4: Continued.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	WLSE
400	AVE	$\hat{\rho}$	2.19928	2.23351	2.12434	1.98656	2.10487	2.22701	2.12101
		$\hat{\delta}$	0.80281	0.81210	0.78777	0.74960	0.78419	0.81309	0.78822
		$\hat{\kappa}$	0.52140	0.50938	0.54236	0.51222	0.54996	0.50616	0.52888
	AVB	$\hat{\rho}$	0.78212	0.76831	0.80115	0.16129	0.81774	0.77148	0.74657
		$\hat{\delta}$	0.17915	0.17629	0.18084	0.04812	0.18478	0.18059	0.17113
		$\hat{\kappa}$	0.13703	0.13600	0.15975	0.04871	0.16807	0.13728	0.14135
	MSE	$\hat{\rho}$	0.87125	0.74539	0.78899	0.16253	0.81562	0.76414	0.69966
		$\hat{\delta}$	0.04720	0.03958	0.04038	0.00823	0.04167	0.04256	0.03742
		$\hat{\kappa}$	0.02671	0.02624	0.03959	0.00944	0.04442	0.02600	0.02909
	MRE	$\hat{\rho}$	0.39106	0.38416	0.40057	0.08064	0.40887	0.38574	0.37329
		$\hat{\delta}$	0.23887	0.23505	0.24112	0.06415	0.24637	0.24079	0.22817
		$\hat{\kappa}$	0.27406	0.27201	0.31950	0.09742	0.33613	0.27455	0.28269

Table 5: Simulation results of the AVE, AVB, MSE, and MRE of BXLE distribution for ($\rho = 1.5, \delta = 2, \kappa = 3$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	WLSE
20	AVE	$\hat{\rho}$	1.48426	1.50143	1.53919	1.39727	1.49729	1.40027	1.51141
		$\hat{\delta}$	1.99585	1.99840	2.00021	1.98168	1.99476	1.97884	1.99683
		$\hat{\kappa}$	3.35961	3.25344	3.30962	3.15830	3.21190	3.23687	3.19768
	AVB	$\hat{\rho}$	0.45024	0.46569	0.47307	0.49942	0.49706	0.52360	0.49561
		$\hat{\delta}$	0.07124	0.07666	0.08052	0.08212	0.08343	0.08790	0.08176
		$\hat{\kappa}$	0.70369	0.68421	0.70003	0.72208	0.73740	0.70854	0.70656
	MSE	$\hat{\rho}$	0.23741	0.25537	0.25762	0.29410	0.28450	0.31937	0.28166
		$\hat{\delta}$	0.00716	0.00859	0.00915	0.00938	0.01012	0.01065	0.00953
		$\hat{\kappa}$	0.61438	0.58270	0.61510	0.64382	0.66424	0.62750	0.61700
	MRE	$\hat{\rho}$	0.30016	0.31046	0.31538	0.33295	0.33137	0.34907	0.33041
		$\hat{\delta}$	0.03562	0.03833	0.04026	0.04106	0.04172	0.04395	0.04088
		$\hat{\kappa}$	0.23456	0.22807	0.23334	0.24069	0.24580	0.23618	0.23552
30	AVE	$\hat{\rho}$	1.49349	1.52758	1.50013	1.40873	1.48982	1.40388	1.48786
		$\hat{\delta}$	1.99440	2.00095	1.99698	1.98237	1.99378	1.97927	1.99263
		$\hat{\kappa}$	3.32788	3.18310	3.29192	3.16789	3.19937	3.23034	3.21322
	AVB	$\hat{\rho}$	0.44356	0.45161	0.45218	0.47739	0.49011	0.49857	0.45999
		$\hat{\delta}$	0.06403	0.06994	0.07372	0.07541	0.07839	0.08068	0.07318
		$\hat{\kappa}$	0.61983	0.61354	0.68297	0.65874	0.69034	0.66390	0.65282
	MSE	$\hat{\rho}$	0.23018	0.23337	0.23855	0.26956	0.27263	0.28904	0.24734
		$\hat{\delta}$	0.00581	0.00670	0.00751	0.00792	0.00839	0.00884	0.00743
		$\hat{\kappa}$	0.51057	0.49647	0.58501	0.55431	0.59566	0.55794	0.54755
	MRE	$\hat{\rho}$	0.29571	0.30107	0.30146	0.31826	0.32674	0.33238	0.30666
		$\hat{\delta}$	0.03202	0.03497	0.03686	0.03771	0.03919	0.04034	0.03659
		$\hat{\kappa}$	0.20661	0.20451	0.22766	0.21958	0.23011	0.22130	0.21761
50	AVE	$\hat{\rho}$	1.52272	1.48816	1.50716	1.44609	1.50200	1.42613	1.48624
		$\hat{\delta}$	1.99826	1.99351	1.99775	1.98939	1.99568	1.98419	1.99475
		$\hat{\kappa}$	3.23116	3.20837	3.24415	3.11142	3.17626	3.18904	3.17970
	AVB	$\hat{\rho}$	0.43070	0.43099	0.44992	0.44334	0.45581	0.46098	0.43779
		$\hat{\delta}$	0.06190	0.06572	0.06765	0.06812	0.06835	0.07124	0.06679
		$\hat{\kappa}$	0.57289	0.58429	0.62487	0.58566	0.62722	0.60344	0.59943
	MSE	$\hat{\rho}$	0.21395	0.21883	0.22988	0.23618	0.23900	0.25058	0.22438
		$\hat{\delta}$	0.00509	0.00574	0.00603	0.00613	0.00623	0.00684	0.00597
		$\hat{\kappa}$	0.44440	0.45749	0.50946	0.46032	0.51462	0.48127	0.47750
	MRE	$\hat{\rho}$	0.28713	0.28733	0.29994	0.29556	0.30387	0.30732	0.29186
		$\hat{\delta}$	0.03095	0.03286	0.03382	0.03406	0.03418	0.03562	0.03340
		$\hat{\kappa}$	0.19096	0.19476	0.20829	0.19522	0.20907	0.20115	0.19981

Table 5: Continued.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	WLSE
100	AVE	$\hat{\rho}$	1.49861	1.49854	1.48977	1.49453	1.48220	1.45097	1.52141
		$\hat{\delta}$	1.99659	1.99520	1.99600	1.99676	1.99332	1.98899	1.99940
		$\hat{\kappa}$	3.18175	3.17224	3.20354	3.07278	3.19864	3.16300	3.13747
	AVB	$\hat{\rho}$	0.37760	0.39828	0.42841	0.37844	0.43741	0.40096	0.40783
		$\hat{\delta}$	0.05390	0.05817	0.06213	0.05587	0.06380	0.05892	0.05851
		$\hat{\kappa}$	0.49406	0.51521	0.57305	0.47351	0.57208	0.51239	0.52015
	MSE	$\hat{\rho}$	0.17340	0.19014	0.21242	0.18343	0.22034	0.19669	0.19731
		$\hat{\delta}$	0.00384	0.00442	0.00486	0.00423	0.00519	0.00459	0.00445
		$\hat{\kappa}$	0.34608	0.37380	0.44334	0.32961	0.43920	0.36842	0.37435
	MRE	$\hat{\rho}$	0.25173	0.26552	0.28560	0.25230	0.29161	0.26730	0.27189
		$\hat{\delta}$	0.16469	0.17174	0.19102	0.15784	0.19069	0.17080	0.17338
		$\hat{\kappa}$	0.02695	0.02908	0.03107	0.02793	0.03190	0.02946	0.02926
200	AVE	$\hat{\rho}$	1.52112	1.51905	1.50244	1.50740	1.51074	1.47311	1.51276
		$\hat{\delta}$	1.99975	1.99985	1.99705	1.99890	1.99808	1.99255	1.99789
		$\hat{\kappa}$	3.11370	3.09869	3.15336	3.03961	3.13256	3.12542	3.11391
	AVB	$\hat{\rho}$	0.33664	0.36135	0.39323	0.30583	0.39643	0.35301	0.36308
		$\hat{\delta}$	0.04711	0.05190	0.05530	0.04539	0.05535	0.05108	0.05125
		$\hat{\kappa}$	0.41955	0.44732	0.50282	0.37172	0.49959	0.44572	0.44781
	MSE	$\hat{\rho}$	0.14564	0.16292	0.18432	0.13545	0.18676	0.15976	0.16363
		$\hat{\delta}$	0.00296	0.00350	0.00386	0.00304	0.00384	0.00348	0.00340
		$\hat{\kappa}$	0.26202	0.28263	0.35606	0.23001	0.34779	0.28964	0.28877
	MRE	$\hat{\rho}$	0.22443	0.24090	0.26215	0.20389	0.26429	0.23534	0.24205
		$\hat{\delta}$	0.02355	0.02595	0.02765	0.02270	0.02768	0.02554	0.02563
		$\hat{\kappa}$	0.13985	0.14911	0.16761	0.12391	0.16653	0.14857	0.14927
400	AVE	$\hat{\rho}$	1.50952	1.51872	1.48852	1.50735	1.51329	1.46800	1.50446
		$\hat{\delta}$	1.99977	2.00038	1.99609	1.99982	1.99935	1.99288	1.99783
		$\hat{\kappa}$	3.07239	3.07818	3.14157	3.02834	3.10499	3.11262	3.08511
	AVB	$\hat{\rho}$	0.27199	0.30058	0.34299	0.23273	0.34738	0.29709	0.29563
		$\hat{\delta}$	0.03884	0.04181	0.04737	0.03348	0.04864	0.04239	0.04145
		$\hat{\kappa}$	0.33349	0.35723	0.43772	0.27538	0.43071	0.37178	0.35158
	MSE	$\hat{\rho}$	0.10212	0.12074	0.14913	0.09413	0.15349	0.11852	0.11873
		$\hat{\delta}$	0.00211	0.00238	0.00294	0.00192	0.00311	0.00247	0.00239
		$\hat{\kappa}$	0.16775	0.19116	0.27750	0.14899	0.27192	0.20760	0.19001
	MRE	$\hat{\rho}$	0.18132	0.20039	0.22866	0.15515	0.23159	0.19806	0.19709
		$\hat{\delta}$	0.01942	0.02091	0.02369	0.01674	0.02432	0.02119	0.02072
		$\hat{\kappa}$	0.11116	0.11908	0.14591	0.09179	0.14357	0.12393	0.11719

Table 6: Simulation results of the AVE, AVB, MSE, and MRE of BXLE distribution for ($\rho = 0.5, \delta = 3, \kappa = 1.5$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	WLSE
20	AVE	$\hat{\rho}$	1.09013	1.09301	1.59993	1.05621	1.26885	1.03708	1.36696
		$\hat{\delta}$	3.75659	3.72276	4.23521	3.65787	3.88774	3.70788	4.01329
		$\hat{\kappa}$	2.49700	2.17462	2.41294	2.16578	2.48830	2.16685	2.19013
	AVB	$\hat{\rho}$	0.89701	0.85443	1.37656	0.83944	1.08015	0.82323	1.13763
		$\hat{\delta}$	1.10098	1.06444	1.53775	1.06360	1.30483	1.12174	1.36835
		$\hat{\kappa}$	1.59407	1.32878	1.74556	1.36212	1.72736	1.34914	1.46963
	MSE	$\hat{\rho}$	2.88732	2.87882	7.28783	2.42481	4.15889	2.50272	8.32081
		$\hat{\delta}$	3.85325	3.46515	6.77092	3.19793	4.71999	5.88239	6.48326
		$\hat{\kappa}$	4.90730	3.52543	6.19279	3.88614	5.63150	3.31593	4.40429
	MRE	$\hat{\rho}$	1.79403	1.70886	2.75312	1.67889	2.16030	1.64646	2.27525
		$\hat{\delta}$	0.36699	0.35481	0.51258	0.35453	0.43494	0.37391	0.45612
		$\hat{\kappa}$	1.06271	0.88585	1.16371	0.90808	1.15158	0.89943	0.97975

Table 6: Continued.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	WLSE
30	AVE	$\hat{\rho}$	0.94954	1.00033	1.30323	0.86314	1.10217	0.88577	1.11377
		$\hat{\delta}$	3.59034	3.65031	3.96094	3.42215	3.74420	3.56056	3.76013
		$\hat{\kappa}$	2.26098	1.89910	2.17864	1.93524	2.17345	1.99736	2.02340
	AVB	$\hat{\rho}$	0.74141	0.72104	1.05990	0.60505	0.86941	0.64025	0.86180
		$\hat{\delta}$	0.93953	0.94579	1.25724	0.77415	1.09003	0.92673	1.09069
		$\hat{\kappa}$	1.28238	1.01758	1.44164	1.03514	1.37325	1.09947	1.21207
	MSE	$\hat{\rho}$	2.25949	2.20718	3.66942	1.39781	2.63789	1.53392	2.77654
		$\hat{\delta}$	3.14915	4.08395	4.36029	1.95861	3.59800	5.64114	4.08607
		$\hat{\kappa}$	3.05310	2.03374	4.07271	2.34656	3.61853	2.29771	2.85553
	MRE	$\hat{\rho}$	1.48282	1.44208	2.11981	1.21010	1.73882	1.28050	1.72360
		$\hat{\delta}$	0.31318	0.31526	0.41908	0.25805	0.36334	0.30891	0.36356
		$\hat{\kappa}$	0.85492	0.67839	0.96110	0.69009	0.91550	0.73298	0.80805
50	AVE	$\hat{\rho}$	0.69172	0.79182	1.00863	0.70484	0.91721	0.74754	0.82745
		$\hat{\delta}$	3.26123	3.39251	3.64936	3.25112	3.55590	3.30188	3.40601
		$\hat{\kappa}$	1.98831	1.68950	2.00656	1.74926	1.97113	1.76822	1.82594
	AVB	$\hat{\rho}$	0.43423	0.47355	0.74483	0.40840	0.65350	0.45431	0.53521
		$\hat{\delta}$	0.56234	0.63452	0.92003	0.54596	0.85673	0.60527	0.68306
		$\hat{\kappa}$	0.89780	0.72426	1.14282	0.75930	1.07921	0.78071	0.87385
	MSE	$\hat{\rho}$	0.74115	0.79329	1.97914	0.55856	1.50196	0.82328	1.15166
		$\hat{\delta}$	1.18160	1.39180	2.76212	0.99183	2.54417	1.45508	1.75732
		$\hat{\kappa}$	1.54985	0.98647	2.50926	1.14006	2.36698	1.14811	1.55161
	MRE	$\hat{\rho}$	0.86846	0.94710	1.48966	0.81681	1.30700	0.90863	1.07042
		$\hat{\delta}$	0.18745	0.21151	0.30668	0.18199	0.28558	0.20176	0.22769
		$\hat{\kappa}$	0.59853	0.48284	0.76188	0.50620	0.71947	0.52047	0.58257
100	AVE	$\hat{\rho}$	0.56807	0.64665	0.71324	0.57865	0.74440	0.59553	0.65737
		$\hat{\delta}$	3.09793	3.18688	3.27587	3.08497	3.30720	3.11328	3.20582
		$\hat{\kappa}$	1.69720	1.62676	1.81135	1.58225	1.73585	1.60584	1.61359
	AVB	$\hat{\rho}$	0.23193	0.30342	0.40499	0.22032	0.42918	0.23957	0.30949
		$\hat{\delta}$	0.31118	0.39824	0.49846	0.29956	0.54175	0.32289	0.40949
		$\hat{\kappa}$	0.50544	0.56005	0.78929	0.44830	0.74598	0.48990	0.54703
	MSE	$\hat{\rho}$	0.18046	0.26910	0.55243	0.14965	0.66605	0.16895	0.34042
		$\hat{\delta}$	0.31402	0.43605	0.84022	0.25285	1.02444	0.29370	0.58959
		$\hat{\kappa}$	0.49210	0.56994	1.26854	0.39231	1.10382	0.46066	0.56457
	MRE	$\hat{\rho}$	0.46387	0.60684	0.80998	0.44065	0.85836	0.47914	0.61898
		$\hat{\delta}$	0.10373	0.13275	0.16615	0.09985	0.18058	0.10763	0.13650
		$\hat{\kappa}$	0.33696	0.37336	0.52619	0.29887	0.49732	0.32660	0.36468
200	AVE	$\hat{\rho}$	0.52163	0.55192	0.58775	0.53757	0.57761	0.53850	0.56014
		$\hat{\delta}$	3.03469	3.06770	3.11509	3.04118	3.09781	3.04023	3.07489
		$\hat{\kappa}$	1.57260	1.56649	1.65780	1.52635	1.65208	1.54260	1.54055
	AVB	$\hat{\rho}$	0.13096	0.17348	0.23565	0.14386	0.23109	0.14610	0.17219
		$\hat{\delta}$	0.17941	0.23151	0.29393	0.20012	0.28508	0.19715	0.22459
		$\hat{\kappa}$	0.29963	0.36415	0.51977	0.30460	0.50699	0.32327	0.35146
	MSE	$\hat{\rho}$	0.03047	0.07002	0.15686	0.04158	0.13620	0.03915	0.06129
		$\hat{\delta}$	0.05648	0.12205	0.25027	0.07665	0.22107	0.06870	0.10343
		$\hat{\kappa}$	0.15729	0.22493	0.56042	0.17224	0.50982	0.18505	0.21295
	MRE	$\hat{\rho}$	0.26191	0.34695	0.47130	0.28772	0.46218	0.29219	0.34437
		$\hat{\delta}$	0.05980	0.07717	0.09798	0.06671	0.09503	0.06572	0.07486
		$\hat{\kappa}$	0.19975	0.24277	0.34651	0.20307	0.33799	0.21551	0.23431
400	AVE	$\hat{\rho}$	0.51200	0.52689	0.53675	0.52540	0.53707	0.52883	0.51915
		$\hat{\delta}$	3.01624	3.03684	3.04806	3.02951	3.04637	3.03206	3.02435
		$\hat{\kappa}$	1.53588	1.52257	1.55591	1.48598	1.55730	1.50231	1.54437
	AVB	$\hat{\rho}$	0.09003	0.11149	0.14263	0.09413	0.14921	0.10498	0.11394
		$\hat{\delta}$	0.12499	0.15018	0.18267	0.13220	0.18794	0.14136	0.15208
		$\hat{\kappa}$	0.20697	0.24149	0.32007	0.19607	0.33507	0.22659	0.26217
	MSE	$\hat{\rho}$	0.01375	0.02398	0.04397	0.01641	0.04258	0.01933	0.02252
		$\hat{\delta}$	0.02570	0.04117	0.07023	0.03113	0.06597	0.03384	0.03974
		$\hat{\kappa}$	0.07078	0.09843	0.18096	0.06585	0.19442	0.08545	0.11580
	MRE	$\hat{\rho}$	0.18006	0.22298	0.28526	0.18825	0.29843	0.20997	0.22788
		$\hat{\delta}$	0.04166	0.05006	0.06089	0.04407	0.06265	0.04712	0.05069
		$\hat{\kappa}$	0.13798	0.16099	0.21338	0.13071	0.22338	0.15106	0.17478

Table 7: Simulation results of the AVE, AVB, MSE, and MRE of BXLE distribution for $(\rho = 2.5, \delta = 0.75, \kappa = 2)$.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	WLSE
20	AVE	$\hat{\rho}$	1.93468	2.02949	1.92525	1.88766	1.93247	1.87049	1.91932
		$\hat{\delta}$	0.71610	0.72405	0.71885	0.71756	0.72157	0.71711	0.71864
		$\hat{\kappa}$	4.58878	3.56315	5.02440	3.97201	4.37834	3.55651	4.10333
	AVB	$\hat{\rho}$	1.03764	0.96413	1.06606	1.00392	1.05027	1.05225	1.04019
		$\hat{\delta}$	0.05603	0.05330	0.05959	0.06013	0.06046	0.06039	0.05869
		$\hat{\kappa}$	2.73667	1.77733	3.23614	2.28332	2.67764	1.83243	2.34900
	MSE	$\hat{\rho}$	1.60674	1.36611	1.71018	1.59658	1.66744	1.60003	1.59770
		$\hat{\delta}$	0.00514	0.00448	0.00546	0.00569	0.00558	0.00550	0.00527
		$\hat{\kappa}$	35.74006	16.09120	47.41465	26.43301	33.13628	14.69811	27.01181
	MRE	$\hat{\rho}$	0.41506	0.38565	0.42642	0.40157	0.42011	0.42090	0.41608
		$\hat{\delta}$	0.07471	0.07106	0.07946	0.08017	0.08061	0.08052	0.07825
		$\hat{\kappa}$	1.36834	0.88867	1.61807	1.14166	1.33882	0.91621	1.17450
30	AVE	$\hat{\rho}$	2.07335	2.11612	2.00259	2.06720	1.92691	2.01205	2.00773
		$\hat{\delta}$	0.72414	0.72853	0.72285	0.72763	0.71952	0.72425	0.72425
		$\hat{\kappa}$	3.31432	2.88686	3.82660	2.76187	3.84179	2.85758	3.23246
	AVB	$\hat{\rho}$	0.89648	0.87194	0.97766	0.82608	1.03461	0.93578	0.95710
		$\hat{\delta}$	0.04775	0.04680	0.05296	0.04633	0.05807	0.05184	0.05307
		$\hat{\kappa}$	1.48215	1.09070	2.02785	1.02366	2.08143	1.12326	1.47502
	MSE	$\hat{\rho}$	1.20422	1.10883	1.45171	1.09246	1.57836	1.27435	1.34196
		$\hat{\delta}$	0.00379	0.00352	0.00448	0.00352	0.00512	0.00420	0.00429
		$\hat{\kappa}$	11.76096	5.59121	19.43984	5.97968	19.49340	4.77205	10.10694
	MRE	$\hat{\rho}$	0.35859	0.34878	0.39106	0.33043	0.41384	0.37431	0.38284
		$\hat{\delta}$	0.06367	0.06240	0.07061	0.06177	0.07742	0.06912	0.07077
		$\hat{\kappa}$	0.74107	0.54535	1.01392	0.51183	1.04072	0.56163	0.73751
50	AVE	$\hat{\rho}$	2.17426	2.20516	2.05871	2.15930	2.00776	2.03987	2.19373
		$\hat{\delta}$	0.73060	0.73417	0.72631	0.73224	0.72417	0.72536	0.73278
		$\hat{\kappa}$	2.64343	2.51874	3.09799	2.40827	3.06424	2.56593	2.61239
	AVB	$\hat{\rho}$	0.78267	0.76088	0.89548	0.72196	0.94080	0.86487	0.80352
		$\hat{\delta}$	0.79162	0.71054	1.26072	0.66060	1.27875	0.76711	0.82421
		$\hat{\kappa}$	0.03975	0.03932	0.04529	0.03996	0.04997	0.04549	0.04171
	MSE	$\hat{\rho}$	0.89336	0.84562	1.17762	0.86059	1.29162	1.06094	0.95492
		$\hat{\delta}$	0.00260	0.00249	0.00330	0.00267	0.00388	0.00317	0.00285
		$\hat{\kappa}$	2.73791	1.90731	8.16773	1.58877	6.84234	1.77905	3.10024
	MRE	$\hat{\rho}$	0.31307	0.30435	0.35819	0.28878	0.37632	0.34595	0.32141
		$\hat{\delta}$	0.05300	0.05242	0.06039	0.05328	0.06663	0.06065	0.05561
		$\hat{\kappa}$	0.39581	0.35527	0.63036	0.33030	0.63937	0.38355	0.41210
100	AVE	$\hat{\rho}$	2.25841	2.26598	2.18105	2.29193	2.16943	2.20190	2.28508
		$\hat{\delta}$	0.73682	0.73771	0.73260	0.74043	0.73263	0.73487	0.73828
		$\hat{\kappa}$	2.28788	2.30145	2.48144	2.17275	2.45347	2.29386	2.29980
	AVB	$\hat{\rho}$	0.66104	0.66707	0.77255	0.55686	0.78640	0.70841	0.66816
		$\hat{\delta}$	0.03198	0.03297	0.03822	0.02917	0.03944	0.03547	0.03311
		$\hat{\kappa}$	0.44198	0.46307	0.64902	0.38714	0.63041	0.47545	0.46638
	MSE	$\hat{\rho}$	0.60785	0.62747	0.84610	0.52706	0.86283	0.70227	0.63059
		$\hat{\delta}$	0.00158	0.00170	0.00228	0.00142	0.00238	0.00192	0.00172
		$\hat{\kappa}$	0.45389	0.55123	1.32059	0.37179	1.10366	0.52812	0.63115
	MRE	$\hat{\rho}$	0.26442	0.26683	0.30902	0.22274	0.31456	0.28336	0.26727
		$\hat{\delta}$	0.04264	0.04397	0.05095	0.03890	0.05258	0.04729	0.04414
		$\hat{\kappa}$	0.22099	0.23154	0.32451	0.19357	0.31521	0.23772	0.23319

Table 7: Continued.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	WLSE
200	AVE	$\hat{\rho}$	2.38319	2.36681	2.32689	2.38031	2.26972	2.26997	2.35139
		$\hat{\delta}$	0.74284	0.74250	0.74039	0.74490	0.73778	0.73805	0.74156
		$\hat{\kappa}$	2.14738	2.16018	2.23583	2.07987	2.26382	2.18058	2.17066
	AVB	$\hat{\rho}$	0.54258	0.56624	0.64209	0.39355	0.66561	0.58906	0.57429
		$\hat{\delta}$	0.02548	0.02693	0.03106	0.02074	0.03193	0.02832	0.02726
		$\hat{\kappa}$	0.28846	0.31529	0.40522	0.24417	0.42483	0.32890	0.32415
	MSE	$\hat{\rho}$	0.39884	0.43566	0.57162	0.31207	0.61496	0.47433	0.45050
		$\hat{\delta}$	0.00097	0.00108	0.00146	0.00080	0.00152	0.00119	0.00111
		$\hat{\kappa}$	0.16655	0.21034	0.38818	0.13295	0.42311	0.21135	0.22558
	MRE	$\hat{\rho}$	0.21703	0.22650	0.25683	0.15742	0.26624	0.23562	0.22972
		$\hat{\delta}$	0.03397	0.03591	0.04141	0.02765	0.04257	0.03776	0.03635
		$\hat{\kappa}$	0.14423	0.15765	0.20261	0.12208	0.21242	0.16445	0.16208
400	AVE	$\hat{\rho}$	2.46565	2.44224	2.38062	2.43642	2.37639	2.38323	2.40852
		$\hat{\delta}$	0.74731	0.74643	0.74325	0.74681	0.74287	0.74408	0.74470
		$\hat{\kappa}$	2.06535	2.07995	2.14420	2.04099	2.14149	2.09305	2.10390
	AVB	$\hat{\rho}$	0.42714	0.46076	0.56325	0.24706	0.54240	0.47087	0.48153
		$\hat{\delta}$	0.01936	0.02167	0.02617	0.01321	0.02534	0.02215	0.02216
		$\hat{\kappa}$	0.20133	0.22996	0.30180	0.15109	0.29189	0.23358	0.24279
	MSE	$\hat{\rho}$	0.24466	0.28492	0.41630	0.17369	0.40917	0.30357	0.31548
		$\hat{\delta}$	0.00054	0.00067	0.00097	0.00041	0.00096	0.00071	0.00072
		$\hat{\kappa}$	0.07472	0.09483	0.18202	0.05741	0.17759	0.09695	0.11244
	MRE	$\hat{\rho}$	0.17086	0.18430	0.22530	0.09883	0.21696	0.18835	0.19261
		$\hat{\delta}$	0.02582	0.02889	0.03490	0.01762	0.03379	0.02953	0.02955
		$\hat{\kappa}$	0.10066	0.11498	0.15090	0.07554	0.14595	0.11679	0.12139

Table 8: Simulation results of the AVE, AVB, MSE, and MRE of BXLE distribution for $(\rho = 3, \delta = 1.5, \kappa = 0.5)$.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	WLSE
20	AVE	$\hat{\rho}$	2.25922	2.27161	2.30386	2.56517	2.31245	2.22016	2.11469
		$\hat{\delta}$	1.28865	1.33928	1.33780	1.46675	1.37558	1.33183	1.31017
		$\hat{\kappa}$	1.44443	1.06799	1.47947	0.89627	1.25029	1.04311	1.34574
	AVB	$\hat{\rho}$	1.24102	1.15163	1.10023	0.65276	1.04342	1.14332	1.28963
		$\hat{\delta}$	0.38019	0.37841	0.36227	0.28477	0.36259	0.38376	0.42134
		$\hat{\kappa}$	0.99257	0.64230	1.04387	0.51867	0.84803	0.63293	0.92532
	MSE	$\hat{\rho}$	2.49321	2.18957	2.28668	1.29495	2.14546	2.24551	2.64590
		$\hat{\delta}$	4.24912	1.78742	5.28595	1.98167	3.57678	1.73329	3.57683
		$\hat{\kappa}$	0.22030	0.20938	0.20249	0.15387	0.20329	0.21702	0.24799
	MRE	$\hat{\rho}$	0.41367	0.38388	0.36674	0.21759	0.34781	0.38111	0.42988
		$\hat{\delta}$	0.25346	0.25227	0.24151	0.18985	0.24172	0.25584	0.28089
		$\hat{\kappa}$	1.98514	1.28459	2.08774	1.03734	1.69606	1.26587	1.85063
30	AVE	$\hat{\rho}$	2.37795	2.35231	2.30591	2.58777	2.26834	2.31186	2.22954
		$\hat{\delta}$	1.32796	1.34465	1.33708	1.43656	1.34068	1.34833	1.31035
		$\hat{\kappa}$	0.99171	0.91347	1.14839	0.75291	1.15454	0.86102	0.99014
	AVB	$\hat{\rho}$	1.12486	1.10341	1.12391	0.59610	1.11783	1.10276	1.21249
		$\hat{\delta}$	0.34291	0.34590	0.35695	0.24323	0.36800	0.36261	0.37609
		$\hat{\kappa}$	0.54142	0.47675	0.70968	0.34657	0.73060	0.44939	0.55049
	MSE	$\hat{\rho}$	1.99694	1.96454	2.20519	1.16656	2.24897	2.02286	2.26525
		$\hat{\delta}$	0.17757	0.17668	0.19166	0.11519	0.20274	0.19130	0.20194
		$\hat{\kappa}$	1.43681	1.00028	2.28948	0.78501	2.48222	0.76471	1.18992
	MRE	$\hat{\rho}$	0.37495	0.36780	0.37464	0.19870	0.37261	0.36759	0.40416
		$\hat{\delta}$	0.22861	0.23060	0.23796	0.16216	0.24533	0.24174	0.25073
		$\hat{\kappa}$	1.08283	0.95351	1.41936	0.69315	1.46121	0.89878	1.10098

Table 8: Continued.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	WLSE
50		$\hat{\rho}$	2.41563	2.44759	2.33970	2.65901	2.34853	2.46812	2.33817
	AVE	$\hat{\delta}$	1.33871	1.36030	1.33053	1.43464	1.34434	1.37902	1.33240
		$\hat{\kappa}$	0.81702	0.72635	0.92425	0.63296	0.88870	0.72036	0.80824
		$\hat{\rho}$	1.07805	1.02476	1.08837	0.50426	1.08137	1.01465	1.13162
	AVB	$\hat{\delta}$	0.31870	0.31530	0.32707	0.19814	0.33465	0.32299	0.34249
		$\hat{\kappa}$	0.36429	0.28073	0.47801	0.20502	0.45484	0.29865	0.36362
		$\hat{\rho}$	1.77579	1.59334	1.99108	0.92871	1.96795	1.64311	1.92550
	MSE	$\hat{\delta}$	0.15294	0.14211	0.16498	0.08597	0.16787	0.15183	0.16718
		$\hat{\kappa}$	0.54310	0.23512	0.97545	0.22440	0.91236	0.30409	0.44341
		$\hat{\rho}$	0.35935	0.34159	0.36279	0.16809	0.36046	0.33822	0.37721
	MRE	$\hat{\delta}$	0.21247	0.21020	0.21804	0.13209	0.22310	0.21533	0.22833
		$\hat{\kappa}$	0.72859	0.56147	0.95601	0.41005	0.90969	0.59730	0.72724
100		$\hat{\rho}$	2.50497	2.54496	2.50903	2.82909	2.35625	2.59121	2.51501
	AVE	$\hat{\delta}$	1.36222	1.38965	1.37429	1.47182	1.33673	1.40815	1.37591
		$\hat{\kappa}$	0.66422	0.64091	0.71243	0.54529	0.74151	0.62469	0.66829
		$\hat{\rho}$	0.95829	0.93129	0.98960	0.28308	1.06864	0.91262	0.97376
	AVB	$\hat{\delta}$	0.27993	0.28231	0.29390	0.12694	0.31529	0.28449	0.28950
		$\hat{\kappa}$	0.20841	0.19472	0.26368	0.09863	0.29472	0.19449	0.21911
		$\hat{\rho}$	1.33183	1.26898	1.53254	0.45669	1.76001	1.23869	1.40130
	MSE	$\hat{\delta}$	0.11403	0.11208	0.12961	0.04257	0.14476	0.11340	0.12035
		$\hat{\kappa}$	0.10638	0.08990	0.20465	0.04885	0.23498	0.09389	0.12633
		$\hat{\rho}$	0.31943	0.31043	0.32987	0.09436	0.35621	0.30421	0.32459
	MRE	$\hat{\delta}$	0.41682	0.38945	0.52735	0.19725	0.58944	0.38899	0.43823
		$\hat{\kappa}$	0.18662	0.18821	0.19593	0.08463	0.21019	0.18966	0.19300
200		$\hat{\rho}$	2.68622	2.63929	2.54449	2.91341	2.52655	2.77992	2.57827
	AVE	$\hat{\delta}$	1.41291	1.40348	1.38300	1.48787	1.38236	1.45466	1.39272
		$\hat{\kappa}$	0.58701	0.59670	0.64583	0.51505	0.64304	0.56015	0.61597
		$\hat{\rho}$	0.79267	0.82070	0.93297	0.13418	0.93781	0.76320	0.86433
	AVB	$\hat{\delta}$	0.22912	0.24048	0.27404	0.07519	0.27461	0.23440	0.25492
		$\hat{\kappa}$	0.13258	0.14285	0.19532	0.04978	0.19404	0.12768	0.16375
		$\hat{\rho}$	0.87772	0.95246	1.28415	0.19687	1.27774	0.83111	1.07875
	MSE	$\hat{\delta}$	0.07403	0.08190	0.10702	0.01894	0.10602	0.07514	0.09153
		$\hat{\kappa}$	0.03721	0.04421	0.09242	0.01387	0.09175	0.03414	0.05970
		$\hat{\rho}$	0.26422	0.27357	0.31099	0.04473	0.31260	0.25440	0.28811
	MRE	$\hat{\delta}$	0.15275	0.16032	0.18270	0.05013	0.18307	0.15627	0.16995
		$\hat{\kappa}$	0.26516	0.28570	0.39063	0.09955	0.38808	0.25536	0.32749
400		$\hat{\rho}$	2.81400	2.78651	2.67628	2.97841	2.68454	2.84482	2.72588
	AVE	$\hat{\delta}$	1.45146	1.44724	1.41857	1.49984	1.42069	1.46642	1.42676
		$\hat{\kappa}$	0.54913	0.55669	0.59135	0.49916	0.58688	0.53957	0.56863
		$\hat{\rho}$	0.65747	0.70065	0.80308	0.03702	0.79883	0.66605	0.73811
	AVB	$\hat{\delta}$	0.19107	0.20580	0.23258	0.03920	0.23358	0.19898	0.21271
		$\hat{\kappa}$	0.09696	0.10719	0.13937	0.02315	0.13700	0.09832	0.11442
		$\hat{\rho}$	0.59368	0.68521	0.92646	0.04280	0.91585	0.60649	0.75292
	MSE	$\hat{\delta}$	0.05023	0.05802	0.07557	0.00509	0.07574	0.05308	0.06228
		$\hat{\kappa}$	0.01849	0.02262	0.04460	0.00308	0.04000	0.01828	0.02615
		$\hat{\rho}$	0.21916	0.23355	0.26769	0.01234	0.26628	0.22202	0.24604
	MRE	$\hat{\delta}$	0.12738	0.13720	0.15505	0.02613	0.15572	0.13265	0.14180
		$\hat{\kappa}$	0.19392	0.21439	0.27874	0.04630	0.27400	0.19663	0.22884

Table 9: Estimated parameters with their standard errors of the BXLE model and other fitted models for first dataset.

Model	Estimated parameters (standard errors)			
BXLE	$\hat{\rho} = 16.6547$ (6215.44)	$\hat{\delta} = 0.01708$ (6.37703)	$\hat{\kappa} = 39.5297$ (7.64793)	
FWME	$\hat{\alpha} = 2.77136$ (1.2456043)	$\hat{\lambda} = 2.88091$ (0.0012536)	$\hat{k} = 3.11453$ (1.256135)	$\hat{a} = 3.84424$ (0.012989)
BE	$\hat{\lambda} = 0.07354$ (0.00071)	$\hat{a} = 24.2815$ (8.98448)	$\hat{b} = 121.998$ (7.64793)	
APExE	$\hat{\alpha} = 46.3239$ (65.8173)	$\hat{a} = 2.52118$ (0.217317)	$\hat{c} = 91.0457$ (52.8461)	
GExE	$\hat{\lambda} = 0.25249$ (0.382442)	$\hat{\alpha} = 35.2218$ (14.7076)	$\hat{\delta} = 27.6803$ (49.284)	
TGE	$\hat{\alpha} = 90.1525$ (38.8898)	$\hat{\lambda} = -0.697529$ (0.20628)	$\hat{\theta} = 2.21536$ (0.187642)	
ExE	$\hat{b} = 89.4374$ (33.3448)	$\hat{a} = 2.01918$ (0.002538)		
TE	$\hat{\beta} = 0.585265$ (0.0672482)	$\hat{\lambda} = -1.00000$ (0.459119)		
LE	$\hat{\beta} = 1.74693 \times 10^{-22}$ (0.375895)	$\hat{\theta} = 0.313899$ (0.162698)		
E	$\hat{a} = 0.40367$ (0.0469257)			
NH	$\hat{\alpha} = 38.5268$ (26.7975)	$\hat{\lambda} = 0.0082434$ (0.0057971)		

Table 10: Estimated parameters with their standard errors of the BXLE model and other fitted models for second dataset.

Model	Estimated parameters (standard errors)			
BXLE	$\hat{\rho} = 0.626149$ (0.970701)	$\hat{\delta} = 0.293385$ (0.432376)	$\hat{\kappa} = 6.24390$ (9.78511)	
FWME	$\hat{\alpha} = 1.90568$ (0.024546)	$\hat{\lambda} = 2.84917$ (0.567416)	$\hat{k} = 2.16087$ (0.005413)	$\hat{a} = 1.71784$ (0.025468)
BE	$\hat{\lambda} = 0.056844$ (0.000868)	$\hat{a} = 5.95815$ (2.13741)	$\hat{b} = 37.5763$ (0.821257)	
APExE	$\hat{\alpha} = 32.2179$ (58.4502)	$\hat{a} = 1.23847$ (0.103739)	$\hat{c} = 4.60025$ (2.54393)	
GExE	$\hat{\lambda} = 0.269724$ (0.257315)	$\hat{\alpha} = 8.07549$ (2.2026)	$\hat{\delta} = 6.15979$ (6.82518)	
TGE	$\hat{\alpha} = 6.18732$ (1.92831)	$\hat{\lambda} = -0.683705$ (0.279297)	$\hat{\theta} = 1.10199$ (0.094658)	
ExE	$\hat{b} = 7.78824$ (33.3448)	$\hat{a} = 1.01317$ (0.002538)		
TE	$\hat{\beta} = 32.2179$ (0.557692)	$\hat{\lambda} = -1.00000$ (0.398622)		
LE	$\hat{\beta} = 4.07405 \times 10^{-10}$ (0.0859971)	$\hat{\theta} = 0.253513$ (0.0475148)		
E	$\hat{a} = 0.381476$ (0.0381476)			
NH	$\hat{\alpha} = 38.3038$ (24.8412)	$\hat{\lambda} = 0.007253$ (0.004764)		

Table 11: Discrimination measures of the BXLE model and other competing models for first dataset.

Model	AKI	CAKI	BAI	HAQUI	ANDA	CRVMI	KOSM	p value
BXLE	110.123	110.466	117.036	112.881	0.380356	0.059098	0.056529	0.972044
FWME	116.084	116.664	125.301	119.761	0.56027	0.0775639	0.0693095	0.927044
BE	112.344	112.687	119.257	115.102	0.565112	0.086484	0.068130	0.882162
APExE	115.308	115.651	122.22	118.066	0.715941	0.0965186	0.0697025	0.864808
GExE	112.777	113.12	119.689	115.534	0.600183	0.0920141	0.0700697	0.860604
TGE	119.542	119.885	126.454	122.3	1.08865	0.15413	0.084345	0.668357
ExE	121.607	121.776	126.215	123.445	1.50244	0.223549	0.0953139	0.512098
TE	237.473	237.642	242.081	239.311	15.8457	3.1266	0.359161	00000
LE	192.302	192.471	196.91	194.14	13.3126	2.65435	0.339298	00000
E	284.259	284.315	286.563	285.178	22.1407	4.67218	0.44947	00000
NH	239.564	239.733	244.172	241.402	23.9619	5.22339	0.468589	00000

Table 12: Discrimination measures of the BXLE model and other competing models for second dataset.

Model	AKI	CAKI	BAI	HAQUI	ANDA	CRVMI	KOSM	p value
BXLE	288.775	289.025	296.591	291.938	0.41295	0.067419	0.060811	0.853353
FWME	300.559	300.98	310.98	304.776	1.10367	0.167555	0.0902804	0.388871
BE	292.486	292.736	300.301	295.649	0.760458	0.15047	0.09349	0.346368
APExE	291.673	291.923	299.489	294.837	0.664769	0.125778	0.0870556	0.43464
GExE	293.482	293.732	301.297	296.645	0.834621	0.164551	0.0963921	0.310696
TGE	294.83	295.08	302.646	297.993	0.927332	0.174829	0.0966358	0.307819
ExE	296.365	296.488	301.575	298.473	1.22463	0.229158	0.107725	0.196182
TE	342.209	342.333	347.419	344.318	8.42304	1.4915	0.21033	0.000287
LE	303.002	303.126	308.212	305.111	3.5464	0.634098	0.138342	0.0435195
E	394.742	394.783	397.347	395.796	17.3021	3.43402	0.320593	00000
NH	347.224	347.348	352.434	349.333	13.8496	2.83065	0.288983	00000

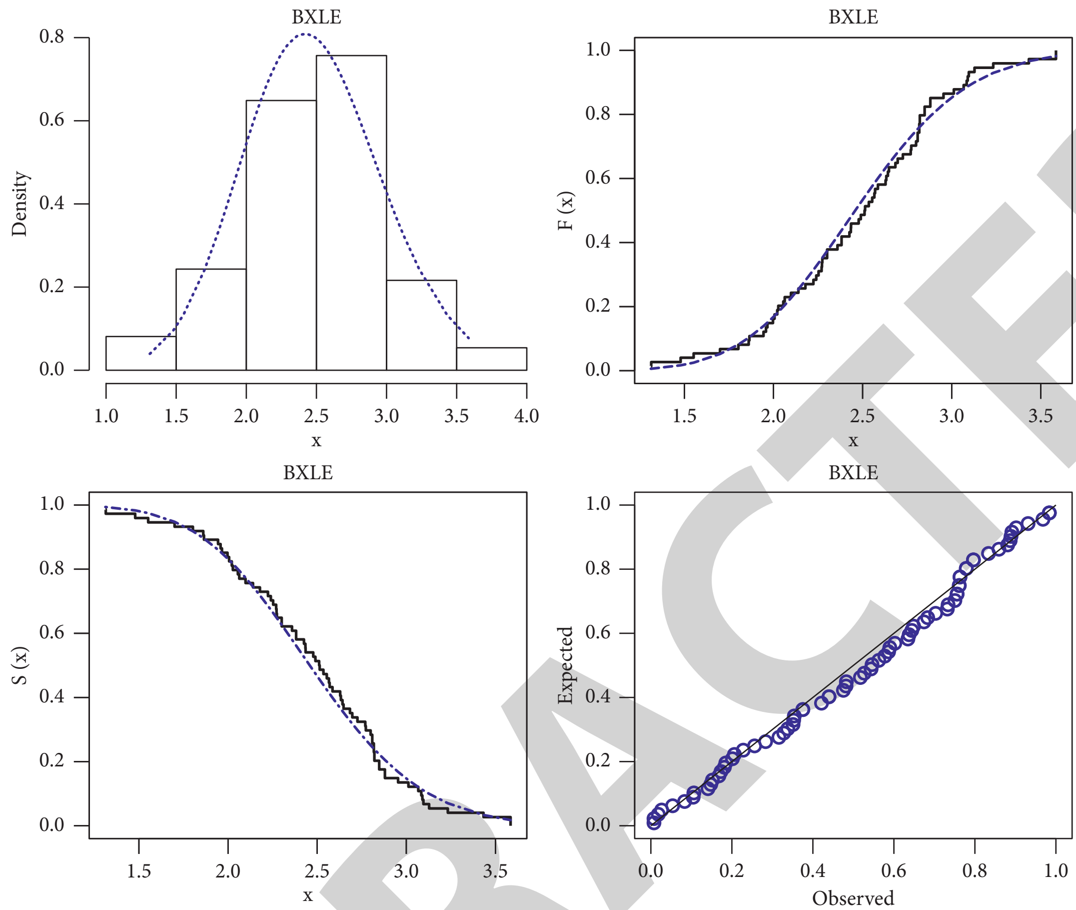

Figure 3: The fitted BXLE PDF, CDF, SF, and P-P plots for first dataset.

Tables 1–8 provide the simulation results for the BXLE parameters utilizing the seven estimate methodologies, including AVE, AVB, MSE, and MRE. The estimates of the BXLE parameters derived from all seven estimation techniques are fully good, that is, they are extremely trustworthy and very near to the real values, with negligible biases, MSE, and MRE in all parameter combinations. For all parameter combinations, all estimators exhibit the consistency property, in which the MSE, AVB, and MRE drop as sample size grows. We find that the MLE, ADE, CVME, LSE, MPSE, PCE, and WLSE approaches do an excellent job at estimating BXLE parameters.

5. Application

In this part, we will look at two real-world datasets. The first dataset has 74 observations and represents gauge lengths of 20 mm [17]. The second set is made up of 100 observations and reflects the breaking stress of carbon fiber [18].

We compare the BXLE model with some other well-known competitive distributions such as the beta *E* (BE) [19], transmuted generalized-E (TGE) [5], exponentiated E (ExE), alpha power exponentiated E (APExE) transmuted E (TE) [23], exponetial (E), Nadarajah-Haghighi (NH) [4], Fréchet Weibull mixture E (FWME) [21], gamma exponentiated E (GExE) [22], and linear E (LE) [23] distributions.

Some discriminatory practice measures, such as Akaike information (AKI), Hannan–Quinn information (HAQUI), Bayesian information (BAI), and consistent Akaike information (CAKI), can be used to compare competing models. Other discrimination measures include the Anderson–Darling (ANDA), Cramér–von Mises (CRVMI), and the p value of the Kolmogorov–Smirnov (KOSM).

The estimated parameters using the maximum likelihood method and their standard errors for the BXLE model and other compared models are reported in Tables 9 and 10 for the two datasets, respectively. The values of discrimination measures are listed in Tables 11 and 12. The values in Tables 11 and 12 indicate that the proposed BXLE distribution provides better fit for the two analyzed datasets than other competing models.

The fitted functions are displayed graphically, including the PDF, CDF, SF, and PP plots, in Figures 3 and 4. These plots supports the numerical values in Tables 11 and 12 that the proposed BXLE model provides the best fit for the two datasets.

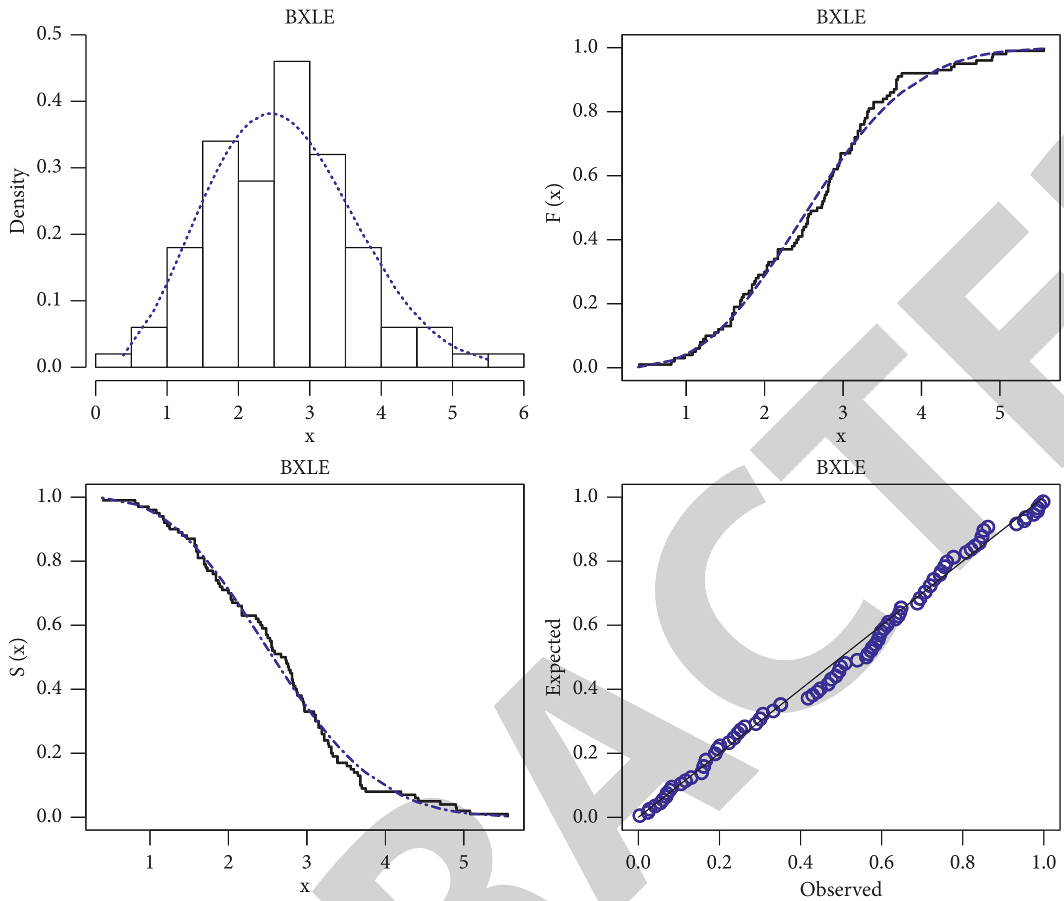

FIGURE 4: The fitted BXLE PDF, CDF, SF, and P-P plots for second dataset.

6. Concluding Remarks

This study proposed a unique three-parameter Burr X logistic-exponential (BXLE) distribution for modeling engineering data and other applications. The BXLE model generalizes and extends the logistic-exponential distribution. The hazard rate of the BXLE distribution might be declining, increasing, bathtub, unimodal, J-shape, or reversed-J shape. In certain circumstances, its mathematical properties were derived. Its density was determined as a mixture of exponential densities. The maximum likelihood estimators, Cramér–von Mises estimators, Anderson–Darling estimators, maximum product of spacing estimators, least-squares estimators, percentile estimators, and weighted least-squares estimators were used to estimate the unknown parameters of the BXLE model. All estimators perform brilliantly in predicting the BXLE parameters, as proved by simulation data. According to our findings, the maximum likelihood approach delivers the best accurate estimations of the parameters of the BXLE distribution. The BXLE distribution's practical importance was proved using two authentic engineering datasets, proving its acceptable fits and benefits over other competing contemporary models.

Data Availability

The datasets used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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