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Research Article

Locally Conformally Flat Doubly Twisted Product Complex Finsler Manifolds

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Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2, F)$ be a doubly twisted product manifold of two strongly pseudoconvex complex Finsler manifolds (M_1, F_1) and (M_2, F_2). In this study, we give a characterization of locally conformally flat doubly twisted product complex Finsler manifold. We also obtain a necessary and sufficient condition for a doubly twisted product of two locally conformally flat complex Finsler manifolds to be locally conformally flat.

1. Introduction

Warped product and twisted product are important methods used to construct manifolds with special curvature properties in Riemann geometry. The warped product manifold was first introduced in 1969 by O'Neill and Bishop to construct Riemannian manifolds with negative curvature [1]. Then, it was extended to real Finsler geometry by the work of Hushmandi et al. (see [2–4]). Later, it was studied by many authors to construct new examples of real Finlser manifold [5–7]. They consider warped product Finsler manifold with scalar flag curvature and some well-known non-Riemannian curvature properties such as Berwald, Landsberg, and S-curvature [8–11].

In [12], He and Zhong extended the warped product to complex Finsler geometry and gave a possible way to construct complex Finsler metric (e.g., weakly complex Berwald metric and complex locally Minkowski metric).

The twisted product manifold, as a generalization of warped product manifold, was mentioned first by Chen [13]. Then, the notion of twisted product was generalized for the pseudo-Riemannian manifold by Ponge and Reckziegel [14]. In [15], Kozma et al. extended the twisted product to real Finsler manifolds. Later, twisted product of Finsler manifolds was studied by many authors [16–19]. In 2021, Deng et al. obtained the necessary and sufficient conditions that the doubly twisted product of real Finsler manifolds is a Berwald manifold [20].

Recently, we extended the twisted product to complex Finsler geometry and gave characterization of a doubly twisted product complex Finsler manifold to be Kähler Finsler manifold (resp. weakly Kähler Finsler manifold, complex Berwald manifold, weakly complex Berwald manifold, and complex Landsberg manifold) [21]. Later, we obtained necessary and sufficient conditions for a doubly twisted product complex Finsler manifold to be locally dually flat [22]. In [23], we gave a characterization for doubly twisted product complex Finsler manifold to be a complex Einstein–Finsler manifold.

The Weyl theorem states that the projective and conformal properties of a Finsler space determine the metric properties uniquely [24, 25]. In Finsler geometry, it is important subject to study conformal transformations of Finsler manifolds. In [26], Aldea proved that the scale function of a conformal transformation between two complex Finsler metrics depends only on the position of the base manifold. Recently, many author studied conformal transformation in complex Finsler geometry [27–30]. In 2021, Li studied the locally conformal pseudo-Kähler Finsler manifolds [31].

Particularly, a Finsler manifold which is conformally related to a Minkowski manifold is called locally conformally flat Finsler manifold. One of the most important problems in Finsler geometry is to study and characterize conformally flat Finsler manifold. In 2012, Matsumoto and Pripathi systematically studied locally conformally flat twisted product Riemann manifolds and proved its component manifolds are locally conformally flat [18]. Later, He also gave a necessary and sufficient condition for a doubly warped product complex Finsler manifolds to be locally conformally flat in [32].

Thus, it is very natural and interesting to ask the following question. Under what condition does a doubly twisted product complex Finsler manifold be locally conformally flat? Under what condition does a doubly twisted product of two locally conformally flat complex Finsler manifolds be locally conformally flat?

Inspired by the above questions, we will investigate the necessary and sufficient conditions for a doubly twisted product complex Finsler manifold to be locally conformally flat. We also give a necessary and sufficient condition that the doubly twisted product of two locally conformally flat complex Finsler manifolds is locally conformally flat.

2. Preliminary

In this section, we briefly recall some basic concepts and notations which we need in this study.

Let M be a complex manifold of complex dimension n. We denote $T^{1,0}M$ as the holomorphic tangent bundle of M and $\widetilde{M}$ as the complement of zero section in $T^{1,0}M$. We denote $z = (z^1, \ldots, z^n)$ as the local holomorphic coordinates on M and $(z, v) = (z^1, \ldots, z^n, v^1, \ldots, v^n)$ as the induced local holomorphic coordinates on the holomorphic tangent bundle $T^{1,0}M$.

Definition 1 (see [33]). A strongly pseudoconvex complex Finsler metric F on a complex manifold M is a continuous function F: $T^{1,0}M \longrightarrow \mathbb{R}^+$ satisfying

(i) $G = F^2$ is smooth on $\widetilde{M}$.

(ii) $F(p, v) > 0$, for all $(p, v) \in \widetilde{M}$.

(iii) $F(p, \zeta v) = |\zeta| F(p, v)$, for all $(p, v) \in T^{1,0}M$ and $\zeta \in \mathbb{C}$.

(iv) The Levi matrix (or complex Hessian matrix),

$$\left(G_{\alpha\overline{\beta}}\right) = \left(\frac{\partial^2 G}{\partial v^\alpha \partial \overline{v^\beta}}\right), \tag{1}$$

is positive definite on $\widetilde{M}$.

In this study, we denote $(G^{\overline{\nu}\beta})$ the inverse matrix of $(G_{\alpha\overline{\nu}})$ such that $G^{\overline{\nu}\beta}G_{\alpha\overline{\nu}} = \delta_\alpha^\beta$. We also use the notion in [33], that is, the derivatives of G with respect to the v-coordinates and z-coordinates are separated by semicolon; for instance,

$$\begin{aligned} G_{\overline{\mu};\nu} &= \frac{\partial^2 G}{\partial z^\nu \partial \overline{v^\mu}}, \\ G_{\alpha;\overline{\nu}} &= \frac{\partial^2 G}{\partial \overline{z^\nu} \partial v^\alpha}. \end{aligned} \tag{2}$$

In the following, we denote ∂_μ as the partial derivative with respect to the local coordinates z^μ on M and $\dot{\partial}_\mu$ as the partial derivative with respect to the fiber coordinates v^μ.

The Chern–Finsler connection was first constructed in [34] and systemically studied in [33]. Let D: $\mathcal{X}(\mathcal{V}^{1,0}) \longrightarrow \mathcal{X}(T_{\mathbb{C}}^*\widetilde{M} \otimes \mathcal{V}^{1,0})$ be the Chern–Finsler connection associated to a strongly pseudoconvex complex Finsler metric F. The Chern–Finsler complex nonlinear connection $\Gamma^\alpha_{;\mu}$ associated to F is given by

$$\Gamma^\alpha_{;\mu} =: G^{\overline{\nu}\alpha} G_{\overline{\nu};\mu}. \tag{3}$$

The connection 1-forms ω^α_β of D are given by

$$\omega^\alpha_\beta = \Gamma^\alpha_{\beta;\mu} dz^\mu + \Gamma^\alpha_{\beta\mu} \psi^\mu, \tag{4}$$

where

$$\begin{aligned} \Gamma^\alpha_{\beta;\mu} &= G^{\overline{\nu}\alpha} \delta_\mu \left(G_{\beta\overline{\nu}}\right) = \dot{\partial}_\beta \left(\Gamma^\alpha_{;\mu}\right), \\ \Gamma^\alpha_{\beta\mu} &= G^{\overline{\nu}\alpha} \dot{\partial}_\mu \left(G_{\beta\overline{\nu}}\right), \end{aligned} \tag{5}$$

and

$$\begin{aligned} \delta_\mu &= \partial_\mu - \Gamma^\alpha_{;\mu} \dot{\partial}_\alpha, \\ \psi^\mu &= dv^\mu + \Gamma^\mu_{;\alpha} dz^\alpha. \end{aligned} \tag{6}$$

The complex Rund connection associated to a strongly pseudoconvex complex Finsler metric F was first introduced in [35] and were systemically studied in [36, 37]. Let $\widehat{D}$: $\mathcal{X}(\mathcal{V}^{1,0}) \longrightarrow \mathcal{X}(T_{\mathbb{C}}^*\widetilde{M} \otimes \mathcal{V}^{1,0})$ be the complex Rund connection; then, the connection 1-forms $\widehat{\omega}$ of $\widehat{D}$ are given by

$$\widehat{\omega}^\alpha_\beta = \Gamma^\alpha_{\beta;\mu} dz^\mu, \tag{7}$$

where $\Gamma^\alpha_{\beta;\mu}$ are defined by equation (5). It is clearly from equation (4) that $\widehat{\omega}^\alpha_\beta$ are just the horizontal part of ω^α_β.

Definition 2 (see [37, 38]). A complex Finsler manifold (M, F) is said to be modeled on a complex Minkowski space if the horizontal connection coefficients $\Gamma^\alpha_{\beta;\mu}$ of the Chern–Finsler connection or complex Rund connection coefficients depend only on the coordinates z of the base manifold M, i.e., $\Gamma^\alpha_{\beta;\mu}(z, v) = \Gamma^\alpha_{\beta;\mu}(z)$.

Definition 3 (see [36]). Let F be a complex Finsler metric on M. If there exits an open cover $\{U, X_U\}$ such that, on each $\pi_T^{-1}(U)$, the function F is a function of the fiber coordinate only, then the complex Finsler metric F will be a complex locally Minkowski metric.

The necessary and sufficient condition of a complex Finsler metric F on M to be a complex locally Minkowski metric is that it is modeled on a complex Minkowski metric and the complex Rund connection coefficients on (M, F) is holomorphic. That is, the horizontal coefficients $\Gamma^\alpha_{\beta;\mu}(z, v)$ of the Chern–Finsler connection or complex Rund connection coefficients satisfy the following conditions:

$$\begin{cases} \Gamma^\alpha_{\beta;\mu}(z, v) = \Gamma^\alpha_{\beta;\mu}(z), \\ \dfrac{\partial\left(\Gamma^\alpha_{\beta;\mu}\right)}{\partial \overline{z^\nu}} = 0. \end{cases} \tag{8}$$

Definition 4 (see [38]). Let F and $\widetilde{F}$ be two strongly pseudoconvex complex Finsler metric on complex manifold M.

A conformal change of F is the change $F \longrightarrow \widetilde{F} = e^{\sigma(z)}F$ for a smooth function $\sigma(z)$ on M.

Definition 5 (see [38]). A complex Finsler manifold (M, F) is said to be locally conformally flat if it is locally conformal to a locally Minkowski space; that is, there exhibits an open covering $\{U_\alpha\}$ with a family $\{\sigma_\alpha\}$ of locally defined smooth functions $\sigma_\alpha\colon U_\alpha \longrightarrow \mathbb{R}$ such that the metric $\widetilde{F} = e^{\sigma_\alpha(z)}F$ is a complex locally Minkowski metric on U_α.

3. Doubly Twisted Product of Complex Finsler Manifolds

Let (M_1, F_1) and (M_2, F_2) be two strongly pseudoconvex complex Finsler manifolds with $\dim_{\mathbb{C}} M_1 = m$ and $\dim_{\mathbb{C}} M_2 = n$; then, $M = M_1 \times M_2$ is a strongly pseudoconvex complex Finsler manifold with $\dim_{\mathbb{C}} M = m + n$.

Let $T^{1,0}M_1$ and $T^{1,0}M_2$ be the holomorphic tangent bundles of M_1 and M_2, respectively. Let $\pi_1\colon M_1 \times M_2 \longrightarrow M_1$ and $\pi_2\colon M_1 \times M_2 \longrightarrow M_2$ be natural projection maps; then, $d\pi_1\colon T^{1,0}(M_1 \times M_2) \longrightarrow T^{1,0}M_1$ and $d\pi_2\colon T^{1,0}(M_1 \times M_2) \longrightarrow T^{1,0}M_2$ be the holomorphic tangent maps induced by π_1 and π_2, respectively. Note that $d\pi_1(z, v) = (z_1, v_1)$ and $d\pi_2(z, v) = (z_2, v_2)$ for every $v = (v_1, v_2) \in T_z^{1,0}(M_1 \times M_2)$ with $v_1 = (v^1, \ldots, v^m) \in T_{z_1}^{1,0}M_1$ and $v_2 = (v^{m+1}, \ldots, v^{m+n}) \in T_{z_2}^{1,0}M_2$.

Definition 6 (see [21]). Let (M_1, F_1) and (M_2, F_2) be two strongly pseudoconvex complex Finsler manifolds and $\lambda_i\colon M_1 \times M_2 \longrightarrow (0, +\infty)\,(i = 1, 2)$ be smooth functions. The doubly twisted product (abbreviated as DTP) complex Finsler manifold of (M_1, F_1) and (M_2, F_2) is the product complex manifold $M = M_1 \times M_2$ endowed with the complex Finsler metric $F\colon \widetilde{M} \longrightarrow \mathbb{R}^+$ given by

$$F^2(z, v) = \lambda_1^2(z)F_1^2(\pi_1(z), d\pi_1(v)) + \lambda_2^2(z)F_2^2(\pi_2(z), d\pi_2(v)), \tag{9}$$

for $z = (z_1, z_2) \in M$ and $v = (v_1, v_2) \in T_z^{1,0}M-$ {zero section}. The functions λ_1 and λ_2 are called twisted functions. The DTP-complex Finsler manifold of (M_1, F_1) and (M_2, F_2) is denoted by $(M_1 \times_{(\lambda_1, \lambda_2)} M_2, F)$.

In the case of $\lambda_1 \equiv 1$ or $\lambda_2 \equiv 1$, the corresponding DTP-complex Finsler manifold is called twisted product complex Finsler manifold. If $\lambda_1 \equiv \lambda_2 \equiv 1$, (M, F) becomes the product complex Finsler manifold. If neither λ_1 nor λ_2 is a constant, then we call (M, F) a nontrivial (proper) DTP-complex Finsler manifold of (M_1, F_1) and (M_2, F_2).

Notation: lowercase Greek indices such as α, β, and γ will run from 1 to $m + n$, whereas lowercase Latin indices such as i, j, and k will run from 1 to m, and lowercase Latin indices with a prime such as i', j', and k' will run from $m + 1$ to $m + n$. Quantities associated to F_1 and F_2 are denoted with upper indices 1 and 2, respectively, such as $\overset{1}{\Gamma}{}^i_{j;k}$, $\overset{2}{\Gamma}{}^{i'}_{j';k'}$.

Denote $g = F_1^2$, $h = F_2^2$, and

$$g_{i\overline{j}} = \frac{\partial^2 g}{\partial v^i \partial \overline{v^j}}, \qquad h_{i'\overline{j'}} = \frac{\partial^2 h}{\partial v^{i'} \partial \overline{v^{j'}}}, \tag{10}$$

so equation (9) is equal to

$$G = F^2 = \lambda_1^2 g + \lambda_2^2 h. \tag{11}$$

Proposition 1. (see [21]). *Let $(M_1 \times_{(\lambda_1, \lambda_2)} M_2, F)$ be a doubly twisted product manifold of complex Finsler manifolds (M_1, F_1) and (M_2, F_2). Then, the fundamental tensor matrix of F is given by*

$$\left(G_{\alpha\overline{\beta}}\right) = \left(\frac{\partial^2 G}{\partial v^\alpha \partial \overline{v^\beta}}\right) = \begin{pmatrix} \lambda_1^2 g_{i\overline{j}} & 0 \\ 0 & \lambda_2^2 h_{i'\overline{j'}} \end{pmatrix}, \tag{12}$$

with its inverse matrix $(G^{\overline{\beta}\alpha})$ given by

$$\left(G^{\overline{\beta}\alpha}\right) = \begin{pmatrix} \lambda_1^{-2} g^{\overline{j}i} & 0 \\ 0 & \lambda_2^{-2} h^{\overline{j'}i'} \end{pmatrix}. \tag{13}$$

Lemma 1 (see [21]). *Let $(M_1 \times_{(\lambda_1, \lambda_2)} M_2, F)$ be a doubly twisted product manifold of two strongly pseudoconvex complex Finsler manifolds (M_1, F_1) and (M_2, F_2). Then, the Chern–Finsler complex nonlinear connection coefficients associated to F are given by*

$$\begin{aligned} \Gamma^i_{;k} &= \overset{1}{\Gamma}{}^i_{;k} + 2\lambda_1^{-1}\frac{\partial \lambda_1}{\partial z^k}v^i, \quad \Gamma^{i'}_{;k} = 2\lambda_2^{-1}\frac{\partial \lambda_2}{\partial z^k}v^{i'}, \\ \Gamma^i_{;k'} &= 2\lambda_1^{-1}\frac{\partial \lambda_1}{\partial z^{k'}}v^i, \quad \Gamma^{i'}_{;k'} = \overset{2}{\Gamma}{}^{i'}_{;k'} + 2\lambda_2^{-1}\frac{\partial \lambda_2}{\partial z^{k'}}v^{i'}. \end{aligned} \tag{14}$$

Proposition 2. (see [21]). *Let $(M_1 \times_{(\lambda_1, \lambda_2)} M_2, F)$ be a doubly twisted product manifold of two strongly pseudoconvex complex Finsler manifolds (M_1, F_1) and (M_2, F_2); the horizontal coefficients of the Chern–Finsler connection associated to F are given by*

$$\begin{aligned} \Gamma^i_{j;k} &= \overset{1}{\Gamma}{}^i_{j;k} + 2\lambda_1^{-1}\frac{\partial \lambda_1}{\partial z^k}\delta^i_j, \quad \Gamma^i_{j;k'} = 2\lambda_1^{-1}\frac{\partial \lambda_1}{\partial z^{k'}}\delta^i_j, \\ \Gamma^{i'}_{j';k'} &= \overset{2}{\Gamma}{}^{i'}_{j';k'} + 2\lambda_2^{-1}\frac{\partial \lambda_2}{\partial z^{k'}}\delta^{i'}_{j'}, \quad \Gamma^{i'}_{j';k} = 2\lambda_2^{-1}\frac{\partial \lambda_2}{\partial z^k}\delta^{i'}_{j'}, \\ \Gamma^i_{j';k} &= \Gamma^i_{j';k'} = \Gamma^{i'}_{j;k} = \Gamma^{i'}_{j;k'} = 0. \end{aligned} \tag{15}$$

4. Locally Conformally Flat Doubly Twisted Product Complex Finsler Manifolds

Let F be a complex Finsler metric on M. A conformal transformation of F is a change $F \longrightarrow e^{\sigma(z)}F$, where

$\sigma(z)\colon M \longrightarrow \mathbb{R}$ is a smooth real function. We denote $e^{\sigma(z)}F$ by $\widetilde{F}$ and use the symbol " $\sim$ " to mark the geometric objects associated to the complex Finsler metric $\widetilde{F}$, e.g., $\widetilde{\Gamma}^{\alpha}_{\beta;\mu}$ are the horizontal coefficients of Chern–Finsler connection associate to $\widetilde{F}$.

Let (M_1, F_1) and (M_2, F_2) be locally conformally flat complex Finsler manifolds. In this section, we shall give the necessary and sufficient conditions for a doubly twisted product manifold $(M_1\times_{(\lambda_1,\lambda_2)}M_2, F)$ of (M_1, F_1) and (M_2, F_2) to be locally conformally flat.

Proposition 3. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2, F)$ be a doubly twisted product manifold of two strongly pseudoconvex complex Finsler manifolds (M_1, F_1) and (M_2, F_2), and $\widetilde{F}$ is conformal transformation of F. Then, the coefficients $\widetilde{\Gamma}^{\alpha}_{;\mu}$ of Chern–Finsler connection associated to $\widetilde{F}$ are given by*

$$\widetilde{\Gamma}^{i}_{;k} = \overset{1}{\Gamma}{}^{i}_{;k} + 2\left(\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^k} + \frac{\partial\sigma}{\partial z^k}\right)v^i,\quad \widetilde{\Gamma}^{i'}_{;k} = 2\left(\lambda_2^{-1}\frac{\partial\lambda_2}{\partial z^k} + \frac{\partial\sigma}{\partial z^k}\right)v^{i'},$$
$$\widetilde{\Gamma}^{i}_{;k'} = 2\left(\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^{k'}} + \frac{\partial\sigma}{\partial z^{k'}}\right)v^i,\quad \widetilde{\Gamma}^{i'}_{;k'} = \overset{2}{\Gamma}{}^{i'}_{;k'} + 2\left(\lambda_2^{-1}\frac{\partial\lambda_2}{\partial z^{k'}} + \frac{\partial\sigma}{\partial z^{k'}}\right)v^{i'}, \tag{16}$$

where $\sigma(z)\colon M \longrightarrow \mathbb{R}$ is a smooth real function.

Proof. Using equation (3), we obtain

$$\widetilde{\Gamma}^{\alpha}_{;\mu} = \widetilde{G}^{\overline{\nu}\alpha}\widetilde{G}_{\overline{\nu};\mu} = \Gamma^{\alpha}_{;\mu} + 2\frac{\partial\sigma}{\partial z^{\mu}}v^{\alpha}. \tag{17}$$

According to Lemma 1, we have

$$\widetilde{\Gamma}^{i}_{;k} = \Gamma^{i}_{;k} + 2\frac{\partial\sigma}{\partial z^k}v^i = \overset{1}{\Gamma}{}^{i}_{;k} + 2\left(\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^k} + \frac{\partial\sigma}{\partial z^k}\right)v^i. \tag{18}$$

Similarly, we can get other equations of Proposition 3. □

Proposition 4. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2, F)$ be a doubly twisted product manifold of two strongly pseudoconvex complex Finsler manifolds (M_1, F_1) and (M_2, F_2), and $\widetilde{F}$ is conformal transformation of F. Then, the horizontal coefficients $\widetilde{\Gamma}^{\alpha}_{\beta;\mu}$ of the Chern–Finsler connection associated to $\widetilde{F}$ are given by*

$$\widetilde{\Gamma}^{i}_{j;k} = \overset{1}{\Gamma}{}^{i}_{j;k} + 2\left(\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^k} + \frac{\partial\sigma}{\partial z^k}\right)\delta^i_j,\quad \widetilde{\Gamma}^{i}_{j;k'} = 2\left(\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^{k'}} + \frac{\partial\sigma}{\partial z^{k'}}\right)\delta^i_j,$$
$$\widetilde{\Gamma}^{i'}_{j';k'} = \overset{2}{\Gamma}{}^{i'}_{j';k'} + 2\left(\lambda_2^{-1}\frac{\partial\lambda_2}{\partial z^{k'}} + \frac{\partial\sigma}{\partial z^{k'}}\right)\delta^{i'}_{j'},\quad \widetilde{\Gamma}^{i'}_{j';k} = 2\left(\lambda_2^{-1}\frac{\partial\lambda_2}{\partial z^{k}} + \frac{\partial\sigma}{\partial z^{k}}\right)\delta^{i'}_{j'},$$
$$\widetilde{\Gamma}^{i}_{j';k} = \widetilde{\Gamma}^{i}_{j';k'} = \widetilde{\Gamma}^{i'}_{j;k} = \widetilde{\Gamma}^{i'}_{j;k'} = 0, \tag{19}$$

where $\sigma(z)\colon M \longrightarrow \mathbb{R}$ is a smooth real function.

Proof. According to equation (5) and Proposition 2, by straightforward computation, we obtain

$$\widetilde{\Gamma}^{i}_{j;k} = \dot{\partial}_j\left(\widetilde{\Gamma}^{i}_{;k}\right) = \Gamma^{i}_{j;k} + 2\frac{\partial\sigma}{\partial z^k}\delta^i_j = \overset{1}{\Gamma}{}^{i}_{j;k} + 2\left(\lambda_1^{-1}\frac{\partial\lambda_1}{\partial z^k} + \frac{\partial\sigma}{\partial z^k}\right)\delta^i_j. \tag{20}$$

Similarly, we can obtain other equations of Proposition 4. □

Theorem 1. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2, F)$ be a doubly twisted product manifold of two strongly pseudoconvex complex Finsler manifolds (M_1, F_1) and (M_2, F_2); then, $(M_1\times_{(\lambda_1,\lambda_2)}M_2, F)$ is locally conformally flat if and only if there exists an open covering $\{U_\alpha\}$ and a family of smooth functions $\{\sigma_\alpha\colon U_\alpha \longrightarrow \mathbb{R}\}$ such that*

$$\begin{cases} \overset{1}{\Gamma}{}^{i}_{j;k}(z,v) = \overset{1}{\Gamma}{}^{i}_{j;k}(z),\quad \overset{2}{\Gamma}{}^{i'}_{j';k'}(z,v) = \overset{2}{\Gamma}{}^{i'}_{j';k'}(z), \\ \dfrac{\partial\overset{1}{\Gamma}{}^{i}_{j;k}}{\partial\overline{z^s}} + 2\left(\dfrac{\partial^2\ln\lambda_1}{\partial z^k\partial\overline{z^s}} + \dfrac{\partial^2\sigma_\alpha}{\partial z^k\partial\overline{z^s}}\right)\delta^i_j = 0, \\ \dfrac{\partial\overset{2}{\Gamma}{}^{i'}_{j';k'}}{\partial\overline{z^{s'}}} + 2\left(\dfrac{\partial^2\ln\lambda_2}{\partial z^{k'}\partial\overline{z^{s'}}} + \dfrac{\partial^2\sigma_\alpha}{\partial z^{k'}\partial\overline{z^{s'}}}\right)\delta^{i'}_{j'} = 0, \\ \dfrac{\partial^2\ln\lambda_1}{\partial z^k\partial\overline{z^{s'}}} + \dfrac{\partial^2\sigma_\alpha}{\partial z^k\partial\overline{z^{s'}}} = \dfrac{\partial^2\ln\lambda_1}{\partial z^{k'}\partial\overline{z^s}} + \dfrac{\partial^2\sigma_\alpha}{\partial z^{k'}\partial\overline{z^s}} = \dfrac{\partial^2\ln\lambda_1}{\partial z^{k'}\partial\overline{z^{s'}}} + \dfrac{\partial^2\sigma_\alpha}{\partial z^{k'}\partial\overline{z^{s'}}} = 0, \\ \dfrac{\partial^2\ln\lambda_2}{\partial z^k\partial\overline{z^s}} + \dfrac{\partial^2\sigma_\alpha}{\partial z^k\partial\overline{z^s}} = \dfrac{\partial^2\ln\lambda_2}{\partial z^k\partial\overline{z^{s'}}} + \dfrac{\partial^2\sigma_\alpha}{\partial z^k\partial\overline{z^{s'}}} = \dfrac{\partial^2\ln\lambda_2}{\partial z^{k'}\partial\overline{z^s}} + \dfrac{\partial^2\sigma_\alpha}{\partial z^{k'}\partial\overline{z^s}} = 0. \end{cases} \tag{21}$$

Proof. According to Definition 5, (M, F) is locally conformally flat if and only if there exists an open covering $\{U_\alpha\}$ and a family of smooth functions $\{\sigma_\alpha\colon U_\alpha \longrightarrow \mathbb{R}\}$ on M such that $\widetilde{F}_\alpha = e^{\sigma_\alpha(z)}$ is a locally Minkowski metric on U_α. Notice that the complex Finsler metric $\widetilde{F}_\alpha$ is a complex locally Minkowski metirc if and only if it is modeled on a complex Minkowski metric and the complex Rund connection coefficients of $\widetilde{F}_\alpha$ is holomorphic. Thus, we obtain

$$\left\{\widetilde{\Gamma}^{\alpha}_{\beta;\mu}(z,v) = \widetilde{\Gamma}^{\alpha}_{\beta;\mu}(z),\right. \tag{22}$$

$$\left\{\frac{\partial\widetilde{\Gamma}^{\alpha}_{\beta;\mu}}{\partial\overline{z^{\nu}}} = 0.\right. \tag{23}$$

Submit the equations of Proposition 4 into equations (22) and (23); after a straightforward computation, we have

$$\begin{cases} \overset{1}{\widetilde{\Gamma}}{}^{i}_{j;k}(z,v) = \overset{1}{\widetilde{\Gamma}}{}^{i}_{j;k}(z), \quad \overset{2}{\widetilde{\Gamma}}{}^{i'}_{j';k'}(z,v) = \overset{2}{\widetilde{\Gamma}}{}^{i'}_{j';k'}(z), \\ \dfrac{\partial \widetilde{\Gamma}^{i}_{j;k}}{\partial \overline{z}^{s}} = \dfrac{\partial \overset{1}{\Gamma}{}^{i}_{j;k}}{\partial \overline{z}^{s}} + 2\left(\dfrac{\partial^2 \ln \lambda_1}{\partial z^k \partial \overline{z}^{s}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^k \partial \overline{z}^{s}}\right)\delta^i_j = 0, \\ \dfrac{\partial \widetilde{\Gamma}^{i}_{j;k}}{\partial \overline{z}^{s'}} = 2\left(\dfrac{\partial^2 \ln \lambda_1}{\partial z^k \partial \overline{z}^{s'}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^k \partial \overline{z}^{s'}}\right)\delta^i_j = 0, \\ \dfrac{\partial \widetilde{\Gamma}^{i'}_{j';k'}}{\partial \overline{z}^{s}} = 2\left(\dfrac{\partial^2 \ln \lambda_2}{\partial z^{k'} \partial \overline{z}^{s}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z}^{s}}\right)\delta^{i'}_{j'} = 0, \\ \dfrac{\partial \widetilde{\Gamma}^{i'}_{j';k'}}{\partial \overline{z}^{s'}} = \dfrac{\partial \overset{2}{\Gamma}{}^{i'}_{j';k'}}{\partial \overline{z}^{s'}} + 2\left(\dfrac{\partial^2 \ln \lambda_2}{\partial z^{k'} \partial \overline{z}^{s'}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z}^{s'}}\right)\delta^{i'}_{j'} = 0, \\ \dfrac{\partial \widetilde{\Gamma}^{i}_{j;k'}}{\partial \overline{z}^{s}} = 2\left(\dfrac{\partial^2 \ln \lambda_1}{\partial z^{k'} \partial \overline{z}^{s}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z}^{s}}\right)\delta^i_j = 0, \\ \dfrac{\partial \widetilde{\Gamma}^{i}_{j;k'}}{\partial \overline{z}^{s'}} = 2\left(\dfrac{\partial^2 \ln \lambda_1}{\partial z^{k'} \partial \overline{z}^{s'}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z}^{s'}}\right)\delta^i_j = 0, \\ \dfrac{\partial \widetilde{\Gamma}^{i'}_{j';k}}{\partial \overline{z}^{s}} = 2\left(\dfrac{\partial^2 \ln \lambda_2}{\partial z^{k} \partial \overline{z}^{s}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k} \partial \overline{z}^{s}}\right)\delta^{i'}_{j'} = 0, \\ \dfrac{\partial \widetilde{\Gamma}^{i'}_{j';k}}{\partial \overline{z}^{s'}} = 2\left(\dfrac{\partial^2 \ln \lambda_2}{\partial z^{k} \partial \overline{z}^{s'}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k} \partial \overline{z}^{s'}}\right)\delta^{i'}_{j'} = 0. \end{cases} \tag{24}$$

Noticing that $\overset{1}{\widetilde{\Gamma}}{}^{i}_{j;k} = \overset{1}{\Gamma}{}^{i}_{j;k} + 2(\partial\sigma_\alpha/\partial z^k)\delta^i_j$, so $\overset{1}{\widetilde{\Gamma}}{}^{i}_{j;k}(z,v) = \overset{1}{\widetilde{\Gamma}}{}^{i}_{j;k}(z)$ if and only if $\overset{1}{\Gamma}{}^{i}_{j;k}(z,v) = \overset{1}{\Gamma}{}^{i}_{j;k}(z)$. Similarly, we know that $\overset{2}{\widetilde{\Gamma}}{}^{i'}_{j';k'}(z,v) = \overset{2}{\widetilde{\Gamma}}{}^{i'}_{j';k'}(z)$ if and only if $\overset{2}{\Gamma}{}^{i'}_{j';k'}(z,v) = \overset{2}{\Gamma}{}^{i'}_{j';k'}(z)$.

Thus, we can obtain equation (21). □

Corollary 1. *Let $(M_1\times_{(\lambda_1,1)}M_2, F)$ be a twisted product manifold of two strongly pseudoconvex complex Finsler manifolds (M_1, F_1) and (M_2, F_2); then, $(M_1\times_{(\lambda_1,1)}M_2, F)$ is locally conformally flat if and only if (M_2, F_2) is locally conformally flat, and there exists an open covering $\{U_\alpha\}$ and a family of smooth functions $\{\sigma_\alpha\colon U_\alpha \longrightarrow \mathbb{R}\}$ such that*

$$\begin{cases} \overset{1}{\Gamma}{}^{i}_{j;k}(z,v) = \overset{1}{\Gamma}{}^{i}_{j;k}(z), \\ \dfrac{\partial \overset{1}{\Gamma}{}^{i}_{j;k}}{\partial \overline{z}^{s}} + 2\dfrac{\partial^2 \ln\lambda_1}{\partial z^k \partial \overline{z}^{s}}\delta^i_j = 0, \\ \dfrac{\partial^2 \ln\lambda_1}{\partial z^{k'} \partial \overline{z}^{s'}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z}^{s'}} = 0, \\ \dfrac{\partial^2 \ln\lambda_1}{\partial z^{k} \overline{z}^{s'}} = \dfrac{\partial^2 \ln\lambda_1}{\partial z^{k'} \overline{z}^{s}} = 0, \\ \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k} \partial \overline{z}^{s}} = \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k} \partial \overline{z}^{s'}} = \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z}^{s}} = 0. \end{cases} \tag{25}$$

Proof. Since $(M_1\times_{(\lambda_1,1)}M_2, F)$ be a twisted product complex Finsler manifold, so

$$\frac{\partial \lambda_2}{\partial z^\alpha} = 0. \tag{26}$$

By plunging equation (26) into equation (21), we obtain

$$\left\{ \overset{1}{\Gamma}{}^{i}_{j;k}(z,v) = \overset{1}{\Gamma}{}^{i}_{j;k}(z), \quad \overset{2}{\Gamma}{}^{i'}_{j';k'}(z,v) = \overset{2}{\Gamma}{}^{i'}_{j';k'}(z), \right. \tag{27}$$

$$\left\{ \frac{\partial \overset{1}{\Gamma}{}^{i}_{j;k}}{\partial \overline{z}^{s}} + 2\left(\frac{\partial^2 \ln \lambda_1}{\partial z^k \partial \overline{z}^{s}} + \frac{\partial^2 \sigma_\alpha}{\partial z^k \partial \overline{z}^{s}}\right)\delta^i_j = 0, \right. \tag{28}$$

$$\left\{ \frac{\partial \overset{2}{\Gamma}{}^{i'}_{j';k'}}{\partial \overline{z}^{s'}} + 2\frac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z}^{s'}}\delta^{i'}_{j'} = 0, \right. \tag{29}$$

$$\left\{ \frac{\partial^2 \ln \lambda_1}{\partial z^k \partial \overline{z}^{s'}} + \frac{\partial^2 \sigma_\alpha}{\partial z^k \partial \overline{z}^{s'}} = \frac{\partial^2 \ln \lambda_1}{\partial z^{k'} \partial \overline{z}^{s}} + \frac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z}^{s}} = \frac{\partial^2 \ln \lambda_1}{\partial z^{k'} \partial \overline{z}^{s'}} \right. + \frac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z}^{s'}} = 0, \tag{30}$$

$$\left\{ \frac{\partial^2 \sigma_\alpha}{\partial z^k \partial \overline{z}^{s}} = \frac{\partial^2 \sigma_\alpha}{\partial z^k \partial \overline{z}^{s'}} = \frac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z}^{s}} = 0. \right. \tag{31}$$

From equation (29) and noticing $\overset{2}{\widetilde{\Gamma}}{}^{i'}_{j';k'}(z,v) = \overset{2}{\widetilde{\Gamma}}{}^{i'}_{j';k'}(z)$, we obtain that (M_2, F_2) is a locally conformally flat manifold.

Submit equations (31) into equations (28) and (30), respectively; we can obtain equation (25). □

Theorem 2. *Let $(M_1\times_{(\lambda_1,\lambda_2)}M_2, F)$ be a doubly twisted product manifold of two locally conformally flat complex*

Finsler manifolds (M_1, F_1) *and* (M_2, F_2)*; then,* $(M_1 \times_{(\lambda_1,\lambda_2)} M_2, F)$ *is locally conformally flat if and only if* $\ln\lambda_1$ *and* $\ln\lambda_2$ *are pluriharmonic functions on* M_1 *and* M_2*, respectively, and there exists an open covering* $\{U_\alpha\}$ *and a family of smooth functions* $\{\sigma_\alpha: U_\alpha \longrightarrow \mathbb{R}\}$ *such that*

$$\begin{cases} \dfrac{\partial^2 \ln\lambda_1}{\partial z^k \partial \overline{z^{s'}}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^k \partial \overline{z^{s'}}} = \dfrac{\partial^2 \ln\lambda_1}{\partial z^{k'} \partial \overline{z^{s}}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z^{s}}} = \dfrac{\partial^2 \ln\lambda_1}{\partial z^{k'} \partial \overline{z^{s'}}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z^{s'}}} = 0, \\ \dfrac{\partial^2 \ln\lambda_2}{\partial z^k \partial \overline{z^{s}}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^k \partial \overline{z^{s}}} = \dfrac{\partial^2 \ln\lambda_2}{\partial z^{k} \partial \overline{z^{s'}}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k} \partial \overline{z^{s'}}} = \dfrac{\partial^2 \ln\lambda_2}{\partial z^{k'} \partial \overline{z^{s}}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z^{s}}} = 0. \end{cases} \quad (32)$$

Proof. According to Theorem 1, $(M_1 \times_{(\lambda_1,\lambda_2)} M_2, F)$ is locally conformally flat if and only if equation (21) holds on U_α.

Since (M_1, F_1) and (M_2, F_2) are locally conformally flat complex Finsler manifolds, thus,

$$\overset{1}{\tilde{\Gamma}}{}^i_{j;k}(z, v) = \overset{1}{\tilde{\Gamma}}{}^i_{j;k}(z), \quad \frac{\partial \overset{1}{\Gamma}{}^i_{j;k}}{\partial \overline{z^s}} + 2\frac{\partial^2 \sigma_\alpha}{\partial z^k \partial \overline{z^s}} \delta^i_j = 0, \quad (33)$$

$$\overset{2}{\tilde{\Gamma}}{}^{i'}_{j';k'}(z, v) = \overset{2}{\tilde{\Gamma}}{}^{i'}_{j';k'}(z), \quad \frac{\partial \overset{2}{\Gamma}{}^{i'}_{j';k'}}{\partial \overline{z^{s'}}} + 2\frac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z^{s'}}} \delta^{i'}_{j'} = 0. \quad (34)$$

Plunging equations (33) and (34) into equation (21), equation (21) can be simplified to

$$\begin{cases} \dfrac{\partial^2 \ln \lambda_1}{\partial z^k \partial \overline{z^s}} = 0, \end{cases} \quad (35)$$

$$\begin{cases} \dfrac{\partial^2 \ln \lambda_2}{\partial z^{k'} \partial \overline{z^{s'}}} = 0, \end{cases} \quad (36)$$

$$\begin{cases} \dfrac{\partial^2 \ln \lambda_1}{\partial z^k \partial \overline{z^{s'}}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^k \partial \overline{z^{s'}}} = \dfrac{\partial^2 \ln \lambda_1}{\partial z^{k'} \partial \overline{z^{s}}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z^{s}}} = \dfrac{\partial^2 \ln \lambda_1}{\partial z^{k'} \partial \overline{z^{s'}}} \\ \quad + \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z^{s'}}} = 0, \end{cases} \quad (37)$$

$$\begin{cases} \dfrac{\partial^2 \ln \lambda_2}{\partial z^k \partial \overline{z^{s}}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^k \partial \overline{z^{s}}} = \dfrac{\partial^2 \ln \lambda_2}{\partial z^{k} \partial \overline{z^{s'}}} + \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k} \partial \overline{z^{s'}}} = \dfrac{\partial^2 \ln \lambda_2}{\partial z^{k'} \partial \overline{z^{s}}} \\ \quad + \dfrac{\partial^2 \sigma_\alpha}{\partial z^{k'} \partial \overline{z^{s}}} = 0. \end{cases} \quad (38)$$

Equations (35) and (36) mean that $\ln\lambda_1$ and $\ln\lambda_2$ are pluriharmonic functions on M_1 and M_2, respectively. Therefore, $(M_1 \times_{(\lambda_1,\lambda_2)} M_2, F)$ is locally conformally flat if and only if $\ln\lambda_1$ and $\ln\lambda_2$ are pluriharmonic functions on M_1 and M_2, respectively, and equation (32) holds on U_α. □

5. Conclusion

This study considered about conformal transformations of a doubly twisted product complex Finsler manifolds and gave a characterization for doubly twisted product complex Finsler manifolds to be locally conformally flat.

Locally conformally flat doubly warped product complex Finsler manifolds were studied by the work of He [32]. He also obtained the relations of locally conformally flatness between doubly warped product manifolds and its components. This study extended the work due to He, where we obtained the relations of locally conformally flatness between doubly twisted product complex Finsler manifolds and its components. Moreover, Theorem 2 gave an answer to question mentioned above in Section 1, that is, the doubly twisted product of two locally conformally flat complex Finsler manifolds is locally conformally flat when $\ln\lambda_1$ and $\ln\lambda_2$ are pluriharmonic functions on M_1 and M_2, respectively, and equation (32) holds. Our approach to this problem depends on the existence of solutions for system (32). We have been trying to find solutions of the PDE systems, but, unfortunately, we have not found a possible way to get some solutions up to now. Investigating possible solutions for PDE systems will be the subject matter of future works.

Data Availability

Previously reported data were used to support this study and are available at 10.4208/jms.v55n2.22.04.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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