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Research Article

Existence and Stability for a Coupled Hybrid System of Fractional Differential Equations with Atangana–Baleanu–Caputo Derivative

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The aim of this article is to investigate a coupled hybrid system of fractional differential equations with the Atangana–Baleanu–Caputo derivative which contains a Mittag–Leffler kernel function in its kernel. We firstly apply the Dhage fixed point principle to obtain the existence of mild solutions. Then, we study the Ulam–Hyers stability of the introduced fractional coupled hybrid system. Finally, an example is presented to exhibit the validity of our results.

1. Introduction

Dhage and Lakshmikantham in 2010 [1] introduced initially first order hybrid differential equations (HDEs) and studied some basic results on the existence and uniqueness of solutions. Furthermore, differential inequalities obtained with respect to HDEs were utilized to examine comparison results and some qualitative properties of the solution. Since the work of [1], many researchers in mathematics and other fields committed to the study of all kinds of HDEs. In particular, some scholars showed that fractional-order hybrid differential equations described the hereditary and memory properties of biology, chemistry, physics etc., better than integer order HDEs. Thus, some scholars would be interesting to the study of fractional-order hybrid differential equations. For instance, Baleanu [2] applied Caputo fractional-order hybrid differential equations to analyze a thermostat model. Other contributions on fractional-order hybrid differential equations involving the Hadamard derivative [3], Riemann derivative [4], Hilfer derivative [5], and Atangana derivative [6, 7] (ABC).

On the other hand, thanks to the fact that a large number of practical and real world phenomena in the fields of biology, physics, chemistry, and computer network, can be modeled by coupled systems of different types of fractional differential equations. Here, we sketch some references, but not a list of all references is included, such as Hadamard type [8], Caputo [9] and Riemann–Liouville types [10], and Ψ-Hilfer type [11].

Motivated by this fact and the work referenced above, the intent of this work is to study the existence and stability for a coupled hybrid system of fractional differential equations with Atangana–Baleanu–Caputo derivative described by the following equation:

$$\begin{cases} {}_0^{ABC}D^{\alpha}\left(\dfrac{x(t)}{g_1(t,x(t),y(t))}\right) = f_1(t,x(t),y(t)), & t\in[0,T], 0<\alpha<1, \\ {}_0^{ABC}D^{\beta}\left(\dfrac{y(t)}{g_2(t,x(t),y(t))}\right) = f_2(t,x(t),y(t)), & 0<\beta<1, \\ a_1\left(\dfrac{x(0)}{g_1(0,x(0),y(0))}\right) + b_1\left(\dfrac{x(T)}{g_1(T,x(T),y(T))}\right) = c_1, \\ a_2\left(\dfrac{x(0)}{g_2(0,x(0),y(0))}\right) + b_2\left(\dfrac{x(T)}{g_2(T,x(T),y(T))}\right) = c_2, \\ f_1(0,x(0),y(0)) = f_2(0,x(0),y(0)) \equiv 0, \end{cases} \tag{1}$$

where ${}_0^{ABC}D^{\alpha}, {}_0^{ABC}D^{\beta}$ denote the Atangana–Baleanu–Caputo fractional derivatives of order α and β, respectively, $g_1, g_2 \in C([0,T]\times\mathbb{R}\times\mathbb{R}, \mathbb{R}, \{0\})$, $f_1, f_2 \in C([0,T]\times \mathbb{R}\times\mathbb{R}, \mathbb{R})$ and $a_i + b_i \neq 0, i = 1,2$. The main contributions of this paper are as follows:

(1) The above-given results [2–7] consider single fractional-order hybrid differential equations, while we consider coupled fractional-order hybrid systems.

(2) The work in [8–11] investigated the existence of solutions or initial value boundary value problem. However, it is well known that stability is one of the important dynamic behaviors for fractional-order hybrid differential equations. Moreover, we should mention that boundary value problem (1) are rather general and include some common cases such as nonseparated boundary and antiperiodic boundary conditions, and so on, by choosing different values of $a_i, b_i, c_i\,(i=1,2)$. Thus, it is meaningful to study existence and stability for (1).

The remainder of the paper is organized as follows: Section 2 contains some necessary concepts, assumptions, and facts. In Section 3, we prove an existence result and the Ulam–Hyers stability for the problem (1). Finally, an example is constructed to show the correctness of our results.

2. Preliminaries

Let $C([0,T],\mathbb{R})$ be the space of all continuous functions on $[0,T]$ endowed with the norm $\|x\| = \sup_{t\in[0,T]}\{x(t)\}$. We denote by $X = AC([0,T],\mathbb{R})$ the space of all absolutely continuous functions. Obviously, the product space $\mathbb{E} = X\times X$ is a Banach space with norm $\|(x,y)\|_{\mathbb{E}} = \|x\| + \|y\|$. In the remainder of this paper, we recall the necessary basic notions and properties related to fractional calculus.

Definition 1 (see [12]). The fractional Riemann–Liouville integral of order $\alpha>0$ for a continuous function x on $[0,b]$ is given by the following equation:

$$\left(I_0^{\alpha}x\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)\mathrm{d}s, \quad \alpha>0,\ b\geq t>0. \tag{2}$$

Definition 2 (see [13, 14]). Let x be differentiable on $(0,b)$, such that $x\in L^1(0,b), 0<b, \alpha\in[0,1]$. Then, the left Atangana–Baleanu–Caputo derivative (ABC) of x for order α is defined by the following equation:

$$\left({}_0^{ABC}D^{\alpha}x\right)(t) = \frac{N(\alpha)}{1-\alpha}\int_0^t x'(s)E_{\alpha}\left(-\alpha\frac{(t-s)^{\alpha}}{1-\alpha}\right)\mathrm{d}s, \tag{3}$$

and in the left ABR sense (Riemann–Liouville type) is defined as follows:

$$\left({}_0^{ABR}D^{\alpha}x\right)(t) = \frac{N(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t x(s)E_{\alpha}\left(-\alpha\frac{(t-s)^{\alpha}}{1-\alpha}\right)\mathrm{d}s, \tag{4}$$

and the fractional integral associated the above operators is

$$\left({}^{AB}I_0^{\alpha}x\right)(t) = \frac{1-\alpha}{N(\alpha)}x(t) + \frac{\alpha}{N(\alpha)}I_0^{\alpha}x(t), \tag{5}$$

where E_{α} is a parameter Mittag–Leffler function defined by $E_{\alpha}(z)\sum_{n=0}^{\infty} z^n/\Gamma(n\alpha+1)$, $N(\alpha)$ is a normalization function satisfying $N(0)=N(1)=1$.

Remark 1. In works [13, 14], it has been verified that $({}^{AB}I_0^{\alpha})({}_0^{ABR}D^{\alpha}x)(t) = x(t)$ and $({}_0^{ABR}D^{\alpha})({}^{AB}I_0^{\alpha}x)(t) = x(t)$. Also, the authors in the work [13] have established that $({}_0^{ABC}D^{\alpha}x)(t) = ({}_0^{ABR}D^{\alpha}x)(t) - N(\alpha)/1-\alpha x(0)E_{\alpha}\ (-\alpha/1-\alpha t^{\alpha})$.

Lemma 1 (see [15]). *If $x(t)$ be well defined on $[0,b]$, then $({}^{AB}I_0^{\alpha})({}^{ABC}I_0^{\alpha}x)(t) = x(t) - \sum_{k=0}^{N} x^{(k)}(0)/k!t^k$.*

Definition 3 (see [16]). An element $(x,y)\in X\times X$ is said to be a coupled fixed point of a mapping $T\colon X\times X\longrightarrow X$ if $T(x,y)=x$ and $T(y,x)=y$.

Lemma 2 (see [17]). *Suppose that S is a nonempty, closed, convex, and bounded subset of the Banach algebra X, $\widehat{S} = S\times S$ and $P,G\colon X\longrightarrow X$ and $F\colon S\longrightarrow X$ are three operators such that,*

(i) P and G are Lipschitzian with a Lipschitz constants σ and δ, respectively

(ii) F is completely continuous

(iii) $y = PyFx + Gy \Rightarrow y\in S, \forall x\in S$

(iv) $4\sigma M + \delta < 1$, *where* $M = sup\{\|Fx\|: x \in S\}$

Then, the operator equation $T(y,x) = PyFx + Gy$ *admits at least one coupled fixed point in* $\widehat{S}$.

Lemma 3 (see [18]). *Let* S^* *be a nonempty, closed, convex, and bounded subset of the Banach space* X *and let* $\mathbb{P}, \mathbb{G}: X \longrightarrow X$ *and* $\mathbb{F}: S^* \longrightarrow X$ *be three operators such that,*

(i) $\mathbb{P}$ *and* $\mathbb{G}$ *are Lipschitzian with a Lipschitz constants* $L_{\mathbb{P}}$ *and* $L_{\mathbb{G}}$, *respectively*

(ii) $\mathbb{F}$ *is completely continuous*

(iii) $y = \mathbb{P}y\mathbb{F}x + \mathbb{G}y \Rightarrow y \in S, \forall x \in S^*$

(iv) $L_{\mathbb{P}}M_{\mathbb{F}} + L_{\mathbb{G}} < 1$, *where* $M_{\mathbb{F}} = sup\{\|\mathbb{F}x\|: x \in S^*\}$

Then, the operator equation $\mathbb{P}y\mathbb{F}x + \mathbb{G}y = y$ *possesses a solution in* S^*.

3. Main Results

To start for verifying the main results, the following assumptions are needed for us in the sequel:

(H1) Two functions $g_1, g_2 \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}, \{0\})$ are such that,

(i) Two functions $x \mapsto x/g_1(t,x,y)$ and $y \mapsto y/g_2(t,x,y)$ are increasing in $\mathbb{R}$ a.e. For each $t \in [0,T]$

(ii) functions $x/g_1(t,x,y), y/g_2(t,x,y) \in AC([0,T],\mathbb{R})$

(H2) There exist constants $L_{f_i} > 0, L_{g_i} > 0$ such that $i = 1, 2$

$$
\begin{aligned}
&\left|f_i(t,x_1,y_2) - f_i(t,x_2,y_2)\right| \le L_{f_i}\left(\left|x_1 - x_2\right| + \left|y_1 - y_2\right|\right),\\
&\left|g_i(t,x_1,y_2) - g_i(t,x_2,y_2)\right| \le L_{g_i}\left(\left|x_1 - x_2\right| + \left|y_1 - y_2\right|\right), \quad \forall (t,x_i,y_i) \in [0,T] \times \mathbb{R} \times \mathbb{R}.
\end{aligned} \tag{6}
$$

(H3) There exist constants $M_{f_i}, M_{g_i} \in \mathbb{R}^+$ such that $i = 1, 2$

$$
\left|f_i(t,x,y)\right| \le M_{f_i}, \left|g_i(t,x,y)\right| \le M_{g_i} \forall (t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}. \tag{7}
$$

(H4) $f_i: [0,T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are absolutely continuous and there exist constants $K_{f_i}(t) \in L^1([0,T],\mathbb{R}^+)$ such that $i = 1, 2$

$$
\left|f_i(t,x,y)\right| \le K_{f_i}(t), \quad \forall t \in [0,T]. \tag{8}
$$

In order to study problem (1), we firstly consider the following problem:

$$
\begin{aligned}
&\left\{ {}_0^{ABC}D^{\alpha}\left(\frac{x(t)}{g_1(t,x(t),y(t))}\right) = \psi_1(t), \quad t \in [0,T], 0 < \alpha < 1, {}_0^{ABC}D^{\beta}\left(\frac{y(t)}{g_2(t,x(t),y(t))}\right) \right.\\
&= \psi_2(t),\ 0 < \beta < 1, a_1\left(\frac{x(0)}{g_1(0,x(0),y(0))}\right) + b_1\left(\frac{x(T)}{g_1(T,x(T),y(T))}\right)\\
&= c_1, a_2\left(\frac{x(0)}{g_2(0,x(0),y(0))}\right) + b_2\left(\frac{x(T)}{g_2(T,x(T),y(T))}\right) = c_2, \psi_1(0) = \psi_2(0) \equiv 0,
\end{aligned} \tag{9}
$$

Similar to Definition 1 in [6], we give a definition of (9)

Definition 4. A function $(x,y) \in AC([0,T],\mathbb{R}) \times AC([0,T],\mathbb{R})$ is called a solution of (9), if functions $x/g_1(t,x,y), y/g_2(t,x,y) \in AC([0,T],\mathbb{R}), \forall x,y \in \mathbb{R}$, $(\psi_1,\psi_2) \in L^1([0,T],\mathbb{R}) \times L^1([0,T],\mathbb{R})$, and (x,y) satisfies (9).

An application of Lemma 4 in [7], Lemma 4 in [9], and Theorem 3.1 in [6], we derive an equivalent fractional integral equation to (9). In order to make this paper readable, we present the proof.Let $(\widehat{x},\widehat{y}) \in \mathbb{E}$ be a solution of (42), $(x,y) \in \mathbb{E}$ be a solution of problem (1) satisfying

Lemma 4. *Let* $0<\alpha,\beta<1$, $\psi_1,\psi_2\in AC([0,T],\mathbb{R})$ *and assume that (H1) holds. Then, the solution of problem (9) if and only if it is a solution of the following system:*

$$\begin{cases} x(t)=g_1(t,x(t),y(t))\left(\dfrac{1-\alpha}{N(\alpha)}\psi_1(t)+\dfrac{\alpha}{N(\alpha)}\left(I_0^{\alpha}\psi_1\right)(t)\right.\\ \left.-\dfrac{1}{a_1+b_1}\left(b_1\left[\dfrac{1-\alpha}{N(\alpha)}\psi_1(T)+\dfrac{\alpha}{N(\alpha)}\left(I_0^{\alpha}\psi_1\right)(T)\right]-c_1\right)\right),\\ y(t)=g_2(t,x(t),y(t))\left(\dfrac{1-\beta}{N(\beta)}\psi_1(t)+\dfrac{\beta}{N(\alpha)}\left(I_0^{\beta}\psi_2\right)(t)\right.\\ \left.-\dfrac{1}{a_2+b_2}\left(b_2\left[\dfrac{1-\beta}{N(\beta)}\psi_2(T)+\dfrac{\beta}{N(\alpha)}\left(I_0^{\beta}\psi_2\right)(T)\right]-c_2\right)\right).\end{cases} \tag{10}$$

Proof. The main idea of the proof comes from Lemma 4 in [7] and Theorem 3.1 in [6]. In the view of Lemma 4 in [7], if (x,y) is a solution of (9), then (x,y) satisfies fractional integral system (10).

Conversely, since

$$\begin{cases} {}_0^{ABC}D^{\alpha}\left(\dfrac{x(t)}{g_1(t,x(t),y(t))}\right)=\psi_1(t), & t\in[0,T],0<\alpha<1,\\ {}_0^{ABC}D^{\beta}\left(\dfrac{y(t)}{g_2(t,x(t),y(t))}\right)=\psi_2(t), & 0<\beta<1.\end{cases} \tag{11}$$

By $(x,y)\in AC([0,T],\mathbb{R})\times AC([0,T],\mathbb{R})$, applying ${}^{AB}I_0^{\alpha}$, ${}^{AB}I_0^{\beta}$ on both sides, and thanks to Remark 1, we obtain the following equation:

$$\begin{cases} \left({}^{AB}I_0^{\alpha}\right){}_0^{ABC}D^{\alpha}\left(\dfrac{x(t)}{g_1(t,x(t),y(t))}\right)=\dfrac{x(t)}{g_1(t,x(t),y(t))}-\dfrac{x(0)}{g_1(0,x(0),y(0))}, t\in[0,T], & 0<\alpha<1,\\ \left({}^{AB}I_0^{\beta}\right){}_0^{ABC}D^{\beta}\left(\dfrac{y(t)}{g_2(t,x(t),y(t))}\right)=\dfrac{y(t)}{g_2(t,x(t),y(t))}-\dfrac{y(0)}{g_2(0,x(0),y(0))}, & 0<\beta<1.\end{cases} \tag{12}$$

Hence

$$\begin{cases} {}^{AB}I_0^{\alpha}\psi_1(t)=\dfrac{x(t)}{g_1(t,x(t),y(t))}-\dfrac{x(0)}{g_1(0,x(0),y(0))}, & t\in[0,T],0<\alpha<1,\\ {}^{AB}I_0^{\beta}\psi_2(t)=\dfrac{y(t)}{g_2(t,x(t),y(t))}-\dfrac{y(0)}{g_2(0,x(0),y(0))}, & 0<\beta<1.\end{cases} \tag{13}$$

Since $\psi_1, \psi_2 \in AC([0,T],\mathbb{R})$, we know that ${}^{AB}I_0^{\alpha}\psi_1(t), {}^{AB}I_0^{\beta}\psi_2(t) \in AC([0,T],\mathbb{R})$. Because $x/g_1(t,x,y), y/g_2(t,x,y) \in AC([0,T],\mathbb{R})$ (see condition (H1)), the two functions in both sides of (13) are continuous and thus (13) hold for every $t \in [0,T]$. It follows from Lemma 1 that

$$\begin{cases} \dfrac{x(t)}{g_1(t,x(t),y(t))} = {}^{AB}I_0^{\alpha}\psi_1(t) + x(0), & t \in [0,T], \\ \dfrac{y(t)}{g_2(t,x(t),y(t))} = {}^{AB}I_0^{\beta}\psi_2(t) + y(0), & t \in [0,T]. \end{cases} \quad (14)$$

By boundary condition (1) and solving the system, we can obtain $x(t)$ and $y(t)$ which have given in (10). □

Theorem 1. *If hypotheses (H1)–(H4) hold and $(L_{g_1} + L_{g_2})(\Delta_1 + \Delta_2) < 1$, then the coupled system (1) has a solution $(x,y) \in \mathbb{E}$, where*

$$\begin{aligned} \Delta_1 &= \left[\left(1 + \frac{|b_1|}{|a_1 + b_1|}\right)\left(\frac{1-\alpha}{N(\alpha)} + \frac{T^{\alpha}}{N(\alpha)\Gamma(\alpha)}\right)\right]\|K_{f_1}\|_{L^1} + \frac{|c_1|}{|a_1 + b_1|}, \\ \Delta_2 &= \left[\left(1 + \frac{|b_2|}{|a_2 + b_2|}\right)\left(\frac{1-\beta}{N(\beta)} + \frac{T^{\beta}}{N(\beta)\Gamma(\beta)}\right)\right]\|K_{f_2}\|_{L^1} + \frac{|c_2|}{|a_2 + b_2|}. \end{aligned} \quad (15)$$

Proof. We choose R^* so that

$$\frac{\left(M_{g_1} + M_{g_2}\right)(\Delta_1 + \Delta_2)}{1 - \left(L_{g_1} + L_{g_2}\right)(\Delta_1 + \Delta_2)} \le R^*, \quad (16)$$

and specify a subset S^* of the Banach space $X \times X$ by

$$S^* = \{(x,y) \in X \times X: \|(x,y)\|_{\mathbb{E}} \le R^*\}. \quad (17)$$

Evidently, S^* is a convex, bounded and closed subset of the Banach space $\mathbb{E}$. According to Lemma 4, if $(x,y) \in S^* \subseteq X \times X$ is a solution of (1), then it is a solution of the next system

$$\begin{cases} x(t) = g_1(t,x(t),y(t))\left(\dfrac{1-\alpha}{N(\alpha)} f_1(t,x(t),y(t)) + \dfrac{\alpha}{N(\alpha)} I_0^{\alpha} f_1(t,x(t),y(t)) \right. \\ \left. -\dfrac{1}{a_1 + b_1}\left(b_1\left[\dfrac{1-\alpha}{N(\alpha)} f_1(T,x(T),y(T)) + \dfrac{\alpha}{N(\alpha)} I_0^{\alpha} f_1(T,x(T),y(T))\right] - c_1\right)\right), \\ y(t) = g_2(t,x(t),y(t))\left(\dfrac{1-\beta}{N(\beta)} f_2(t,x(t),y(t)) + \dfrac{\beta}{N(\beta)} I_0^{\beta} f_2(t,x(t),y(t)) \right. \\ \left. -\dfrac{1}{a_2 + b_2}\left(b_2\left[\dfrac{1-\beta}{N(\beta)} f_2(T,x(T),y(T)) + \dfrac{\beta}{N(\beta)} I_0^{\beta} f_2(T,x(T),y(T))\right] - c_2\right)\right). \end{cases} \quad (18)$$

For $i = 1,2$, we define operators $\mathbb{P} = (\mathbb{P}_1, \mathbb{P}_1)$: $\mathbb{E} \longrightarrow \mathbb{E}, \mathbb{F} = (\mathbb{F}_1, \mathbb{F}_2)$: $S^* \longrightarrow \mathbb{E}$ by

$$
\begin{aligned}
&\mathbb{P}_i(x,y)(t) = g_i(t,x(t),y(t)),\\
&\begin{cases}
\mathbb{F}_1(x,y)(t) = \dfrac{1-\alpha}{N(\alpha)} f_1(t,x(t),y(t)) + \dfrac{\alpha}{N(\alpha)} I_0^{\alpha} f_1(t,x(t),y(t))\\
-\dfrac{1}{a_1+b_1}\left(b_1\left[\dfrac{1-\alpha}{N(\alpha)} f_1(T,x(T),y(T)) + \dfrac{\alpha}{N(\alpha)} I_0^{\alpha} f_1(T,x(T),y(T))\right] - c_1\right),\\
\mathbb{F}_2(x,y)(t) = \dfrac{1-\beta}{N(\beta)} f_2(t,x(t),y(t)) + \dfrac{\beta}{N(\beta)} I_0^{\beta} f_2(t,x(t),y(t))\\
-\dfrac{1}{a_2+b_2}\left(b_2\left[\dfrac{1-\beta}{N(\beta)} f_2(T,x(T),y(T)) + \dfrac{\beta}{N(\beta)} I_0^{\beta} f_2(T,x(T),y(T))\right] - c_2\right).
\end{cases}
\end{aligned}
\tag{19}
$$

Then, the coupled hybrid system of (18) transformed into the coupled system of operator equations as follows:

$$\mathbb{P}(x,y)(t)\mathbb{F}(x,y)(t) = (x,y)(t), \quad t \in [0,T]. \tag{20}$$

This implies that

$$
\begin{cases}
\mathbb{P}_1(x,y)(t)\mathbb{F}_1(x,y)(t) = x(t), & t \in [0,T],\\
\mathbb{P}_2(x,y)(t)\mathbb{F}_2(x,y)(t) = y(t), & t \in [0,T].
\end{cases}
\tag{21}
$$

By Lemma 3, we divide our proof into four steps: □

Step 1. $\mathbb{P} = (\mathbb{P}_1, \mathbb{P}_1)$ are Lipschitz operators with constants $L_{\mathbb{P}} = (L_{g_1} + L_{g_2})$ on $\mathbb{E}$. Let $(x_i, y_i), \in \mathbb{E}, i = 1,2$. Then it follow from (H2) that

$$
\begin{aligned}
&\left|\mathbb{P}_1(x_1,y_1)(t) - \mathbb{P}_1(x_2,y_2)(t)\right|\\
&= \left|g_1(t,x_1(t),y_1(t)) - g_1(t,x_2(t),y_2(t))\right|\\
&\le L_{g_1}\left(\left|x_1(t)-x_2(t)\right| + \left|y_1(t)-y_2(t)\right|\right)\\
&\le L_{g_1}\left(\left\|x_1-x_2\right\| + \left\|y_1-y_2\right\|\right).
\end{aligned}
\tag{22}
$$

Thus, we operate the supremum norm over t and obtain

$$\left\|\mathbb{P}_1(x_1,y_1) - \mathbb{P}_1(x_2,y_2)\right\| \le L_{g_1}\left(\left\|x_1-x_2\right\| + \left\|y_1-y_2\right\|\right). \tag{23}$$

Along the same lines, one has

$$\left\|\mathbb{P}_2(x_1,y_1) - \mathbb{P}_2(x_2,y_2)\right\| \le L_{g_2}\left(\left\|x_1-x_2\right\| + \left\|y_1-y_2\right\|\right). \tag{24}$$

Thus, we use the definition of operator $\mathbb{P}$ to get

$$
\begin{aligned}
&\left\|\mathbb{P}(x_1,y_1) - \mathbb{P}(x_2,y_2)\right\|\\
&= \left\|(\mathbb{P}_1(x_1,y_1), \mathbb{P}_2(x_1,y_1)) - (\mathbb{P}_1(x_2,y_2), \mathbb{P}_2(x_2,y_2))\right\|\\
&= \left\|\mathbb{P}_1(x_1,y_1) - \mathbb{P}_1(x_2,y_2), \mathbb{P}_2(x_1,y_1) - \mathbb{P}_2(x_2,y_2)\right\|\\
&\le \left\|\mathbb{P}_1(x_1,y_1) - \mathbb{P}_1(x_2,y_2)\right\| + \left\|\mathbb{P}_2(x_1,y_1) - \mathbb{P}_2(x_2,y_2)\right\|\\
&\le \left(L_{g_1}\left(\left\|x_1-x_2\right\| + \left\|y_1-y_2\right\|\right) + L_{g_2}\left\|x_1-x_2\right\| + \left\|y_1-y_2\right\|\right)\\
&= \left(L_{g_1} + L_{g_2}\right)\left(\left\|x_1-x_2\right\| + \left\|y_1-y_2\right\|\right)\\
&= L_{\mathbb{P}}\left\|(x_1,y_1) - (x_2,y_2)\right\|.
\end{aligned}
\tag{25}
$$

Therefore, we can confirm that $\mathbb{P}$ is also Lipschitzian with Lipschitz constant $L_{\mathbb{P}} = (L_{g_1} + L_{g_2})$.

Step 2. Now, we show that $\mathbb{F} = (\mathbb{F}_1, \mathbb{F}_2)$ is a continuous and compact operator from S^* into $\mathbb{E}$. We firstly prove the continuity of $\mathbb{F}$, let $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ be a sequence in S^* converging to a point $(x,y) \in S^*$. Then, Lebesgue dominated convergence theorem yields

$$
\begin{aligned}
\lim_{n\longrightarrow\infty}\mathbb{F}_1(x_n,y_n)(t) &= \frac{1-\alpha}{N(\alpha)}\lim_{n\longrightarrow\infty} f_1(t,x_n(t),y_n(t)) + \frac{\alpha}{N(\alpha)}\lim_{n\longrightarrow\infty} I_0^{\alpha} f_1(t,x_n(t),y_n(t))\\
&\quad -\frac{1}{a_1+b_1}\left(b_1\left[\frac{1-\alpha}{N(\alpha)}\lim_{n\longrightarrow\infty} f_1(T,x_n(T),y_n(T)) + \frac{\alpha}{N(\alpha)}\lim_{n\longrightarrow\infty} I_0^{\alpha} f_1(T,x_n(T),y_n(T))\right] - c_1\right)\\
&= \frac{1-\alpha}{N(\alpha)}\lim_{n\longrightarrow\infty} f_1(t,x_n(t),y_n(t))\\
&\quad + \frac{\alpha}{N(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lim_{n\longrightarrow\infty} f_1(s,x_n(s),y_n(s))\mathrm{d}s\\
&\quad -\frac{1}{a_1+b_1}\left(b_1\left[\frac{1-\alpha}{N(\alpha)}\lim_{n\longrightarrow\infty} f_1(T,x_n(T),y_n(T))\right.\right.\\
&\quad \left.\left.+\frac{\alpha}{N(\alpha)}\int_0^T (T-s)^{\alpha-1}\lim_{n\longrightarrow\infty} f_1(s,x_n(s),y_n(s))\mathrm{d}s\right] - c_1\right) = \mathbb{F}_1(x,y)(t).
\end{aligned}
\tag{26}
$$

Similarly, we prove

$$\lim_{n\longrightarrow\infty}\mathbb{F}_2(x_n, y_n)(t) = \mathbb{F}_2(x, y)(t). \tag{27}$$

Therefore, $\mathbb{F}(x_n, y_n) = (\mathbb{F}_1(x_n, y_n), \mathbb{F}_2(x_n, y_n))$ converges to $\mathbb{F}(x, y)$.

In what follows, the compactness of $\mathbb{F}$ is explored on S^*. Firstly, we prove the uniform boundedness. Let $(x, y) \in S^*$. Then using (H4), one has for $t \in [0, T]$,

$$\begin{aligned}
\left|\mathbb{F}_1(x, y)(t)\right| &\leq \frac{1-\alpha}{N(\alpha)}\left|f_1(t, x(t), y(t))\right| + \frac{\alpha}{N(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^t\left|(t-s)^{\alpha-1}f_1(s, x(s), y(s))\right|ds \\
&\quad - \frac{1}{a_1+b_1}\left(b_1\left[\frac{1-\alpha}{N(\alpha)}\left|f_1(T, x(T), y(T))\right| + \frac{\alpha}{N(\alpha)}\int_0^T\left|(T-s)^{\alpha-1}f_1(s, x(s), y(s))ds\right|\right] + \left|c_1\right|\right) \\
&\leq \frac{1-\alpha}{N(\alpha)}\left\|K_{f_1}\right\|_{L^1} + \frac{T^\alpha}{N(\alpha)\Gamma(\alpha)}\left\|K_{f_1}\right\|_{L^1} \\
&\quad + \frac{1}{\left|a_1+b_1\right|}\left(\frac{\left|b_1\right|(1-\alpha)}{N(\alpha)}\left\|K_{f_1}\right\|_{L^1} + \frac{\left|b_1\right|T^\alpha}{N(\alpha)\Gamma(\alpha)}\left\|K_{f_1}\right\|_{L^1} + \left|c_1\right|\right).
\end{aligned} \tag{28}$$

Consequently, one has

$$\left\|\mathbb{F}_1(x, y)(t)\right\| \leq \left[\left(1+\frac{\left|b_1\right|}{\left|a_1+b_1\right|}\right)\left(\frac{1-\alpha}{N(\alpha)} + \frac{T^\alpha}{N(\alpha)\Gamma(\alpha)}\right)\right]\left\|K_{f_1}\right\|_{L^1} + \frac{\left|c_1\right|}{\left|a_1+b_1\right|} = \Delta_1. \tag{29}$$

Similarly, we prove

$$\left\|\mathbb{F}_2(x, y)(t)\right\| \leq \left[\left(1+\frac{\left|b_2\right|}{\left|a_2+b_2\right|}\right)\left(\frac{1-\beta}{N(\beta)} + \frac{T^\beta}{N(\beta)\Gamma(\beta)}\right)\right]\left\|K_{f_2}\right\|_{L^1} + \frac{\left|c_2\right|}{\left|a_2+b_2\right|} = \Delta_2. \tag{30}$$

Consequently, we get

$$\left\|\mathbb{F}(x, y)(t)\right\| = \left\|\mathbb{F}_1(x, y)(t)\right\| + \left\|\mathbb{F}_2(x, y)(t)\right\| \leq \Delta_1 + \Delta_2. \tag{31}$$

Hence, it yields that $\mathbb{F}$ is uniformly bounded. Next we prove that $\mathbb{F}$ is equicontinious, Let $t_1, t_2 \in [0, T], t_1 < t_2$, any $(x, y) \in S^*$, one has

$$\begin{aligned}
\left|\mathbb{F}_1(x, y)(t_2) - \mathbb{F}_1(x, y)(t_1)\right| &\leq \frac{1-\alpha}{N(\alpha)}\left|f_1(t_2, x(t_2), y(t_2)) - f_1(t_1, x(t_1), y(t_1))\right| \\
&\quad + \left|\frac{\alpha}{N(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right]f_1(s, x(s), y(s))ds\right. \\
&\quad \left. + \frac{\alpha}{N(\alpha)}\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f_1(s, x(s), y(s))ds\right|.
\end{aligned} \tag{32}$$

It is easy to see that $|f_1(t_2, x(t_2), y(t_2)) - f_1(t_1, x(t_1), y(t_1))| \longrightarrow 0$ as $|t_1 - t_2| \longrightarrow 0$, in view of (H4), we obtain

$$\begin{aligned}\left|\mathbb{F}_1(x,y)(t_2)-\mathbb{F}_1(x,y)(t_1)\right| &\le \frac{\alpha}{N(\alpha)}\frac{\|K_{f_1}\|_{L^1}}{\Gamma(\alpha+1)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\mathrm{d}s\\ &+\frac{\alpha}{N(\alpha)}\frac{\|K_{f_1}\|_{L^1}}{\Gamma(\alpha+1)}\int_0^{t_2}(t_2-s)^{\alpha-1}\mathrm{d}s\\ &\le \frac{\|K_{f_1}\|_{L^1}}{\Gamma(\alpha+1)}\left[(t_2-t_1)^{\alpha}+t_2^{\alpha}-t_1^{\alpha}\right].\end{aligned}\tag{33}$$

Hence, for $\varepsilon>0$, there exists $\delta>0$ such that $|t_1-t_2|<\delta\Rightarrow|\mathbb{F}_1(x,y)(t_2)-\mathbb{F}_1(x,y)(t_1)|<\varepsilon$. Similarly, we prove

$$\left|\mathbb{F}_2(x,y)(t_2)-\mathbb{F}_2(x,y)(t_1)\right|\longrightarrow 0, t_2\longrightarrow t_1.\tag{34}$$

Consequently, we get

$$\left\|\mathbb{F}(x,y)(t_2)-\mathbb{F}(x,y)(t_1)\right\|\longrightarrow 0.\tag{35}$$

This implies that $\mathbb{F}$ is equicontinious on S^*, and so $\mathbb{F}$ is relatively compact. In consequence, we apply the Arzelá–Ascoli theorem to show that $\mathbb{F}$ is a complete continuous.

Step 3. we show that the item (iii) of Lemma 3 is satisfied. Let $(x,y)\in S^*$ satisfy (20).

$$(x,y)=\left(\mathbb{P}_1(x,y)\mathbb{F}_1(x,y),\mathbb{P}_2(x,y)\mathbb{F}_2(x,y)\right).\tag{36}$$

Then, one has

$$\begin{aligned}|x(t)| &= \left|\mathbb{P}_1(x,y)\mathbb{F}_1(x,y)\right|\\ &\le \left|g_1(t,x(t),y(t))\right|\left\{\frac{1-\alpha}{N(\alpha)}\left|f_1(t,x(t),y(t))\right|+\frac{\alpha}{N(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^t\left|(t-s)^{\alpha-1}f_1(s,x(s),y(s))\right|\mathrm{d}s\right.\\ &\left.+\frac{1}{a_1+b_1}\left(b_1\left[\frac{1-\alpha}{N(\alpha)}\left|f_1(T,x(T),y(T))\right|+\frac{\alpha}{N(\alpha)}\int_0^T\left|(T-s)^{\alpha-1}f_1(s,x(s),y(s))\mathrm{d}s\right|\right]+\left|c_1\right|\right)\right\}\\ &\le\left[g_1(t,x(t),y(t))-g_1(t,0,0)+g_1(t,0,0)\right]\\ &\times\left[\left(\frac{1-\alpha}{N(\alpha)}+\frac{T^{\alpha}}{N(\alpha)\Gamma(\alpha)}+\frac{|b_1|}{|a_1+b_1|}\left(\frac{1-\alpha}{N(\alpha)}+\frac{T^{\alpha}}{N(\alpha)\Gamma(\alpha)}\right)\right)\|K_{f_1}\|_{L^1}+\frac{|c_1|}{|a_1+b_1|}\right]\\ &\le\left[L_{g_1}(|x(t)|+|y(t)|)+M_{g_1}\right]\\ &\times\left[\left(\frac{1-\alpha}{N(\alpha)}+\frac{T^{\alpha}}{N(\alpha)\Gamma(\alpha)}+\frac{|b_1|}{|a_1+b_1|}\left(\frac{1-\alpha}{N(\alpha)}+\frac{T^{\alpha}}{N(\alpha)\Gamma(\alpha)}\right)\right)\|K_{f_1}\|_{L^1}+\frac{|c_1|}{|a_1+b_1|}\right].\end{aligned}\tag{37}$$

Hence, we have

$$\begin{aligned}\|x\| &\le\left[L_{g_1}(\|x\|+\|y\|)+M_{g_1}\right]\times\left[\left(\frac{1-\alpha}{N(\alpha)}+\frac{T^{\alpha}}{N(\alpha)\Gamma(\alpha)}\right.\right.\\ &\left.\left.+\frac{|b_1|}{|a_1+b_1|}\left(\frac{1-\alpha}{N(\alpha)}+\frac{T^{\alpha}}{N(\alpha)\Gamma(\alpha)}\right)\right)\|K_{f_1}\|_{L^1}+\frac{|c_1|}{|a_1+b_1|}\right]\\ &=\left[L_{g_1}(\|x\|+\|y\|)+M_{g_1}\right]\times\Delta_1.\end{aligned}\tag{38}$$

Similarly, we prove

$$\|y\|\le\left[L_{g_2}(\|x\|+\|y\|)+M_{g_2}\right]\times\Delta_2.\tag{39}$$

It follows from (38) and (39) that

$$\|x\|+\|y\|\le\frac{\left(M_{g_1}+M_{g_2}\right)(\Delta_1+\Delta_2)}{1-\left(L_{g_1}+L_{g_2}\right)(\Delta_1+\Delta_2)}\le R^*.\tag{40}$$

Formula (40) implies that item (iii) of Lemma 3 holds.

Step 4. we show that the condition (iv) of Lemma 3 is fulfilled. Since

$$\begin{aligned} M_{\mathbb{F}} &= \sup\{\|\mathbb{F}(x,y)\|: (x,y)\in S^*\} \\ &= \sup\left\{\left\|\mathbb{F}_1(x,y)\right\| + \left\|\mathbb{F}_1(x,y)\right\|: (x,y)\in S^*\right\} \\ &\le \Delta_1 + \Delta_2. \end{aligned} \tag{41}$$

Thus, we have $L_{\mathbb{P}} M_{\mathbb{F}} = (L_{g_1} + L_{g_2})(\Delta_1 + \Delta_2) < 1$, that is, condition (iv) of Lemma 3 holds.

From steps 1 to step 4, it follows that all the conditions of Lemma 3, coupled system of hybrid (1) has a solution.

In what follows, we show the Ulam-Hyers stability of (1).

Definition 5. coupled system of hybrid (1) is Ulam–Hyers stable, if for all $\vartheta = \max\{\vartheta_1, \vartheta_2\} > 0$ and for all $(\widehat{x}, \widehat{y}) \in \mathbb{E}$ of (42)

$$\begin{cases} \left| {}_0^{ABC}D^{\alpha}\left(\dfrac{x(t)}{g_1(t,x(t),y(t))}\right) = f_1(t,x(t),y(t))\right| \le \vartheta_1, & t\in[0,T], 0<\alpha<1, \\[2ex] \left| {}_0^{ABC}D^{\beta}\left(\dfrac{y(t)}{g_2(t,x(t),y(t))}\right) = f_2(t,x(t),y(t))\right| \le \vartheta_2, & 0<\beta<1, \end{cases} \tag{42}$$

there is $\lambda = \max\{\lambda_1, \lambda_2\} > 0$ and a solution $(x, y) \in \mathbb{E}$ of (1) such that,

$$\|(\widehat{x}, \widehat{y}) - (x, y)\| \le \lambda\vartheta. \tag{43}$$

Remark 2. $(\widehat{x}, \widehat{y}) \in \mathbb{E}$ is called a solution of (42) iff there is a function $(x, y) \in \mathbb{E} \le$ (depending on $(\widehat{x}, \widehat{y})$) satisfying
$|u_1(t)| \le \vartheta_1$ and $|u_2(t)| \le \vartheta_2, t \in [0, T]$;

$$\begin{cases} {}_0^{ABC}D^{\alpha}\left(\dfrac{x(t)}{g_1(t,x(t),y(t))}\right) = f_1(t,x(t),y(t)) + u_1(t), & \forall t\in[0,T], 0<\alpha<1, \\[2ex] {}_0^{ABC}D^{\beta}\left(\dfrac{y(t)}{g_2(t,x(t),y(t))}\right) = f_2(t,x(t),y(t)) + u_2(t), & \forall t\in[0,T], 0<\beta<1. \end{cases} \tag{44}$$

Lemma 5. *For $(\widehat{x}, \widehat{y}) \in \mathbb{E}$ solution of (42), the following inequality holds:*

$$\begin{cases} \left|\dfrac{\widehat{x}(t)}{g_1(t,\widehat{x}(t),\widehat{y}(t))} - \varphi_1(0) - {}^{AB}I_0^{\alpha} f_1(t,\widehat{x}(t),\widehat{y}(t))\right| = \left|\left({}^{AB}I_0^{\alpha}u_1\right)(t)\right| \\[2ex] \le \vartheta_1\left(\dfrac{1-\alpha}{N(\alpha)} + \dfrac{T^{\alpha}}{N(\alpha)\Gamma(\alpha)}\right)\varphi_1(0) = \dfrac{\widehat{x}(0)}{g_1(0,\widehat{x}(0),\widehat{y}(0))}; \\[2ex] \left|\dfrac{\widehat{y}(t)}{g_2(t,\widehat{x}(t),\widehat{y}(t))} - \varphi_2(0) - {}^{AB}I_0^{\alpha} f_2(t,\widehat{x}(t),\widehat{y}(t))\right| = \left|\left({}^{AB}I_0^{\alpha}u_2\right)(t)\right| \\[2ex] \le \vartheta_2\left(\dfrac{1-\alpha}{N(\beta)} + \dfrac{T^{\beta}}{N(\beta)\Gamma(\beta)}\right), \varphi_2(0) = \dfrac{\widehat{y}(0)}{g_2(0,\widehat{x}(0),\widehat{y}(0))}. \end{cases} \tag{45}$$

Theorem 2. *If (H1)–(H4), (42) $\Theta = (1-\Lambda_1)(1-\Lambda_2) - \Lambda_2\Lambda_1 \neq 0$ are fulfilled, then (1) is Ulam-Hyers stable, where* $\Lambda_1 = L_{f_1} M_{g_1}(1 - \alpha/N(\alpha) + T^{\alpha}/N(\alpha)\Gamma(\alpha+1))$, $\Lambda_2 = L_{f_2} M_{g_2}(1 - \beta/N(\beta) + T^{\beta}/N(\beta)\Gamma(\beta+1))$.

Proof. Let $(\widehat{x}, \widehat{y}) \in \mathbb{E}$ be a solution of (42), $(x, y) \in \mathbb{E}$ be a solution of problem (1) satisfying

$$\begin{cases} \dfrac{x(0)}{g_1(t, x(0), y(0))} = \varphi_1(0) = \dfrac{1}{a_1 + b_1}\left(-b_1\left[{}^{AB}I_0^{\alpha} f_1(T, x(T), y(T))\right] + c_1\right), \\ \dfrac{y(0)}{g_2(t, x(0), y(0))} = \varphi_2(0) = \dfrac{1}{a_2 + b_2}\left(-b_2\left[{}^{AB}I_0^{\beta} f_2(T, x(T), y(T))\right] + c_2\right). \end{cases} \tag{46}$$

□

By Definition 2 and Lemma 2, conditions (H3), (H4), $L_{f_1} \leq N(\alpha)/1 - \alpha$, we have

$$\begin{aligned} |\widehat{x}(t) - x(t)| &\leq M_{g_1}\left|\frac{\widehat{x}(t)}{g_1(t, \widehat{x}(t), \widehat{y}(t))} - \frac{x(t)}{g_1(t, x(t), y(t))}\right| \\ &\leq M_{g_1}\Big|\frac{\widehat{x}(t)}{(f_1)} - {}^{AB}I_0^{\alpha} f_1(t, \widehat{x}(t), \widehat{y}(t)) \\ &\quad + {}^{AB}I_0^{\alpha} f_1(t, \widehat{x}(t), \widehat{y}(t)) - \varphi_1(0) - {}^{AB}I_0^{\alpha} f_1(t, x(t), y(t))\Big| \\ &\leq M_{g_1}\left|\frac{\widehat{x}(t)}{f_1(t, \widehat{x}(t), \widehat{y}(t))} - {}^{AB}I_0^{\alpha} f_1(t, \widehat{x}(t), \widehat{y}(t)) - \varphi_1(0)\right| \\ &\quad + M_{g_1}\left|{}^{AB}I_0^{\alpha} f_1(t, \widehat{x}(t), \widehat{y}(t)) - {}^{AB}I_0^{\alpha} f_1(t, x(t), y(t))\right| \\ &\leq \vartheta_1\left(\frac{1-\alpha}{N(\alpha)} + \frac{T^{\alpha}}{N(\alpha)\Gamma(\alpha)}\right) + L_{f_1} M_{g_1}^{AB} I_0^{\alpha}\left(|\widehat{x}(t) - x(t)| + |\widehat{y}(t) - y(t)|\right) \\ &\leq \vartheta_1\left(\frac{1-\alpha}{N(\alpha)} + \frac{T^{\alpha}}{N(\alpha)\Gamma(\alpha)}\right) \\ &\quad + L_{f_1} M_{g_1}\left[\left(\frac{1-\alpha}{N(\alpha)} + \frac{T^{\alpha}}{N(\alpha)\Gamma(\alpha+1)}\right)\left(\|\widehat{x} - x\| + \|\widehat{y} - y\|\right)\right] \\ &\leq \vartheta_1\left(\frac{1-\alpha}{N(\alpha)} + \frac{T^{\alpha}}{N(\alpha)\Gamma(\alpha)}\right) + \Lambda_1\left(\|\widehat{x} - x\| + \|\widehat{y} - y\|\right). \end{aligned} \tag{47}$$

Using (47), we get

$$(1 - \Lambda_1)\|\widehat{x} - x\| + \Lambda_1\|\widehat{y} - y\| \leq \vartheta_1\left(\frac{1-\alpha}{N(\alpha)} + \frac{T^{\alpha}}{N(\alpha)\Gamma(\alpha)}\right). \tag{48}$$

Similar the proof of (48), we can obtain

$$(1 - \Lambda_2)\|\widehat{y} - y\| + \Lambda_1\|\widehat{x} - x\| \leq \vartheta_2\left(\frac{1-\beta}{N(\beta)} + \frac{T^{\beta}}{N(\beta)\Gamma(\beta)}\right). \tag{49}$$

It follows from (48) and (49) that

$$\|\widehat{x} - x\| \leq \frac{1 - \Lambda_1}{\Theta}\vartheta_1\left(\frac{1-\alpha}{N(\alpha)} + \frac{T^{\alpha}}{N(\alpha)\Gamma(\alpha)}\right) + \frac{\Lambda_1}{\Theta}\vartheta_2\left(\frac{1-\beta}{N(\beta)} + \frac{T^{\beta}}{N(\beta)\Gamma(\beta)}\right), \tag{50}$$

$$\|\widehat{y} - y\| \leq \frac{\Lambda_2}{\Theta}\vartheta_1\left(\frac{1-\alpha}{N(\alpha)} + \frac{T^{\alpha}}{N(\alpha)\Gamma(\alpha)}\right) + \frac{1 - \Lambda_2}{\Theta}\vartheta_2\left(\frac{1-\beta}{N(\beta)} + \frac{T^{\beta}}{N(\beta)\Gamma(\beta)}\right), \tag{51}$$

Using (50) and (51), we get

$$\begin{aligned}\|\hat{x}-x\|+\|\hat{y}-y\| &\leq \left(\frac{1-\Lambda_1}{\Theta}+\frac{\Lambda_2}{\Theta}\right)\vartheta_1\left(\frac{1-\alpha}{N(\alpha)}+\frac{T^{\alpha}}{N(\alpha)\Gamma(\alpha)}\right)\\ &+\left(\frac{\Lambda_1}{+}\frac{1-\Lambda_2}{\Theta}\right)\vartheta_2\left(\frac{1-\beta}{N(\beta)}+\frac{T^{\beta}}{N(\beta)\Gamma(\beta)}\right)\\ &=\left(\frac{1-\Lambda_1}{\Theta}+\frac{\Lambda_2}{\Theta}\right)\vartheta_1\Omega_1+\left(\frac{\Lambda_1}{+}\frac{1-\Lambda_2}{\Theta}\right)\vartheta_2\Omega_2,\end{aligned} \tag{52}$$

where $\Omega_1 = (1-\alpha/N(\alpha)+T^{\alpha}/N(\alpha)\Gamma(\alpha)), \Omega_2 = (1-\beta/N(\beta) + T^{\beta}/N(\beta)\Gamma(\beta))$. For $\vartheta = \max\{\vartheta_1, \vartheta_2\}$ and $\lambda = ((1-\Lambda_1+\Lambda_2)\Omega_1 + (\Lambda_1+1-\Lambda_2)/\Theta)$, we can get

$$\|(\hat{x},\hat{y})-(x,y)\| = \|\hat{x}-x\|+\|\hat{y}-y\| \leq \lambda\vartheta. \tag{53}$$

Therefore, by means of Definition 2, the solution of problem (1) is Ulam–Hyers stable.

4. An Example

We consider the next problem

$$\begin{cases} {}_0^{ABC}D^{1/2}\left(\dfrac{x(t)}{1+(\sin(x(t))/32)+(\sin(y(t))/32)}\right) = \dfrac{t}{9}\left(\dfrac{x(t)}{1+|x(t)|}+\dfrac{y(t)}{1+|y(t)|}\right) t\in[0,T],\\ \dfrac{1}{2}x(0)+\dfrac{1}{2}x(\pi)=0,\end{cases}$$
$$\begin{cases} {}_0^{ABC}D^{1/2}\left(\dfrac{y(t)}{1+(\sin(x(t))/32)+(\sin(y(t))/32)}\right) = \dfrac{t}{9}\left(\dfrac{x(t)}{1+|x(t)|}+\dfrac{y(t)}{2+y(t)}\right) t\in[0,T],\\ \dfrac{1}{2}y(0)+\dfrac{1}{2}y(\pi)=0.\end{cases} \tag{54}$$

Obviously, for $i=1,2$, $L_{g_i}=1/32, K_{f_i}(t)=t$, $M_{g_i}=M_{f_i}=1$, and taking $N(\alpha)=N(\beta)=1$ as a normalization function, all hypotheses of Theorem 2 are satisfied.

In fact, since $\|t/9\|_{L^1}=\pi^2/18$, we can find that $L_{\mathbb{P}}M_{\mathbb{F}} = (L_{g_1}+L_{g_2})(\Delta_1+\Delta_2) = \pi^2/18\times 7/4\times 1/16 < 1$.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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