Research Article

Fekete–Szegö Problems for Certain Classes of Meromorphic Functions Involving $q$-Al-Oboudi Differential Operator

M.K. Aouf and Fatma Z. El-Emam

1Department of Mathematics, Faculty of Science Mansoura University, Mansoura 35516, Egypt
2Delta Higher Institute for Engineering and Technology Mansoura, Mansoura, Egypt

Correspondence should be addressed to Fatma Z. El-Emam; fatma_elemam@yahoo.com

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1. Introduction

For two analytic functions $f$ and $g$ in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$, if there is a Schwarz function $w(z)$ with $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{U}$), such that $f(z) = g(w(z))$, ($z \in \mathbb{U}$). If $g$ is univalent, then $f < g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

A function $f$ is meromorphic if it is analytic throughout a domain $D$, except possibly for poles in $D$ (see [40]).

Let $\Sigma$ denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k,$$

which are analytic in the open punctured unit disc $\mathbb{U}^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \} = \mathbb{U}/\{0\}$. A function $f \in \Sigma$ is said to be a meromorphic starlike function of order $\zeta$, denoted by $\mathcal{S}^*(\zeta)$, if

$$-\Re \left[ \frac{zf'(z)}{f(z)} \right] > \zeta, 0 \leq \zeta < 1.$$

The class $\mathcal{S}^*(\zeta)$ was studied by Pommerenke [29], Miller [23], and many others (see [9, 25]).

Let $\varphi(z)$ be an analytic function with a positive real part on $\mathbb{U}$ satisfying $\varphi(0) = 1$ and $\varphi'(0) > 0$ which maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis.

Let $\mathcal{S}_b^\varphi$ be the class of functions $f \in \Sigma$ for which

$$1 - \frac{1}{b} \left[ \frac{zf'(z)}{f(z)} + 1 \right] \varphi(z), \quad b \in \mathbb{C}^* = \mathbb{C}/\{0\}.$$

The class $\mathcal{S}_b^\varphi$ was introduced and studied by Mohammed and Darus [26] (see also Reddy and Sharma [30], with $\gamma = 1$).

We note that for suitable choices of $b$ and $\varphi(z)$, we obtain the following subclasses:

1. $\mathcal{S}_1^\varphi = \mathcal{S}^*(\varphi)$ (see [4], with $\alpha = 1$ and [33]);
2. $\mathcal{S}^\varphi_\alpha = (1 + z)/(1 - z) = F^\varphi_\alpha$ (see [6]);
3. $\mathcal{S}^\varphi_\alpha ((1 + (1 - 2\zeta)z)/(1 - z)) = \mathcal{S}^\varphi_\zeta, \quad (0 \leq \zeta < 1)$ (see [29]);
4. $\mathcal{S}^\varphi_\alpha (1 + Az)/(1 - Az) = K_1(A,B), \quad (0 \leq B \leq 1, -B \leq A < B)$ (see [17]);
5. $\mathcal{S}^\varphi_\alpha (1 - p)e^{i\theta} + (1 + z)/(1 - z) = \mathcal{S}^\varphi_\alpha (\rho), \quad (0 \leq \rho < 1, |\theta| < (\pi/2))$ (see [16, 31]).
Let $M^*_b(\varphi)$ be the class of functions $f \in \Sigma$ for which
\[ 1 - \frac{1}{b} \left[ \frac{z^f(z)}{f'(z)} + 2 \right] < \varphi(z), \quad b \in \mathbb{C}^*. \quad (5) \]

We note that

1. $M^*_b (1 + z)/(1 - z) = G^*(b)$ (see Aouf [6]);
2. $M^*_{(1-\rho)e^{i\theta}}(\varphi) = M^*_{p,\rho}(\varphi), \quad (0 \leq \rho < 1, |\theta| < (\pi/2), \quad f \in \sum; (-e^{i\theta}(1 + z f^\prime)(z)/f'(z)) - \rho \cos \theta - i \sin \theta/((1 - \rho) \cos \theta < \varphi(z), z \in U^*).$

In geometric function theory, operators play an important role. Many authors present differential and integral operators, for example ([1], [20, 32, 37]). For a function $f \in \sum$ given by (2), the $\delta$-derivative of a function $f(z)$ is defined by [3, 11] (see also [14, 15])
\[ D_\delta^* f(z) = \frac{f(z) - f(\delta z)}{(1 - \delta)z} \quad (6) \]
where
\[ [k]_{\delta} = \frac{1 - \delta^k}{1 - \delta} \quad (7) \]

As $\delta \rightarrow 1^-$, we have $[k]_{\delta} \rightarrow k$ and $\lim_{\delta \rightarrow 1^-} D_\delta^* f(z) = f'(z)$.

Due to its use in numerous fields of mathematics and physics, the $\delta$-derivative operator $D_\delta^*$ has fascinated and inspired many researchers. Jackson [14] was among the key contributors of all the scientists who introduced and developed the $\delta$-calculus theory. In 1991, Ismail [13] was the first to demonstrate a crucial link between geometric function theory and the $\delta$-derivative operator, but a solid and comprehensive foundation was provided in 1989 in a book chapter by Srivastava [34]. Several recent works on this operator can be found in ([7, 18, 19, 35, 36]).

For $f(z) \in \sum_0, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\}, \lambda \geq 0$ and $0 < \delta < 1$, we define the following operator $D_{\delta,\lambda}^*$ as follows:
\[ D_{\delta,\lambda}^* f(z) = f(z), \quad D_{\delta,\lambda}^{*n} f(z) = (1 - \lambda)f(z) + \frac{\lambda}{z} D_{\delta,\lambda}^{*n-1} f(z) \quad (n \in \mathbb{N}). \quad (8) \]

From (2) and (8), we obtain
\[ D_{\delta,\lambda}^{*n} f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left[ 1 + \lambda((k + 2)\alpha - 1) \right] n_{\alpha} z^k, \quad (n \in \mathbb{N}_0) \]
\[ = \frac{1}{z} + \sum_{k=0}^{\infty} [1 + \lambda \delta(k+1)] n_{\alpha} z^k, \quad n \in \mathbb{N}_0. \quad (9) \]

From (9), it is easy to see that, for $f(z) \in \sum_0, \lambda^2 z D_{\delta,\lambda}^{*n} (D_{\delta,\lambda}^{*n} f(z)) = D_{\delta,\lambda}^{*n+1} f(z) - (\lambda^2 + 1) D_{\delta,\lambda}^{*n} f(z), \quad (\lambda > 0). \quad (10) \]

We note that

1. $\lim_{\alpha \rightarrow 1^-} D_{\delta,\lambda}^{*n} f(z) = D_{\delta,\lambda}^{*n} f(z) = (1/z) + \sum_{k=0}^{\infty} [1 + \lambda((k + 1)) n_{\alpha} z^k \quad (5), \quad (k = 1, \ldots, m)]. \quad (p = 1).$
2. $\lim_{\alpha \rightarrow \frac{1}{2}} D_{\delta,\lambda}^{*n} f(z) = D_{\delta,\lambda}^{*n} f(z) = (1/z) + \sum_{k=0}^{\infty} (k + 1)) n_{\alpha} z^k \quad (2), \quad (k = 1, \ldots, m)]. \quad (p = 1).$

Making use of $D_{\delta,\lambda}^{*n}$, we define the following class $\Sigma_{\delta,\lambda,\alpha}^* (b, \varphi)$ as follows:

**Definition 1.** For $n \in \mathbb{N}_0, \lambda \geq 0, 0 < \delta < 1, \ b \in \mathbb{C}^*$, and $0 \leq \alpha < (\delta \delta + 1)$, we say that a function $f \in \sum$ is in the class $\Sigma_{\delta,\lambda,\alpha}^* (b, \varphi)$ if and only if
\[ 1 + \frac{1}{b} \left[ \frac{-(1 - (\alpha/\delta)) \delta z D_{\delta,\lambda}^{*n} (D_{\delta,\lambda}^{*n} f(z)) + a \delta z D_{\delta,\lambda}^{*n} (z D_{\delta,\lambda}^{*n} f(z))}{(1 - (\alpha/\delta)) D_{\delta,\lambda}^{*n} f(z) - a z D_{\delta,\lambda}^{*n} (D_{\delta,\lambda}^{*n} f(z))} - 1 \right] < \varphi(z). \quad (11) \]

Noting that

1. $\Sigma_{\delta,\lambda,\alpha}^{*n} (b, \varphi) = \Sigma_{\delta,\lambda,\alpha}^{*n} (b, \varphi) = \{ f \in \sum_0: 1 - (1/b) \ [ (z D_{\delta,\lambda}^{*n} f(z))/D_{\delta,\lambda}^{*n} f(z)] + 1 \} \varphi(z), \ z \in U^*; \quad (1) \]
2. $\lim_{\delta \rightarrow 1^-} \Sigma_{\delta,\lambda,\alpha}^{*n} (b, \varphi) = \Sigma_{\lambda,\alpha}^{*n} (b, \varphi) = \{ f \in \sum_0: 1 - (1/b) [(z D_{\delta,\lambda}^{*n} f(z))/D_{\delta,\lambda}^{*n} f(z)] + 1 \} \varphi(z), \ z \in U^*; \quad (2) \]
3. $\Sigma_{\lambda,\alpha}^{*n} (b, \varphi) = \Sigma_{\delta,\lambda,\alpha}^{*n} (b, \varphi) = \{ f \in \sum_0: 1 - (1/b) \ [ (z D_{\delta,\lambda}^{*n} f(z))/D_{\delta,\lambda}^{*n} f(z)] + 1 \} \varphi(z), \ z \in U^*; \quad (3) \]

\[ \{ [\delta z D_{\delta,\lambda}^{*n} (f(z))/f(z)] + 1 \} \varphi(z), \ z \in U^*; \quad (4) \]

\[ \Sigma_{\delta,\lambda,\alpha}^{*n} (1, \varphi) = \Sigma_{\delta,\lambda,\alpha}^{*n} (\varphi) \quad (5) \]

\[ \Sigma_{\lambda,\alpha}^{*n} (1 - \rho e^{-i\theta} \cos \theta, \varphi) = \Sigma_{\lambda,\alpha}^{*n} (\rho, \theta, \varphi), \quad (0 \leq \rho < 1, |\theta| < \pi/2) = \{ f \in \sum_0: [(-e^{i\theta} (\delta z D_{\delta,\lambda}^{*n} f(z))/D_{\delta,\lambda}^{*n} f(z)] + 1 \} \varphi(z), \ z \in U^*; \quad (6) \]

\[ \phi(z), \ z \in U^*; \quad (7). \]
The result is sharp for the functions given by

\[ p(z) = \frac{1 + z^2}{1 - z^2}, \]

\[ p(z) = \frac{1 + z}{1 - z}. \]

**Lemma 2** (see [22]). If \( h(z) = 1 + c_1 z + c_2 z^2 + \cdots \) is a function with positive real part in \( \mathbb{U} \), then

\[
|c_2 - v c_1^2| \leq \left\{ \begin{array}{ll}
0 & \text{if } 0 \leq v \leq 1, \\
4v - 2 & \text{if } v > 1.
\end{array} \right.
\]
Proof. If \( f(z) \in \sum_{k=0}^{n} (b, \varphi) \), then there is a Schwarz function \( w(z) \) in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) in \( U \), such that

\[
1 + \frac{1}{b} \left[ -\frac{(1 - (a\delta))\delta zD^*_\delta(D^*_\delta f(z)) + a\delta zD^*_\delta zD^*_\delta f(z) - azD^*_\delta (D^*_\delta f(z))}{(1 - (a\delta))D^*_\delta f(z) - azD^*_\delta (D^*_\delta f(z))} - 1 \right] = \varphi(w(z)).
\]  

(21)

If we set

\[
h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots.
\]  

(22)

Using (21)–(23), we get

\[
p(z) = 1 + \frac{1}{b} \left[ -\frac{(1 - (a\delta))\delta zD^*_\delta(D^*_\delta f(z)) + a\delta zD^*_\delta zD^*_\delta f(z) - azD^*_\delta (D^*_\delta f(z))}{(1 - (a\delta))D^*_\delta f(z) - azD^*_\delta (D^*_\delta f(z))} - 1 \right]
\]  

(23)

\[
= 1 + b_1 z + b_2 z^2 + \cdots.
\]

Then, from (2) and (23), we see that \( b_1 = -(1/b)(1 - (a\delta))D^*_\delta f(z) [1 + \lambda \delta]a_0 \) and \( b_2 = (1/b)(1 - (a\delta))^2 [1 + \lambda \delta]a_0 \) \( a_0^2 - (\delta + 1)(1 - a - (a\delta))[1 + \lambda \delta(\delta + 1)]a_1 \), or, equivalently, we obtain

\[
a_0 = -\frac{\delta d_1 c_1 b}{2(\delta - \alpha)[1 + \lambda \delta]^n},
\]

\[
a_1 = -\frac{\delta d_1 c_2 b}{2(\delta + 1)[\delta - \alpha(\delta + 1)][1 + \lambda \delta(\delta + 1)]^n} \left[ c_2 - \frac{c_1^2}{2} \left( 1 - \frac{d_1}{d_2} + d_1 b \right) \right].
\]

Therefore

From (24) and (26), we have

\[
b_1 = \frac{1}{2} d_1 c_1,
\]

\[
b_2 = \frac{1}{2} d_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} d_2 c_1^2.
\]

(27)

Applying Lemma 1, we obtain the result (19). Also, if \( d_1 = 0 \), then

\[
a_0 = 0,
\]

\[
a_1 = -\frac{\delta d_1 c_2^2 b}{4(\delta + 1)[\delta - \alpha(\delta + 1)][1 + \lambda \delta(\delta + 1)]^n}.
\]

(30)
Since $\Re \{ p(z) \} > 0$, then $|c_1| \leq 2$ (see [28]), hence 
$$
|a_1| \leq \frac{\delta}{(\delta + 1)[1 + \lambda \delta (\delta + 1)]^n} \left| \frac{d_1 b}{\delta - \alpha (\delta + 1)} \right| \tag{31}
$$
this proves (20). The result is sharp for the functions 
$$
1 + \frac{1}{b} \left[ \frac{-(1 - (a/\delta)) \delta z D_1^* \left( D_{1,\delta}^* f(z) \right) + a \delta z D_1^* \left( z D_1^* \left( D_{1,\delta}^* f(z) \right) \right)}{(1 - (a/\delta)) D_{1,\delta}^* f(z) - az D_1^* \left( D_{1,\delta}^* f(z) \right)} - 1 \right] = \phi\left( z^2 \right),
$$
which completes the proof of Theorem 1.

Taking $\alpha = 0$ in Theorem 1, we get 
$$
|a_1| \leq \frac{d_1 |b|}{(\delta + 1)[1 + \lambda \delta (\delta + 1)]^n} \max \left\{ 1, \left| \frac{d_2}{d_1} - \left[ \frac{1 - \mu (\delta + 1)[1 + \lambda \delta (\delta + 1)]^n} {1 + \lambda \delta (\delta + 1)} \right] d_1 b \right| \right\}, \quad d_1 \neq 0, 
$$
$$
|a_1| \leq \frac{|d_2 b|}{(\delta + 1)[1 + \lambda \delta (\delta + 1)]^n}, \quad d_1 = 0.
$$

The result is sharp.

Taking $\alpha = n = 0$ in Theorem 1, we get 

**Corollary 1.** Let $f(z)$ be defined by (2) and $\phi(z) = 1 + d_1 z + d_2 z^2 + \cdots (d_1 \geq 0)$. If $f(z) \in \sum_{\delta}^n (b, \phi)$ and $\mu \in C$, then 

$$
\left| a_1 - \mu a_0 \right| \leq \frac{d_1 |b|}{(\delta + 1)[1 + \lambda \delta (\delta + 1)]^n} \max \left\{ 1, \left| \frac{d_2}{d_1} - \left[ \frac{1 - \mu (\delta + 1)[1 + \lambda \delta (\delta + 1)]^n} {1 + \lambda \delta (\delta + 1)} \right] d_1 b \right| \right\}, \quad d_1 \neq 0, 
$$
$$
\left| a_1 \right| \leq \frac{|d_2 b|}{(\delta + 1)[1 + \lambda \delta (\delta + 1)]^n}, \quad d_1 = 0.
$$

The result is sharp.

**Remark 1.**

(1) For $b = 1$ in Corollary 2, we get the result obtained by [12], [Theorem 2.3].

(2) For $\delta \rightarrow 1^+$ in Corollary 2, we get the results obtained by [26, 30].

(3) For $\delta \rightarrow 1^+$ and $b = 1$ in Corollary 2, we get the results obtained by [33] and [4], [Theorem 5.2].

Taking $n = 0$ in Theorem 1, we get 

**Corollary 3.** Let $f(z)$ be defined by (2) and $\phi(z) = 1 + d_1 z + d_2 z^2 + \cdots (d_1 \geq 0)$. If $f(z) \in \sum_{\delta, \alpha}^n (b, \phi)$ and $\mu \in C$, then 

$$
\left| a_1 - \mu a_0 \right| \leq \frac{d_1 \delta}{(\delta + 1)[\delta - \alpha (\delta + 1)]^n} \max \left\{ 1, \left| \frac{d_2}{d_1} - \left[ \frac{1 - \mu \delta (\delta + 1)[\delta - \alpha (\delta + 1)]} {\delta - \alpha (\delta + 1)} \right] d_1 b \right| \right\}, \quad d_1 \neq 0, 
$$
$$
\left| a_1 \right| \leq \frac{\delta}{(\delta + 1)} \left| \frac{d_2 b}{\delta - \alpha (\delta + 1)} \right|, \quad d_1 = 0.
$$
The result is sharp.

Remark 2.

(1) Taking \( b = 1 \) in Corollary 3, we get the result obtained by [12], [Theorem 2.8].

(2) Letting \( \delta \longrightarrow 1^- \) in Corollary 3, we get the result obtained by [30].

\[
|a_1 - \mu a_0^2| \leq \frac{d_1 \delta}{(\delta + 1)[1 + \lambda \delta (\delta + 1)]^n} \left| \frac{1}{\delta - \alpha (\delta + 1)} \right|^\mu \\
\times \max \left\{ 1, \left[ 1 - \frac{\delta (\delta + 1)}{(\delta - \alpha)^2 (1 + \lambda \delta)^{2n}} \right] d_1 \right\}, \quad d_1 \neq 0.
\]

The result is sharp.

Remark 3. Letting \( \delta \longrightarrow 1^- \) and taking \( \kappa = 0 \) and \( \varphi(z) = (1 + z)/(1 - z) \) in Corollary 5, we get the result obtained by [27], [Example 1.1].

\[
|a_1 - \mu a_0^2| \leq \frac{d_1 (1 - \rho) \cos \theta}{(\delta + 1)[1 + \lambda \delta (\delta + 1)]^n} \\
\times \max \left\{ 1, \left[ 1 - \frac{\delta (\delta + 1)}{(\delta - \alpha)^2 (1 + \lambda \delta)^{2n}} \right] d_1 (1 - \rho) \cos \theta \right\}, \quad d_1 \neq 0.
\]

The result is sharp.

Corollary 4. Let \( f(z) \) be defined by (2) and \( \varphi(z) = 1 + d_1 z + d_2 z^2 + \cdots (d_1 \geq 0) \). If \( f(z) \in \sum_{\kappa, \lambda, \mu}(\varphi) \) and \( \mu \in \mathbb{C} \), then

\[
|a_1 - \mu a_0^2| \leq \frac{\delta}{(\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n} \\
\times \max \left\{ 1, \left[ 1 - \frac{\delta (\delta + 1)}{(\delta - \alpha)^2 (1 + \lambda \delta)^{2n}} \right] d_1 \right\}, \quad d_1 \neq 0.
\]

By using Lemma 2, we can obtain the following theorem.

Theorem 2. For real \( \mu \), let \( \varphi(z) = 1 + d_1 z + d_2 z^2 + \cdots (d_1 > 0, i \in 1, 2) \). If \( f(z) \) given by (2) belongs to the class \( \sum_{\kappa, \lambda, \mu}(1, \varphi) = \sum_{\kappa, \lambda, \mu}(\varphi) \), then

\[
|a_1 - \mu a_0^2| \leq \frac{\delta}{(\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n} \\
\times \max \left\{ 1, \left[ 1 - \frac{\delta (\delta + 1)}{(\delta - \alpha)^2 (1 + \lambda \delta)^{2n}} \right] d_1 \right\}, \quad d_1 \neq 0.
\]

\[
|a_1 - \mu a_0^2| \leq \frac{\delta d_1}{(\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n} \\
\times \max \left\{ 1, \left[ 1 - \frac{\delta (\delta + 1)}{(\delta - \alpha)^2 (1 + \lambda \delta)^{2n}} \right] d_1 \right\}, \quad d_1 \neq 0.
\]

\[
|a_1 - \mu a_0^2| \leq \frac{\delta}{(\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n} \\
\times \max \left\{ 1, \left[ 1 - \frac{\delta (\delta + 1)}{(\delta - \alpha)^2 (1 + \lambda \delta)^{2n}} \right] d_1 \right\}, \quad d_1 \neq 0.
\]

\[
|a_1 - \mu a_0^2| \leq \frac{\delta}{(\delta + 1)[\delta - \alpha (\delta + 1)][1 + \lambda \delta (\delta + 1)]^n} \\
\times \max \left\{ 1, \left[ 1 - \frac{\delta (\delta + 1)}{(\delta - \alpha)^2 (1 + \lambda \delta)^{2n}} \right] d_1 \right\}, \quad d_1 \neq 0.
\]
where

\[
\sigma_1 = \frac{(\delta - \alpha)^2 [1 + \lambda \delta]^{2n} [-d_1^2 - d_2^2]}{\delta (\delta + 1) \alpha [\delta - \alpha (\delta + 1)] [1 + \lambda \delta (\delta + 1)] d_1^2}, \tag{38}
\]

\[
\sigma_2 = \frac{(\delta - \alpha)^2 [1 + \lambda \delta]^{2n} [d_1^2 - d_2^2]}{\delta (\delta + 1) \alpha [\delta - \alpha (\delta + 1)] [1 + \lambda \delta (\delta + 1)] d_1^2}.
\]

The result is sharp. Further, let \( \sigma_3 = ((\delta - \alpha)^2 [1 + \lambda \delta]^{2n} [-d_2 + d_1^2] (\delta + 1) [\delta - \alpha (\delta + 1)] [1 + \lambda \delta (\delta + 1)] d_1^2). \)

(i) If \( \sigma_1 \leq \sigma_3, \) then

\[
|a_1 - \mu a_0^2| + \frac{(\delta - \alpha)^2 [1 + \lambda \delta]^{2n}}{\delta (\delta + 1) [\delta - \alpha (\delta + 1)] [1 + \lambda \delta (\delta + 1)] d_1^2} \times \left\{ (d_1 + d_2) - \left[ 1 - \mu \frac{\delta (\delta + 1) [\delta - \alpha (\delta + 1)] [1 + \lambda \delta (\delta + 1)] d_1^2}{(\delta - \alpha)^2 [1 + \lambda \delta]^{2n}} \right] d_1^2 \right\} |a_0| \tag{39}
\]

(ii) If \( \sigma_3 \leq \mu < \sigma_2, \) then

\[
|a_1 - \mu a_0^2| + \frac{(\delta - \alpha)^2 [1 + \lambda \delta]^{2n}}{\delta (\delta + 1) [\delta - \alpha (\delta + 1)] [1 + \lambda \delta (\delta + 1)] d_1^2} \times \left\{ (d_1 - d_2) + \left[ 1 - \mu \frac{\delta (\delta + 1) [\delta - \alpha (\delta + 1)] [1 + \lambda \delta (\delta + 1)] d_1^2}{(\delta - \alpha)^2 [1 + \lambda \delta]^{2n}} \right] d_1^2 \right\} |a_0| \tag{40}
\]

Proof. First, let \( \mu \leq \sigma_1. \) Then

\[
|a_1 - \mu a_0^2| \leq \frac{\delta d_1}{(\delta + 1) [\delta - \alpha (\delta + 1)] [1 + \lambda \delta (\delta + 1)] d_1^2} \left\{ -d_2 + \left[ 1 - \mu \frac{\delta (\delta + 1) [\delta - \alpha (\delta + 1)] [1 + \lambda \delta (\delta + 1)] d_1^2}{(\delta - \alpha)^2 [1 + \lambda \delta]^{2n}} \right] d_1^2 \right\} \tag{41}
\]

Let, \( \sigma_1 \leq \mu \leq \sigma_2. \) Then, we obtain

\[
|a_1 - \mu a_0^2| \leq \frac{\delta d_1}{(\delta + 1) [\delta - \alpha (\delta + 1)] [1 + \lambda \delta (\delta + 1)] d_1^2} \left\{ -d_2 + \left[ 1 - \mu \frac{\delta (\delta + 1) [\delta - \alpha (\delta + 1)] [1 + \lambda \delta (\delta + 1)] d_1^2}{(\delta - \alpha)^2 [1 + \lambda \delta]^{2n}} \right] d_1^2 \right\} \tag{42}
\]
Finally, if $\mu \geq \sigma_2$. Then

$$|a_1 - \mu a_0^2| \leq \frac{\delta d_1}{(\delta + 1)[\delta + \alpha(\delta + 1)]^{n^2}} \left\{ \begin{array}{l} d_2 - \left[ 1 - \frac{\mu}{[\delta + (\delta + 1)]^{n^2}} \right] d_1 \right\}$$

$$= \frac{\delta}{(\delta + 1)[\delta + \alpha(\delta + 1)]^{n^2}} \left\{ \begin{array}{l} d_2 - \left[ 1 - \frac{\mu}{[\delta + (\delta + 1)]^{n^2}} \right] d_1 \right\}.$$  \hspace{1cm} (43)

The sharpness is an immediate consequence of Lemma 2. This completes the proof of Theorem 2. \qed

Remark 4. Taking $n = 0$ in Theorem 2, we get the result obtained by [12], [Theorem 2.10].

Taking $\alpha = 0$ in Theorem 2, we get

**Corollary 6.** For real $\mu$, let $\varphi(z) = 1 + d_1 z + d_2 z^2 + \cdots$ $(d_i > 0, i = 1, 2)$. If $f(z)$ given by (2) belongs to the class $\Sigma_{1,0}^{*}$, then

$$|a_1 - \mu a_0^2| \leq \frac{1}{(\delta + 1)[1 + \lambda \delta(\delta + 1)]^{n^2}}$$

$$= \frac{d_1}{(\delta + 1)[1 + \lambda \delta(\delta + 1)]^{n^2}} \left\{ \begin{array}{l} d_2 - \left[ 1 - \frac{\mu}{[\delta + (\delta + 1)]^{n^2}} \right] d_1 \right\}, \text{ if } \mu \leq \sigma_4$$

$$\frac{1}{(\delta + 1)[1 + \lambda \delta(\delta + 1)]^{n^2}}$$

$$= \frac{d_1}{(\delta + 1)[1 + \lambda \delta(\delta + 1)]^{n^2}} \left\{ \begin{array}{l} d_2 - \left[ 1 - \frac{\mu}{[\delta + (\delta + 1)]^{n^2}} \right] d_1 \right\}, \text{ if } \sigma_4 \leq \mu \leq \sigma_5$$

$$\frac{1}{(\delta + 1)[1 + \lambda \delta(\delta + 1)]^{n^2}}$$

$$= \frac{d_1}{(\delta + 1)[1 + \lambda \delta(\delta + 1)]^{n^2}} \left\{ \begin{array}{l} d_2 - \left[ 1 - \frac{\mu}{[\delta + (\delta + 1)]^{n^2}} \right] d_1 \right\}, \text{ if } \mu \geq \sigma_5,$$  \hspace{1cm} (44)

where

$$\sigma_4 = \frac{[1 + \lambda \delta]^{n^2}[-d_2 - d_2 + d_1^2]}{(\delta + 1)[1 + \lambda \delta(\delta + 1)]^{n^2} d_1^2}$$

$$\sigma_5 = \frac{[1 + \lambda \delta]^{n^2}[-d_2 + d_2 + d_1^2]}{(\delta + 1)[1 + \lambda \delta(\delta + 1)]^{n^2} d_1^2}$$

The result is sharp. Further, let $\sigma_6 = \{[1 + \lambda \delta]^{n^2}[-d_2 - d_2 + d_1^2]/(\delta + 1)[1 + \lambda \delta(\delta + 1)]^{n^2} d_1^2\}$.

(i) If $\sigma_4 \leq \mu < \sigma_6$, then

$$|a_1 - \mu a_0^2| \leq \frac{[1 + \lambda \delta]^{n^2} d_1}{(\delta + 1)[1 + \lambda \delta(\delta + 1)]^{n^2} d_1^2} \times \left\{ \begin{array}{l} d_1 + d_2 - \left[ 1 - \frac{\mu}{[\delta + (\delta + 1)]^{n^2}} \right] d_1 \right\} |a_0|^2$$

$$\leq \frac{d_1}{(\delta + 1)[1 + \lambda \delta(\delta + 1)]^{n^2}}.$$  \hspace{1cm} (45)
If $\sigma_6 \leq \mu < \sigma_7$, then
\[
|a_1 - \mu a_0^2| + \frac{\left[1 + \lambda \delta\right]^{2n}}{(\delta + 1)[1 + \lambda \delta(\delta + 1)]^n} \sum_{i=3}^{\infty} d_i \left(\frac{1}{\lambda \delta(\delta + 1)}\right)^i |a_0|^2 
\leq \frac{d_1}{(\delta + 1)[1 + \lambda \delta(\delta + 1)]^n}
\]
(46)

Remark 5.

(1) Taking $n = 0$ in Corollary 6, we get the result obtained by [12].
(2) Letting $\delta \rightarrow 1^-$ and taking $n = 0$ in Corollary 6, we get the result obtained by [4].

3. Conclusion

In the fields of combinatorics and quantum calculus, the $\delta$-derivative introduced by Frank Hilton Jackson [14] plays an important role in the theory of functions of a complex variable and other fields of mathematics. In this paper, we define a new differential operator for meromorphic functions. By using this new operator, we define and study a new family of meromorphic functions. Several properties of the abovementioned family of functions are investigated, including coefficient inequalities and Fekete–Szegő functionals.

Data Availability

No data were used to support this study

Conflicts of Interest

The authors declare that there are no conflicts of interest.

References


