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*Research Article*

# Finite Groups Which Are the Union of Autocentralizers of Some Automorphisms

**Esmat Alamshahi,[1] Mohammad Reza R. Moghaddam ,[1,2,3] and Mohammad Amin Rostamyari [2]**

[1]*Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran*
[2]*Department of Mathematics, Khayyam University, Mashhad, Iran*
[3]*Department of Pure Mathematics, Centre of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O.Box 1159, Mashhad, 91775, Iran*

Correspondence should be addressed to Mohammad Reza R. Moghaddam; rezam@ferdowsi.um.ac.ir

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Let $\alpha$ be an automorphism of a given group $G$; then, $C_G(\alpha) = \{g \in G | \alpha(g) = g\}$ is called the autocentralizer of $\alpha$ in $G$. In this work, we study finite groups $G$, which can be written as the union of autocentralizers of some automorphisms of $G$. In particular, if a group $G$ is 5-$A$-centralizers, then we determine the absolute central quotient $G/L(G)$ of $G$, where $A = \mathrm{Aut}(G)$. Finally, it is shown that autocommutative property of a group is equivalent to the one, in which every $A$-centralizer of its nontrivial elements is abelian.

## 1. Introduction

Let $A = \mathrm{Aut}(G)$ be the group of all automorphisms of a given group $G$ and $\alpha \in \mathrm{A}$. The *autocentralizer* or *A-centralizer* of $\alpha$ in $G$ is the subgroup of $G$ consisting of all elements of $G$ which are fixed by $\alpha$ and denoted by $C_G(\alpha)$, i.e.,

$$C_G(\alpha) = \{g \in G | g^{\alpha} = \alpha(g) = g\}. \tag{1}$$

Also, we denote the set of all such autocentralizes in $G$ by $C_G(A)$. Note that, if $C_G(\alpha) = \{e\}$, where $e$ is the identity element of $G$, then $\alpha$ is called fixed-point-free automorphism. We remark that, for the identity automorphism id. of $A$, we have $G = C_G(\mathrm{id})$, and hence, $G \in C_G(A)$.

For any element $g \in G$ and $\alpha \in \mathrm{A}$, the element,

$$[g, \alpha] = g^{-1} g^{\alpha} = g^{-1} \alpha(g), \tag{2}$$

is the autocommutator of $g$ and $\alpha$. Clearly, if $\alpha = \phi_x$ is the inner automorphism induced by the element $x$ of $G$, then the autocommutator $[g, \phi_x]$ coincides with the ordinary commutator $[g, x]$. The set $L(G)$, consisting of all elements $g$ in $G$ which are fixed by every automorphism of $G$, is a central characteristic subgroup of $G$, which is called absolute centre or the autocentre of $G$ (see [1–5], for more information). Hence, we may write

$$L(G) = \{g \in G\colon\ [g, \alpha] = 1, \forall \alpha \in \mathrm{A}\} = \bigcap_{\alpha \in \mathrm{A}} C_G(\alpha). \tag{3}$$

We denote the number of distinct autocentralizers of automorphisms in $G$ by $|C_G(A)|$, and now a question arises that “how many distinct autocentralizers can a group have?”

This article is motivated by providing some answers to the above question.

## 2. Preliminary Results

It is clear that, in a given group $G$, there may exist an element $x$, which is not fixed by none of nontrivial automorphisms of $G$. For example, consider cyclic groups of order $p$, for odd prime $p$.

In the present article, we consider all finite groups which do not have the above property, so the following result is obvious.

**Lemma 1.** *A group $G$ is the union of autocentralizers of all nontrivial automorphisms of $A$ in $G, L(G)$, i.e., $G = \cup_{id\neq\alpha\in A} C_{G,L(G)}(\alpha)$.*

*Proof.* Clearly, $\cup_{\mathrm{id}\neq\alpha\in\mathrm{A}} C_G, L(G)(\alpha) \subseteq G$. Now, if $x \in G, L(G)$, then, by the above assumption, there exists $\alpha \in \mathrm{A}$, such that $x \in C_G, L(G)(\alpha)$. Therefore, $G \subseteq \cup_{\mathrm{id}\neq\alpha\in\mathrm{A}} C_G, L(G)(\alpha)$, which completes the proof. □

*Remark 1.* Note that if $|C_G(A)| = 1$, then (3) implies that $L(G) = G$, so $[x, \alpha] = 1$, for all $x \in G$ and $\alpha \in \mathrm{A}$. Hence, $A = \{\mathrm{id}\}$, and consequently, $G = \langle 1 \rangle$ or $\mathbb{Z}_2$. Clearly, the converse is also true. Therefore, $|C_G(A)| = 1$ if and only if $G = \langle 1 \rangle$ or $\mathbb{Z}_2$.

**Lemma 2.** *Let $G$ be a nontrivial group except $\mathbb{Z}_2$; then, $|C_G(A)| \geq 4$.*

*Proof.* By Remark 1, $|C_G(A)| \neq 1$. If $|C_G(A)| = 2$, then $G$ is the proper subgroup of itself, which is impossible. Suppose that $|C_G(A)| = 3$; then, $C_G(A) = \{G, C_G(\alpha_1), C_G(\alpha_2)\}$, for some automorphisms $\alpha_1$ and $\alpha_2 \in \mathrm{A}$, where $C_G(\alpha_1)$ and $C_G(\alpha_2)$ are proper subgroups of $G$. Therefore, $G = C_G(\alpha_1) \cup C_G(\alpha_2)$, which is impossible, since a group cannot be written as the union of two proper subgroups. Thus, $|C_G(A)| \geq 4$. □

A group $G$ is called $n$-$A$-centralizer if $|C_G(A)| = n$ (see also [6]).

*Example 1*

(i) Consider $D_8 = \langle x, y | x^4 = y^2 = 1, x^y = x^{-1} \rangle$, the Dihedral group of order 8; then, clearly, the set of all automorphisms of $D_8$ is

$$\alpha_{ij} : \begin{cases} x \mapsto x^i, & i = 1, 3, \\ y \mapsto x^j y, & j = 1, 2, 3, 4. \end{cases} \tag{4}$$

Thus, one can calculate that

$$\begin{aligned} &C_{D_8}(\alpha_{11}) = C_{D_8}(\alpha_{12}) = C_{D_8}(\alpha_{13}) = \langle x \rangle, \\ &C_{D_8}(\alpha_{31}) = C_{D_8}(\alpha_{33}) = \{1, x^2\}, \\ &C_{D_8}(\alpha_{14}) = D_8, \\ &C_{D_8}(\alpha_{32}) = \{1, x^2, xy, x^3 y\}, \\ &C_{D_8}(\alpha_{34}) = \{1, x^2, y, x^2 y\}. \end{aligned} \tag{5}$$

Therefore, $D_8$ is 4-$A$-centralizer, as $C_{D_8}(\alpha_{31})$ and $C_{D_8}(\alpha_{32})$ are not distinct.

Similarly, one can easily see that $|C_{Q_8}(A)| = 4$.

(ii) Consider $S_3 = \langle x, y | x^3 = y^2 = 1, x^y = x^{-1} \rangle$, the symmetric group of order 6; then, one can calculate that $S_3$ has 5 $A$-centralizers as follows:

$$S_3, \{1, y\}, \{1, x, x^2\}, \{1, x^2 y\}, \{1, xy\}. \tag{6}$$

## 3. Counting $A$-Centalizers in Groups

For an arbitrary element $x$ of a given group $G$, the set $C_G(x) = \{y \in G\colon [y, x] = [y, \phi_x] = 1\}$, where $\phi_x$ is the inner automorphism of $G$, is called the centralizer of $x$ in $G$. The set of all such centralizers in $G$ is denoted by $\mathrm{Cent}(G)$. Clearly, $G$ is abelian if and only if $|\mathrm{Cent}(G)| = 1$.

Many authors have studied the influence of the number of centralizers of a finite group $G$ on the structure of the group. In 1994, Belcastro and Sherman [7] proved that $|\mathrm{Cent}(G)| \geq 4$, for any nonabelian finite group $G$. They also showed that $G$ has 4 centralizers if and only if $G/Z(G) \cong C_2 \times C_2$ and $G$ has 5 centralizers if and only if $G/Z(G) \cong C_3 \times C_3$ or $S_3$. In [8], Ashrafi proved that if $G$ has 6 centralizers, then $G/Z(G) \cong D_8, A_4, C_2 \times C_2 \times C_2$ or $C_2 \times C_2 \times C_2 \times C_2$, where $A_4$ is the alternating group of degree 4.

In this section, we compute $|C_G(A)|$ under certain conditions on the group $G$.

**Proposition 1.** *Let $D_{2n}$ be the Dihedral group of order $2n$; then,*

$$\left|C_{D_{2n}}(A)\right| = \begin{cases} n + 2, & n \text{ is odd}, \\ \dfrac{n}{2} + 2, & n \text{ is even}. \end{cases} \tag{7}$$

*Proof.* Clearly, for $n \geq 3$, the group $D_{2n}$ has the following presentation:

$$\begin{aligned} D_{2n} &= \langle x, y | x^n = y^2 = 1, x^y = x^{-1} \rangle \\ &= \{1, x, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1} y\}. \end{aligned} \tag{8}$$

Then, $A = \left\{\alpha_{ij} : \begin{cases} x \mapsto x^i \\ y \mapsto x^j y \end{cases} \middle| \gcd(i, n) = 1, 0 \leq j \leq n - 1 \right\}$, so $C_{D_{2n}}(\alpha_{10}) = D_{2n}$. Assume that $n$ is odd; then, we have the following $A$-centralizers:

(i) $C_{D_{2n}}(\alpha_{1j}) = \langle x \rangle$, as $[x^k, \alpha_{1j}] = x^{-k} x^k = 1$, $[y, \alpha_{1j}] = y^{-1} x^j y = x^{-j} \neq 1$, and $[x^k y, \alpha_{1j}] = y^{-1} x^{-k} x^k x^j y = x^{-j} \neq 1$, for every $0 < k \leq n - 1$.

(ii) $C_{D_{2n}}(\alpha_{i0}) = \{1, y\}$, as $[x^k, \alpha_{i0}] = x^{-k} x^{ik} \neq 1$, $[y, \alpha_{i0}] = y^{-1} y = 1$, and $[x^k y, \alpha_{i0}] = y^{-1} x^{-k} x^{ik} y = x^{k-ik} \neq 1$, for every $0 < k \leq n - 1$ and even $i$.

(iii) $C_{D_{2n}}(\alpha_{21}) = \{1, x^{n-1} y\}$, as $[x^k, \alpha_{21}] = x^{-k} x^{2k} \neq 1$, for every $0 < k \leq n - 1$, $[y, \alpha_{21}] = y^{-1} x y = x^{-1} \neq 1$, and $[x^s y, \alpha_{21}] = y^{-1} x^{-s} x^{2s} x y = x^{s+1} = 1$, when $s = n - 1$.

By similar argument, one has $C_{D_{2n}}(\alpha_{22}) = \{1, x^{n-2} y\}, C_{D_{2n}}(\alpha_{23}) = \{1, x^{n-3} y\}, \ldots, C_{D_{2n}}(\alpha_{2(n-1)}) = \{1, xy\}$, so $|C_{D_{2n}}(A)| = n + 2$ if $n$ is odd.

Now, assume that $n$ is even; then, we have the following cases

(iv) $C_{D_{2n}}(\alpha_{1j}) = \langle x\rangle$, as $[x^k, \alpha_{1j}] = x^{-k}x^k = 1$, $[y, \alpha_{1j}] = y^{-1}x^j y = x^{-j} \neq 1$, and $[x^k y, \alpha_{1j}] = y^{-1}x^{-k}x^k x^j y = x^{-j} \neq 1$, for every $0 < k \leq n-1$.

(v) $C_{D_{2n}}(\alpha_{(n-1)0}) = \{1, y, x^{(n/2)}, x^{(n/2)}y\}$, as $[x^k, \alpha_{(n-1)0}] = x^{-k}x^{nk-k} = 1$, when $k = n/2$, $[y, \alpha_{(n-1)0}] = y^{-1}y = 1$, and $[x^k y, \alpha_{(n-1)0}] = y^{-1}x^{-k}x^{nk-k}y = x^{2k-nk} = 1$, when $k = (n/2)$.

Using similar argument for $C_{D_{2n}}(\alpha_{(n-1)j})$, when $j$ is even, we have the following $A$-centralizers $C_{D_{2n}}(\alpha_{(n-1)2}) = \{1, xy, x^{(n/2)}, x^{(n/2)+1}y\}$, $C_{D_{2n}}(\alpha_{(n-1)4}) = \{1, x^2y, x^{(n/2)}, x^{(n/2)+2}y\}$, $\ldots, C_{D_{2n}}(\alpha_{(n-1)(n-2)}) = \{1, x^{(n/2)-1}y, x^{(n/2)}, x^{n-1}y\}$, so $|C_{D_{2n}}(A)| = (n/2) + 2$ if $n$ is even. □

The following result of [9] is useful in our further investigations.

**Theorem 1** (see Theorem 3.1 in [9]). *Let $G$ be a group with $G/L(G) \cong C_p \times C_p$, for any prime number $p$. Then, $G$ is isomorphic to one of the following groups:*

*(1) $C_p \times C_p$*
*(2) $C_p \times C_p \times C_2$ ($p$ is odd)*
*(3) $C_4 \times C_2$*
*(4) $D_8$*
*(5) $Q_8$*
*(6) $\langle x, y\colon x^4 = y^4 = 1, x^y = x^{-1}\rangle$*

Using the above theorem, we have the following.

**Theorem 2.** *Let $G$ be a group such that $G/L(G) \cong C_p \times C_p$, for any prime number $p$. Then, $|C_G(A)| = p + 2$.*

*Proof.* Let $p = 2$ and $G/L(G) \cong C_2 \times C_2$; then, Proposition 1 implies that $G \cong C_2 \times C_2 = \langle x, y\colon x^2 = 1, y^2 = 1, [x, y] = 1\rangle$. Clearly, $A = \mathrm{Aut}(C_2 \times C_2) \cong S_3$, and it is easily checked that $|C_{C_2\times C_2}(A)| = 4$.

Assume $G = C_4 \times C_2 = \langle x, y\colon x^4 = y^2 = 1, [x, y] = 1\rangle$; then, clearly,

$$\mathrm{Aut}(C_4 \times C_2) = \left\{\alpha_{ij} : \left\{\begin{array}{l} x\mapsto x \text{ or } x^3 \text{ or } xy \text{ or } x^3y \\ y\mapsto y \text{ or } x^2y \end{array}\right.\right\}. \quad (9)$$

Hence, one can calculate that the group $C_4 \times C_2$ has the following $A$-centralizers:

$$C_4 \times C_2, \langle x\rangle, \left\{1, x^2, xy, x^3y\right\}, \left\{1, x^2, y, x^2y\right\}. \quad (10)$$

In the cases $G \cong D_8$ and $Q_8$, example 1 and Proposition 1 show that $G$ is 4-$A$-centralizer.

If $G = \langle x, y\colon x^4 = y^4 = 1, x^y = x^{-1}\rangle$, then with relatively simple and long calculations, we obtain that $G$ has the following $A$-centralizers:

$$\begin{aligned} &G, \left\{1, x^2, y^2, x^2y^2, x, x^3, xy^2, x^3y^2\right\}, \\ &\left\{1, x^2, y^2, x^2y^2, y, x^2y, y^3, x^2y^3\right\}, \\ &\left\{1, x^2, y^2, x^2y^2, xy, x^3y, xy^3, x^3y^3\right\}. \end{aligned} \quad (11)$$

Now, let $G \cong C_p \times C_p \times C_2$ and $p$ be an odd prime number. Then, it is clear that $|C_G(\alpha)| = 2, 2p$ or $2p^2$, for every $\alpha \in A$. Thus, we have the following $A$-centralizers:

$$\begin{aligned} G = \langle x, y, z\colon & x^p = y^p = z^2 = 1, [x, y] = [x, z] = [y, z] = 1\rangle, \\ &\cdot \{1, z\}, \\ &\cdot \left\{1, z, x, x^2, \ldots, x^{p-1}, zx, zx^2, \ldots, zx^{p-1}\right\}, \\ &\cdot \left\{1, z, y, y^2, \ldots, y^{p-1}, zy, zy^2, \ldots, zy^{p-1}\right\}, \\ &\cdot \left\{1, z, xy^i, \left(xy^i\right)^2, \ldots, \left(xy^i\right)^{p-1}, zxy^i, \ldots, z\left(xy^i\right)^{p-1}\right\}, \end{aligned} \quad (12)$$

where $1 \leq i \leq p-1$, so $G$ has $p + 2$ distinct $A$-centralizers. □

The following theorem of [10] is needed to prove our next result.

**Theorem 3** (see Theorem 1 in [10]). *A group $G$ is the nontrivial union of three subgroups if and only if it is homomorphic to the Klein four-group.*

**Proposition 2.** *Let $G$ be a group; then, $|C_G(A)| = 4$ if and only if $G/L(G) \cong C_2 \times C_2$.*

*Proof.* Using Theorem 1, it is enough to show that $|C_G(A)| = 4$, which implies $G/L(G) \cong C_2 \times C_2$.

Suppose $|C_G(A)| = 4$; then, $C_G(A) = \{G, C_G(\alpha_1), C_G(\alpha_2), C_G(\alpha_3)\}$, for some $\alpha_1, \alpha_2, \alpha_3 \in A$. Hence, by Lemma 1, $G = \cup_{i=1}^{3} C_G(\alpha_i)$. Consider $C_G(\alpha_1\alpha_2)$, which will be one of the $A$-centralizers $G, C_G(\alpha_1), C_G(\alpha_2)$, or $C_G(\alpha_3)$. If $C_G(\alpha_1\alpha_2) = G$, then $\alpha_2(x) = \alpha^{-1}(x)$, for all $x \in G$, so $C_G(\alpha_1) = C_G(\alpha_2)$, which is a contradiction.

Assume $C_G(\alpha_1\alpha_2) = C_G(\alpha_1)$, then for any $x \in C_G(\alpha_1\alpha_2) = C_G(\alpha_1)$, we have $\alpha_1\alpha_2(x) = \alpha_1(x) = x$. Therefore $\alpha_2(x) = \alpha_1^{-1}(x) = x$, and hence, $C_G(\alpha_1\alpha_2) = C_G(\alpha_1) \subseteq C_G(\alpha_2)$, which is again a contradiction. Similarly, if $C_G(\alpha_1\alpha_2) = C_G(\alpha_2)$, then we have a contradiction, so $C_G(\alpha_1\alpha_2) = C_G(\alpha_2\alpha_1)$ must be equal with $C_G(\alpha_3)$.

Now, Theorem 3 implies that $(G/\cap_{i=1}^{3} C_G(\alpha_i)) = (G/L(G))$ is isomorphic with Klein four-group. □

**Theorem 4.** *Let $G$ be a finite group with $|C_G(A)| = 5$; then, $G/L(G) \cong S_3, C_3 \times C_3, D_{12}, C_2 \times C_6, C_3 \rtimes C_4$ or $A_4$.*

*Proof.* Assume $|C_G(A)| = 5$ and $C_G(\alpha_i)$ is $A$-centralizer, for some automorphisms $\alpha_1, \ldots, \alpha_4$ of $G$. Hence, Lemma 1 implies that $G = C_G(\alpha_1) \cup C_G(\alpha_2) \cup C_G(\alpha_3) \cup C_G(\alpha_4)$.

Now, consider $C_G(\alpha_1\alpha_2)$, which should be one of $G, C_G(\alpha_1), C_G(\alpha_2), C_G(\alpha_3)$, or $C_G(\alpha_4)$. Thus, we have the following cases:

(a) If $C_G(\alpha_1\alpha_2) = G$, then $\alpha_2(x) = \alpha^{-1}(x)$, for all $x \in G$, so $C_G(\alpha_1) = C_G(\alpha_2)$, which is a contradiction.

(b) If $C_G(\alpha_1\alpha_2) = C_G(\alpha_1)$, then, for all $x \in C_G(\alpha_1\alpha_2) = C_G(\alpha_1)$, we have $\alpha_1\alpha_2(x) = \alpha_1(x) = x$. Therefore, $\alpha_2(x) = \alpha_1^{-1}(x) = x$, and hence, $C_G(\alpha_1\alpha_2) = C_G(\alpha_1) \subseteq C_G(\alpha_2)$ and gives a contradiction.

Similarly, if $C_G(\alpha_1\alpha_2) = C_G(\alpha_2)$, then we obtain a contradiction, so $C_G(\alpha_1\alpha_2)$ can be equal to either $C_G(\alpha_3)$ or $C_G(\alpha_4)$.

(i) Assume that $C_G(\alpha_1), C_G(\alpha_2), C_G(\alpha_1\alpha_2) = C_G(\alpha_1\alpha_2)$, and $C_G(\alpha_4)$ are the $A$-centralizers of $G$. Then, using similar argument as in parts (a) and (b), we have $C_G(\alpha_1\alpha_4) = C_G(\alpha_2)$ and $C_G(\alpha_2\alpha_4) = C_G(\alpha_1)$. On the contrary, $L(G) \subseteq C_G(\alpha_1) \cap C_G(\alpha_2)$. Thus, for all $x \in C_G(\alpha_1) \cap C_G(\alpha_2)$, we have

$$\begin{aligned} &\alpha_1(x) = \alpha_2\alpha_4(x) = x \Rightarrow \alpha_2^{-1}(x) = \alpha_4(x), \\ &\alpha_2(x) = \alpha_1\alpha_4(x) = x \Rightarrow \alpha_2^{-1}(x) = x, \end{aligned} \tag{13}$$

and hence, $x \in C_G(\alpha_4)$. Also, $\alpha_1\alpha_2(x) = \alpha_1(x) = x$ and $\alpha_2\alpha_1(x) = \alpha_2(x) = x$ imply that $x \in C_G(\alpha_1\alpha_2) = C_G(\alpha_2\alpha_1)$. Therefore, $x \in L(G)$ and $L(G) = C_G(\alpha_1) \cap C_G(\alpha_2)$.

(ii) Assume that $C_G(\alpha_1), C_G(\alpha_2), C_G(\alpha_1\alpha_2)$, and $C_G(\alpha_2\alpha_1)$ are the $A$-centralizers of $G$. Then, similar argument as in part (i) implies that $L(G) = C_G(\alpha_1) \cap C_G(\alpha_2)$.

Hence, $\cap_{i=1}^{4} C_G(\alpha_i) = L(G) = C_G(\alpha_i) \cap C_G(\alpha_j) = C_G(\alpha_i) \cap C_G(\alpha_j) \cap C_G(\alpha_k)$, for $1 \le i, j, k \le 4$, when $|C_G(A)| = 5$.

Now, for computing the value of $|L(G)|$, we show that if $C_G(\alpha_i)$ and $C_G(\alpha_j)$ are arbitrary distinct proper $A$-centralizers of $G$, for $1 \le i \ne j \le 4$; then,

$$\frac{|C_G(\alpha_i)||C_G(\alpha_j)|}{|G|} \le |L(G)| \le \frac{|G|}{6} \cdot (*). \tag{14}$$

Clearly, $(|C_G(\alpha_i)||C_G(\alpha_j)|/|C_G(\alpha_i)C_G(\alpha_j)|) = |C_G(\alpha_i) \cap C_G(\alpha_j)|$. As $C_G(\alpha_i)C_G(\alpha_j) \subseteq G$, we have $(1/|C_G(\alpha_i)C_G(\alpha_j)|) \ge (1/|G|)$. Therefore, $|C_G(\alpha_i) \cap C_G(\alpha_j)| \ge (|C_G(\alpha_i)||C_G(\alpha_j)|/|G|)$ implies that $|L(G)| \ge (|C_G(\alpha_i)||C_G(\alpha_j)|/|G|)$. On the contrary, one observes that

$$\begin{aligned} |G| &= |C_G(\alpha_1)| + |C_G(\alpha_2)| + |C_G(\alpha_3)| + |C_G(\alpha_4)| - 3|L(G)| \\ &\ge 2|L(G)| + 2|L(G)| + 2|L(G)| + 2|L(G)| - 3|L(G)| = 5|L(G)|, \end{aligned} \tag{15}$$

and hence, $(|G|/|L(G)|) \ge 5$. Assume $G/|L(G) = 5$; then, $(G/|L(G)|)$ is cyclic and Theorem 2.2 of [9] implies that $G \cong \mathbb{Z}_5$ or $\mathbb{Z}_{10}$, which both give contradictions as $\mathbb{Z}_5$ has fixed-point-free automorphism, and $\mathbb{Z}_{10}$ does not conform to the conditions given at the beginning of the second section. Therefore, $(|G|/|L(G)|) \ge 6$.

Now, without loss of generality, we may assume that $|C_G(\alpha_1)| \ge |C_G(\alpha_2)| \ge |C_G(\alpha_3)| \ge |C_G(\alpha_4)|$. Suppose $|C_G(\alpha_1)| \le (|G|/4)$; then, we have

$$\begin{aligned} |G| &= |C_G(\alpha_1)| + |C_G(\alpha_2)| + |C_G(\alpha_3)| + |C_G(\alpha_4)| - 3|L(G)| \\ &\le \frac{|G|}{4} + \frac{|G|}{4} + \frac{|G|}{4} + \frac{|G|}{4} - 3|L(G)| = |G| - 3|L(G)|, \end{aligned} \tag{16}$$

which is a contradiction. Hence, $|C_G(\alpha_1)| = (|G|/2)$ or $(|G|/3)$. If $|C_G(\alpha_1)| = (|G|/2)$, we obtain

$$\begin{aligned} |G| &= |C_G(\alpha_1)| + |C_G(\alpha_2)| + |C_G(\alpha_3)| + |C_G(\alpha_4)| - 3|L(G)| \\ &= \frac{|G|}{2} + |C_G(\alpha_2)| + |C_G(\alpha_3)| + |C_G(\alpha_4)| - 3|L(G)|. \end{aligned} \tag{17}$$

One can easily calculate that

$$\frac{|G|}{2} < |C_G(\alpha_2)| + |C_G(\alpha_3)| + |C_G(\alpha_4)| \le 3|C_G(\alpha_2)|, \tag{18}$$

so $(|G|/6) < |C_G(\alpha_2)|$.

Now, applying (3) to $C_G(\alpha_1)$ and $C_G(\alpha_2)$, we have $(|C_G(\alpha_1)||C_G(\alpha_2)|/|G|) \le (|G|/6)$, and hence, $|C_G(\alpha_2)| \le (2|G|/6)$. That is, $(|G|/6) < |C_G(\alpha_2)| \le (|G|/3)$, so $|C_G(\alpha_2)| = (|G|/5), (|G|/4)$ or $(|G|/3)$. The property $(|C_G(\alpha_1)||C_G(\alpha_2)|/|G|) \le |L(G)| \le (|G|/6)$ implies that $(|G|/10) \le |L(G)| \le (|G|/6)$. Therefore, the value of $|L(G)|$ must be one of $(|G|/6), (|G|/7), (|G|/8), (|G|/9)$, or $(|G|/10)$.

Now, if $|L(G)| = (|G|/7)$, then $|L(G)|$ divides $|C_G(\alpha_1)|$, and hence, $2|7$ is impossible. Similarly, $|L(G)| \ne (|G|/9)$. Hence, we have the following cases:

$$|L(G)| = \frac{|G|}{6} \Rightarrow \frac{|G|}{|L(G)|} = 6 \Rightarrow \frac{G}{L(G)} \cong C_6 \text{ or } S_3. \tag{19}$$

If $(G/L(G)) \cong S_3$, then Theorem 3.5 of [9] and example 1 (iii) imply that $G \cong S_3$ and $|C_{S_3}(A)| = 5$. Again Theorem 2.2 of [9] implies that if $(G/L(G)) \cong C_6$, then $G \cong C_{12}$, which does not conform to the conditions given at the beginning of Section 2.

Let $|L(G)| = (|G|/8)$; then, as $|L(G)|$ divides $|C_G(\alpha_2)|$ if $|C_G(\alpha_2)| = (|G|/3)$, then $3|8$, and if $|C_G(\alpha_2)| = (|G|/5)$, then $5|8$, which both give contradictions. Therefore, $|C_G(\alpha_2)| = (|G|/4)$. On the contrary, the property $|G| = |C_G(\alpha_1)| + |C_G(\alpha_2)| + |C_G(\alpha_3)| + |C_G(\alpha_4)| - 3|L(G)|$ implies that $(|G|/4) = |C_G(\alpha_3)| + |C_G(\alpha_4)| - 3(|G|/8)$, and hence, $(5|G|/8) = |C_G(\alpha_3)| + |C_G(\alpha_4)|$. As $|C_G(\alpha_3)|, |C_G(\alpha_4)| \le (|G|/4)$, we obtain $(5|G|/8) = |C_G(\alpha_3)| + |C_G(\alpha_4)| \le (|G|/2)$, which is again a contradiction. So, $|L(G)|$ cannot be equal to $(|G|/8)$.

Finally, assume that $|L(G)| = (|G|/10)$ and $|L(G)|$ divides $|C_G(\alpha_2)|$. If $|C_G(\alpha_2)| = (|G|/3)$, then $3|10$, and if $|C_G(\alpha_2)| = (|G|/4)$, then $4|10$, which are both impossible. Therefore, $|C_G(\alpha_2)| = (|G|/5)$. Now, again $|G| = |C_G(\alpha_1)| + |C_G(\alpha_2)| + |C_G(\alpha_3)| + |C_G(\alpha_4)| - 3|L(G)|$ implies that

$|C_G(\alpha_3)| + |C_G(\alpha_4)| = (6|G|/10)$. Also, $|C_G(\alpha_2)| \geq |C_G(\alpha_3)| \geq |C_G(\alpha_4)|$ implies that $(6|G|/10) = |C_G(\alpha_3)| + |C_G(\alpha_4)| = (2|G|/5)$, which is a contradiction. Hence, $|L(G)| \neq (|G|/10)$.

Now, assume that $|C_G(\alpha_1)| = (|G|/3)$. In this case, using

$$|G| = |C_G(\alpha_1)| + |C_G(\alpha_2)| + |C_G(\alpha_3)| + |C_G(\alpha_4)| - 3|L(G)|, \tag{20}$$

we have $(2|G|/3) < |C_G(\alpha_2)| + |C_G(\alpha_3)| + |C_G(\alpha_4)| \leq 3|C_G(\alpha_2)|$. Thus, $|C_G(\alpha_2)| > (2|G|/9)$. On the contrary, $|C_G(\alpha_1)| \geq |C_G(\alpha_2)|$, so $(2|G|/9) < |C_G(\alpha_2)| \leq (|G|/3)$. Therefore, $|C_G(\alpha_2)| = (|G|/3)$ or $(|G|/4)$. Again applying $(*)$ on $C_G(\alpha_1)$ and $C_G(\alpha_2)$, we obtain

$$\frac{|C_G(\alpha_1)||C_G(\alpha_2)|}{|G|} \leq |L(G)| \leq \frac{|G|}{6}. \tag{21}$$

Thus, $(|G|/12) \leq |L(G)| \leq (|G|/6)$, and hence, $|L(G)| = (|G|/6), (|G|/7)$, $(|G|/8), (|G|/9), (|G|/10), (|G|/11)$, or $(|G|/12)$.

Assume that $|L(G)| = (|G|/7)$, and as $|L(G)|$ divides $|C_G(\alpha_1)|$, we must have $3|7$, which is impossible. Similarly, $|L(G)| \neq (|G|/8), (|G|/10)$, and $(|G|/11)$. Also, assume that $|L(G)| = (|G|/6)$, $|C_G(\alpha_1)| = (|G|/3)$, and $|C_G(\alpha_2)| = (|G|/4)$ or $(|G|/3)$; then, again

$$|G| = |C_G(\alpha_1)| + |C_G(\alpha_2)| + |C_G(\alpha_3)| + |C_G(\alpha_4)| - 3|L(G)| \tag{22}$$

implies that $(11|G|/12) = |C_G(\alpha_3)| + |C_G(\alpha_4)| \leq (|G|/2)$ or $(5|G|/6) = |C_G(\alpha_3)| + |C_G(\alpha_4)| \leq (2|G|/3)$, respectively, which are both impossible. Hence, $|L(G)| \neq (|G|/6)$, so we have one of the following cases:

$$|L(G)| = \frac{|G|}{12} \Rightarrow \frac{|G|}{|L(G)|} = 12 \Rightarrow \frac{G}{L(G)} \cong C_{12}, A_4, D_{12}, C_3 \rtimes C_4, C_2 \times C_6, \tag{23}$$

or

$$|L(G)| = \frac{|G|}{9} \Rightarrow \frac{|G|}{|L(G)|} = 9 \Rightarrow \frac{G}{L(G)} \cong C_9, C_3 \times C_3. \tag{24}$$

□

## 4. Groups with Abelian *A*-Centralizers

The concept of commutative transitive groups was first introduced and studied by Weisner [11] in 1925.

In this section, we introduce the new concept of *autocommutative transitive* groups, which is a generalized version of commutative transitive groups. Also, we study a group $G$, in which every $A$-centralizer of a nontrivial element of $G$ is abelian. We show that such groups are equivalent to autocommutative transitive groups.

*Definition 1.* A group $G$ is autocommutative transitive (henceforth $A-\mathrm{CT}$) if $[x,\alpha] = 1$ and $[\alpha, y] = 1$ imply that $[x, y] = 1$, for any nontrivial elements $x, y$ in $G$ and every $\alpha \in \mathrm{A}$.

If $\alpha$ runs over the inner automorphisms of $G$, then one has the usual commutative transitive groups.

**Lemma 3.** *For any group G, the following statements are equivalent:*

*(i) G is A-CT group*

*(ii) The A-centralizers of nontrivial automorphisms of G are abelian*

*Proof*

(i) ⇒ (ii) Let $G$ be A-CT group. For any nonidentity automorphism element $\alpha \in \mathrm{A}$, if $x, y \in C_G(\alpha)$, we have $[x,\alpha] = 1$ and $[\alpha, y] = 1$. The definition of A-CT implies that $[x, y] = 1$. Hence, $C_G(\alpha)$ is abelian.

(ii) ⇒ (i) Assume $x, y$ are nontrivial elements of $G$, with $[x,\alpha] = 1$ and $[\alpha, y] = 1$, for every $\alpha \in \mathrm{A}$. Obviously, $x, y \in C_G(\alpha)$, by the assumption $C_G(\alpha)$ is abelian, and hence, $[x, y] = 1$. Thus, $G$ is autocommutative transitive. □

Using the above lemma, we have the following.

**Corollary 1.** *Let G be a finite A-CT group and $\{x_1, \ldots, x_r\}$ be a set of pairwise noncommuting elements of G with maximal size. Then, $|C_G(A)| \geq r + 1$.*

## Data Availability

The datasets used and analysed during the study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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