Research Article

Convergence for a Fixed Point of Evolution Families in Banach Space via Iterative Process

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1. Introduction

Let $\mathcal{R}$ be a Banach space and $B$ be a nonempty, bounded, closed, and convex subset of $\mathcal{R}$. Let us consider a pointwise Lipschitzian evaluation family of nonlinear mappings, that is, a family of mappings $\Phi(\xi, \lambda): B \to B$ satisfying the conditions $\Phi(\xi, \lambda)\mu = \mu, \Phi(\xi, \lambda)\phi(\lambda, \xi)\mu = \Phi(\xi, \lambda)\mu, \lambda \to \Phi(\xi, \lambda)x$ is strongly continuous for every $x \in B$, and each $\Phi(\xi, \lambda)$ is pointwise Lipschitzian. The latter means that there exists a family of functions $\alpha_i: B \to [0, \infty)$ such that $\|\Phi(\xi, \lambda)\mu - \Phi(\xi, \lambda)v\| \leq \alpha_i(\mu - v)$. The existence of common fixed points for function of contraction and nonexpansive mappings have been investigated since the early 1960s, see Bruck[1, 2] Lim [3], Browder [4], Demarr [5], Belluce and Kirk [6, 7]. The asymptotic approach for finding common points of semigroups of Lipschitzian (but not pointwise Lipschitzian) mappings has been also investigated for some time, see Xu and Tan [8] and the references there in. Kirk and Xu [4] proved the existence of fixed points for asymptotic pointwise contraction and asymptotic pointwise nonexpansive mappings in Banach space, and latter on, Khamsi and Hussain extend these results to metric spaces [9], and Kozłowski and Khamsi to modular function spaces [10, 11]. The generalization of known iterative fixed point construction process of pointwise asymptotically nonexpansive mapping has been studied by several authors, such as the Mann process [12, 13] or the Ishikawa process for detail see [14]. The iterative fixed point construction process for asymptotically nonexpansive mapping exists in Hilbert space, Banach space, and Metric spaces, see for further detail [8, 9, 15–28] and the works referred there in. The modified Mann iterative process for a fixed point of asymptotically nonexpansive mappings in uniformly convex Banach space $\mathcal{R}$ which possess the Opial property of the weak convergence and the strong convergence for asymptotically nonexpansive mapping has been proved by Schu [20]. Xu and Tan [19] proved for weak convergence of the modified Ishikawa and modified Mann iterative process for asymptotically nonexpansive mapping on Banach space $\mathcal{R}$ that satisfy the Opial property. The generalized Ishikawa and Mann process weakly converges to a common fixed point of a pointwise nonexpansive mapping $\Phi(\xi, \lambda): B \to B (B$ is a nonempty, bounded, closed, and convex subset of a Banach space $\mathcal{R}$) has been proved by Kozłowski in [29] which satisfying the Opial conditions.

In [30], Kozłowski and Braily Sims examine the convergence of generalized Mann and Ishikawa iteration process for $T(t): B \to B$. In this paper, we proved more
2. Definitions and Preliminary Results

Definition 1. A family $\mathcal{U}$ on $B$ is pointwise Lipschitzian if there exists a family of mappings $\alpha(t)$: $[0, \infty) \to [0, \infty)$ such that
\[
\|\Phi(\zeta, \xi)\mu - \Phi(\zeta, \xi)\nu \| \leq \alpha(t)\|\mu - \nu\| \text{ for all } \zeta, \xi \in B. \quad (1)
\]

If $\alpha(t) \leq 1$ for each $t$, then the family is said to be a pointwise contraction family. If $\alpha(t) \leq 1$ for all $t$, then the family is called nonexpansive pointwise evolution family.

Definition 2. A pointwise Lipschitzian evolution family denoted by $\mathcal{U}$ and defined as $\mathcal{U} = \{\Phi(\zeta, \xi) ; \zeta \geq \xi \geq 0\}$ is said to be pointwise nonexpansive if $\sup \lim_{n \to \infty} \alpha_n(t) \leq 1$ for each $t \in B$. By letting $a_0 = 1$ and $a_n(t) = \max(\alpha_n(t), 1)$ for $n > 0$, we say that $\Phi(\zeta, \xi)$ is asymptotically nonexpansive if
\[
\|\Phi(\zeta, \xi)\mu - \Phi(\zeta, \xi)\nu \| \leq \alpha_n(t)\|\mu - \nu\| \text{ for all } \mu, \nu \in B, n \in N. \quad (2)
\]

where
\[
\lim_{n \to \infty} \alpha_n(t) = 1, \quad \alpha_n(t) \geq 1 \text{ where } t \in B, n \in N.
\]

By defining $\beta_n(t) = \alpha_n(t) - 1$, we have
\[
\lim_{n \to \infty} \beta_n(t) = 0. \quad (4)
\]

Definition 3. By $\mathcal{D}(B)$, we will represent the class of all asymptotic pointwise nonexpansive evolution families on $B$ such that $M_n = \sup \{\alpha_n(s); s \in B\} < \infty, \forall n \in N,$
\[
\lim_{n \to \infty} \sup M_n = 1. \quad (5)
\]

Lemma 1 (see [31]). Let $\mathcal{R}$ be a uniformly convex Banach space and $\mathcal{U}$ be an asymptotically nonexpansive pointwise Lipschitzian evolution family on $B$. Then, the set $F(\mathcal{U})$ of common fixed points is closed and convex.

$$\sum_{j=1}^{m-1} \sup \{\alpha_n(\mu) ; \mu \in B\} \leq M,$$

$$\|\Phi(m\zeta, 0)\mu_n - \mu_n\| = \|\Phi(m\zeta, 0)\mu_n - \Phi([m-1] \zeta, 0) + \Phi([m-1] \zeta, 0)

- \Phi([m-2] \zeta, 0) + \Phi([m-2] \zeta, 0) - \cdots - \Phi(2 \zeta, 0)\mu_n + \Phi(2 \zeta, 0)\mu_n - \Phi(\zeta, 0)\mu_n + \Phi(\zeta, 0)\mu_n - \mu_n\|

\leq \|\Phi(m\zeta, 0)\mu_n - \Phi([m-1] \zeta, 0)\| + \|\Phi([m-1] \zeta, 0) - \Phi([m-2] \zeta, 0)\| + \cdots + \|\Phi(\zeta, 0)\mu_n - \mu_n\|. \quad (9)$$

Then, by using the inequality, we have

Lemma 2 (see [15]). Assume $\{r_k\}$ be a bounded sequence of real numbers and $\{d_{kn}\}$ be double index sequence of real number satisfying
\[
\lim_{k \to \infty} \sup \lim_{n \to \infty} \sup d_{kn} \leq 0, \quad (6)
\]
and $r_{k,n} \leq r_k + d_{kn}$ for each $k, n \geq 1$. Then, the sequence $\{r_k\}$ converged to $r$, where $r \in \mathbb{R}$.

Definition 4. A subset $A \subset N$ is said to be a generating set for $N$ if for each $u \in N$, $u > 0$ there exists $m \in N$, and $\zeta, r \in A$ such that $u = m\zeta + r$.

The next lemmas will be used in the proof of our main result.

Lemma 3 (see [20, 32]). If $\mathcal{R}$ is uniformly convex Banach space, let $\{c_n\} \subset (0, 1)$ be bounded away from 0 to 1, and $\{u_n\}, \{v_n\} \subset \mathcal{R}$ be such that
\[
\lim_{n \to \infty} \sup \|u_n\| \leq a, \lim_{n \to \infty} \sup \|v_n\| \leq a, \quad (7)
\]
and $\lim_{n \to \infty} \|c_n u_n + (1 - c_n)v_n\| = a$, then $\lim_{n \to \infty} \|u_n - v_n\| = 0$.

3. Main Result

A sequence $\{c_n\} \subset (0, 1)$ is called bounded away from 0 if exists a number $a \in (0, 1)$ such that $c_n > a$, for all natural $n$. Similarly, $\{c_n\} \subset (0, 1)$ is called bounded away from 1 if exists a number $b \in (0, 1)$ such that $c_n < b$ for every natural number of $n$.

Lemma 4. Let $\mathcal{L} = \{\Phi(\zeta, 0) ; \zeta \geq 0\}$ be a subset of an evolution family $\mathcal{U}$ on a Banach space $\mathcal{R}$. If for each $m \in N$, there exists $\alpha_n(m)$ such that
\[
\|\Phi(m\zeta, 0)\mu_n - \Phi([m-1] \zeta, 0)\| \leq \alpha_n(m)\|\mu - \nu\| \text{ for all } \zeta \geq 0. \quad (8)
\]

Let $\|\Phi(\zeta, 0)\mu_n - \mu_n\| \to 0$, then $\|\Phi(m\zeta, 0)\mu_n - \mu_n\| \to 0$ for all $m \in N$.

Proof 1. It follows from the fact that every $\alpha_n$ is a bounded function that there exists a finite number $M > 0$, such that
\[ \| \Phi (m\zeta, 0)\mu_n - \mu_n \| \leq \alpha_{m-1}\| \Phi (\zeta, 0)\mu_n - \mu_n \| + \alpha_{m-2}\| \Phi (\zeta, 0)\mu_n - \mu_n \| + \cdots + \alpha_2\| \Phi (\zeta, 0)\mu_n - \mu_n \| \\
+ \alpha_1\| \Phi (\zeta, 0)\mu_n - \mu_n \| + 1\| \Phi (\zeta, 0)\mu_n - \mu_n \|. \] (10)

or

\[ \| \Phi (m\zeta, 0)\mu_n - \mu_n \| \leq \left[ \alpha_{m-1} + \alpha_{m-2} + \cdots + \alpha_1 + 1 \right] \| \Phi (\zeta, 0)\mu_n - \mu_n \| \\
= \sum_{j=1}^{m-1} (\alpha_{j+1}) \| \Phi (\zeta, 0)\mu_n - \mu_n \| \leq (M + 1)\| \Phi (\zeta, 0)\mu_n - \mu_n \| \longrightarrow 0, \] (11)

\[ \lim_{n \to \infty} \| \Phi (m\zeta, 0)\mu_n - \mu_n \| = 0. \]

This completes the proof. \( \square \)

**Lemma 5.** If \( B \) be a nonempty, bounded, closed, and convex subset of a Banach space \( \mathcal{R} \) and \( \{ \mu_n \} \subset B \) be an approximate fixed point sequence of \( L \) for each \( n \in A \), then the sequence \( \{ \mu_n \} \) is an approximate fixed point sequence of \( \Phi (m\zeta + r, 0) \).

**Proof 2.** Consider

\[ \| \Phi (m\zeta + r, 0)\mu_n - \mu_n \| \leq \| \Phi (m\zeta + r, 0)\mu_n - \Phi [(m-1)\zeta, 0]\mu_n + \Phi [(m-1)\zeta, 0]\mu_n - \mu_n \| \\
\leq \| \Phi (m\zeta + r, 0)\mu_n - \Phi [(m-1)\zeta, 0]\mu_n \| + \| \Phi [(m-1)\zeta, 0]\mu_n - \mu_n \| \longrightarrow 0. \] (12)

implies that

\[ \| \Phi (m\zeta + r, 0)\mu_n - \mu_n \| \leq \alpha (\mu)\| \Phi (\zeta, 0)\mu_n - \mu_n \| + \| \Phi [(m-1)\zeta, 0]\mu_n - \mu_n \| \longrightarrow 0. \] (13)

\[ \lim_{n \to \infty} \sup \| \mu_n - \mu \| < \lim_{n \to \infty} \sup \| \mu_n - \eta \|. \] (15)

The result follows by lemma 4 and the boundedness of \( \alpha (t) \).

Let us recall the definition of the Opial property which will play an important role in our paper. \( \square \)

**Definition 5.** A Banach space \( \mathcal{R} \) is said to have the Opial property if for every sequence \( \{ \mu_n \} \subset \mathcal{R} \) weakly converging to a point \( \mu \in \mathcal{R} \) (denoted by \( \mu_n \rightharpoonup \mu \)) and for each \( \nu \in \mathcal{R} \) such that \( \mu \neq \nu \) there holds

\[ \lim_{n \to \infty} \inf \| \mu_n - \mu \| < \lim_{n \to \infty} \inf \| \mu_n - \nu \|. \] (14)

or equivalently

\[ \| \Phi (m\zeta, 0)\mu_n - \mu_n \| \leq \| \Phi (m\zeta, 0)\mu_n - \Phi [(m-1)\zeta, 0]\mu_n \| + \| \Phi [(m-1)\zeta, 0]\mu_n - \mu_n \| + \| \mu_n - \mu \| \\
\leq [\alpha_{m-1} + \alpha_{m-2} + \cdots + \alpha_1 + 1] \| \Phi (\zeta, 0)\mu_n - \mu_n \| + \| \mu_n - \mu \| \\
\leq \left( \sum_{j=1}^{m-1} \alpha_{j+1} \right) \| \Phi (\zeta, 0)\mu_n - \mu_n \| + \| \mu_n - \mu \| \leq (M + 1)\| \Phi (\zeta, 0)\mu_n - \mu_n \| + \| \mu_n - \mu \|. \] (16)

Since all functions \( \alpha_i \) are bounded and hence \( \| \Phi (\zeta, 0)\mu_n - \mu_n \| \longrightarrow 0 \), it follows that

\[ \lim_{n \to \infty} \sup \| \Phi (m\zeta, 0)\mu_n - \mu_n \| \leq \lim_{n \to \infty} \sup \| \mu_n - \mu \| \]

\[ = \phi (\mu). \] (17)

By Lemma 4, we have

\[ \phi (\mu) \leq \lim_{n \to \infty} \sup \| \mu_n - \Phi (m\zeta, 0)\mu_n \| \\
+ \lim_{n \to \infty} \sup \| \Phi (m\zeta, 0)\mu_n - \mu \| \]

\[ = \lim_{n \to \infty} \sup \| \Phi (m\zeta, 0)\mu_n - \mu \|. \] (18)

Hence,
\[
\phi(\mu) = \lim_{n \to \infty} \sup_{\mu_n} \| \Phi(m\xi, 0)\mu_n - \mu \|.
\]  

(19)

Since \( \Phi(\xi, 0) \) is asymptotic pointwise nonexpansive, therefore,
\[
\phi(\Phi(m\xi, 0)\mu) \leq a_{m\xi}(\mu)\phi(\mu),
\]

(20)

for each \( \mu \in B \). By applying this to \( w \) and passing with \( m \to \infty \), we have
\[
\lim_{m \to \infty} \phi(\Phi(m\xi, 0)\mu) \leq \phi(w), (A).
\]

(21)

Since \( \mu_n \to w \), for \( \mu, w \in \mathcal{R} \) with \( \mu \neq w \), then by Opial property of \( \mathcal{R} \), we have
\[
\phi(w) \equiv \lim_{n \to \infty} \sup_{\mu_n} \| \mu_n - w \| < \lim_{n \to \infty} \sup_{\mu_n} \| \mu_n - \mu \| = \phi(\mu).
\]

(22)

By applying on both sides \( \lim_{n \to \infty} \sup \) and \( \phi(w) = \inf \{ \phi(\mu); \mu \in B \} \), we get
\[
\phi(w)^2 \leq \frac{1}{2} \phi(w)^2 + \frac{1}{2} \phi(\Phi(m\xi, 0)\mu)^2
\]
\[
- \frac{1}{4} \psi(\| \Phi(m\xi, 0)\mu - w \|),
\]

(26)

\[
\Rightarrow \psi(\| \Phi(m\xi, 0)\mu - w \|)
\]
\[
\leq 2\phi(\Phi(m\xi, 0)\mu)^2 - 2\phi(w)^2.
\]

Let \( n \to \infty \), we get that
\[
\lim_{n \to \infty} \psi(\| \Phi(m\xi, 0)\mu - w \|) = 0.
\]

(27)

Due to the property of \( \psi \), \( \Phi(m\xi, 0)\mu \to w \) for each fix \( k \in N \) using the same argument, we have
\[
\Phi(k\xi, k\xi)U(m\xi, 0)w \to U(k\xi, 0)w,
\]

(28)

or
\[
w \in F(\Phi(k\xi, 0)].
\]

(29)

\[
\Rightarrow \phi(w) = \inf \{ \phi(\mu); \mu \in B \}
\] comparing this with (A) give us
\[
\lim_{n \to \infty} \phi(\Phi(m\xi, 0)\mu) = \phi(w).
\]

By Theorem 2, [33] and Proposition 3.4, [4] for every \( d > 0 \), there exists a continuous map
\[
\psi: \mathbb{R}^+ \to \mathbb{R}^+.
\]

(23)

such that \( \psi(n) = 0 \) if and only if \( n = 0 \). This yields
\[
\| \alpha \mu + (1 - \alpha)\nu \| \leq \alpha\| \mu \| + (1 - \alpha)\| \psi(\| \mu - \nu \|),
\]

(24)

for each \( \alpha \in \mathbb{R}^+ \), for every \( \mu, \nu \in \mathcal{R} \) such that \( \| \mu \| \leq d, \| \nu \| \leq d \).

Put \( \mu = \mu_n - w, \nu = \mu_n - \Phi(m\xi, 0)\mu \)

and \( \alpha = 1/2 \), we get
\[
\Rightarrow \phi(w) = \inf \{ \phi(\mu); \mu \in B \}
\] evolution family \( \mathcal{F} \), the iterative formula is defined by the following:
\[
\mu_{k+1} = c_k\Phi(\xi_k, 0)\mu_k + (1 - c_k)\mu_k,
\]

(30)

where \( \mu_k \in B \) is chosen arbitrarily, (i) \( \{ c_k \} \) is bounded in 0 and 1. (ii) \( \lim_{k \to \infty} \xi_k = 0 \). (iii) \( \sum_{n=1}^{\infty} b_n c_n < \infty \) for all \( \mu \in B \).

\[
\text{Definition 7. If } \lim_{k \to \infty} \sup_{\mu} c_k = 1, \text{ then the above process }
\]

\[
\text{GKM} (\mathcal{L}, \{ c_k \}, \{ t_k \}) \text{ is well defined.}
\]

(31)

The following lemmas are important for the Mann convergence theorems.

\[
\text{Lemma 6. Let } B \text{ is a closed, bounded, and convex subset of a Banach space } \mathcal{R}. \text{ Let } w \in F(\mathcal{L}), \text{ and } \{ \mu_k \} \text{ be a sequence generated by GKM} (\mathcal{L}, \{ c_k \}, \{ t_k \}). \text{ Then, } \exists \text{ an element } r \in \mathcal{R} \text{ such that } \lim_{k \to \infty} \| \mu_k - w \| = r.
\]

\[
\text{Proof 4. Let } w \in F(\Phi(\xi, 0)). \text{ Then,}
\]
\[
\| \mu_{k+1} - w \| = c_k\Phi(\xi_k, 0)\mu_k + (1 - c_k)\| \mu_k - w \|
\]
\[
= c_k\Phi(\xi_k, 0)\mu_k + (1 - c_k)\mu_k - c_k w + c_k w - w.
\]

(32)

This implies that
\[ \| \mu_{k+1} - w \| = \| c_k \Phi ( \zeta_k, 0 ) \mu_k - c_k \Phi ( \zeta_k, 0 ) w + (1 - c_k) \mu_k - c_k w + c_k w - w \|
\leq \| c_k \Phi ( \zeta_k, 0 ) \mu_k - c_k \Phi ( \zeta_k, 0 ) w \| + \| (1 - c_k) \mu_k - c_k w + c_k w - w \|
\leq c_k (1 + b_{t_k} (w)) \| \mu_k - w \| + \| (1 - c_k) (\mu_k - w) \|
\leq c_k b_{t_k} (w) \| (\mu_k - w) \| + \| (\mu_k - w) \|
\leq b_{t_k} (w) \text{diam}(B) + \| (\mu_k - w) \| . \]  

It follows for each natural number \( n \in \mathbb{N} \),
\[ \| \mu_{kn} - w \| \leq c_k \| \Phi ( \zeta_k, 0 ) \mu_k - w \| + (1 - c_k) \| \mu_k - w \|
\leq c_k (1 + b_{t_k} (w)) \| \mu_k - w \| + (1 - c_k) \| \mu_k - w \|
\leq c_k b_{t_k} \| \mu_k - w \| + \| (1 - c_k) (\mu_k - w) \|
\leq c_k b_{t_k} \text{diam}(B) + \| (\mu_k - w) \|
\leq \sum_{i=k}^{k+2} b_{t_i} \text{diam}(B) + \| (\mu_k - w) \| . \]  

Now, for any \( n \in \mathbb{N} \),
\[ \| \mu_{kn} - w \| \leq \| \mu_k - w \| + \text{diam}(B) \sum_{i=k}^{k+1} b_{t_i} (w) . \]  

Let \( \| \mu - w \| = r_p \) for every natural number \( p \), we have
\[ d_{kn} = \text{diam}(B) \sum_{i=k}^{k+1} b_{t_i} (w) . \]  

We observe that \( \lim_{k \to \infty} \text{sup} \lim_{n \to \infty} d_{kn} = 0 \). By lemma 2, there exists an element \( r \in \mathbb{R} \), such that \( \lim_{k \to \infty} \| \mu_k - w \| = r \).  

\[ \lim_{k \to \infty} \sup_{k \to \infty} \| \Phi ( \zeta_k, 0 ) (\mu_k) - w \| = \lim_{k \to \infty} \sup_{k \to \infty} \| \Phi ( \zeta_k, 0 ) (\mu_k) - \Phi ( \zeta_k, 0 ) (w) \|
\leq \lim_{k \to \infty} \sup_{k \to \infty} \| \mu_k - w \| = r . \]  

We observe that
\[ \lim_{k \to \infty} c_k (\Phi ( \zeta_k, 0 ) - w) + (1 - c_k) (\mu_k - w) \]  
\[ = \lim_{k \to \infty} c_k v_k + (1 - c_k) u_k . \]  
By Lemma 3 apply for \( \Phi ( \zeta_k, 0 ) - w = v_k \) and \( \mu_k - w = u_k \)
\[ \Rightarrow \lim_{k \to \infty} \| v_k - u_k \| = 0 , \]
\[ \lim_{k \to \infty} \| \Phi ( \zeta_k, 0 ) (\mu_k) - \mu_k + u_k \| = 0 , \]
\[ \lim_{k \to \infty} \| \Phi ( \zeta_k, 0 ) (\mu_k) - \mu_k \| = 0 . \]  

Now, the construction of sequence \( \{ \mu_k \} \) is equivalent to
\[ \lim_{k \to \infty} \| \mu_{k+1} - \mu_k \| = \lim_{k \to \infty} c_k (\Phi ( \zeta_k, 0 ) \mu_k + (1 - c_k) \mu_k - \mu_k \| \to 0 , \]
\[ \lim_{k \to \infty} \| c_k (\Phi ( \zeta_k, 0 ) \mu_k - \mu_k \| \to 0 , \]
\[ \Rightarrow \lim_{k \to \infty} \| \mu_{k+1} - \mu_k \| = 0 . \]  

This completes the proof.
Now, we are to show the consequential lemma which proves that under the appropriate assumption of sequence, \( \{\mu_k\} \) generated by the following:

\[
\text{GKM}(\mathcal{Z}, \{c_k\}, \{\zeta_k\}).
\]  

(43)

The iterative process to become an approximate fixed point sequence will give the final determination of proving the process for convergence.

Lemma 8. Let \( B \) be a convex and bounded subset of the Banach space \( \mathcal{R} \). Let the GKM \((\mathcal{Z}, \{c_k\}, \{\zeta_k\})\) is well defined, and \( A \) is a subset of \( N \) such that for all \( s \in A \) a strictly increasing sequence \( \{j_k\} \in \mathbb{N} \) with Opial property. Let the sequence \( \mu_k \) with Opial property. Let the sequence \( \mu_k \) such that for all \( s \in A \) an approximate fixed points sequence for each bounded linear operators \( \Phi (m\zeta, 0)\), for \( s \in A \) and \( m \in \mathbb{N} \), that is,

\[
\lim_{k \to \infty} \|\Phi (m\zeta, 0)\mu_k - \mu_k\| = 0.
\]  

(44)

If \( A \subset N \), then

\[
\lim_{k \to \infty} \|\Phi (\zeta, 0)\mu_k - \mu_k\| = 0, \quad \zeta \in N.
\]  

(45)

Proof. 6. By Lemma 4, it is sufficient to show that \( \lim_{k \to \infty} \|\Phi (m\zeta, 0)\mu_k - \mu_k\| = 0 \) for \( m = 1 \). To this end, let \( s \in A \), and noted that \( \|\mu_k - \mu_{k+1}\| \to 0 \) as \( k \to \infty \). We have

\[
\|\mu_k - \mu_{k+1}\| \leq \|\mu_k - \mu_k\| + \|\mu_k - \mu_{k+1}\| + \|\mu_{k+1} - \mu_k\| + \|\Phi (\zeta, 0)\mu_k - \mu_k\|.
\]  

(46)

In view of Lemma 7, we observe that \( \mu_k - \Phi (\zeta, 0)\mu_k \to 0 \) as \( k \to \infty \). Indeed,

\[
\lim_{k \to \infty} \|\Phi (m\zeta, 0)\mu_k - \mu_k\| = 0.
\]  

(47)

Also,

\[
\lim_{k \to \infty} \|\Phi (\zeta, 0)\mu_k - \mu_k\| = 0, \quad \zeta \in N.
\]  

(48)

Theorem 2. If \( B \subset \mathcal{R} \) which is closed, convex, and bounded, with Opial property. Let the sequence \( \{\mu_k\} \), GKM \((\mathcal{Z}, \{c_k\}, \{\zeta_k\})\) iteration process is well defined and is an approximate fixed point sequence for each \( s \in A \subset N \), then a sequence \( \{\mu_k\} \) weakly converges to the common fixed point \( w \in F (\Phi (\zeta, 0)) \).

Proof. 7. Let \( \gamma, \omega \in B \), be two weak limit points of the sequence \( \{\mu_k\} \). Then, \( \exists \) two subsequences \( \{\gamma_k\} \) and \( \{\omega_k\} \) of \( \{\mu_k\} \), such that \( \gamma_k \to \gamma \) and \( \omega_k \to \omega \). For any fix \( s \in A \), and since \( \{\mu_k\} \) is approximate fixed point sequence for \( s \), it satisfies

\[
\lim_{k \to \infty} \|\Phi (\zeta, 0)\mu_k - \mu_k\| = 0.
\]  

(49)

By the Demiclosedness principle theorem, we have \( \Phi (\zeta, 0)\gamma = \gamma \) and \( \Phi (\zeta, 0)\omega = \omega \). Now, by Lemma 6, the
following limits $\lim_{k \to \infty} \|\mu_k - v\| = r_1$ and $\lim_{k \to \infty} \|\mu_k - \omega\| = r_2$ exist. We claim that $v = \omega$. On the contrary base let $v \neq \omega$, then by Opial property

$$r_1 = \liminf_{k \to \infty} \|v_k - v\| < \liminf_{k \to \infty} \|v_k - \omega\| = r_2$$

$$= \liminf_{k \to \infty} \|\omega_k - \omega\| < \liminf_{k \to \infty} \|\omega_k - v\| = r_1,$$

which is an absurd hence $v = \omega$. This shows that the sequence $\{v_k\}$ has mostly one weak limit point $w \in B$. Since $B$ is weakly sequentially compact, $\{\mu_k\}$ has only one weak limit point, that is, $\mu \in B$. Again, using the Demi-closedness principle, we get that $\Phi(\zeta,0)\mu = \mu$. Since $s \in A$ is chosen to be arbitrary and by the construction of $\mu$ does not depend on $s$, where $A \subset N$, we conclude that $\Phi(\zeta,0)\mu = \mu$ for each $\zeta \in J$.

Now, we can use the result for some more distinct situations. First, we start the discrete case.

**Theorem 3.** If $B \subset \mathcal{R}$ which is nonempty, convex, closed, and bounded with Opial property. Let $\mathcal{D}$ be an evolution family with discrete generating set $A = \{a_1, a_2, a_3, \ldots\}$, and $GKM(\mathcal{D}, [c_k], [\zeta_k])$ iterative process with $m \leq \text{card}(A)$. If for each $n \in \mathbb{N}$, $\exists$ a quasiperiodic strictly increasing sequence $\{j_k(m)\}$ of natural numbers with quasiperiod $p_k$, and for each natural number $k \in \mathbb{N}$, $\exists$ a quasiperiodic strictly increasing sequence $\{j_k(m)\}$ of natural numbers with quasiperiod $p_k$.

Then, the sequence $\{\mu_k\}$ generated by $GKM(\mathcal{D}, [c_k], [\zeta_k])$ is the approximate fixed point sequence for each $s \in A \subset N$, and $\{\mu_k\}$ weakly converges to a common fixed point $w \in F(\mathcal{D})$.

**Proof.** To prove the result, we will verify both the conditions of Lemma 8. The condition (ii) of Lemma 8 is clearly satisfied because $\zeta_{j_k(m)} - \zeta_{j_k(m)} - a_m = 0$. To show (i) we noted by quasiperiodicity of $\{j_k(m)\}$ for each natural number $k$, $\exists$ $j_k(m)$ such that $|k - j_k(m)| \leq p_k$. Assuming that $k - p_k \leq j_k(m) \leq k$. Fix $\varepsilon > 0$. Noted that by Lemma 7, $\|\mu_k - \mu_s\| < \varepsilon/p_k$ for $k$. Therefore, for natural number $k$ sufficiently large, it is true that

$$\|\mu_k - \mu_s\| \leq \|\mu_k - \mu_{k-1}\| + \cdots + \|\mu_{j_k(m)+1} - \mu_{j_k(m)}\|$$

$$\leq \frac{\varepsilon}{p_k} = \varepsilon.$$

This shows that (i) is also satisfied; hence, the sequence $\{\mu_k\}$ generated by $GKM(\mathcal{D}, [c_k], [\zeta_k])$ is the approximate fixed point sequence for each $s \in A \subset N$. By Theorem 2, the sequence $\{\mu_k\}$ is weakly converges to the common fixed point $\mu \in \mathcal{D}$.

**Remark 1.** We noted that Theorem 1 in [29] is a special case of Theorem 3 with $A = \{1\}$.

**Data Availability**

The data that support the findings of this study are available from the corresponding author upon reasonable request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest regarding this work.

**References**


