Research Article

Similarity Measures Based on *q*-Rung Linear Diophantine Fuzzy Sets and Their Application in Logistics and Supply Chain Management

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With the frequent occurrence of emergency events, decision-making (DM) plays an increasingly significant role in coping with them and has become an important and the challenging research focus recently. It is critical for decision makers to make accurate and reasonable emergency judgments in a short period as poor decisions can result in enormous economic losses and an unstable social order. As a consequence, this work offers a new DM approach based on novel distance and similarity measures using q-rung linear Diophantine fuzzy (q-RLDF) information to assure that DM problems may be addressed successfully and fast. One of the useful methods for determining the degree of similarity between the objects is the similarity measure. In this paper, we propose some new q-rung linear Diophantine fuzzy (q-ROLDF) distances and similarity measures. The Jaccard similarity measure, exponential similarity measure, and cosine and cotangent function-based similarity measures are proposed for q-LDFSs. The defined similarity measures are applied to the logistics and supply chain management problem, and the results are discussed. A comparison of new similarity measures is developed, and the proposed work's advantages are discussed.

1. Introduction

Models of knowledge-based decision-making attract substantial attention in industry and academia. A significant number of original research and thesis studies were undertaken to create effective decision support systems (DSSs) to promote managerial decision. DSS is classified as a specific class of electronic data system which supports decision-making (DM) management activities. In the early 1970s, the DSS idea was developed by Scott Morton's work. The methodology aims to evaluate strategic decisions in a complex and poorly organized environment, providing guidance to decision makers. DSSs have some huge benefits in decision-making by helping decision makers in their challenges and improve the effectiveness of the decision-making process [1].

For the purpose of logistics support, the better selection of logistic providers (LPs) is now important with the growth of the supply chain theories. Over the past decades, there has been a drastic change in the course of DSS [2]. Computer and technical experts have made great efforts to systematic processes of decision-making in the engineering and manufacturing industries [3]. Zha et al. [4], for example, developed a compromise decision-supporting issue approach and the fuzzy synthetic decision model (FSDM). Implementation of the DSS of multicriteria decision-making (MCDM) in supply chains is a constant challenge [5–7]. The reliability of the decision criteria in many decision-making problems (DMPs) is strictly dependent as external weights on the stakeholders and customer expectations. The delivery of decisions through solid and numerical values is difficult for clients. Fuzzy

linguistic variables enable us to ensure judgmental consistency.

Zadeh [8] developed the concept of fuzzy set (FS) in 1965, which addressed the membership degree (MD), and Atanassov [9] generalized the concept of FS by including nonmembership degree (NMD). IFS is a significant generalization of fuzzy sets that has been developed as a useful tool for understanding uncertain data. Pattern detection, decision-making, cluster analysis, and a variety of other areas are all derived from the IFS. In IFS theory, an element's MD and NMD must be between 0 and 1, and the number of MD and NMD must be between 0 and 1. As a result, in some practical applications, the values of MD and NMD of an element whose number of squares of MD and NMD may be greater than one are assigned to decision makers or decision experts. In such instances, the IFS has failed to clarify true or uncertain facts. Yager addressed this form of vague data by adding a new concept Pythagorean fuzzy set (PFS). Yager increased the space of MD and NMD by requiring that the square sum of MD and NMD lies between 0 and 1. PFSs [10] have found the complicated and indeterminacy problems in the assessment detail. Similar to the IFS, the PFS considered the function of MD u and NMD v, but with the condition that the square sum of MD and NMD must lie between 0 and 1. This means that the PFs are more generalized and comprehensive than the IFS, making them more useful to decision makers. If an expert assigns MD 0.7 and NMD 0.6 to an object, this information does not satisfy the IFS condition, but it is clear that these data can only be conveyed using PFS. As a result, IF decision-making and SMs are distinct from PF decision-making and SMs. Finally, this shows that PFS can covenant more effectively than IFS. Many researchers and investigators have evolved various decision-making methods and techniques [11–15]. Verma [16] defined generalized SMs for PFSs and their applications to MADM. The extended version of the PF-TOPSIS method was studied and applied to MCDM by Zhang and Xu [17]. Peng and Yang [18] introduced the new operations of subtraction and division and studied their fundamental properties. Reformat and Yager built a recommender framework based on collaboration under PFS information in [19]. PFSs are studied by various authors, who introduced various models for group decision-making problems [20–23]. In [24–28], various aggregation operators based on Bonferroni mean (BM) operators, BM with geometric mean, and other aggregation operators were described. Du [29] defined Minkowski-type distance measures for generalized orthopair fuzzy sets. Ejegwa [30] developed a modified Zhang and Xu's distance measure for PFSs and its application to pattern recognition problems.

Due to the restrictions on the MD and NMD in the IFS and PFS, the information for an object does not clarify, and both concepts have failed to convey the information about an object in certain cases. As a result, in [31], Yager introduced a new concept called q-rung orthopair fuzzy set (q-ROFS), which expands the MD and MND space by relaxing the condition to sum of the qth power of MD u and NMD v, i.e., $(u)^q + (v)^q \leq 1$, where q is equal to or less than 1. This means that if $q = 1$, then q-ROFS will be reduced to the IFS,

and if $q = 2$, q-ROFS will be reduced to the PFS. Clearly, q-ROFS is a more generalized framework than the IFS and PFS for explaining unclear and ambiguous details. Liu and Wang described the algebraic sum and product operation for q-ROFSs in [32] and studied their application in MCDM. In [33], aggregation operators for q-ROFSs based on Maclaurin symmetric mean (MSM) were created. Heronian mean operator [34], partitioned Bonferroni mean operator [35], and power Bonferroni mean operator [36] were developed by the investigator and studied. Liu and Xu developed an MCDM technique for green supplier selection problems under q-ROFSs in [37].

The similarity measure (SM) is a powerful method for determining the degree of similarity and dissimilarity between two items in multicriteria decision-making (MCDM) theory and pattern recognition. For the last few decades, the theory and implementations of SMs have been studied. The authors of [38–41] predicted a wide range of SMs for two IFSs. Liu and Cheng's SMs were modified by Mitchel [42] and also applied to MCDM. Park et al. [43] evaluated the generalized form of fuzzy Hamming distance measure (HDsM), developed IFHDM, and proposed various SMs to apply to the MCDM problem. Torra and Narukawa created some new SMs based on Hausdorff distance and discussed some applications in [44]. Using geometric aggregation operators, Xia and Xu defined the concept of distance measure (DsM) and SM and applied them to the MCDM problem in [45]. Ye [46] developed IF cosine SMs using the cosine feature and applied them to the MCDM problem. Hung [47] specified the likelihood-based SM for the IFS and applied it to medical diagnosis. Shi and Ye provided a modified form of CSMs in [48]. Maoying defined the cotangent SM between two IFSs for MD [49]. Rajarajeswari and Uma described a modified version of cotangent SMs [50], in which they considered MD, NMD, and indeterminacy degrees designated in the IFS. Szmidt introduced the distance measure of IFSs and also developed SMs. In [51–54], some different and generalized distance measures and SMs of IFSs were found. Li and Lu [55] defined some novel similarity and distance measures of PFSs and their applications. Verma and Merigo [56] proposed generalized SMs for PFSs and their applications to MADM. Zeng et al. [57] proposed q-rung orthopair fuzzy weighted induced logarithmic distance measures and their application in MADM. Akram et al. [58] defined a novel MCGDM analysis under m-polar fuzzy soft expert sets. Akram et al. [59] proposed attribute reduction algorithms for m-polar fuzzy relation decision systems. Ali and Sarwar [60] developed a novel technique for group DM under the fuzzy parameterized-rung orthopair fuzzy soft expert framework. Ali and Akram [61] proposed a DM method based on fuzzy N-soft expert sets.

Wang et al. [62] defined the SMs for two q-ROFSs and applied them to pattern recognition and MCDM problems. Peng and Liu established distance, information, and similarity measures, as well as their relationships, in [63]. The Minkowski-type DM, which includes HM, ED, and Chebyshev DsM, was developed and debated in MCDM by Du [29]. Ali addressed two other methods in [64], developing

the notions of possibility and confidence, as well as credibility and certainty, in q-ROFSs. The proposed work's key motivation is to analyze each extension of the fuzzy set (FS) in detail; in intuitionistic fuzzy sets, the two memberships explain the object's ambiguity, but the IFS fails to explain the described real-life problem. Consider a real-world problem where the MD and NMD values are greater than 0.5, i.e., 0.6 and 0.7; then, $0.6 + 0.7 > 1$, in this case, using the Pythagorean fuzzy sets to describe the real-world problem. When the PFS fails to define some information in a real-world problem, it does not explain the unknown information. As a result, it is failing to clarify uncertainty by using the q-rung orthopair fuzzy sets. As a result, the definition of a linear Diophantine fuzzy set (LDFS) was created to cover the importance of membership and nonmembership functions. Raiz and Hashmi introduced the principle of the linear Diophantine fuzzy set (LDFS) in [65], and they demonstrated that it is more generalized than IFS, PFS, and q-ROFSs. The IFS, PFS, and q-ROFSs are less descriptive and efficient than the LDFS. The key benefit of the LDFS is the availability of comparable parameters (RPs). The MD and NMD have more space than the IFS, PFS, and q-ROFSs due to these RPs.

The RPs in the LDFS are constrained and bounded to a finite space, i.e., the number of RPs must be less than or equal to 1. We work on distance and similarity measures of q-rung linear Diophantine fuzzy sets to extend the theory and applications of q-RLDFS. The distance measure for q-RLDFNs will be constructed, and their relations with the similarity measure will be studied. We continue to develop different types of similarity measures and define the Jaccard similarity measure (JMS) of the q-RLDFS, which tells us whether the information about the two q-RLDFSs is similar or different. For two q-RLDFSs, we also expand the JSM to weighted and generalized weighted JSM. Also, for two q-RLDFSs, the exponential similarity measure (ESM) was developed and extended for weighted and generalized weighted ESM. The proposed similarity study's applications are discussed at the end of the article. The proposed similarity measure was used to solve the logistics and supply chain management problem. At the conclusion of the paper, we use the suggested similarity measures to solve a real-world problem of supply chain management. We solve the logistic provider supply chain management problem and show that the proposed SMs are superior to other SMs, currently in use.

The layout is structured as follows: some basic concepts of FS, IFS, q-ROFS, and LDFS are offered in Section 2. In Section 3, we define the novel distance measure of q-rung linear Diophantine fuzzy sets and the Hamming DM. In Section 4, we introduce a similarity measure of q-rung linear Diophantine fuzzy sets and different types of similarity measure such as Jaccard SM and exponential SM. In Section 5, we present the cosine similarity measure for q-RLDFSs and cotangent function similarity measure for q-RLDFSs. In Section 6, we address the decision frames, a case study for q-RLFSs, and a numerical example given to demonstrate the application of the proposed method by using the proposed algorithms. In Section 7, we discuss the comparison between the existing methods and proposed method. In Section 8, this work is eventually outlined.

2. Basic Concepts

The basic concept of fuzzy sets and their extensions are offered in the current section.

Definition 1 (see [8]). Let X be an arbitrary nonempty set. A fuzzy set (FS) L is defined as

$$L = \{(x, u_L(x)) | x \in X\}, \tag{1}$$

where the function u_L is a mapping from $X \longrightarrow [0, 1]$, and for every $x \in X$, $0 \le u_L(x) \le 1$, and the function $u_L(x)$ is said to be the MD of x in X.

Definition 2 (see [9]). Let X be an arbitrary nonempty set. An IFS B in X is defined as

$$B = \{(x, u_B(x), v_B(x)) | x \in X\}, \tag{2}$$

where $u_B \longrightarrow [0, 1]$ and $v_B \longrightarrow [0, 1]$ are called, respectively, MD and NMD functions such that $\forall x \in X : 0 \le u_B(x) + v_B(x) \le 1$. Indeterminacy degree can be defined as

$$\pi_B(x) = 1 - u_B(x) - v_B(x), \forall x \in X. \tag{3}$$

Definition 3 (see [10]). Let X be an arbitrary nonempty set. A q-ROFS is denoted by B_P and mathematically defined as

$$B = \{(x, u_B(x), v_B(x)) | x \in X\}, \tag{4}$$

where $u_B(x)$ and $v_B(x)$ are MD and NMD functions : $X \longrightarrow [0, 1]$ with subject to $(u_B(x)^q) + (v_B(x)^q) \le 1$. The hesitancy MD is denoted by

$$\pi_B(x) = \sqrt{1 - (u_B(x)^q) - (v_B(x)^q)}. \tag{5}$$

q-ROFSs also have certain restrictions on MDs and NMDs.

Definition 4 (see [65]). Let X be an arbitrary nonempty set. A LDFS is denoted by G_D and mathematically defined as

$$G = \{x, \langle u_G(x), v_G(x) \rangle, \langle a, b \rangle : x \in X\}, \tag{6}$$

where $u_G(x), v_G(x), a, b \in [0, 1]$ are MD, NMD, and RPs, respectively, and satisfy the condition $0 \le au_G(x) + bv_G(x) \le 1, \forall x \in X$ with $0 \le a + b \le 1$. The following is the hesitancy degree:

$$\pi_G(x) = 1 - (a)u_G(x) - (b)v_G(x). \tag{7}$$

Definition 5 (see [66]). Let X be an arbitrary nonempty set. Then, the q-rung linear Diophantine fuzzy set (q-RLDFS) is represented by T and mathematically defined as

$$T = \{x, \langle u_T(x), v_T(x) \rangle, \langle a, b \rangle : x \in X\}, \tag{8}$$

where $u_T, v_T, a, b \in [0, 1]$ are MD, NMD, and reference parameters (RPs), respectively. These functions fulfill the restriction $0 \leq (a)^q u_T + (b)^q v_T \leq 1, \forall x \in X, q \geqslant 1$, with $0 \leq a^q + b^q \leq 1, q \geq 1$. The percentage of hesitation can be measured as follows:

$$\pi_T = \sqrt[q]{1 - \left(\left(a^{q)} u_T(x) + \left(b^{q)} v_T(x) \right) \right)}, \tag{9}$$

and q-rung linear Diophantine fuzzy number (q-RLDFN) is defined as

$$T = \{\langle u_T, v_T \rangle, \langle a, b \rangle\}. \tag{10}$$

3. Distance Measure of q-Rung Linear Diophantine Fuzzy Sets

In this section, we describe the distance measure (DsM) between two q-RLDFSs, as well as their basic properties and their relation.

Definition 6. Consider a family of q-RLDFSs. Then, a mapping $d(\Upsilon_1, \Upsilon_2): q - RL \; DF(\Upsilon) \times q - RL \; DF(\Upsilon) \longrightarrow [0, 1]$ is said to be distance measure (DM), where $\Upsilon_1, \Upsilon_2, \Upsilon_3 \in q - RL \; DF(\Upsilon)$, if the following conditions hold:

(1) $0 \leq d(\Upsilon_1, \Upsilon_2) \leq 1$.

(2) $d(\Upsilon_1, \Upsilon_2) \leq d(\Upsilon_2, \Upsilon_1)$.

(3) $d(\Upsilon_1, \Upsilon_2) = 1 \Leftrightarrow \Upsilon_1 = \Upsilon_2$.

(4) $d(\Upsilon_1, \Upsilon_2)^c = 1 \Leftrightarrow \Upsilon_1$ is a crisp set.

(5) Let $\Upsilon_1 \subseteq \Upsilon_2 \subseteq \Upsilon_3$. Then, $d(\Upsilon_1, \Upsilon_3) \leq d(\Upsilon_1, \Upsilon_2) + d(\Upsilon_2, \Upsilon_3)$.

Now, we define normalized Hamming distance and normalized Euclidean distance measure between two q-RLDFSs Υ_1 and Υ_2 in the following.

Definition 7. Consider two nonempty q-RLDFNs $\Upsilon_1 = (\langle u_{\Upsilon_1}(x_i), v_{\Upsilon_1}(x_i) \rangle, \langle a_i, b_i \rangle)$ and $\Upsilon_2 = (\langle u_{\Upsilon_2}(x_i), v_{\Upsilon_2}(x_i) \rangle, \langle c_i, d_i \rangle)$ on the discrete nonempty set $\wp = (\rho_1, \rho_2, \ldots, \rho_n)$. Then, the normalized Hamming distance measure is denoted by $\mathrm{NHD}(\Upsilon_1, \Upsilon_2)$ and given as

$$\mathrm{NHD}(\Upsilon_1, \Upsilon_2) = \frac{1}{4n} \sum_{i=1}^{n} \left\{ \begin{array}{l} \left| u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i) \right| \\ + \left| v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i) \right| \\ + \left| a_i^q - c_i^q \right| + \left| b_i^q - d_i^q \right| \end{array} \right\}. \tag{11}$$

Definition 8. Consider two nonempty q-RLDFNs $\Upsilon_1 = (\langle u_{\Upsilon_1}(x_i), v_{\Upsilon_1}(x_i) \rangle, \langle a_i, b_i \rangle)$ and $\Upsilon_2 = (\langle u_{\Upsilon_2}(x_i), v_{\Upsilon_2}(x_i) \rangle, \langle c_i, d_i \rangle)$ on the discrete nonempty set $\wp = (\rho_1, \rho_2, \ldots, \rho_n)$. Then, the normalized Euclidean distance is denoted by $\mathrm{NED}(\Upsilon_1, \Upsilon_2)$ and given as

$$\mathrm{NED}(\Upsilon_1, \Upsilon_2) = \sqrt{\frac{1}{4n} \sum_{i=1}^{n} \left\{ \begin{array}{l} \left(u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i) \right)^2 \\ + \left(v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i) \right)^2 \\ + \left(a_i^q - c_i^q \right)^2 + \left(b_i^q - d_i^q \right)^2 \end{array} \right\}}. \tag{12}$$

We further generalize the HDM and EDM for two q-RLDFSs Υ_1 and Υ_2 using a parameter $\lambda \geq 1$ into equations (13) and (14), respectively. The generalized distance measure (GDsM) of two q-RLDFSs Υ_1 and Υ_2 is given as

$$\mathrm{GDM}(\Upsilon_1, \Upsilon_2) = \left\{ \frac{1}{4n} \sum_{i=1}^{n} \left(\begin{array}{l} \left| u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i) \right|^\lambda \\ + \left| v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i) \right|^\lambda \\ + \left| a_i^q - c_i^q \right|^\lambda + \left| b_i^q - d_i^q \right|^\lambda \end{array} \right) \right\}^{1/\lambda}. \tag{13}$$

Remark 1. If we put $\lambda = 1$, then $\mathrm{GDM}(\Upsilon_1, \Upsilon_2)$ is reduced to $\mathrm{HD}(\Upsilon_1, \Upsilon_2)$.

Remark 2. If we put $\lambda = 2$, then $\mathrm{GDM}(\Upsilon_1, \Upsilon_2)$ is reduced to $\mathrm{HD}(\Upsilon_1, \Upsilon_2)$.

Next, we use the weight of each x_i with the weight vector $w = (w_1, \ldots, w_n)$ subject to $w_i \in [0, 1]$ such that $\sum_{i=1}^{n} w_i = 1$. Then, we propose a new DsM, so called the generalized weighted distance measure (GWDsM) of Υ_1 and Υ_2 which is denoted by $\mathrm{GWDM}(\Upsilon_1, \Upsilon_2)$ and defined as

$$\mathrm{GWDM}(\Upsilon_1, \Upsilon_2) = \left\{ \sum_{i=1}^{n} w_i \left(\begin{array}{l} \left| u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i) \right|^\lambda \\ + \left| v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i) \right|^\lambda \\ + \left| a_i^q - c_i^q \right|^\lambda + \left| b_i^q - d_i^q \right|^\lambda \end{array} \right) \right\}^{1/\lambda}. \tag{14}$$

4. Similarity Measure of q-Rung Linear Diophantine Fuzzy Sets

In this section, we define different types of similarity measure (SM) of q-RLDFS and also discuss the basic properties of the proposed SMs. We will use the notion of the Jaccard, exponential, cosine, and cotangent functions and develop the similarity measures between q-RLDFSs.

First, we give the definition of similarity measures between two q-RLDFSs.

Definition 9. Consider a family of q-RLDFNs. Then, a mapping $\mathrm{SM}(\Upsilon_1, \Upsilon_2): q - \mathrm{RLDF}(\Upsilon) \times q - \mathrm{RLDF}(\Upsilon) \longrightarrow$

$[0, 1]$ is said to be a similarity measure (SM), where $\Upsilon_1, \Upsilon_2, \Upsilon_3 \in q - \text{RLDF}(\Upsilon)$, if the following conditions hold:

(1) $0 \le \text{SM}(\Upsilon_1, \Upsilon_2) \le 1$.

(2) $\text{SM}(\Upsilon_1, \Upsilon_2) \le \text{SM}(\Upsilon_2, \Upsilon_1)$.

(3) $\text{SM}(\Upsilon_1, \Upsilon_2) = 1 \Leftrightarrow \Upsilon_1 = \Upsilon_2$.

(4) $\text{SM}(\Upsilon_1, \Upsilon_2)^c = 1 \Leftrightarrow \Upsilon_1$ is a crisp set.

(5) Let $\Upsilon_1 \subseteq \Upsilon_2 \subseteq \Upsilon_3$. Then, $\text{SM}(\Upsilon_1, \Upsilon_3) \le \min\{\text{SM}(\Upsilon_1, \Upsilon_2) + \text{SM}(\Upsilon_2, \Upsilon_3)\}$.

information of similarity of two q-RLDFSs is near to 1, then both are similar to each other, and if the information of similarity is near to close to 0, then both are dissimilar from each other. The JSM is very informative for similarity and dissimilarity for q-RLDFSs, so the JSM is useful in the decision-making problem and pattern recognition.

(1) $0 \le \text{JSM}(\Upsilon_1, \Upsilon_2) \le 1$

(2) $\text{JSM}(\Upsilon_1, \Upsilon_2) = \text{JSM}(\Upsilon_2, \Upsilon_1)$

(3) $\text{JSM}(\Upsilon_1, \Upsilon_2) = 1$ iff $\Upsilon_1 = \Upsilon_2$

4.1. Jaccard Similarity Measure of q-RLDFSs. This section consists of a novel SM between q-RLDFSs which is called the Jaccard similarity measure (JSM) between two q-RLDFSs. The JSM tells us the information whether the two q-RLDFSs are similar or distinct. The JSM provides information of similarity from 0 to 1 for the two q-RLDFSs. If the

Definition 10. Consider two nonempty q-RLDFNs $\Upsilon_1 = (\langle u_{\Upsilon_1}(x_i), v_{\Upsilon_1}(x_i)\rangle, \langle a_i, b_i\rangle)$ and $\Upsilon_2 = (\langle u_{\Upsilon_2}(x_i), v_{\Upsilon_2}(x_i)\rangle, \langle c_i, d_i\rangle)$ on the discrete nonempty set $\wp = (\rho_1, \rho_2, \ldots, \rho_n)$. Then, the Jaccard similarity measure (JSM) is denoted by $\text{JSM}(\Upsilon_1, \Upsilon_2)$ and given as

$$\text{JSM}(\Upsilon_1, \Upsilon_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{u_{\Upsilon_1}^q(x)u_{\Upsilon_2}^q(x_i) + v_{\Upsilon_1}^q(x_i)v_{\Upsilon_2}^q(x_i) + a_i^q c_i^q + b_i^q d_i^q}{\left[\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 + \left(a_i^q\right)^2 + \left(c_i^q\right)^2\right]}$$

$$+ \left[\left(u_{\Upsilon_2}^q(x_i)\right)^2 + \left(v_{\Upsilon_2}^q(x_i)\right)^2 + \left(b_i^q\right)^2 + \left(d_i^q\right)^2\right]$$

$$- \left[u_{\Upsilon_1}^q(x_i)u_{\Upsilon_2}^q(x_i) + v_{\Upsilon_1}^q(x_i)v_{\Upsilon_2}^q(x_i) + a_i^q c_i^q + b_i^q d_i^q\right] \tag{15}$$

Theorem 1. *Consider two nonempty q-RLDFNs $\Upsilon_1 = (\langle u_{\Upsilon_1}(x_i), v_{\Upsilon_1}(x_i)\rangle, \langle a_i, b_i\rangle)$ and $\Upsilon_2 = (\langle u_{\Upsilon_2}(x_i), v_{\Upsilon_2}(x_i)\rangle, \langle c_i, d_i\rangle)$ on the discrete nonempty set $\wp = (\rho_1, \rho_2, \ldots, \rho_n)$. Then, $\text{JSM}(\Upsilon_1, \Upsilon_2)$ defined in equation (11) satisfies the following conditions:*

Proof

(1) As Υ_1, Υ_2 are two q-RLDFNs and we know that $x^2 + y^2 \ge 2xy$,

$$u_{\Upsilon_1}^q(x_i)u_{\Upsilon_2}^q(x_i) + v_{\Upsilon_1}^q(x_i)v_{\Upsilon_2}^q(x_i) + a_i^q c_i^q + b_i^q d_i^q \ge 0, \tag{16}$$

and

$$\left\{\begin{array}{l} \left[\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 + \left(a_i^q\right)^2 + \left(c_i^q\right)^2\right] \\ + \left[\left(u_{\Upsilon_2}^q(x_i)\right)^2 + \left(v_{\Upsilon_2}^q(x_i)\right)^2 + \left(b_i^q\right)^2 + \left(d_i^q\right)^2\right] \\ - \left[u_{\Upsilon_1}^q(x_i)u_{\Upsilon_2}^q(x_i) + v_{\Upsilon_1}^q(x_i)v_{\Upsilon_2}^q(x_i) + a_i^q c_i^q + b_i^q d_i^q\right] \end{array}\right\} \ge 0. \tag{17}$$

Thus, $\text{JSM}(\Upsilon_1, \Upsilon_2) \ge 0$, and summing up to n and normalizing, we have $\text{JSM}(\Upsilon_1, \Upsilon_2) \le 1$. Hence, $0 \le \text{JSM}(\Upsilon_1, \Upsilon_2) \le 1$.

(2) The proof of $\text{JSM}(\Upsilon_1, \Upsilon_2) = \text{JSM}(\Upsilon_2, \Upsilon_1)$ is trivial.

(3) Suppose that $\Upsilon_1 = \Upsilon_2$; then, $u_{\Upsilon_1}^q(x_i) = u_{\Upsilon_2}^q(x_i)$, $v_{\Upsilon_1}^q(x_i) = v_{\Upsilon_2}^q(x_i)$, $a_i^q = c_i^q$, $b_i^q = d_i^q$. Then, equation (11) becomes

$$\text{JSM}(\Upsilon_1, \Upsilon_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{u_{\Upsilon_1}^q(x)u_{\Upsilon_1}^q(x_i) + v_{\Upsilon_1}^q(x_i)v_{\Upsilon_1}^q(x_i) + a_i^q a_i^q + b_i^q b_i^q}{\left[\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 + \left(a_i^q\right)^2 + \left(a_i^q\right)^2\right]}$$

$$+ \left[\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 + \left(b_i^q\right)^2 + \left(b_i^q\right)^2\right]$$

$$- \left[u_{\Upsilon_1}^q(x_i)u_{\Upsilon_1}^q(x_i) + v_{\Upsilon_1}^q(x_i)v_{\Upsilon_1}^q(x_i) + a_i^q a_i^q + b_i^q b_i^q\right]$$

$$
\begin{aligned}
&= \frac{1}{n} \sum_{i=1}^{n} \frac{\left(u_{\Upsilon_1}^q(x)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 + \left(a_i^q\right)^2 + \left(b_i^q\right)^2}{\left[\begin{array}{c}\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 \\ + \left(a_i^q\right)^2 + \left(a_i^q\right)^2\end{array}\right]} \\
&\qquad + \left[\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 + \left(b_i^q\right)^2 + \left(b_i^q\right)^2\right] \\
&\qquad - \left[\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 + \left(a_i^q\right)^2 + \left(b_i^q\right)^2\right] \\
&= \frac{1}{n} \sum_{i=1}^{n} \frac{\left(u_{\Upsilon_1}^q(x)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 + \left(a_i^q\right)^2 + \left(b_i^q\right)^2}{\left[\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 + \left(a_i^q\right)^2 + \left(a_i^q\right)^2\right]} \\
&= 1.
\end{aligned}
\tag{18}
$$

Conversely, the proof is easy.

Next, we use the weight of each x_i with the weight vector $w = (w_1, w_2, \ldots, w_n)^T$ subject to $w_i \in [0,1]$ such that $\sum_{i=1}^{n} w_i = 1$. Then, we propose a new SM, called weighted Jaccard similarity measure of Υ_1 and Υ_2 which is denoted by $\mathrm{JSM}(\Upsilon_1, \Upsilon_2)$ and defined in the following equation:

$$
\mathrm{WJSM}(\Upsilon_1, \Upsilon_2) = \sum_{i=1}^{n} w_i \frac{u_{\Upsilon_1}^q(x) u_{\Upsilon_2}^q(x_i) + v_{\Upsilon_1}^q(x_i) v_{\Upsilon_2}^q(x_i) + a_i^q c_i^q + b_i^q d_i^q}{\left[\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 + \left(a_i^q\right)^2 + \left(c_i^q\right)^2\right]} \cdot \\
\qquad + \left[\left(u_{\Upsilon_2}^q(x_i)\right)^2 + \left(v_{\Upsilon_2}^q(x_i)\right)^2 + \left(b_i^q\right)^2 + \left(d_i^q\right)^2\right] \\
\qquad - \left[u_{\Upsilon_1}^q(x_i) u_{\Upsilon_2}^q(x_i) + v_{\Upsilon_1}^q(x_i) v_{\Upsilon_2}^q(x_i) + a_i^q c_i^q + b_i^q d_i^q\right]
\tag{19}
$$

The proposed $\mathrm{WJSM}(\Upsilon_1, \Upsilon_2)$ also satisfies the conditions of Theorem 1. □

4.2. Exponential Similarity Measure of q-RLDFSs.

In this section, the exponential function with Euclidean and Hamming distances between two q-RLDFSs Υ_1 and Υ_2 is used to provide a new similarity measure between the two q-RLDFSs. Shepard predicted a universal law regarding distance and similarity measures in 1987. Both are related to an exponential function, and using the similarity measure based on the expansion function, the exponential distance measure is defined as

$$
\mathrm{ESM}_d(\Upsilon_1, \Upsilon_2) = E^{-d(\Upsilon_1, \Upsilon_2)}.
\tag{20}
$$

Now, we will talk about the properties of the distance measure, also known as axioms. Every distance measure must satisfy the following axioms:

(1) Equality: $d(\Upsilon_1, \Upsilon_2) = d(\Upsilon_2, \Upsilon_1)$ for every Υ_1 and Υ_2. Thus, $\mathrm{ESM}_d(\Upsilon_1, \Upsilon_2) = \mathrm{ESM}_d(\Upsilon_2, \Upsilon_1)$ for every Υ_1 and Υ_2.

(2) Minimality: $d(\Upsilon_1, \Upsilon_2) > d(\Upsilon_1, \Upsilon_1)$ for every $\Upsilon_1 \neq \Upsilon_2$. Thus, $\mathrm{ESM}_d(\Upsilon_1, \Upsilon_2) < \mathrm{ESM}_d(\Upsilon_1, \Upsilon_1)$ for every $\Upsilon_1 \neq \Upsilon_2$.

(3) Symmetry: $d(\Upsilon_1, \Upsilon_2) = d(\Upsilon_2, \Upsilon_1)$ for every Υ_1 and Υ_2. Thus, $\mathrm{ESM}_d(\Upsilon_1, \Upsilon_2) = \mathrm{ESM}_d(\Upsilon_2, \Upsilon_1)$ for every Υ_1 and Υ_2.

(4) Triangle inequality: $d(\Upsilon_1, \Upsilon_2) + d(\Upsilon_2, \Upsilon_3) \geq d(\Upsilon_1, \Upsilon_3)$ for all Υ_1, Υ_2 and Υ_3. Thus, we can say that the dissimilarities of any three q-RLDFSs satisfy the above properties. We can also say that if Υ_1 is similar to Υ_2 and Υ_2 is similar to Υ_3, then Υ_1 should be similar to Υ_3.

Now, in equation (20), we use the normalized Hamming distance (NHD) of two q-RLDFSs to get a new kind of SM based on NHD.

(1) $0 \leq \mathrm{ESM}_{\mathrm{NHD}}(\Upsilon_1, \Upsilon_2) \leq 1$

(2) $\mathrm{ESM}_{\mathrm{NHD}}(\Upsilon_1, \Upsilon_2) = \mathrm{ESM}_{\mathrm{NHD}}(\Upsilon_1, \Upsilon_2)$

(3) $\mathrm{ESM}_{\mathrm{NHD}}(\Upsilon_1, \Upsilon_2) = 1$ iff $\Upsilon_1 = \Upsilon_2$

(1) $0 \leq \mathrm{ESM}_{\mathrm{NED}}(\Upsilon_1, \Upsilon_2) \leq 1$

(2) $\mathrm{ESM}_{\mathrm{NED}}(\Upsilon_1, \Upsilon_2) = \mathrm{ESM}_{\mathrm{NED}}(\Upsilon_1, \Upsilon_2)$

(3) $\mathrm{ESM}_{\mathrm{NED}}(\Upsilon_1, \Upsilon_2) = 1$ iff $\Upsilon_1 = \Upsilon_2$

Definition 11. Consider two nonempty q-RLDFNs $\Upsilon_1 = (\langle u_{\Upsilon_1}(x_i), v_{\Upsilon_1}(x_i)\rangle, \langle a_i, b_i\rangle)$ and $\Upsilon_2 = (\langle u_{\Upsilon_2}(x_i), v_{\Upsilon_2}(x_i)\rangle, \langle c_i, d_i\rangle)$ on the discrete nonempty set $\wp = (\rho_1, \rho_2, \ldots, \rho_n)$. Then, ESM based on NHD is generated by using equation (20) and given as

$$\text{ESM}_{\text{NHD}}(\Upsilon_1, \Upsilon_2) = e^{-(1/4n)\sum\limits_{i=1}^{n}\left\{\begin{array}{c} \left|u^q_{\Upsilon_1}(x_i) - u^q_{\Upsilon_2}(x_i)\right| \\ +\left|v^q_{\Upsilon_1}(x_i) - v^q_{\Upsilon_2}(x_i)\right| + \left|a^q_i - c^q_i\right| + \left|b^q_i - d^q_i\right| \end{array}\right\}}. \tag{21}$$

Theorem 2. *Consider two nonempty q-RLDFNs $\Upsilon_1 = (\langle u_{\Upsilon_1}(x_i), v_{\Upsilon_1}(x_i)\rangle, \langle a_i, b_i\rangle)$ and $\Upsilon_2 = (\langle u_{\Upsilon_2}(x_i), v_{\Upsilon_2}(x_i)\rangle, \langle c_i, d_i\rangle)$ on the discrete nonempty set $\wp = (\rho_1, \rho_2, \ldots, \rho_n)$. Then, $\text{ESM}_{\text{NHD}}(\Upsilon_1, \Upsilon_2)$ defined in equation (21) satisfies the following conditions:*

Proof.

(1) We know from the definition of q-RLDFS that $0 \le u_{\Upsilon_1}(x_i) + v_{\Upsilon_1}(x_i) + a_i + b_i \le 1$ and $0 \le u_{\Upsilon_2}(x_i) + v_{\Upsilon_2}(x_i) + a_i + b_i \le 1$. Thus, we can write as

$$\frac{1}{4n}\sum_{i=1}^{n}\left\{\begin{array}{c} \left|u^q_{\Upsilon_1}(x_i) - u^q_{\Upsilon_2}(x_i)\right| \\ +\left|v^q_{\Upsilon_1}(x_i) - v^q_{\Upsilon_2}(x_i)\right| + \left|a^q_i - c^q_i\right| + \left|b^q_i - d^q_i\right| \end{array}\right\} \ge 0, \tag{22}$$

and

$$-\frac{1}{4n}\sum_{i=1}^{n}\left\{\left|u^q_{\Upsilon_1}(x_i) - u^q_{\Upsilon_2}(x_i)\right| + \left|v^q_{\Upsilon_1}(x_i) - v^q_{\Upsilon_2}(x_i)\right| + \left|a^q_i - c^q_i\right| + \left|b^q_i - d^q_i\right|\right\} \le 0. \tag{23}$$

We know about the exponential properties that the zero power of e is equal to 1, and the value of the negative power of e is less than 1. In this case, we have

$$0 \le e^{-(1/4n)\sum\limits_{i=1}^{n}\left\{\left|u^q_{\Upsilon_1}(x_i) - u^q_{\Upsilon_2}(x_i)\right| + \left|v^q_{\Upsilon_1}(x_i) - v^q_{\Upsilon_2}(x_i)\right| + \left|a^q_i - c^q_i\right| + \left|b^q_i - d^q_i\right|\right\}} \le 1. \tag{24}$$

Thus, we obtained that $0 \le \text{ESM}_{\text{NHD}}(\Upsilon_1, \Upsilon_2) \le 1$.

(2) The proof of this property is trivial.

(3) Suppose that $\Upsilon_1 = \Upsilon_2$; then, $u^q_{\Upsilon_1}(x_i) = u^q_{\Upsilon_2}(x_i)$, $v^q_{\Upsilon_1}(x_i) = v^q_{\Upsilon_2}(x_i), a^q_i = c^q_i, b^q_i = d^q_i$. Thus, equation (14) becomes

$$\begin{aligned}\text{ESM}_{\text{NHD}}(\Upsilon_1, \Upsilon_2) \\ = e^{-(1/4n)\sum\limits_{i=1}^{n}\left\{\left|u^q_{\Upsilon_1}(x_i) - u^q_{\Upsilon_2}(x_i)\right| + \left|v^q_{\Upsilon_1}(x_i) - v^q_{\Upsilon_2}(x_i)\right| + \left|a^q_i - c^q_i\right| + \left|b^q_i - d^q_i\right|\right\}} \\ = e^0 \\ = 1. \end{aligned} \tag{25}$$

Hence, $ESM_{NHD}(\Upsilon_1, \Upsilon_2) = 1$.

Next, using the normalized Euclidean distance (NED) of two q-RLDFSs in equation (21), we will get the new type of SM based on ED. □

Definition 12. Consider two nonempty q-RLDFNs $\Upsilon_1 = (\langle u_{\Upsilon_1}(x_i), v_{\Upsilon_1}(x_i)\rangle, \langle a_i, b_i\rangle)$ and $\Upsilon_2 = (\langle u_{\Upsilon_2}(x_i), v_{\Upsilon_2}(x_i)\rangle, \langle c_i, d_i\rangle)$ on the discrete nonempty set $\wp = (\rho_1, \rho_2, \ldots, \rho_n)$. Then, ESM based on NED is generated. So, the ESM based on NED is given in the following equation:

$$ESM_{NED}(\Upsilon_1, \Upsilon_2) = e^{-\sqrt{(1/4n)\sum\limits_{i=1}^{n}\left\{\begin{array}{c}\left(u^q_{\Upsilon_1}(x_i) - u^q_{\Upsilon_2}(x_i)\right)^2 + \left(v^q_{\Upsilon_1}(x_i) - v^q_{\Upsilon_2}(x_i)\right)^2 \\ +\left(a^q_i - c^q_i\right)^2 + \left(b^q_i - d^q_i\right)^2\end{array}\right\}}}. \tag{26}$$

Theorem 12. *Consider two nonempty q-RLDFNs $Y_1 = (\langle u_{Y_1}(x_i), v_{Y_1}(x_i)\rangle, \langle a_i, b_i\rangle)$ and $Y_2 = (\langle u_{Y_2}(x_i), v_{Y_2}(x_i)\rangle, \langle c_i, d_i\rangle)$ on the discrete nonempty set $\wp = (\rho_1, \rho_2, \ldots, \rho_n)$. Then, $ESM_{NED}(Y_1, Y_2)$ defined in equation (26) satisfies the following conditions:*

$$e^{\sqrt{(1/4n)\sum_{i=1}^{n}\left\{\left(u_{Y_1}^q(x_i)-u_{Y_2}^q(x_i)\right)^2+\left(v_{Y_1}^q(x_i)-v_{Y_2}^q(x_i)\right)^2+\left(a_i^q-c_i^q\right)^2+\left(b_i^q-d_i^q\right)^2\right\}}} \geq 0, \tag{27}$$

and

$$e^{-\sqrt{(1/4n)\sum_{i=1}^{n}\left\{\begin{array}{c}\left(u_{Y_1}^q(x_i)-u_{Y_2}^q(x_i)\right)^2+\left(v_{Y_1}^q(x_i)-v_{Y_2}^q(x_i)\right)^2\\+\left(a_i^q-c_i^q\right)^2+\left(b_i^q-d_i^q\right)^2\end{array}\right\}}} \leq 0. \tag{28}$$

We also know about the exponential properties that the zero power of e is equal to 1, and the value of the negative power of e is less than 1. Then, we have

$$\begin{aligned}
&ESM_{NED}(Y_1, Y_2)\\
&= e^{-\sqrt{(1/4n)\sum_{i=1}^{n}\left\{\begin{array}{c}\left(u_{Y_1}^q(x_i)-u_{Y_2}^q(x_i)\right)^2+\left(v_{Y_1}^q(x_i)-v_{Y_2}^q(x_i)\right)^2\\+\left(a_i^q-c_i^q\right)^2+\left(b_i^q-d_i^q\right)^2\end{array}\right\}}}\\
&= e^0\\
&= 1.
\end{aligned} \tag{29}$$

Hence, $ESM_{ED}(Y_1, Y_2) = 1$.

(2) The proof of this property is followed by using the symmetric property of NED. So, we omit the proof here.

$$\begin{aligned}
&ESM_{NED}(Y_1, Y_2)\\
&= e^{-\sqrt{(1/4n)\sum_{i=1}^{n}\left\{\begin{array}{c}\left(u_{Y_1}^q(x_i)-u_{Y_2}^q(x_i)\right)^2+\left(v_{Y_1}^q(x_i)-v_{Y_2}^q(x_i)\right)^2\\+\left(a_i^q-c_i^q\right)^2+\left(b_i^q-d_i^q\right)^2\end{array}\right\}}}\\
&= e^0\\
&= 1.
\end{aligned} \tag{30}$$

Hence, $ESM_{NED}(Y_1, Y_2) = 1$.

Next, we use the weight of each x_i with the weight vector $w = (w_1, w_2, \ldots, w_n)^T$ subject to $w_i \in [0,1]$ such that $\sum_{i=1}^{n} w_i = 1$.

Proof

(1) Given that Y_1 and Y_2 are the two q-RLDFNs and as $0 \leq u_{Y_1}(x_i) + v_{Y_1}(x_i) + a_i + b_i \leq 1$ and $0 \leq u_{Y_2}(x_i) + v_{Y_2}(x_i) + a_i + b_i \leq 1$, it implies that

(3) Suppose that $Y_1 = Y_2$; then, $u_{Y_1}^q(x_i) = u_{Y_2}^q(x_i)$, $v_{Y_1}^q(x_i) = v_{Y_2}^q(x_i), a_i^q = c_i^q, b_i^q = d_i^q$. Thus, equation (26) becomes

Then, we propose a new ESM, called the weighted exponential similarity measure (WESM) of Y_1 and Y_2 which is denoted by $WESM_{NHD}(Y_1, Y_2)$ and $WESM_{NED}(Y_1, Y_2)$, respectively. □

Definition 13. Consider two nonempty q-RLDFNs $\Upsilon_1 = (\langle u_{\Upsilon_1}(x_i), v_{\Upsilon_1}(x_i)\rangle, \langle a_i, b_i\rangle)$ and $\Upsilon_2 = (\langle u_{\Upsilon_2}(x_i), v_{\Upsilon_2}(x_i)\rangle, \langle c_i, d_i\rangle)$ on the discrete nonempty set $\wp = (\rho_1, \rho_2, \ldots, \rho_n)$.

Then, ESM based on NHD and NED is denoted by $\text{WESM}_{\text{NHD}}(\Upsilon_1, \Upsilon_2)$ and $\text{WESM}_{\text{NED}}(\Upsilon_1, \Upsilon_2)$, given in equations (31) and (32), respectively.

$$\text{WESM}_{\text{NHD}}(\Upsilon_1, \Upsilon_2) = e^{-\left\{(1/4n)\sum_{i=1}^{n} w_i\left(\begin{bmatrix} \left|u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i)\right| + \left|v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i)\right| \\ + \left|a_i^q - c_i^q\right| + \left|b_i^q - d_i^q\right| \end{bmatrix}\right)\right\}}. \tag{31}$$

The weighted ESM based on NED is given as

$$\text{WESM}_{\text{NED}}(\Upsilon_1, \Upsilon_2) = e^{-\left\{\sqrt{(1/4n)\sum_{i=1}^{n} w_i\left(\begin{bmatrix} \left(u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i)\right)^2 + \left(v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i)\right)^2 \\ + \left(a_i^q - c_i^q\right)^2 + \left(b_i^q - d_i^q\right)^2 \end{bmatrix}\right)}\right\}}. \tag{32}$$

5. Similarity Measure of q-RLDFS Based on Cosine and Cotangent Functions

This section is used for the SM based on cosine and cotangent functions for q-RLDFSs. The proposed SM is also known as a cosine similarity measure (CSM) and cotangent similarity measure (cotSM) between two q-RLDFSs Υ_1 and Υ_2. The CSM is generated by using the cosine function, and cotSM is generated by using the cotangent function. This section has two sections: one is for CSM and another is for cotSM.

5.1. Cosine Similarity Measure of q-RLDFSs. In this section, the authors use the cosine function to develop a CSM for two q-RLDFSs Υ_1 and Υ_2. We also develop the weighted CSM for two q-RLDFSs Υ_1 and Υ_2 and also prove some properties of the CSM and WCSM with the distance measure.

Definition 14. Consider two nonempty q-RLDFNs $\Upsilon_1 = (\langle u_{\Upsilon_1}(x_i), v_{\Upsilon_1}(x_i)\rangle, \langle a_i, b_i\rangle)$ and $\Upsilon_2 = (\langle u_{\Upsilon_2}(x_i), v_{\Upsilon_2}(x_i)\rangle, \langle c_i, d_i\rangle)$ on the discrete nonempty set $\wp = (\rho_1, \rho_2, \ldots, \rho_n)$. Then, the cosine similarity measure is denoted by CSM (Υ_1, Υ_2) and given as

$$\text{CSM}(\Upsilon_1, \Upsilon_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{\begin{matrix} u_{\Upsilon_1}^q(x)u_{\Upsilon_2}^q(x_i) + v_{\Upsilon_1}^q(x_i)v_{\Upsilon_2}^q(x_i) \\ + a_i^q c_i^q + b_i^q d_i^q \end{matrix}}{\sqrt{\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 + \left(a_i^q\right)^2 + \left(c_i^q\right)^2} \sqrt{\left(u_{\Upsilon_2}^q(x_i)\right)^2 + \left(v_{\Upsilon_2}^q(x_i)\right)^2 + \left(b_i^q\right)^2 + \left(d_i^q\right)^2}}. \tag{33}$$

The CSM of three q-RLDFSs Υ_1, Υ_2, and Υ_3 satisfied the following properties:

(1) $0 \le \text{CSM}(\Upsilon_1, \Upsilon_2) \le 1$.

(2) $\text{CSM}(\Upsilon_1, \Upsilon_2) \le \text{CSM}(\Upsilon_2, \Upsilon_1)$.

(3) $\text{CSM}(\Upsilon_1, \Upsilon_2) = 1 \Leftrightarrow \Upsilon_1 = \Upsilon_2$.

(4) Let $\Upsilon_1 \subseteq \Upsilon_2 \subseteq \Upsilon_3$. Then, $\text{CSM}(\Upsilon_1, \Upsilon_3) \le \text{CSM}(\Upsilon_1, \Upsilon_2)$ and $\text{CSM}(\Upsilon_1, \Upsilon_3) \le \text{CSM}(\Upsilon_2, \Upsilon_3)$.

Proof. The first two properties are obvious, so we omit the proof here and prove the last two properties.

(3): assume that $\Upsilon_1 = \Upsilon_2$; then, $u_{\Upsilon_1}^q(x) = u_{\Upsilon_2}^q(x_i)$, $v_{\Upsilon_1}^q(x) = v_{\Upsilon_2}^q(x_i)$, $a^q = b^q$ and $c^q = d^q$ for all $1 \le i \le n$. We have

$$\begin{aligned}\text{CSM}(\Upsilon_1, \Upsilon_1) &= \frac{1}{n}\sum_{i=1}^{n} \frac{u_{\Upsilon_1}^q(x_i)u_{\Upsilon_1}^q(x) + v_{\Upsilon_1}^q(x_i)v_{\Upsilon_1}^q(x_i) + a_i^q a_i^q + b_i^q b_i^q}{\sqrt{\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 + \left(a_i^q\right)^2 + \left(a_i^q\right)^2} \sqrt{\left(u_{\Upsilon_2}^q(x_i)\right)^2 + \left(v_{\Upsilon_2}^q(x_i)\right)^2 + \left(b_i^q\right)^2 + \left(b_i^q\right)^2}} \\ &= \frac{1}{n}\sum_{i=1}^{n} \frac{\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 + \left(a_i^q\right)^2 + \left(b_i^q\right)^2}{\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 + \left(a_i^q\right)^2 + \left(c_i^q\right)^2} \\ &= 1.\end{aligned} \tag{34}$$

(4): now, to prove the last property, let $\Upsilon_1 \subseteq \Upsilon_2 \subseteq \Upsilon_3$, so the angle of Υ_1, Υ_3 is larger than the angle of Υ_1, Υ_2 and

Υ_1, Υ_3. Thus, from the fact, we can write as $CSM(\Upsilon_1, \Upsilon_3) \le CSM(\Upsilon_1, \Upsilon_2)$ and $CSM(\Upsilon_1, \Upsilon_3) \le CSM(\Upsilon_2, \Upsilon_3)$.

In the following, we present a relation between the distance measure (DsM) and CSM. We explore the DsM between two q-RLDFSs using the CSM.   □

Definition 15. Consider two nonempty q-RLDFNs $\Upsilon_1 = (\langle u_{\Upsilon_1}(x_i), v_{\Upsilon_1}(x_i)\rangle, \langle a_i, b_i\rangle)$ and $\Upsilon_2 = (\langle u_{\Upsilon_2}(x_i), v_{\Upsilon_2}(x_i)\rangle, \langle c_i, d_i\rangle)$ on the discrete nonempty $\wp = (\rho_1, \rho_2, \ldots, \rho_n)$. Then, the DsM between two q-RLDFSs is given by the following equation:

$$d(\Upsilon_1, \Upsilon_2) = \cos^{-1}(CSM(\Upsilon_1, \Upsilon_2)). \tag{35}$$

Now, we will prove that equation (35) satisfies the conditions of the DsM:

(1) $0 \le CSM(\Upsilon_1, \Upsilon_2) \le 1$; then, $d(\Upsilon_1, \Upsilon_2) \ge 0$

(2) $CSM(\Upsilon_1, \Upsilon_2) = CSM(\Upsilon_2, \Upsilon_1)$; then, $d(\Upsilon_1, \Upsilon_2) = d(\Upsilon_2, \Upsilon_1)$

(3) $CSM(\Upsilon_1, \Upsilon_2) = 1$; if $\Upsilon_1 = \Upsilon_2$, then $d(\Upsilon_1, \Upsilon_2) = 0$

(4) If $\Upsilon_1 \subseteq \Upsilon_2 \subseteq \Upsilon_3$, then $d(\Upsilon_1, \Upsilon_3) \le d(\Upsilon_1, \Upsilon_2) + (\Upsilon_2, \Upsilon_3)$

Proof. The first three properties of DsM are obvious; we only prove the last property of equation (35); consider $\Upsilon_1, \Upsilon_2,$ and Υ_3 such that $\Upsilon_1 \subseteq \Upsilon_2 \subseteq \Upsilon_3$. Then, DsM of $\Upsilon_1, \Upsilon_2,$ and Υ_3 is given as

$$\begin{aligned} d(\Upsilon_1, \Upsilon_2) &= \cos^{-1}(CSM(\Upsilon_1, \Upsilon_2)), \\ d(\Upsilon_2, \Upsilon_3) &= \cos^{-1}(CSM(\Upsilon_2, \Upsilon_3)), \\ d(\Upsilon_1, \Upsilon_3) &= \cos^{-1}(CSM(\Upsilon_1, \Upsilon_3)). \end{aligned} \tag{36}$$

Therefore, we can write the following equations from the above equations as

$$CSM(\Upsilon_1, \Upsilon_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{u_{\Upsilon_1}^q(x)u_{\Upsilon_2}^q(x_i) + v_{\Upsilon_1}^q(x_i)v_{\Upsilon_2}^q(x_i) + a_i^q c_i^q + b_i^q d_i^q}{\sqrt{\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 + \left(a_i^q\right)^2 + \left(c_i^q\right)^2}}, \tag{37}$$
$$\sqrt{\left(u_{\Upsilon_2}^q(x_i)\right)^2 + \left(v_{\Upsilon_2}^q(x_i)\right)^2 + \left(b_i^q\right)^2 + \left(d_i^q\right)^2}$$

$$CSM(\Upsilon_2, \Upsilon_3) = \frac{1}{n}\sum_{i=1}^{n} \frac{u_{\Upsilon_2}^q(x)u_{\Upsilon_3}^q(x_i) + v_{\Upsilon_2}(x_i)v_{\Upsilon_3}(x_i) + c_i^q e_i^q + d_i^q f_i^q}{\sqrt{\left(u_{\Upsilon_2}^q(x_i)\right)^2 + \left(v_{\Upsilon_3}^q(x_i)\right)^2 + \left(c_i^q\right)^2 + \left(e_i^q\right)^2}}, \tag{38}$$
$$\sqrt{\left(u_{\Upsilon_2}^q(x_i)\right)^2 + \left(v_{\Upsilon_3}^q(x_i)\right)^2 + \left(d_i^q\right)^2 + \left(f_i^q\right)^2}$$

$$CSM(\Upsilon_1, \Upsilon_3) = \frac{1}{n}\sum_{i=1}^{n} \frac{u_{\Upsilon_1}^q(x)u_{\Upsilon_3}^q(x_i) + v_{\Upsilon_1}(x_i)v_{\Upsilon_3}(x_i) + a_i^q e_i^q + b_i^q f_i^q}{\sqrt{\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_3}^q(x_i)\right)^2 + \left(a_i^q\right)^2 + \left(e_i^q\right)^2}}. \tag{39}$$
$$\sqrt{\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_2}^q(x_i)\right)^2 + \left(b_i^q\right)^2 + \left(f_i^q\right)^2}$$

Now, consider weight w_i for each x_i with the weight vector for all values of x_i. Then, we propose the weighted CSM for two q-RLDFSs.   □

Definition 16. Consider two nonempty q-RLDFNs $\Upsilon_1 = (\langle u_{\Upsilon_1}(x_i), v_{\Upsilon_1}(x_i)\rangle, \langle a_i, b_i\rangle)$ and $\Upsilon_2 = (\langle u_{\Upsilon_2}(x_i), v_{\Upsilon_2}(x_i)\rangle, \langle c_i, d_i\rangle)$ on the discrete nonempty set $\wp = (\rho_1, \rho_2, \ldots, \rho_n)$. Then, the weighted cosine similarity measure is denoted by $WCSM(\Upsilon_1, \Upsilon_2)$ and given as

$$WCSM(\Upsilon_1, \Upsilon_2) = \sum_{i=1}^{n} w_i \frac{u_{\Upsilon_1}^q(x)u_{\Upsilon_2}^q(x_i) + v_{\Upsilon_1}(x_i)v_{\Upsilon_2}(x_i) + a_i^q c_i^q + b_i^q d_i^q}{\sqrt{\left(u_{\Upsilon_1}^q(x_i)\right)^2 + \left(v_{\Upsilon_1}^q(x_i)\right)^2 + \left(a_i^q\right)^2 + \left(c_i^q\right)^2}}, \tag{40}$$
$$\sqrt{\left(u_{\Upsilon_2}^q(x_i)\right)^2 + \left(v_{\Upsilon_2}^q(x_i)\right)^2 + \left(b_i^q\right)^2 + \left(d_i^q\right)^2}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is a weight vector of each x_i such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. If we take $w_i = 1/n$, then equation (40) becomes equation (33). The WCSM is a generalized form of the CSM. The WCSM is also satisfied for two q-RLDFSs.

The CSM of three q-RLDFNs $\Upsilon_1, \Upsilon_2,$ and Υ_3 satisfied the following properties:

(1) $0 \le WCSM(\Upsilon_1, \Upsilon_2) \le 1$

(2) $WCSM(\Upsilon_1, \Upsilon_2) \le WCSM(\Upsilon_2, \Upsilon_1)$

(3) $\text{WCSM}(\Upsilon_1, \Upsilon_2) = 1 \Leftrightarrow \Upsilon_1 = \Upsilon_2$

Proof. The proof follows from the proof of the properties of equation (35).

Now, we will use the cosine function to propose new SMs and WSMs based on the cosine function. The similarity measure based on the cosine function is given in equations (41) and (42). $\qquad\square$

Definition 17. The SMs based on the cosine function for two q-RLDFNs Υ_1 and Υ_2 are denoted by $\text{SM}_{\cos}(\Upsilon_1, \Upsilon_2)$ and defined by the following equation:

$$\text{SM}_{\cos^1}(\Upsilon_1, \Upsilon_2) = \frac{1}{n} \sum_{i=1}^{n} \cos\left[\frac{\pi}{2} \left(\begin{array}{c} \left| u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i) \right| \\ \vee \left| v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i) \right| \\ \vee \left| a_i^q - c_i^q \right| \vee \left| b_i^q - d_i^q \right| \end{array} \right) \right],$$

(41)

$$\text{SM}_{\cos^2}(\Upsilon_1, \Upsilon_2) = \frac{1}{n} \sum_{i=1}^{n} \cos\left[\frac{\pi}{4} \left(\begin{array}{c} \left| u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i) \right| \\ + \left| v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i) \right| \\ + \left| a_i^q - c_i^q \right| + \left| b_i^q - d_i^q \right| \end{array} \right) \right].$$

(42)

Also, we prove the following properties for equations (41) and (42):

(1) $0 \leq \text{SM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_2) \leq 1$

(2) $\text{SM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_2) = \text{SM}_{\cos^{1,2}}(\Upsilon_2, \Upsilon_1)$

(3) $\text{SM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_2) = 1$ iff $\Upsilon_1 = \Upsilon_2$

(4) If $\Upsilon_1 \subseteq \Upsilon_2 \subseteq \Upsilon_3$, then $\text{SM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_3) \leq \text{SM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_2)$ and $\text{SM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_3) \leq \text{SM}_{\cos^{1,2}}(\Upsilon_2, \Upsilon_3)$

Proof.

(1) Since the cosine function values lie between the closed interval 0 and 1 and based on the cosine function, the q-RLDF cosine similarity measures are also within $[0, 1]$. Thus, the values of $\text{SM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_2)$ must lie between 0 and 1.

(2) This is easily proven that $\text{SM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_2) = \text{SM}_{\cos^{1,2}}(\Upsilon_2, \Upsilon_1)$.

(3) This is easily proven such that $\text{SM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_2) = 1$.

(4) Consider three q-RLDFNs, such as $\Upsilon_1 \subseteq \Upsilon_2 \subseteq \Upsilon_3$; then,

$$u_{\Upsilon_1}^q(x_i) \leq u_{\Upsilon_2}^q(x_i) \leq u_{\Upsilon_3}^q(x_i),$$
$$v_{\Upsilon_1}^q(x_i) \geq v_{\Upsilon_2}^q(x_i) \geq v_{\Upsilon_3}^q(x_i), \qquad (43)$$
$$a_i^q \leq c_i^q \leq e_i^q \text{ and } b_i^q \geq d_i^q \geq f_i^q.$$

Then,

$$\left| u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i) \right| \leq \left| u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i) \right|,$$
$$\left| u_{\Upsilon_2}^q(x_i) - u_{\Upsilon_3}^q(x_i) \right| \leq \left| u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_3}^q(x_i) \right|,$$
$$\left| v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i) \right| \leq \left| v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i) \right|,$$
$$\left| v_{\Upsilon_2}^q(x_i) - v_{\Upsilon_3}^q(x_i) \right| \leq \left| v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_3}^q(x_i) \right|,$$
$$\left| a_i^q - c_i^q \right| \leq \left| a_i^q - e_i^q \right| \text{ and } \left| c_i^q - e_i^q \right| \leq \left| a_i^q - e_i^q \right|,$$
$$\left| b_i^q - d_i^q \right| \leq \left| b_i^q - f_i^q \right| \text{ and } \left| d_i^q - f_i^q \right| \leq \left| b_i^q - f_i^q \right|.$$

(44)

We know that the cosine function is a decreasing function in $[0, \pi/2]$. Thus, $\text{SM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_3) \leq \text{SM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_2)$ and $\text{SM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_3) \leq \text{SM}_{\cos^{1,2}}(\Upsilon_2, \Upsilon_3)$.

Now, consider weight w_i for each x_i with the weight vector for all values of x_i. Then, we propose the weighted SM for two q-RLDFNs based on the cosine function. $\qquad\square$

Definition 18. Consider two nonempty q-RLDFNs $\Upsilon_1 = (\langle u_{\Upsilon_1}(x_i), v_{\Upsilon_1}(x_i) \rangle, \langle a_i, b_i \rangle)$ and $\Upsilon_2 = (\langle u_{\Upsilon_2}(x_i), v_{\Upsilon_2}(x_i) \rangle, \langle c_i, d_i \rangle)$ on the discrete nonempty set $\wp = (\rho_1, \rho_2, \ldots, \rho_n)$. Then, the weighted cosine similarity measure based on the cosine function is denoted by $\text{WSM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_2)$ and given as

$$\text{WSM}_{\cos^1}(\Upsilon_1, \Upsilon_2) = \sum_{i=1}^{n} w_i \cos\left[\frac{\pi}{2} \left(\begin{array}{c} \left| u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i) \right| \\ \vee \left| v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i) \right| \\ \vee \left| a_i^q - c_i^q \right| \vee \left| b_i^q - d_i^q \right| \end{array} \right) \right],$$

(45)

$$\text{WSM}_{\cos^2}(\Upsilon_1, \Upsilon_2) = \sum_{i=1}^{n} w_i \cos\left[\frac{\pi}{4} \left(\begin{array}{c} \left| u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i) \right| \\ + \left| v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i) \right| \\ + \left| a_i^q - c_i^q \right| + \left| b_i^q - d_i^q \right| \end{array} \right) \right]$$

(46)

where $w = (w_1, w_2, \ldots, w_n)^T$ is a weight vector of each x_i such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. If we take $w_i = 1/n$, then equations (45) and (46) become equations (41) and (42). The WCSM is a generalized form of the CSM. The WCSM is also satisfied for two q-RLDFNs.

The WCSM of three q-RLDFNs Υ_1, Υ_2, and Υ_3 satisfied the following properties:

(1) $0 \leq \text{WSM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_2) \leq 1$

(2) $\text{WSM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_2) = \text{WSM}_{\cos^{1,2}}(\Upsilon_2, \Upsilon_1)$

(3) $\text{WSM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_2) = 1 \Leftrightarrow \Upsilon_1 = \Upsilon_2$

(4) If $\Upsilon_1 \subseteq \Upsilon_2 \subseteq \Upsilon_3$, then $\text{WSM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_3) \leq \text{WSM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_2)$ and $\text{WSM}_{\cos^{1,2}}(\Upsilon_1, \Upsilon_3) \leq \text{WSM}_{\cos^{1,2}}(\Upsilon_2, \Upsilon_3)$

5.2. Cotangent Function Similarity Measure of q-RLDFSs. In this section, we use the cotangent function to develop a cotangent similarity measure (cotSM) for two q-RLDFNs Υ_1 and Υ_2. We also develop the weighted cotSM for two q-RLDFNs Υ_1 and Υ_2 and also prove some properties of the cot SM and WcotSM with distance measures.

Definition 19. Consider two nonempty q-RLDFNs $\Upsilon_1 = (\langle u_{\Upsilon_1}(x_i), v_{\Upsilon_1}(x_i)\rangle, \langle a_i, b_i\rangle)$ and $\Upsilon_2 = (\langle u_{\Upsilon_2}(x_i), v_{\Upsilon_2}(x_i)\rangle, \langle c_i, d_i\rangle)$ on the discrete nonempty set $\wp = (\rho_1, \rho_2, \ldots, \rho_n)$.

The SMs based on the cotangent function for two q-RLDFNs Υ_1 and Υ_2 are denoted by $\mathrm{SM}_{\mathrm{cot}}(\Upsilon_1, \Upsilon_2)$ and defined by the following equation:

$$\mathrm{SM}_{\mathrm{cot}^1}(\Upsilon_1, \Upsilon_2) = \frac{1}{n}\sum_{i=1}^{n}\cot\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\begin{array}{c} \left|u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i)\right| \\ \vee\left|v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i)\right| \\ \vee\left|a_i^q - c_i^q\right|\vee\left|b_i^q - d_i^q\right| \end{array}\right)\right], \tag{47}$$

$$\mathrm{SM}_{\mathrm{cot}^2}(\Upsilon_1, \Upsilon_2) = \frac{1}{n}\sum_{i=1}^{n}\cot\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\begin{array}{c} \left|u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i)\right| \\ +\left|v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i)\right| \\ +\left|a_i^q - c_i^q\right| + \left|b_i^q - d_i^q\right| \end{array}\right)\right] \tag{48}$$

Equations (46) and (48) satisfied the following properties:

(1) $0 \leq \mathrm{SM}_{\mathrm{cot}^{1,2}}(\Upsilon_1, \Upsilon_2) \leq 1$

(2) $\mathrm{SM}_{\mathrm{cot}^{1,2}}(\Upsilon_1, \Upsilon_2) = \mathrm{SM}_{\mathrm{cot}^{1,2}}(\Upsilon_2, \Upsilon_1)$

(3) $\mathrm{SM}_{\mathrm{cot}^{1,2}}(\Upsilon_1, \Upsilon_2) = 1$ iff $\Upsilon_1 = \Upsilon_2$

(4) If $\Upsilon_1 \subseteq \Upsilon_2 \subseteq \Upsilon_3$, then $\mathrm{SM}_{\mathrm{cot}^{1,2}}(\Upsilon_1, \Upsilon_3) \leq \mathrm{SM}_{\mathrm{cot}^{1,2}}(\Upsilon_1, \Upsilon_2)$ and $\mathrm{SM}_{\mathrm{cot}^{1,2}}(\Upsilon_1, \Upsilon_3) \leq \mathrm{SM}_{\mathrm{cot}^{1,2}}(\Upsilon_2, \Upsilon_3)$

Proof.

(1) Since the cosine function values lie between the closed interval 0 and 1 and based on the cosine function, the q-RLDFN cosine similarity measures are also within $[0, 1]$. Thus, the values of $\mathrm{SM}_{\mathrm{cot}^{1,2}}(\Upsilon_1, \Upsilon_2)$ must lie between 0 and 1.

(2) This is easily proven such that $\mathrm{SM}_{\mathrm{cot}^{1,2}}(\Upsilon_1, \Upsilon_2) = \mathrm{SM}_{\mathrm{cot}^{1,2}}(\Upsilon_2, \Upsilon_1)$.

(3) This is easily proven such that $\mathrm{SM}_{\mathrm{cot}^{1,2}}(\Upsilon_1, \Upsilon_2) = 1$.

(4) Consider three q-RLDFNs, such as $\Upsilon_1 \subseteq \Upsilon_2 \subseteq \Upsilon_3$. Then,

$$u_{\Upsilon_1}^q(x_i) \leq u_{\Upsilon_2}^q(x_i) \leq u_{\Upsilon_3}^q(x_i),$$
$$v_{\Upsilon_1}^q(x_i) \geq v_{\Upsilon_2}^q(x_i) \geq v_{\Upsilon_3}^q(x_i), \tag{49}$$
$$a_i^q \leq c_i^q \leq e_i^q \text{ and } b_i^q \geq d_i^q \geq f_i^q.$$

Then,

$$\left|u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i)\right| \leq \left|u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_3}^q(x_i)\right|,$$
$$\left|u_{\Upsilon_2}^q(x_i) - u_{\Upsilon_3}^q(x_i)\right| \leq \left|u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_3}^q(x_i)\right|,$$
$$\left|v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i)\right| \leq \left|v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_3}^q(x_i)\right|,$$
$$\left|v_{\Upsilon_2}^q(x_i) - v_{\Upsilon_3}^q(x_i)\right| \leq \left|v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_3}^q(x_i)\right|,$$
$$\left|a_i^q - c_i^q\right| \leq \left|a_i^q - e_i^q\right| \text{ and } \left|c_i^q - e_i^q\right| \leq \left|a_i^q - e_i^q\right|,$$
$$\left|b_i^q - d_i^q\right| \leq \left|b_i^q - f_i^q\right| \text{ and } \left|d_i^q - f_i^q\right| \leq \left|b_i^q - f_i^q\right|. \tag{50}$$

We know that the cotangent function is a decreasing function in $[0, \pi/4]$. Thus, $\mathrm{SM}_{\mathrm{cot}^{1,2}}(\Upsilon_1, \Upsilon_3) \leq \mathrm{SM}_{\mathrm{cot}^{1,2}}(\Upsilon_1, \Upsilon_2)$ and $\mathrm{SM}_{\mathrm{cot}^{1,2}}(\Upsilon_1, \Upsilon_3) \leq \mathrm{SM}_{\mathrm{cot}^{1,2}}(\Upsilon_2, \Upsilon_3)$.

Now, consider weight w_i for each x_i, with the weight vector for all values of x_i. Then, we propose the weighted SM for two q-RLDFSs based on the cotangent function. $\square$

Definition 20. Consider two nonempty q-RLDFNs $\Upsilon_1 = (\langle u_{\Upsilon_1}(x_i), v_{\Upsilon_1}(x_i)\rangle, \langle a_i, b_i\rangle)$ and $\Upsilon_2 = (\langle u_{\Upsilon_2}(x_i), v_{\Upsilon_2}(x_i)\rangle, \langle c_i, d_i\rangle)$ on the discrete nonempty set $\wp = (\rho_1, \rho_2, \ldots, \rho_n)$. Then, the weighted cosine similarity measure based on the cotangent function is denoted by $\mathrm{WSM}_{\mathrm{cot}^{1,2}}(\Upsilon_1, \Upsilon_2)$ and given as

$$\mathrm{WSM}_{\mathrm{cot}^1}(\Upsilon_1, \Upsilon_2) = \sum_{i=1}^{n}w_i\cot\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\begin{array}{c} \left|u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i)\right| \\ \vee\left|v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i)\right| \\ \vee\left|a_i^q - c_i^q\right|\vee\left|b_i^q - d_i^q\right| \end{array}\right)\right], \tag{51}$$

$$\text{WSM}_{\cot^2}(\Upsilon_1, \Upsilon_2) = \sum_{i=1}^{n} w_i \cot\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\begin{array}{c}\left|u_{\Upsilon_1}^q(x_i) - u_{\Upsilon_2}^q(x_i)\right| \\ +\left|v_{\Upsilon_1}^q(x_i) - v_{\Upsilon_2}^q(x_i)\right| \\ +\left|a_i^q - c_i^q\right| + \left|b_i^q - d_i^q\right|\end{array}\right)\right], \tag{52}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is a weight vector of each x_i such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. If we take $w_i = 1/n$, then equations (51) and (52) become equations (47) and (48). The WcotSM is a generalized form of cotSM. The WcotSM is also satisfied for two q-RLDFNs.

The WcotSM of three q-RLDFNs Υ_1, Υ_2, and Υ_3 satisfied the following properties:

(1) $0 \leq \text{WSM}_{\cot^{1,2}}(\Upsilon_1, \Upsilon_2) \leq 1$

(2) $\text{WSM}_{\cot^{1,2}}(\Upsilon_1, \Upsilon_2) = \text{WSM}_{\cot^{1,2}}(\Upsilon_2, \Upsilon_1)$

(3) $\text{WSM}_{\cot^{1,2}}(\Upsilon_1, \Upsilon_2) = 1$ iff $\Upsilon_1 = \Upsilon_2$

(4) If $\Upsilon_1 \subseteq \Upsilon_2 \subseteq \Upsilon_3$, then $\text{WSM}_{\cot^{1,2}}(\Upsilon_1, \Upsilon_3) \leq \text{WSM}_{\cot^{1,2}}(\Upsilon_1, \Upsilon_2)$ and $\text{WSM}_{\cot^{1,2}}(\Upsilon_1, \Upsilon_3) \leq \text{WSM}_{\cot^{1,2}}(\Upsilon_2, \Upsilon_3)$

6. Applications of Proposed Similarity Measures

In the literature, some similarity measures of different fuzzy sets have been developed in different fields, i.e., decision-making, clustering analysis, data mining, pattern recognition, and supply chain management. This section provides the detailed description of applications of the proposed SM of the present paper. We are going to develop general multicriteria decision support systems for logistics and supply chain management. In case of logistics and provider selection, the detailed description of the algorithm is given in the following steps.

Let $A = (A_1, A_2, \ldots, A_m)$ be the collection of m alternatives and $C = (C_1, C_2, \ldots, C_n)$ be the set of criteria. The criteria $C_j (j = 1, 2, \ldots, n)$ weight vector is $w = (w_1, w_2, \ldots, w_n)^T$, satisfying that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, for $j = 1, 2, \ldots, n$. Suppose that $E = (E_1, E_2, \ldots, E_p)$ is the set of experts; every expert gives their assessment in the positive and negative grade to each criterion $C_j (j = 1, 2, \ldots, n)$ on the alternative $A_i (i = 1, 2, \ldots, m)$, respectively. So, we construct the decision matrix in the form of q-RLDFNs.

Then, the following steps are used to solve the MCDM problems, applying the cosine and cotangent similarity measures of q-RLDFNs.

Step 1: in this step, we construct a committee of the experts

Step 2: in this step, we collect information in the form of q-RLDFNs and make a decision matrix of the alternatives

Step 3: the proposed similarity measures are used to find the collective values of the decision matrix

Step 4: the alternatives are ranked according to the highest value of the alternatives, and the best one is chosen

6.1. Logistics Providers (LPs) and Supply Chain Management Problem. In this section, we have addressed a case study for selecting logistics providers (LPs). In our decision and ranking of alternatives, we use the proposed similarity measures. We connect our results with certain previous methods.

6.1.1. Case Study with the Numerical Approach. For the purpose of logistics support, the better selection of logistics providers (LPs) is now important to the growth and advancement of the supply chain theories. Over the past decades, there has been a drastic change in the course of decision support systems.

A supply chain is considered as a method with a full range of activities in which raw materials are converted into finally completed goods and then distributed by distribution, logistics, and retail to customers. As a supply chain management system, all interfunctional processes such as ordering, distribution, product delivery, and reverse logistics are taken into account [67–69].

Such systems allow experts in the supply chain to promote information flow, highly qualified decisions, and through internal coordination to increase the demand for goods and services [70]. In order to control the cost of the products in a textile factory, a DSS has aided decision makers in choosing effective ways to minimize overall cost of production.

Supply chain management can be considered sustainable when extended to all related aspects of the supply chain: energy conservation, final goods assurance, after-sales service, ethical job problems, reuse, and reverse logistics. In this way, arguing consumer expectations and then translating them into the supply chain considerations in this work may not only shift the supply chain towards a global and sustainable appearance but also strengthen the relevance of customer-based analysis. In the current group decision model, the task is to develop an alternative LP ranking based on the proposed similarity measures. The proposed method steps include a window for implementing costumer-driven weighting procedure for the criteria of decision. This factor allows for the decision process to satisfy external stakeholders and customers. We considered four decision makers to send us their opinions on the above logistics providers regarding the related factors for constructing a decision system. First, the necessary and the corresponding technical criteria for determining the logistics providers (LPs) of third parties are defined and implemented. The alternatives are true logistics and transport companies operating in France and Europe as a whole.

How to control the flow of our information? Where should our houses be located? What a framework should be

introduced for transport and logistic? A managed and flexible supply chain is of paramount importance not only because of communication between all departments of the business (marketing, logistics, procurement, etc.) but also because of the extension of communication beyond its boundaries (suppliers, subcontractors, LPs, etc.).

Multiactivities of over 400 businesses, a network of almost 1500 members, ASLOG is now the leading French network of supply chain professionals. Its objectives include providing forward-looking goals, developing standards and qualifications, measuring and assessing logistic output, and eventually producing research distribution in collaboration with the benchmarking of best practices.

Step 1: Mathez, Bansard, Gefco, and Schneider are chosen to be four logistics provider companies, i.e., alternatives A_m, and examined by the suggested methods.

A_1: Mathez group's main activities can be expanded from air, sea, and road freight, also packaging, supply and distribution management and optimizing, port agent, cargo, and cruise ship management. The main activities of the Mathez group can be expanded from air, sea, and road freight, packaging, deliver and distribution processing, port controller, and transport of goods.

A_2: Bansard International offers a comprehensive platform for transport by sea, air, and road via agreements with leading airports to meet our clients' needs. Thus, Bansard International serves the main ports and airports. Moreover, this firm offers door-to-door support services to promote custom activities. They provide services such as air, marine, and road transport, logistics, and construction plants.

A_3: Gefco offers effective logistics solutions worldwide for its industrial clients. The company is able to fulfil all criteria for the supply chain optimization process, land transport, distribution, shipment planning, and air flows.

A_4: the company Schneider prepares and manages the transport of all kinds of goods between Europe and Switzerland. This company provides a type of daily activities for transport by air, sea, and road, reverse transportation, vegetables transportation, etc.

This case study focuses on applying the decision support model in the supply chain which aims to demonstrate how it can choose a third party LP as a partner from the pool of potential suppliers. The model's goal is to include the interests of consumers and stakeholders in the decision process so as to provide a global and detailed framework. Suppose the six technical factors denote technical requirements (decision criteria $C_1, C_2, \ldots, C_6$) which satisfy the four alternative LPs. On the contrary, the following criteria with weight vector $w = (0.11, 0.31, 0.15, 0.17, 0.14, 0.12)^T$ are listed for LP evaluation as follows:

C_1: financial stability

C_2: quality system

C_3: delivery condition

C_4: services

C_5: flexibility of the system

C_6: company stability

Step 1: in this step, we invite a team of experts to give their assessments

Step 2: now, every expert gives their assessments about the alternatives in the form of q-RLDFNs based on the six criteria given in Table 1

Step 3: now, we used the proposed similarity measures to find the collective values of the decision matrix, as given in Table 2

Based on the estimated results in Table 2, the degree of similarity between A_2 and A is the highest, as determined by the other eight similarity measures. This indicates that all ten similarity measures think the alternative A_2 is closest to be the best alternative A based on the principle of maximum similarity between q-rung linear Diophantine fuzzy sets (q-RLDFSs). In other words, A_2 is the best alternative.

In Figure 1, we graphically show the ranking of the alternative given in Table 2.

In practical DM problems, it is important to take the weights of elements into account if we take weights of criteria as $w = (0.11, 0.31, 0.15, 0.17, 0.14, 0.12)^T$, respectively. Then, the weighted similarity measures proposed in this paper have been utilized to compute the similarity, and the results are listed in Table 3 (suppose $q = 3$).

Based on the results in Table 3, the degree of similarity between A_2 and A is the highest, as determined by the other eight similarity measures. This shows that, in all eight similarity measures, the alternative A_2 is closest to be the best alternative A based on the principle of maximum similarity between q-RLDFNs. Thus, A_2 is the best alternative.

In Figure 2, we graphically show the ranking of the alternative given in Table 3.

7. Comparison and Discussion

A comparison of our defined similarity measures with certain existing approaches in context [29, 40, 52, 53, 62, 63, 71] has been done to illustrate the superiority of our defined similarity measures. Table 4 shows that the existing SMs are inaccessible when using q-ROF aggregation operators to solve the developed illustrated case of Section 6. However, while the approaches described in [65, 66] contain Diophantine fuzzy information, they are inaccessible for solving the proposed information. Based on the study of Table 4, we can see that existing SMs lack rough information, and these approaches are unable to solve and rank the created case. As a result, the new method is more capable and effective than the existing approaches.

To demonstrate the effectiveness and scientific nature of our suggested approaches, we will compare them to the existing decision-making approaches such as [32–36, 57] with the q-rung orthopair fuzzy aggregation operators. The

TABLE 1: The q-RLDFNs' information given by the expert.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	$\begin{pmatrix}\langle 0.5,0.7\rangle\\\langle 0.6,0.4\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.8,0.3\rangle\\\langle 0.7,0.4\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.9,0.2\rangle\\\langle 0.8,0.4\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6,0.7\rangle\\\langle 0.3,0.6\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.8,0.6\rangle\\\langle 0.4,0.5\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5,0.7\rangle\\\langle 0.6,0.4\rangle\end{pmatrix}$
A_2	$\begin{pmatrix}\langle 0.8,0.6\rangle\\\langle 0.5,0.3\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.9,0.7\rangle\\\langle 0.2,0.6\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.3,0.8\rangle\\\langle 0.6,0.5\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.8,0.6\rangle\\\langle 0.5,0.3\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.3,0.8\rangle\\0.6,0.5\end{pmatrix}$	$\begin{pmatrix}\langle 0.9,0.4\rangle\\\langle 0.5,0.3\rangle\end{pmatrix}$
A_3	$\begin{pmatrix}\langle 0.7,0.6\rangle\\\langle 0.4,0.5\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.9,0.1\rangle\\\langle 0.8,0.3\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5,0.9\rangle\\\langle 0.8,0.2\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5,0.9\rangle\\\langle 0.8,0.2\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.8,0.3\rangle\\\langle 0.7,0.4\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.8,0.7\rangle\\\langle 0.2,0.6\rangle\end{pmatrix}$
A_4	$\begin{pmatrix}\langle 0.9,0.4\rangle\\\langle 0.6,0.3\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5,0.6\rangle\\\langle 0.3,0.9\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6,0.7\rangle\\\langle 0.3,0.6\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.9,0.2\rangle\\\langle 0.7,0.4\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5,0.6\rangle\\\langle 0.3,0.9\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.9,0.1\rangle\\\langle 0.8,0.3\rangle\end{pmatrix}$
A	$\begin{pmatrix}\langle 0.8,0.5\rangle\\\langle 0.7,0.2\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.4,0.8\rangle\\\langle 0.6,0.5\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5,0.8\rangle\\\langle 0.7,0.4\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.4,0.7\rangle\\\langle 0.6,0.3\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6,0.5\rangle\\\langle 0.7,0.4\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.7,0.2\rangle\\\langle 0.4,0.6\rangle\end{pmatrix}$

TABLE 2: The similarity measures between A_i an d A.

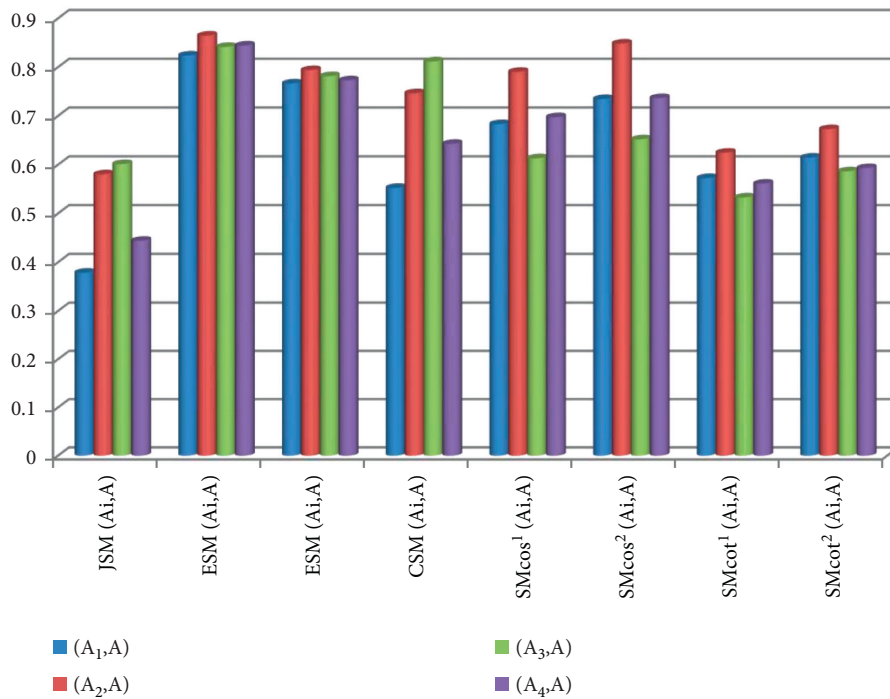| Similarity measure	(A_1, A)	(A_2, A)	(A_3, A)	(A_4, A)
JSM (A_i, A)	0.378	0.580	0.601	0.443
ESM$_{NHD}$ (A_i, A)	0.823	0.864	0.841	0.843
ESM$_{NED}$ (A_i, A)	0.766	0.793	0.781	0.772
CSM (A_i, A)	0.552	0.746	0.811	0.642
SM$_{cos^1}$ (A_i, A)	0.683	0.790	0.612	0.697
SM$_{cos^2}$ (A_i, A)	0.734	0.848	0.651	0.736
SM$_{cot^1}$ (A_i, A)	0.572	0.624	0.532	0.560
SM$_{cot^2}$ (A_i, A)	0.614	0.672	0.586	0.592

FIGURE 1: The ranking of the alternative given in Table 2.

TABLE 3: The weighted similarity measures between A_i an d A.

Similarity measure	(A_1, A)	(A_2, A)	(A_3, A)	(A_4, A)
$\text{WJSM}(A_i, A)$	0.359	0.529	0.528	0.423
$\text{WESM}_{\text{NHD}}(A_i, A)$	0.964	0.969	0.964	0.970
$\text{WESM}_{\text{NED}}(A_i, A)$	0.892	0.900	0.886	0.895
$\text{WCSM}(A_i, A)$	0.522	0.797	0.733	0.622
$\text{WSM}_{\cos^1}(A_i, A)$	0.721	0.793	0.772	0.668
$\text{WSM}_{\cos^2}(A_i, A)$	0.761	0.849	0.813	0.709
$\text{WSM}_{\cot^1}(A_i, A)$	0.535	0.582	0.562	0.486
$\text{WSM}_{\cot^2}(A_i, A)$	0.580	0.635	0.599	0.540
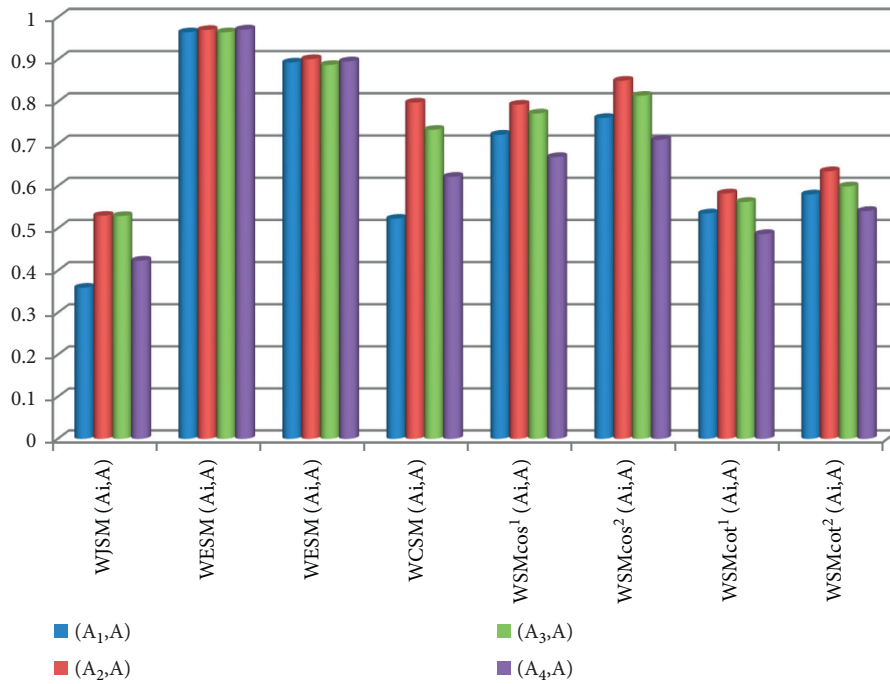
FIGURE 2: The ranking of the alternative given in Table 3.

TABLE 4: Different methods and their ranking.

Methods	Appraisal scores of alternatives	Ranking
Du [29]	Inaccessible	×
Peng and Liu [63]	Inaccessible	×
Son and Phong [52]	Inaccessible	×
Wu et al. [71]	Inaccessible	×
Wei and Wei [53]	Inaccessible	×
Wang et al. [62]	Inaccessible	×
Ye [40]	Inaccessible	×
Methods	Remarks	Parameterization
q-RLDFS [66]	q − RLDFS covers this situation, $0 \le (a)^q \mu_{D(x)} + (b)^q v_{D(x)} \le 1$,	Yes
LDFS [65]	LDFS covers this situation, $0 \le (a)\mu_{D(x)} + (b)v_{D(x)} \le 1$,	No

aggregated results of a comparative examination of existing methods using our example are listed in Table 5 with criteria weight vector $w = (0.11, 0.31, 0.15, 0.17, 0.14, 0.12)^T$.

To date, we can see that cosine and cotangent SMs with q-ROFSs have been examined by a large number of researchers. However, as previously indicated, some particular scenarios

TABLE 5: Score values and ranking order of the existing methods.

	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	Ranking
Liu and Wang [32]	0.692	0.743	0.729	0.663	$A_2 > A_3 > A_1 > A_4$
Liu and Liu [36]	0.596	0.679	0.668	0.579	$A_2 > A_3 > A_1 > A_4$
Wei et al. [34]	0.532	0.573	0.554	0.509	$A_2 > A_3 > A_1 > A_4$
Wei et al. [33]	0.822	0.885	0.837	0.793	$A_2 > A_3 > A_1 > A_4$
Yang and Pang [35]	0.233	0.294	0.276	0.247	$A_2 > A_3 > A_4 > A_1$
Zeng et al. [57]	0.442	0.480	0.458	0.413	$A_2 > A_3 > A_1 > A_4$

are not covered by q-ROFSs. As a result, algorithms based on SMs with a q-ROFS are incapable of dealing with such problems. The cosine and cotangent similarity measures with q-ROF information are special cases of the SMs described in this study with q-rung linear Diophantine fuzzy information. Because of the qth power of reference parameters, this notion is impressive in that it covers the valuation spaces of IFSs, PyFSs, q-ROFSs, and LDFSs. As a result, our specified SMs are better suited and generalized to cope with the real-world situation than the current ones.

8. Conclusion

Based on q-rung linear Diophantine fuzzy sets (q-RLDFSs), we proposed another form of similarity measures by considering the function of membership degree (MD), nonmembership degree (NMD), and reference parameters (RPs) in q-RLDFSs. The q-RLDFS model is more effective and versatile than other techniques due to the use of RPs. In addition, for the q-RLDFSs, we presented a family of similarity measures such as Jaccard SMs, exponential SMs, and cosine and cotangent SMs between q-RLDFSs by considering MD, NMD, and RPs. The concept of the qth power of reference parameters will provide a more versatile and efficient basis for fuzzy system modeling and decision-making under uncertainty, covering the space of existing structures as well as the space of MD, NMD, and RPs. It is important to note that if we increase the value of q, the Diophantine space expands, giving the large boundary limits to express a wider range of fuzzy data. We implemented the proposed MCDM methods under the guidance of a case study for the selection of the best logistics provider. We conclude that the current DM method is appropriate and stable and can be effectively implemented for multicriteria group decision-making problems.

In the future, study on q-RLDFSs could focus on various types of decision-making challenges, such as staff selection, investment, machine selection, project selection, and production systems.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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