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Research Article

Generalized Auto-Convolution Volterra Integral Equations: Numerical Treatments

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In this paper, we use the operational Tau method based on orthogonal polynomials to achieve a numerical solution of generalized autoconvolution Volterra integral equations. Displaying a lower triangular matrix for basis functions, the corresponding solution is represented in matrix form, and an infinite upper triangular Toeplitz matrix is used to show the matrix representation of the integral part of the autoconvolution integral equation. We also investigate solvability of the obtained nonlinear system with infinite dimensional space and examine the convergence analysis of this method under the L^2- norm. Finally, the efficiency of the operational Tau method is studied by numerical examples.

1. Introduction

Nonlinear Volterra integral equations of the second kind as special case of the functional equations are one of the most important topics in applied mathematics. This functional equation has many applications in engineering sciences, physics, and mathematical modelling of the growth of a population [1]. Hence, finding a numerical solution with proper precision for this kind of integral equations is very important. In recent years, researchers have proposed several methods for solving nonlinear Volterra integral equations. For example, in the reference [2], the following nonlinear Volterra integral equations are solved by the Chebyshev polynomials, and its convergence analysis has been studied

$$u(t)=f(t)+\int_{t_0}^{t}k(t,s)G(u(s))ds,\ t\in D:=\left[t_0,t_f\right]. \tag{1}$$

Also, in [3], Tau method is used for solving the following nonlinear integral equations:

$$u(t)=f(t)+\int_{t_0}^{t}k(t,s)G(s,u(s))ds,\ t\in[0,T]. \tag{2}$$

A nonlinear VIE of the form

$$u(t)=f(t)+\lambda\int_{0}^{t}u(t-s)u(s)ds,\ t\in[0,T],\ \lambda\in\mathscr{R}, \tag{3}$$

is called an autoconvolution VIE (AVIE) of the second kind. In [4], using the Laplace transform, the existence of a continuous and bounded solution of the basic autoconvolution VIE is established. The reader will find good introductions for numerical approaches to other kinds of Volterra integral equations in [5–8]. Here, we consider a generalised autoconvolution Volterra integral equation as follows:

$$u(t)=f(t)+\int_{0}^{t}k(t,s)u(t-s)u(s)ds,\ t\in I:=[0,T], \tag{4}$$

such that $u(t)$ is an unknown solution and $f(t)$ and $k(t,s)$ are given and smooth functions. It should be noted that the linear barycentric rational quadrature method in reference [9] and collocation method by Zhang et al. [10] are used for solving these equations. Piecewise polynomial collocation methods for a class of third-kind autoconvolution Volterra integral equations have been considerd in [11]. Autoconvolution Volterra integral equations of the second kind (4) have some applications, namely, its use for the computation of special functions like the generalised Mittag-Leffler function and in the identification of memory kernels in the theory of viscoelasticity [4, 12]. Also, autoconvolution VIEs of the first kind,

$$f(t) = \int_0^t u(t-s)u(s)ds, \tag{5}$$

as ill-posed problems, arise in stochastic, probability theory, and spectroscopy [13–16]. In [10], the equation (4) was solved by using the collocation method according to piecewise polynomials, and the (super) convergence of the mentioned method has been studied. In this paper, we study the operational Tau method for numerical analysis of this equation.

This article is arranged as follows:

In Section 2, existence, uniqueness theorem and regularity conditions are given for the solution of AVIE (4), and in the sequel, the operational Tau method as numerical method is suggested to solve this equation. Section 3 includes solvability of the obtained nonlinear system with infinite dimensional space and the convergence analysis of the Tau numerical method. Finally, in Section 4, by various examples, we study the efficiency of the operational Tau method.

2. Main Results

2.1. Existence and Uniqueness Theorem. In this section, we present the existence and uniqueness theorem and regularity properties for the solution of generalized autoconvolution Volterra integral equation of the second kind (4). Existence and uniqueness theorem is investigated by two steps. Firstly, the local existence and uniqueness of a continuous solution on some small interval is established, and then, on the remaining interval, the classical theory of the Volterra integral equation is applied. We can summarise these results in the following theorem:

Theorem 1 (see [10]). *Assume that the given functions $f(t)$ and $k(t,s)$ in autoconvolution Volterra integral equation (4) satisfy $f \in C(I)$ and $k \in C(D)$, where $D := \{(t,s): 0 \le s \le t \le T\}$. Then, the autocovolution Volterra integral equation (4) has a unique solution $u \in C(I)$.*

Also, the regularity of the solution to the generalised autoconvolution VIE (4) can be deduced from the regularity of the Picard iterates and resolvent kernel of the given kernel $k(t,s)$ such that $K \in C^p(D)$ for some $p \ge 1$; the corresponding iterated kernels inherit this regularity [1].

Theorem 2 (see [10]). *Assume that the given functions in autoconvolution Volterra integral equation (4) satisfy $f \in C^p(I)$ and $k \in C^p(D)$ for $p \ge 1$. Then, the solution u of (4) possesses the regularity $u \in C^p(I)$.*

In other words, Theorem 2 shows that the regularity of order p from given functions can lead to regularity of order p for the solution of equation (4).

2.2. Numerical Computation of Tau Approximation for Generalized Volterra Integral Equations. In this section, we use the Tau method to calculate the approximate solution of the generalized AVIE (4). The operational Tau solution is a polynomial of degree N,

$$u_N(t) = \sum_{i=0}^{N} \hat{a}_i \varphi_i(t) = \hat{a}_N \varphi_t = \hat{a}_N \varphi X_t, \tag{6}$$

where $\{\varphi_i(t)\}_{i=1}^{\infty}$ is a set of orthogonal basis functions and φ is a nonsingular lower triangular matrix which satisfies $\varphi_t = \varphi X_t$. Also, $X_t = [1, t, t^2, \cdots]^T$ is a standard basis vector and $\hat{a}_N$ is defined by $\hat{a}_N = [a_0, a_1, \cdots, a_N, 0, 0, \cdots]$.

Similarly, for $u_N(s-t)$ we have:

$$u_N(s-t) = \sum_{i=0}^{N} \hat{a}_i \varphi_i(s-t) = \hat{a}_N \varphi_{s-t} = \hat{a}_N \hat{\varphi}_s X_t, \tag{7}$$

where $\hat{\varphi}_s$ is a nonsingular lower triangular matrix according to s which satisfies $\varphi_{s-t} = \hat{\varphi}_s X_t$. Let $f(t)$ be given polynomial with

$$f(t) = \sum_{i=0}^{N_f} f_i t^i = \hat{f} X_t, \tag{8}$$

where $\hat{f} = [f_0, f_1, \cdots, f_{N_f}, 0, 0, \cdots]$ (note that if the smooth function $f(t)$ is not polynomial, it may be approximated by polynomial).

For the matrix representation of the term $[u(s).u(t-s)]$, we consider the following lemma similar to Lemma 1 in [3, 17]:

Lemma 1. *Assume that $u(s) = \sum_{i=0}^{\infty} \hat{a}_i \varphi_i(s) = \hat{a}\varphi X_s$ and $u(t-s) = \sum_{i=0}^{\infty} \hat{a}_i \varphi_i(t-s) = \hat{a}\hat{\varphi}_t X_s$ are the polynomials with*

$$\begin{aligned} \hat{a} &= [\hat{a}_0, \hat{a}_1, \hat{a}_2, \cdots], \\ \varphi &= \left[\varphi_{ij}\right]_{i,j=0}^{\infty}, \\ \hat{\varphi}_t &= \left[\hat{\varphi}_{ij}\right]_{i,j=0}^{\infty}, \\ X_s &= \left[1, s, s^2, \cdots\right]^T, \end{aligned} \tag{9}$$

then

$$u(s).u(t-s) = \hat{a}\varphi\hat{B}X_s, \tag{10}$$

where $\hat{B}$ is an infinite upper triangular Toeplitz matrix with the following structure

$$\widehat{B}=\begin{pmatrix} \widehat{a}\widehat{\varphi}_0 & \widehat{a}\widehat{\varphi}_1 & \widehat{a}\widehat{\varphi}_2 & \cdots \\ 0 & \widehat{a}\widehat{\varphi}_0 & \widehat{a}\widehat{\varphi}_1 & \cdots \\ 0 & 0 & \widehat{a}\widehat{\varphi}_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{11}$$

and $\widehat{\varphi}_i$*'s is calculated by* $\widehat{\varphi}_i=[\widehat{\varphi}_{ij}]_{j=0}^{\infty}$.

Proof. By using the above assumptions, we have

$$u(s).u(t-s)=\widehat{a}\varphi X_s.\widehat{a}\widehat{\varphi}_t X_s=\widehat{a}\varphi(X_s.(\widehat{a}\widehat{\varphi}_t X_s)). \tag{12}$$

We show that

$$X_s.(\widehat{a}\widehat{\varphi}_t X_s)=\widehat{\mathrm{B}}\mathrm{X}_s, \tag{13}$$

for this purpose, we have

$$X_s(\widehat{a}\widehat{\varphi}_t X_s)=X_s\left(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\widehat{a}_i\widehat{\varphi}_{ij}s^j\right)=\left[\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\widehat{a}_i\widehat{\varphi}_{ij}s^{j+r}\right]_{r=0}^{\infty}. \tag{14}$$

On the other hand

$$\widehat{\mathrm{B}}\mathrm{X}_s=\left[\sum_{l=0}^{\infty}\widehat{B}_{rl}s^l\right]_{r=0}^{\infty}=\left[\sum_{l=0}^{\infty}\left(\sum_{i=0}^{\infty}\widehat{a}_i\widehat{\varphi}_{i,l-r}\right)s^l\right]_{r=0}^{\infty}. \tag{15}$$

Since for $r>l$ we have $\widehat{B}_{rl}=0$, so we can rewrite the above equation as follows:

$$\widehat{\mathrm{B}}\mathrm{X}_s=\left[\sum_{l=r}^{\infty}\left(\sum_{i=0}^{\infty}\widehat{a}_i\widehat{\varphi}_{i,l-r}\right)s^l\right]_{r=0}^{\infty}, \tag{16}$$

so, we can write

$$\widehat{\mathrm{B}}\mathrm{X}_s=\left[\sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\widehat{a}_i\widehat{\varphi}_{il}s^{l+r}\right]_{r=0}^{\infty}. \tag{17}$$

According to (14) and (17), the proof of Lemma 1 is completed. □

To obtain the matrix multiplication representation of the integral part of equation (4), we first consider expansion $k(t,s)$ as follows:

$$k(t,s)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}k_{ij}\varphi_i(t)\varphi_j(s)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\widehat{k}_{ij}t^is^j, \tag{18}$$

where k_{ij} can be obtained by the following formula:

$$k_{ij}=\frac{\left\langle\varphi_i(s),\left\langle k(t,s),\varphi_j(t)\right\rangle\right\rangle}{\|\varphi_i\|^2\left\|\varphi_j\right\|^2}. \tag{19}$$

Now, using Lemma 1 and the relation (18), the integral term of (4) can be written as

$$\int_0^t k(t,s)u(t-s)u(s)ds=\widehat{a}\varphi\widehat{\mathrm{B}}\mathcal{N}\mathrm{X}_t, \tag{20}$$

where

$$\mathcal{N}=\begin{pmatrix} 0 & \widehat{k}_{00} & \widehat{k}_{01}+\frac{1}{2}\widehat{k}_{10} & \widehat{k}_{02}+\frac{1}{2}\widehat{k}_{11}+\frac{1}{3}\widehat{k}_{20} & \cdots \\ 0 & 0 & \frac{1}{2}\widehat{k}_{00} & \frac{1}{2}\widehat{k}_{01}+\frac{1}{3}\widehat{k}_{10} & \cdots \\ 0 & 0 & 0 & \frac{1}{3}\widehat{k}_{00} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & \cdots & 0 & \frac{1}{n}\widehat{k}_{00} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{21}$$

and $\widehat{B}$ is the calculated Toeplitz matrix in Lemma 1. Finally, the following algebraic form of the equation (4) is obtained by (6) and (20) as

$$\widehat{a}_N Y\varphi_t=\widehat{f}\varphi^{-1}\varphi_t, \tag{22}$$

where $Y=I-\varphi\widehat{B}\mathcal{N}\varphi^{-1}$. Due to orthogonality of $\{\varphi_i(t)\}_{i=0}^{\infty}$, projection of (22) on the basis functions $\{\varphi_i(t)\}_{i=0}^{N}$ leads to the nonlinear algebraic system of the equations, and the unknown vector $\widehat{a}_N$ can be determined by solving this nonlinear algebraic system. Solvability of the Tau nonlinear algebraic system will be investigated in the next section.

3. Convergence Phenomenon

In this section, we are going to study solvability of the Tau nonlinear system with infinite dimensional space and convergence results of the proposed numerical method based on the orthogonal ultraspherical polynomials. To do this, we first consider Sobolev space:

3.1. Sobolev Space. We first define domain $\Omega=(0,1)$ in $\mathbb{R}^d$ with $d=1,2$, and we assume that

$$\mathbf{x}=(x_1,\cdots,x_d)\in\mathbb{R}^d. \tag{23}$$

Also we denote by $L_w^2(\Omega)$ the space of square integrable functions in Ω such that by the following norm it is a Banach space,

$$\|f\|_{L_w^2(\Omega)}=\left(\int_\Omega|f(\mathbf{x})|^2w(\mathbf{x})d\mathbf{x}\right)^{1/2}. \tag{24}$$

Definition 1 (see [18]). Suppose that $p\geq 0$ is an integer, the Sobolev space $W_w^p(\Omega)$ is the space of all L_w^2 functions that contains derivatives f of order p, in other words

$$W_w^p(\Omega) = \{f \in L_w^2(\Omega)\,;\, D^\alpha f \in L_w^2(\Omega), |\alpha| \le p\}, \tag{25}$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$ is a nonnegative multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$ and $D^\alpha f$ is defined by

$$D^\alpha f = \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_d} f}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_d^{\alpha_d}}. \tag{26}$$

Now, we can claim that this space is a Hilbert space by the following inner product

$$(u, v)_p = \sum_{|\alpha|\le p}\int_\Omega D^\alpha u(\mathbf{x}) D^\alpha v(\mathbf{x}) w(\mathbf{x}) d\mathbf{x}, \tag{27}$$

which induces the following norm:

$$\|f\|_{W_w^p(\Omega)} = \left(\sum_{|\alpha|\le p}\|D^\alpha f\|^2_{L_w^2(\Omega)}\right)^{1/2}. \tag{28}$$

Lemma 2. Sobolev inequality [18, 19]. *Let $(a, b) \subset \mathbb{R}$ be a bounded interval of the real line. For each function $v \in W_w^1(a, b)$ the following inequality holds:*

$$\|v\|_\infty \le C\|v\|^{1/2}_{L_w^2(a,b)}\|v\|^{1/2}_{W_w^1(a,b)}, \tag{29}$$

where the constant C depends only on the interval $[a, b]$.

3.2. Ultraspherical Polynomials. Jacobi polynomials $P_N^{(\alpha,\beta)}(x)$ of indices $\alpha, \beta > -1$ and degree N [18, 20] are explicitly defined by

$$P_N^{(\alpha,\beta)}(x) = \frac{1}{2^N}\sum_{l=0}^{N}\begin{pmatrix} N+\alpha \\ l\end{pmatrix}\begin{pmatrix} N+\beta \\ N-l\end{pmatrix}(x-1)^l(x+l)^{N-l}. \tag{30}$$

The Rodriguez formula provides an alternative representation, namely,

$$P_N^{(\alpha,\beta)}(x) = \frac{(-1)^N}{2^N N!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^N}{dx^N}\cdot\left((1-x)^{\alpha+N}(1+x)^{\beta+N}\right), \tag{31}$$

and Jacobi polynomials satisfy the following recursion relation:

$$\begin{aligned} &P_0^{(\alpha,\beta)}(x) = 1, P_1^{(\alpha,\beta)}(x) = \frac{1}{2}[((\alpha-\beta)) + (\alpha+\beta+2)x], \\ &a_{1,N}P_{N+1}^{(\alpha,\beta)}(x) = a_{2,N}P_N^{(\alpha,\beta)}(x) - a_{3,N}P_{N-1}^{(\alpha,\beta)}(x), \end{aligned} \tag{32}$$

where

$$\begin{aligned} a_{1,N} &= 2(N+1)(N+\alpha+\beta+1)(2N+\alpha+\beta), \\ a_{2,N} &= (2N+\alpha+\beta+1)\left(\alpha^2-\beta^2\right) \\ &\quad + x\Gamma(2N+\alpha+\beta+3)/\Gamma(2N+\alpha+\beta), \\ a_{3,N} &= 2(N+\alpha)(N+\beta)(2N+\alpha+\beta+2). \end{aligned} \tag{33}$$

Jacobi polynomials are orthogonal over the interval $(-1, 1)$ with the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$. The ultraspherical polynomials are simply Jacobi polynomials with $\alpha = \beta$ and normalized differently:

$$P_N^{(\alpha)}(x) = \frac{\Gamma(\alpha+1)\Gamma(N+2\alpha+1)}{\Gamma(2\alpha+1)\Gamma(N+\alpha+1)}P_N^{(\alpha,\alpha)}(x), \tag{34}$$

where $\Gamma(.)$ is the gamma function. The relation between Legendre, Chebyshev, and Gegenbauer polynomials with the ultraspherical polynomials is

$$\begin{aligned} P_N(x) &= P_N^{(0)}(x), \\ T_N(x) &= \frac{P_N^{(-1/2,-1/2)}(x)}{P_N^{(-1/2,-1/2)}(1)}, \\ C_N^{(\lambda)} &= P_N^{(\lambda-1/2)}(x). \end{aligned} \tag{35}$$

The following theorems provide the basic approximation results for the ultraspherical polynomial expansions:

Theorem 3 (see [21]). *For any $u(t) \in W_w^p(a, b), p \ge 0$, there exists a constant C and independent of N, such that*

$$\|u - \mathscr{P}_N u\|_{L_w^2(a,b)} \le CN^{-p}\|u\|_{W_w^p(a,b)}, \tag{36}$$

where $\mathscr{P}_N$ is the orthogonal projection operator as

$$\begin{aligned} \mathscr{P}_N u(x) &= \sum_{n=0}^{N}\widehat{u}_n P_n^{(\alpha)}(x), \\ \widehat{u}_n &= \frac{\left\langle u, P_n^{(\alpha)}\right\rangle_w}{\left\langle P_n^{(\alpha)}, P_n^{(\alpha)}\right\rangle_w}. \end{aligned} \tag{37}$$

Theorem 4 (see [21]). *For any $u(t) \in W_w^p(a, b), p \ge 0$, there exists a constant C and independent of N, such that*

$$\|u - \mathscr{P}_N u\|_{W_w^q(a,b)} \le CN^{\sigma(q,p)}\|u\|_{W_w^p(a,b)}, \tag{38}$$

where

$$\sigma(q,p)=\begin{cases}\frac{3}{2}q-p, 0\leq q\leq 1,\\ 2q-p-\frac{1}{2}, q\geq 1,\end{cases} \tag{39}$$

and $0\leq q\leq p$.

3.3. Convergence Results. Now, we investigate the main result of this section by the following theorem:

Theorem 5. *Let* $\mathscr{P}_N(u(t))=u_N(t)$ *is the Tau-ultraspherical polynomial approximation of the exact solution* $u(t)$ *of the autoconvolution Volterra integral equation (4) proposed in Subsection 2.2, and that the given functions in (4) be sufficiently smooth. Assume that* $u\in W_w^p(0,1), p\geq 1$. *Then, for sufficiently large* N, *we have*

$$\begin{aligned}&\|u_N(t)-u(t)\|_{L_w^2(0,1)}\longrightarrow 0,\\ &\|u_N(t)-u(t)\|_{L_w^2(0,1)}\simeq\mathcal{O}\left(N^{3/4-p}\right).\end{aligned} \tag{40}$$

Proof. According to the proposed operational Tau method, for sufficiently large N, we can write

$$u_N(t)=f(t)+\int_0^t k_{N,N}(t,s)u_N(t-s)u_N(s)ds. \tag{41}$$

By subtracting (41) from (4), it follows that

$$\begin{aligned}e_N(u(t))=&\int_0^t k(t,s)u(t-s)u(s)ds\\ &-\int_0^t k_{N,N}(t,s)u_N(t-s)u_N(s)ds,\end{aligned} \tag{42}$$

where $e_N(u(t))=u_N(t)-u(t)$. So we can write

$$\begin{aligned}e_N(u(t))=&\left(\int_0^t k(t,s)u(t-s)u(s)ds-\int_0^t k(t,s)u_N(t-s)u_N(s)ds\right)\\ &+\left(\int_0^t k(t,s)u_N(t-s)u_N(s)ds-\int_0^t k_{N,N}(t,s)u_N(t-s)u_N(s)ds\right).\end{aligned} \tag{43}$$

By defining S_1 and S_2 as follows:

$$\begin{aligned}S_1&:=\int_0^t k(t,s)u(t-s)u(s)ds-\int_0^t k(t,s)u_N(t-s)u_N(s)ds,\\ S_2&:=\int_0^t k(t,s)u_N(t-s)u_N(s)ds-\int_0^t k_{N,N}(t,s)u_N(t-s)u_N(s)ds,\end{aligned} \tag{44}$$

the equation (43) can be written as

$$e_N(u(t))=S_1+S_2, \tag{45}$$

and also

$$\|e_N(u(t))\|_{L_w^2(0,1)}\leq\|S_1\|_{L^2w(0,1)}+\|S_2\|_{L_w^2(0,1)}. \tag{46}$$

In the sequel, we show that $\|S_1\|_{L_w^2(0,1)}$ and $\|S_2\|_{L_w^2(0,1)}$ in inequality (46) tend to zero as $N\longrightarrow\infty$. For this aim, we consider two parts:

(i) -Part I:

$$\begin{aligned}\|S_1\|_{L_w^2(0,1)}=&\left\|\int_0^t k(t,s)(u(t-s)u(s)-u_N(t-s)u_N(s))ds\right\|_{L_w^2(0,1)}\\ \leq&\left\|\int_0^t k(t,s)(u(t-s)-u_N(t-s))(u(s)-u_N(s))ds\right\|_{L_w^2(0,1)}\\ &+\left\|\int_0^t k(t,s)(u(t-s)u_N(s)+u_N(t-s)u(s)\right.\\ &\left.-2u_N(t-s)u_N(s))ds\right\|_{L_w^2(0,1)}\\ \leq&\left\|\int_0^t k(t,s)(u(t-s)-u_N(t-s))(u(s)-u_N(s))ds\right\|_{L_w^2(0,1)}\\ &+\left\|\int_0^t k(t,s)(u_N(s)(u(t-s)-u_N(t-s))\right.\\ &\left.+u_N(t-s)(u(s)-u_N(s)))ds\right\|_{L_w^2(0,1)}\\ \leq&\|K\|_\infty(\|e_N(u(t-s))\|_\infty\cdot\|e_N(u(s))\|_\infty\\ &+\|u_N(s)\|_\infty\cdot\|e_N(u(t-s))\|_\infty\\ &+\|u_N(t-s)\|_\infty\cdot\|e_N(u(s))\|_\infty),\end{aligned} \tag{47}$$

where

$$\|K\|_\infty=\max_{t\in[0,1]}\int_0^t|k(t,s)ds|<\infty. \tag{48}$$

Therefore

$$\begin{aligned}\|S_1\|_{L_w^2(0,1)}\leq C\big(&\|e_N(u(t-s))\|_\infty\|e_N(u(s))\|_\infty\\ &+\|e_N(u(t-s))\|_\infty\left(\|e_N(u(s))\|_\infty+\|u(s)\|_\infty\right)\\ &+\|e_N(u(s))\|_\infty\left(\|e_N(u(t-s))\|_\infty+\|u(t-s)\|_\infty\right)\big).\end{aligned} \tag{49}$$

Then, from the Sobolev inequality in Lemma 2, we conclude

$$\|e_N(u(t-s))\|_\infty\leq C\|e_N(u(t-s))\|_{L_w^2(0,1)}^{1/2}\|e_N(u(t-s))\|_{W_w^1(0,1)}^{1/2}. \tag{50}$$

Now from the relations (36) and (38), the above equation can be written as follows:

$$\begin{aligned}\|e_N(u(t-s))\|_\infty\leq C&\left(\mathrm{C}N^{-p}\|u(t-s)\|_{W_w^p(0,1)}\right)^{1/2}\\ &\cdot\left(\mathrm{C}N^{3/2-p}\|u(t-s)\|_{W_w^p(0,1)}\right)^{1/2}.\end{aligned} \tag{51}$$

Similarly,

$$\|e_N(u(s))\|_\infty \le C\left(\mathrm{CN}^{-p}\|u(s)\|_{W^p_w(0,1)}\right)^{1/2} \cdot \left(\mathrm{CN}^{3/2-p}\|u(s)\|_{W^p_w(0,1)}\right)^{1/2}. \tag{52}$$

Therefore when $N \longrightarrow \infty$, $\|e_N u(t-s)\|_\infty$ and $\|e_N u(s)\|_\infty$ tend to zero. That means $\|S_1\|_{L^2_w(0,1)} \longrightarrow 0$ as $N \longrightarrow \infty$.

(ii) Part II:

From the generalized Hardy's inequality (see e.g., Lemma 3 from [22]), we have

$$\|S_2\|_{L^2_w(0,1)} \le \left\|e_{N,N}(k(t,s))\right\|_{L^2_w(\Omega)}\|u_N(s)\|_{L^2_w(0,1)}\|u_N(t-s)\|_{L^2_w(0,1)}, \tag{53}$$

where $\Omega = (0,1)^2$ and $e_{N,N}(k(t,s)) = k(t,s) - k_{N,N}(t,s)$. Also, (53) can be written

$$S_{2L^2_w(0,1)} \le e_{N,N}(k(t,s))_{L^2_w(\Omega)}\left(e_N(u(s))_{L^2_w(0,1)} + u(s)_{L^2_w(0,1)}\right) \cdot \left(e_N(u(t-s))_{L^2_w(0,1)} + u(t-s)_{L^2_w(0,1)}\right). \tag{54}$$

From (36), it follows that

$$\left\|e_{N,N}(k(t,s))\right\|_{L^2_w(\Omega)} \le \mathrm{CN}^{-p}\|k(t,s)\|_{H^p_w(\Omega)}. \tag{55}$$

Thus, we obtain $\|S_2\|_{L^2_w(0,1)} \longrightarrow 0$ as $N \longrightarrow \infty$. Finally, from the above relations, the convergence result of the Tau method is obtained. Also, the obtained results show that the rate of convergence of $u_N(t)$ to $u(t)$ depends on the value of regularity of $u(t)$. □

3.4. Solvability of the Tau Nonlinear Algebraic System. For investigation of the solvability of nonlinear system with infinite dimensional space which is obtained by the Tau method, we firstly consider some notation and definitions from [23, 24].

Let $\mathscr{B}$ be Banach space with a norm $\|u\|, u \in \mathscr{B}$. Consider the nonlinear equations as

$$u = Tu, \tag{56}$$

$$u = T_N u, N \in \mathbb{N}, \tag{57}$$

where $T : \mathscr{B}_0 \longrightarrow \mathscr{B}$ and $T_N : \mathscr{B}_0 \longrightarrow \mathscr{B}$ are nonlinear operators defined on an open set $\mathscr{B}_0 \subset \mathscr{B}$. Operator $T : \mathscr{B}_0 \longrightarrow \mathscr{B}$ is called Frechet differentiable at $u^0 \in \mathscr{B}_0$ if there exists a linear operator $T'(u^0) \in \mathscr{L}(\mathscr{B}, \mathscr{B})$ such that

$$\frac{\left\|Tu - Tu^0 - T'(u^0)(u-u^0)\right\|}{\|u-u^0\|} \longrightarrow 0, \text{ as } \left\|u-u^0\right\| \longrightarrow 0, \tag{58}$$

in this case $T'(u^0)$ is the (unique) Frechet derivative of T at u^0.

Lemma 3 (see [23]). *Let the following conditions be fulfilled:*

(1) The equation (56) has a solution $u^* \in \mathscr{B}_0$, and the operator T is Frechet differentiable at u^*

(2) There is a positive δ, such that the operators $T_N (N \in \mathbb{N})$ are Frechet differentiable in the ball $\|u-u^*\| \le \delta$ which is assumed to be contained in $\mathscr{B}_0$, and for any $\varepsilon > 0$ there is a $\delta_\varepsilon \in (0,\delta]$ such that for every $N \in \mathbb{N}$,

$$\left\|T'_N(u) - T'_N(u^*)\right\|_{\mathscr{L}(\mathscr{B},\mathscr{B})} \le \varepsilon, \textit{whenever } \|u-u^*\| \le \delta_\varepsilon \tag{59}$$

(3) $\|T_N u^* - Tu^*\| \longrightarrow 0$ as $N \longrightarrow \infty$

(4) $T'_N(u^*) \longrightarrow T'(u^*)$ compactly, whereby $T'_N(u^*) \in \mathscr{L}(\mathscr{B},\mathscr{B})(N \in \mathbb{N})$ are compact and the homogeneous equation $u = T'(u^*)u$ has in $\mathscr{B}$ only the trivial solution

Then there exist $N_0 \in \mathbb{N}$ and $\delta_0 \in (0,\delta]$, such that equation (57) has for $N \ge N_0$ a unique solution u_N in the ball $\|u-u^*\| \le \delta_0$. Thereby $u_N \longrightarrow u^*$ as $N \longrightarrow \infty$ and the following error estimate holds:

$$\|u_N - u^*\| \le C\|T_N u^* - Tu^*\|, N \ge N_0. \tag{60}$$

Here C is a positive constant not depending on N.

Now, we rewrite the equation (4) as

$$u = Tu = f(t) + \int_0^t K(t,s,u(t-s),u(s))ds, t \in [0,1], u(t) \in W^p_w(0,1), p \ge 0, \tag{61}$$

where $K(t,s,u(t-s),u(s)) = k(t,s)u(t-s)u(s)$. Considering the operational Tau method proposed in Section 2, we have

$$u_N = f(t) + \int_0^t k_{N,N}(t,s)u_N(t-s)u_N(s)ds = T_N u_N. \tag{62}$$

Let u^* be a solution of the equation (61). If u and u^0 are included in the ball $\|u-u^*\|_{L^2_w(0,1)} < \delta$ with a positive δ, then by the Taylor formula

$$
\begin{aligned}
&(\mathrm{Tu})(t)-\left(\mathrm{Tu}^{0}\right)(t)\\
&\quad=\int_{0}^{t} K(t,s,u(t-s),u(s))-K\left(t,s,u^{0}(t-s),u^{0}(s)\right)ds\\
&\quad=\int_{0}^{t}\frac{\partial K}{\partial u}\left(t,s,u^{0}(t-s),u^{0}(s)\right)\left(u(s)-u^{0}(s)\right)\\
&\qquad+\frac{1}{2}\frac{\partial^{2}K}{\partial u^{2}}\left(t,s,u^{0}(t-s),u^{0}(s)+\tau\left(u(s)-u^{0}(s)\right)\right)\\
&\qquad\cdot\left(u(s)-u^{0}(s)\right)^{2}ds,
\end{aligned}
\tag{63}
$$

where $\tau\in[0,1]$. From this, it follows that

$$
\left(T'\left(u^{0}\right)u\right)(t)=\int_{0}^{t}\frac{\partial K}{\partial u}\left(t,s,u^{0}(t-s),u^{0}(s)\right)u(t-s)u(s)ds.
\tag{64}
$$

Also, from Theorem 5, we have

$$
\|u-u_{N}\|_{L_{w}^{2}(0,1)}\longrightarrow 0,\ N\longrightarrow\infty.
\tag{65}
$$

Since integral operator in (64) is compact, by considering $\|S_1\|_{L_w^2(0,1)}, \|S_2\|_{L_w^2(0,1)} \longrightarrow 0$ from the proof of the Theorem 5, the operators T and T_N satisfy the conditions 1-4 of Lemma 3. Hence, in agreement with Lemma 3, there exist $N_0\in\mathbb{N}$ and $\delta_0\in(0,\delta)$; such that, for all $N\geq N_0$, nonlinear equation (62) possesses a unique solution u_N in the ball $\|u-u^*\|_{L_w^2(0,1)}\leq\delta_0$, and

$$
\|u_{N}-u^{*}\|_{L_{w}^{2}(0,1)}\longrightarrow 0,\ N\longrightarrow\infty.
\tag{66}
$$

4. Numerical Results

In this part of the paper, we apply the operational Tau method for solving the integral equation (4) by using the Chebyshev and Legendre polynomials as basis functions. All the computations were performed by Wolfram Mathematica 9.0.

Example 1. Consider the autoconvolution Volterra integral equation as follows:

$$
\begin{aligned}
u(t)&=\beta te^{-\gamma t}-\beta^{2}e^{-\gamma t}[t+t\cos t-2\sin t]\\
&\quad-\int_{0}^{t}k(t,s)u(t-s)u(s)ds,\ t\in[0,1],
\end{aligned}
\tag{67}
$$

with $k(t,s)=\cos(t-s)$. This integral equation has an exact and analytic solution $u(t)=\beta te^{-\gamma t}$.

In our computation, we chose $\beta=\gamma=1$ and $\beta=\gamma=50$. You can find the plots of the exact solution $u(t)=\beta te^{-\gamma t}$ for $\beta=\gamma=1$ and $\beta=\gamma=50$ in Figure 1. We use the Legendre and Chebyshev polynomials as basis functions and solve this integral equation by the operational Tau method. Also, the L_w^2 errors for different values of N have been reported in

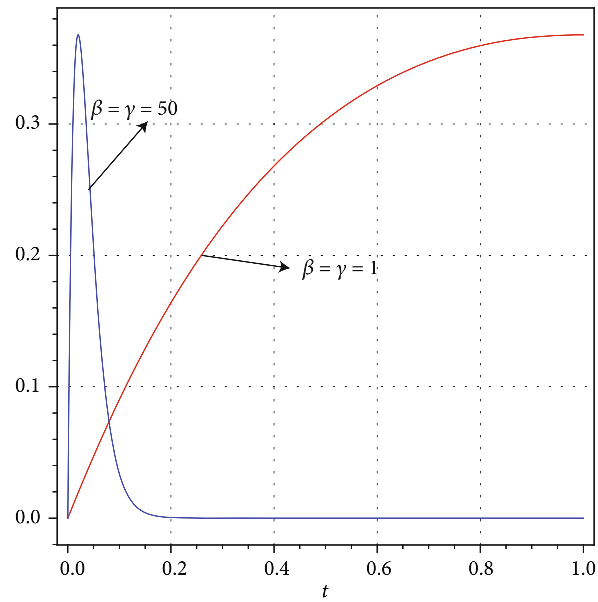

Figure 1: The plots of exact solution $u(t)=\beta te^{-\gamma t}$ for $\beta=\gamma=1$ and $\beta=\gamma=50$.

Table 1: L_w^2 errors of the operational Tau methods with $\beta=\gamma=1$ for Example 1.

N	Tau-Legendre	Tau-Chebyshev	Order
4	4.46×10^{-5}	5.93×10^{-5}	
5	2.20×10^{-6}	2.99×10^{-6}	13.48
6	9.16×10^{-8}	1.25×10^{-7}	17.43
7	3.26×10^{-9}	4.52×10^{-9}	21.63
8	1.06×10^{-10}	1.58×10^{-10}	25.65

Table 2: L_w^2 errors of the operational Tau methods with $\beta=\gamma=50$ for Example 1.

N	Tau-Legendre	Tau-Chebyshev
4	7.04×10^{-2}	7.01×10^{-2}
5	6.82×10^{-2}	6.61×10^{-2}
6	6.19×10^{-2}	6.09×10^{-2}
7	7.54×10^{-2}	6.29×10^{-2}
8	1.17×10^{-1}	6.92×10^{-2}

Tables 1 and 2. It is noticed that the proposed numerical scheme leads to a nonlinear algebraic system, and then we solve it numerically by the Newton method. Also, we report the convergence order of the Tau-Legendre method by

$$
p\approx-\frac{\log\left(e_{N_1}/e_{N_2}\right)}{\log\left(N_1/N_2\right)},
\tag{68}
$$

where e_{N_1} and e_{N_2} are the errors for N_1 and N_2, respectively. It displays the exponential rate of convergence for the

Table 3: L_w^2 errors of the operational Tau methods for Example 2.

N	Tau-Legendre	Tau-Chebyshev
4	1.40×10^{-5}	1.88×10^{-5}
5	3.11×10^{-7}	4.31×10^{-7}
6	2.00×10^{-8}	2.76×10^{-8}
7	3.39×10^{-10}	4.78×10^{-10}
8	1.71×10^{-11}	2.40×10^{-11}

Table 4: L_w^2 errors of the operational Tau methods for Example 3.

N	Tau-Legendre	Tau-Chebyshev
4	1.86×10^{-1}	6.26×10^{-1}
5	8.61×10^{-2}	1.52×10^{-1}
6	8.64×10^{-3}	1.28×10^{-2}
7	8.55×10^{-3}	1.11×10^{-2}
8	6.35×10^{-4}	8.82×10^{-4}

problem with complete smooth solution and confirms the prediction of Theorem 5.

Example 2. Consider the AVIE as follows:

$$\begin{aligned} u(t) &= \sin(t) - \frac{1}{3}(\cos(t) - 1)\sin(t) \\ &\quad + \int_0^t k(t,s)u(t-s)u(s)ds, \; t \in [0,1], \end{aligned} \tag{69}$$

where $k(t,s) = -\cos(t-s)$. This integral equation has an exact solution $u(t) = \sin(t)$. The errors for several values of N have been reported in Table 3.

Example 3 (see [25]). The integral equation

$$u(t) = \frac{1}{2}\sin t + \frac{1}{2}\int_0^t u(t-s)u(s)ds, \; t \in [0,10], \tag{70}$$

possesses the oscillatory solution $u(t) = J_1(t)$, where $J_1(t)$ is the Bessel function of order 1. The computational results have been reported in Table 4.

5. Conclusion

In this paper, we have considered the regularity properties and existence, uniqueness solution of the autoconvolution Volterra integral equation. The operational Tau method was used for solving these equations, and convergence analysis was investigated. Also, by applying this numerical method for some examples, we observed that this method is more accurate due to the complete smoothness of the exact solutions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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