Hindawi
Journal of Mathematics
Volume 2022, Article ID 5224289, 16 pages
https://doi.org/10.1155/2022/5224289

Research Article

Solitary Wave Solutions to the Modified Zakharov–Kuznetsov and the $(2+1)$-Dimensional Calogero–Bogoyavlenskii–Schiff Models in Mathematical Physics

M. Al-Amin,[1] **M. Nurul Islam**,[1] **Onur Alp İlhan**,[2] **M. Ali Akbar**,[3] **and Danyal Soybaş**[2]

[1]*Department of Mathematics, Islamic University, Kushtia, Bangladesh*
[2]*Department of Mathematics, Faculty of Education, Erciyes University, 38039-Melikgazi, Kayseri, Turkey*
[3]*Department of Applied Mathematics, University of Rajshahi, Rajshahi, Bangladesh*

Correspondence should be addressed to M. Ali Akbar; ali.akbar@ru.ac.bd

Received 5 August 2022; Revised 6 October 2022; Accepted 19 October 2022; Published 31 October 2022

Academic Editor: Firdous A. Shah

The modified Zakharov–Kuznetsov (mZK) and the $(2+1)$-dimensional Calogero–Bogoyavlenskii–Schiff (CBS) models convey a significant role to instruct the internal structure of tangible composite phenomena in the domain of two-dimensional discrete electrical lattice, plasma physics, wave behaviors of deep oceans, nonlinear optics, etc. In this article, the dynamic, companionable, and further broad-spectrum exact solitary solitons are extracted to the formerly stated nonlinear models by the aid of the recently enhanced auxiliary equation method through the traveling wave transformation. The implication of the soliton solutions attained with arbitrary constants can be substantial to interpret the involuted phenomena. The established soliton solutions show that the approach is broad-spectrum, efficient, and algebraic computing friendly and it may be used to classify a variety of wave shapes. We analyze the achieved solitons by sketching figures for distinct values of the associated parameters by the aid of the Wolfram Mathematica program.

1. Introduction

Nonlinear evolution equations (NLEEs) are the fundamental tools to model most of the phenomena that arise in engineering, natural sciences, and mathematical physics, such as propagation of shallow water waves, optical fibers, signal processing, condensed matter, electromagnetic, plasma physics, and fluid mechanics. The solitary wave solutions of NLEEs are used to analyze many natural events. This is owing to the fact that NLEEs contain indefinite multivariable functions and their derivatives. Therefore, many researchers have shown their interest in studying and researching on this topic. The traveling waves are the waves which propagate with respect to time and are gained much attention to the recent researchers. Therefore, a large assortment of mathematical methods which are efficient, powerful, and reliable were suggested for findings the exact traveling wave solutions of the nonlinear differential equations (NLDEs) as for instance, the method of exp-function [1], the (G'/G)-expansion method [2–5], the finite difference approach [6], the extended tanh-function approach [7], the Jacobi elliptic function method [8], the Bernoulli sub-equation function [9], the MSE method [10–12], the first integral scheme [13], the variational iteration scheme [14], the modified Kudryashov scheme [15], the Hirota's bilinear form [16], the Shehu transform scheme [17], the modified trial equation approach [18], the fractional sub-equation approach [19], the Laplace transform scheme [20], the sine-Gordon equation (SGE) [21, 22], the modified extended direct algebraic technique [23], the auxiliary equation method [24–27], etc.

The weakly nonlinear ion-acoustic waves in strongly magnetized lossless plasma in two-dimension are described by the Zakharov–Kuznetsov (ZK) equation, was first introduced by Zakharov and Kuznetsov in 1974. But, when more realistic situations arise, the nonisothermal electrons are governed by the ZK equation, the equation is changed into a modified form, known as the modified Zakharov–Kuznetsov (mZK) equation [28–30]. The mZK equation was first introduced by Munro and Parkes in 1999 [28]. The mZK equation is also important in the field of 2D discrete electrical lattice, plasma physics, etc. Bogoyavlenskii and Schiff constructed the Calogero–Bogoyavlenskii–Schiff (CBS) equation, where Bogoyavlenskii used the modified Lax formalism [31]. And, Schiff derived the CBS equation by reducing the self-dual Yang–Mills equation [32]. The $(2+1)$-dimensional CBS equation [33, 34] is useful in research on wave behavior in deep sea, nonlinear optics, and other fields. In addition, the researchers showed that the CBS equation possesses soliton as well as N-soliton solutions which are smooth in one coordinate. In the literature, the modified mZK equation was investigated through the first integral approach [35], the extended tanh-function method [36], the Lie symmetry scheme [37], the exp-function technique [38], the unified method [39], and some other techniques. The modified simple equation approach [40], the Hirota's bilinear approach [41], the improved (G'/G)-expansion scheme [42], the $\exp(-\phi(\zeta))$-expansion method [43], the bilinear method [44], and some other techniques were used to examine the CBS equation.

The auxiliary equation (AE) method is a relatively new technique. It is observed that the AE method is readily applicable to a large variety of NLEEs as well as coupled NLEEs. Moreover, it not only generates regular solutions but also singular ones involving csch and coth functions. Due to this advantage the AE method, it gains considerable interest to the researchers. To our optimum comprehension, the exact solutions of the modified ZK and CBS equations have yet not been developed using the auxiliary equation approach [24–27]. Therefore, the aim of this study is to examine the soliton solutions which means the solutions of a widespread class of weakly NLDEs describing a physical system of the modified Zakharov–Kuznetsov (mZK) equation and the $(2+1)$-dimensional Calogero–Bogoyavlenskii–Schiff (CBS) equation via the auxiliary equation method. With the appropriateness and simplicity of this method, we achieve several realistic and further generic solutions to the equations. Inserting specific values of arbitrary factors various wave solitons are created and these attained solitons are not established in the previous literature. We portray the diagram of attain solutions and illustrate their physical signification.

The organization of this work is as follows: in Section 2, the auxiliary equation method is described. We study the stated nonlinear evolution equations and examine solutions of the equations in Section 3. Explain the physical importance of the obtained solutions in Section 4 and compare the results in Section 5. Finally, the conclusion is presented.

2. Description of the Method

The auxiliary equation (AE) method, which is promising, powerful, and proficient, has been briefly described in this section.

Consider a general higher dimensional NLEE:

$$H\Big(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{yy}, \dots\Big) = 0, \tag{1}$$

where $u = u(t, x, y, z)$ be the unknown wave function and$H$ is the polynomial of $u(t, x, y, z)$ and its diverse partial derivatives, where the utmost order derivatives and nonlinear terms are concerned. By executing the consequent steps we can evaluate the solution of (1) by the aid of the AE method:

Step 1: Suppose the wave variable in the form:

$$u(t, x, y, z) = V(\phi),\ \phi = x + y + z \pm \omega t, \tag{2}$$

where ω denotes wave propagation velocity. The transformation (2) assists us to transform (1) into the following equation:

$$L(V, V', V'', V''', \dots) = 0, \tag{3}$$

where L is a polynomial in $V(\phi)$ and the derivatives for $V(\phi)$, in which $V'(\phi) = \mathrm{d}V/\mathrm{d}\phi$.

Step 2: Here, Equation (3) could be integrated one or more times as per possibility.

Step 3: As per AE method the subsequent solution of Equation (3) can be revealed as follows:

$$u(\phi) = \sum_{i=0}^{N} c_i a^{\mathrm{if}(\phi)}. \tag{4}$$

Here, the constants c_i, a have to evaluated, where $c_N \neq 0$ (N is a positive integer) and $f(\phi)$ satisfies the subsequent auxiliary equation:

$$f'(\phi) = \frac{1}{\ln a}\left\{\mathrm{la}^{-f(\phi)} + m + \mathrm{na}^{f(\phi)}\right\}. \tag{5}$$

The prime species the derivative with regard to ϕ; l, m, n are parameters.

Here, the transformation $\psi = a^{f(\xi)}$ converts the auxiliary equation Equation (5) to a Riccati equation in ψ as follows:

$$\psi' + n\psi^2 + m\psi + l = 0. \tag{6}$$

The Riccati equation (6) gives a set of generic solutions that provide the exact solutions to the nonlinear equation (3). It is worth noting that series extension of (4) is a particular type of rational function and also a polynomial. Therefore, an interrelation was found between the considered approach and the transformed rational function technique. The AE method is used in the present study since it is straightforward, easy to compute, and adaptable to user-friendly computation software like Maple to examine closed-form soliton solutions.

Step 4: The value of N seem in (4) can be acquired by considering the homogeneous balance between the utmost order exponent and the derivative occurring in (3).

Step 5: Inserting Equation (4) jointly with (5) into (3) and the score of N calculated in Step 4 yields polynomial in $a^{\mathrm{if}(\phi)}$, $(i = 1, 2, 3 \ldots)$. Collecting all the terms of like power of $a^{\mathrm{if}(\phi)}$, and equating the coefficient to zero yield a class of equations (algebraic) for c_i, ω, l, m and n and solving this set of equations, we get the values of the indefinite parameters. Since the solution of Equation (5) is identified, putting the values of $c_i\,(i = 1, 2, 3 \ldots), l, m$ and n in Equation (4), we attain wide-ranging, more comprehensive and newer closed-form solution of Equation (3).

Step 6: Equation (5) delivers different generic solutions of NLDEs for the different values of l, m and n as well as their interrelation and provide the exact solutions of nonlinear evolution (3). Thus, it can be determined adequate exact solutions to the nonlinear models using the AE method.

The AE method is used to examine the exact soliton solutions of NLEEs. The key idea of this method is to take full advantage of NLEEs which yields useful, fresher, and further general exact wave solutions. Principally, the AE method is a direct algebraic method, effective algorithm, further generalized to establish several exact traveling wave solutions for NLEEs and can be utilized to arrange the wave velocity. Furthermore, by this method, some physically important of NLEEs are investigated with the aid of symbolic computation.

3. Mathematical Analysis of the Wave Solutions

In this part, we establish scores of advanced, standard, and wide-spectral closed-form traveling wave solutions to the modified Zakharov–Kuznetsov (mZK) equation and the (2 + 1)-dimensional Calogero–Bogoyavlenskii–Schiff (CBS) equation by executing the thriving AE method.

3.1. The mZK Model. Herein, we examine a variety of newer closed-form soliton solutions to the mZK model by using the AE method. This is a noteworthy model which has vast applications in engineering and mathematical physics, namely, the effects of ionic temperature, ion density gradient, presence of third species (dust), composite phenomena in the domain of two-dimensional discrete electrical lattice, oblique propagation, and others. The mZK model is an integrable model which is in the form [28–30]:

$$u_t + u^2 u_x + u_{xxx} + u_{xyy} = 0. \tag{7}$$

Using the compound transformation $u(\psi) = U(t, x, y)$, where $\psi = x + y - \omega t$, (7) turns into a nonlinear equation and integrating, it becomes the subsequent form:

$$-\omega U + U^3/3 + 2U'' = 0. \tag{8}$$

By means of homogeneous balancing of the uppermost order nonlinear and linear terms appearing in (8), we find $N = 1$. Therefore, the solution structure of (8) is as follows:

$$u = c_0 + c_1 a^{f(\psi)}. \tag{9}$$

Inserting (9) jointly with (5) and (8) and gathering all the terms of similar power of $a^{\mathrm{if}(\psi)}$ and equating the coefficients to zero yields a class of (algebraic) equations (these are not shown for simplicity) and addressing these via the computation software Maple, gives the solutions:

$$\begin{aligned} c_0 &= \pm m\sqrt{-3}, \\ c_1 &= \pm 2n\sqrt{-3}, \\ \omega &= 4\ln - m^2, \end{aligned} \tag{10}$$

where ω is traveling wave velocity and l, m, n are constants.

Substituting the values assembled in (10) into (9) and applying the conditions of the AE method, we establish the subsequent solutions of (7):

While $m^2 - 4\ln < 0$ and $n \neq 0$, the resulting solutions are as follows:

$$v(\psi) = \pm\sqrt{3}\left(\sqrt{m^2 - 4\ln}\tan\left(\frac{\sqrt{4\ln - m^2}}{2}\psi\right)\right), \tag{11}$$

$$v(\psi) = \pm\sqrt{3}\left(\sqrt{m^2 - 4\ln}\cot\left(\frac{\sqrt{4\ln - m^2}}{2}\psi\right)\right). \tag{12}$$

When $m^2 - 4\ln > 0$ and $n \neq 0$, the obtain the solutions are

$$v(\psi) = \mp\sqrt{3}\left(\sqrt{4\ln - m^2}\tan h\left(\frac{\sqrt{m^2 - 4\ln}}{2}\psi\right)\right), \tag{13}$$

$$v(\psi) = \mp\sqrt{3}\left(\sqrt{4\ln - m^2}\cot h\left(\frac{\sqrt{m^2 - 4\ln}}{2}\psi\right)\right). \tag{14}$$

For $m^2 + 4l^2 < 0$, $n \neq 0$ and $n = -l$, gives the solutions.

$$v(\psi) = \pm\sqrt{3}\left(\sqrt{(m^2 + 4l^2)}\tan\left(\frac{\sqrt{-(m^2 + 4l^2)}}{2}\psi\right)\right), \tag{15}$$

$$v(\psi) = \pm\sqrt{3}\left(\sqrt{(m^2 + 4l^2)}\cot\left(\frac{\sqrt{-(m^2 + 4l^2)}}{2}\psi\right)\right). \tag{16}$$

When $m^2 + 4l^2 > 0$, $n \neq 0$ and $n = -l$, we attain the solutions

$$v(\psi) = \mp\sqrt{3}\left(\sqrt{-(m^2 + 4l^2)}\tan h\left(\frac{\sqrt{(m^2 + 4l^2)}}{2}\psi\right)\right), \tag{17}$$

$$v(\psi) = \mp\sqrt{3}\left(\sqrt{-(m^2 + 4l^2)}\cot h\left(\frac{\sqrt{(m^2 + 4l^2)}}{2}\psi\right)\right). \tag{18}$$

While $m^2 - 4l^2 < 0$ and $n = l$, we achieve the solution.

$$v(\psi) = \pm\sqrt{3}\left(\sqrt{\left(m^2-4l^2\right)}\tan\left(\frac{\sqrt{-\left(m^2-4l^2\right)}}{2}\psi\right)\right), \quad (19)$$

$$v(\psi) = \mp\sqrt{3}\left(\sqrt{\left(m^2-4l^2\right)}\cot\left(\frac{\sqrt{-\left(m-4l^2\right)}}{2}\psi\right)\right). \quad (20)$$

By means of $m^2 - 4l^2 > 0$ and $n = l$, leads the solutions.

$$v(\psi) = \mp\sqrt{3}\left(\sqrt{-\left(m^2-4l^2\right)}\tan h\left(\frac{\sqrt{-\left(m^2-4l^2\right)}}{2}\psi\right)\right), \quad (21)$$

$$v(\psi) = \mp\sqrt{3}\left(\sqrt{-\left(m^2-4l^2\right)}\cot h\left(\frac{\sqrt{-\left(m^2-4l^2\right)}}{2}\psi\right)\right). \quad (22)$$

While $m^2 = 4$ ln, the wave velocity becomes zero, therefore, the soliton solution does not exist in this case.

When ln < 0, $m = 0$ and $n \neq 0$, we derive,

$$v(\psi) = \pm i\sqrt{3}\left\{m + 2n\left(-\sqrt{-l/n}\tan h\left(\sqrt{-\ln}\,\psi\right)\right)\right\}, \quad (23)$$

$$v(\psi) = \pm i\sqrt{3}\left\{m + 2n\left(-\sqrt{-l/n}\cot h\left(\sqrt{-\ln}\,\psi\right)\right)\right\}. \quad (24)$$

For $m = 0$ and $l = -n$, we ascertain the soliton.

$$v(\psi) = \pm i\sqrt{3}\left(m - 2l\left(\frac{1+e^{2l\psi}}{-1+e^{2l\psi}}\right)\right). \quad (25)$$

When $l = n = 0$, the soliton reach into trivial form and it is physical significance less. So the solution dropped here.

While $l = m = f$ and $n = 0$, we achieve a trivial soliton solution and that is negligible here.

For $m = n = f$ and $l = 0$, the solutions turns into,

$$v(\psi) = \pm i\sqrt{3}\left(m + 2f\left(\frac{e^{f\psi}}{1-e^{f\psi}}\right)\right). \quad (26)$$

When $m = l + n$, we get a trivial solution, which is not represented here.

For $m = -(l+n)$, we attain the trivial solution, therefore it is omitted here because it has no physical significance.

When $l = 0$, we obtain the exponential function solution which is significant.

$$v(\psi) = \pm i\sqrt{3}\left(m + 2n\left(\frac{\mathrm{me}^{m\psi}}{1-\mathrm{ne}^{m\psi}}\right)\right). \quad (27)$$

For $n = m = l \neq 0$, we accomplish the solutions.

$$v(\psi) = \pm i\sqrt{3}\left\{m + 2l\left(\frac{1}{2}\left(\sqrt{3}\tan\left(\frac{\sqrt{3}}{2}l\psi\right)-1\right)\right)\right\}. \quad (28)$$

For $n = l = 0$ and $l = m = 0$, the obtained solution is in trivial form and therefore omitted here.

While $n = l$ and $m = 0$, we determine.

$$v(\psi) = \pm i\sqrt{3}\left(m + 2l\left(\tan\left(l\psi\right)\right)\right). \quad (29)$$

When $n = 0$, we achieve a trivial form soliton solution, which is also skipped here.

In all solutions, $\psi = x + y - \omega t$, where ω is the traveling wave velocity and l, m, n are free parameters.

It is remarkable to observe that the established wave solutions of the considered models which are further general and for the individual values of the arbitrary parameters some exact solutions are reinstated and some other solutions are ascertained which are not found in the earlier study.

3.2. The CBS Model. In this subsection, we determine novel, useful, and further general closed-form solitary wave solutions to the $(2+1)$-dimensional CBS model by the aid of the auxiliary equation method. The model has a vital role in mathematical physics and engineering such as, explore several nonlinear dynamics of interaction phenomena in fluids and plasmas fields, nonlinear wave in optics, wave behaviors of deep oceans, and more. The CBS model [33, 34] which is an integrable model is as follows:

$$u_{xt} + 4u_x u_{xy} + 2u_{xx}u_y + u_{xxxy} = 0. \quad (30)$$

The traveling wave variable $u(\xi) = V(x, y, t)$, where $\xi = x + y - \omega t$ converts the model (30) into a nonlinear equation and after integration with zero integral constant it turns into the following equation:

$$-\omega V' + 3\left(V'\right)^2 + V' = 0. \quad (31)$$

Now, using the balance number $N = 1$, the form of the solution of (30) is written as follows:

$$v = c_0 + c_1 a^{f(\xi)}. \quad (32)$$

Setting the solution (32), and (5) into (31) and summing up the coefficients of identical powers of $a^{\mathrm{if}(\xi)}$ and assigning them to zero, we find out a set of algebraic equations (for simplicity which are not assembled here) for c_0, c_1, l, m, n. Unraveling the system of equations (algebraic) by applying computation software (Maple), it provides the following solutions:

$$\begin{aligned} c_0 &= 0,\\ c_1 &= -2n,\\ \omega &= 4\ \ln - m^2, \end{aligned} \quad (33)$$

where ω is wave velocity and l, m, n are free parameters.

Now, applying the results (33) and (32) and by means of the constraints on the free parameters, we establish the following solutions of (30):

While $m^2 - 4$ ln < 0 and $n \neq 0$, we attain the singular periodic solutions:

$$v(\xi) = m - \sqrt{4\ \ln - m^2}\tan\left(\frac{\sqrt{4\ \ln - m^2}}{2}\xi\right), \quad (34)$$

$$v(\xi) = m - \sqrt{4\ \ln - m^2}\cot\left(\frac{\sqrt{4\ \ln - m^2}}{2}\xi\right). \quad (35)$$

When $m^2 - 4$ ln > 0 and $n \neq 0$, we attain kink and singular kink shape solutions.

$$v(\xi) = m + \sqrt{m^2 - 4\,\ln}\,\tan h\left(\frac{\sqrt{m^2 - 4\,\ln}}{2}\xi\right), \quad (36)$$

$$v(\xi) = m + \sqrt{m^2 - 4\,\ln}\,\cot h\left(\frac{\sqrt{m^2 - 4\,\ln}}{2}\xi\right). \quad (37)$$

For $m^2 + 4l^2 < 0$, $n \neq 0$ and $n = -l$, we reach the soliton solutions.

$$v(\xi) = m - \sqrt{-\left(m^2 + 4l^2\right)}\tan\left(\frac{\sqrt{-\left(m^2 + 4l^2\right)}}{2}\xi\right), \quad (38)$$

$$v(\xi) = m + \sqrt{-\left(m^2 + 4l^2\right)}\cot\left(\frac{\sqrt{-\left(m^2 + 4l^2\right)}}{2}\xi\right). \quad (39)$$

When $m^2 + 4l^2 > 0$, $n \neq 0$ and $n = -l$, we achieve the solutions.

$$v(\xi) = m + \sqrt{\left(m^2 + 4l^2\right)}\tan h\left(\frac{\sqrt{\left(m^2 + 4l^2\right)}}{2}\xi\right), \quad (40)$$

$$v(\xi) = m + \sqrt{\left(m^2 + 4l^2\right)}\cot h\left(\frac{\sqrt{\left(m^2 + 4l^2\right)}}{2}\xi\right). \quad (41)$$

While $m^2 - 4l^2 < 0$ and $n = l$, the soliton reach into,

$$v(\xi) = m - \sqrt{-\left(m^2 - 4l^2\right)}\tan\left(\frac{\sqrt{-\left(m^2 - 4l^2\right)}}{2}\xi\right), \quad (42)$$

$$v(\xi) = m + \sqrt{-\left(m^2 - 4l^2\right)}\cot\left(\frac{\sqrt{-\left(m^2 - 4l^2\right)}}{2}\xi\right). \quad (43)$$

By means of $m^2 - 4l^2 > 0$ and $n = l$, gives the solutions.

$$v(\xi) = m + \sqrt{\left(m^2 - 4l^2\right)}\tan h\left(\frac{\sqrt{-\left(m^2 - 4l^2\right)}}{2}\xi\right), \quad (44)$$

$$v(\xi) = m + \sqrt{\left(m^2 - 4l^2\right)}\cot h\left(\frac{\sqrt{-\left(m^2 - 4l^2\right)}}{2}\xi\right). \quad (45)$$

While $m^2 = 4\,\ln$, the wave velocity becomes zero, so the soliton solution does not exist for this case.

When $\ln < 0$, $m = 0$ and $n \neq 0$, we attain the standard kink and singular kink soliton.

$$v(\xi) = 2n\sqrt{-l/n}\tan h\left(\sqrt{-\ln}\;\xi\right), \quad (46)$$

$$v(\xi) = 2n\sqrt{-l/n}\cot h\left(\sqrt{-\ln}\;\xi\right). \quad (47)$$

For $m = 0$ and $l = -n$, we obtain exponential function solutions.

$$v(\xi) = 2l\frac{1 + e^{2l\xi}}{-1 + e^{2l\xi}}. \quad (48)$$

When $l = n = 0$, or $l = m = f$ and $n = 0$, introducing a trivial solution thus has no physical significance.

For $m = n = f$ and $l = 0$, we accomplish the following solution:

$$v(\xi) = -2f\frac{e^{f\xi}}{1 - e^{f\xi}}. \quad (49)$$

When $m = l + n$, and/or $m = -(l + n)$, we have the afterwards trivial soliton solution and that is negligible.

While $l = 0$, the solution turns into,

$$v(\xi) = -2n\left(\frac{\mathrm{me}^{m\xi}}{1 - \mathrm{ne}^{m\xi}}\right). \quad (50)$$

For $n = m = l \neq 0$, we accomplish the solution.

$$v(\xi) = -2l\left(\frac{(\sqrt{3}\,\tan(l\xi\sqrt{3}/2) - 1)}{2}\right). \quad (51)$$

For $n = m = 0$, the resulting soliton is a trivial solution and has no physically significance. So the result dropped here.

When $l = m = 0$, we get the trivial solution, which is not essential to show here.

While $n = l$ and $m = 0$, we determine

$$v(\xi) = -2l\,\tan(l\xi). \quad (52)$$

When $n = 0$, the established soliton is not present here because the resulting solution is physically significance less.

For all solutions, $\xi = x + y - \omega t$, where ω is traveling wave velocity and l, m, n are indefinite constants.

The above-established solutions of the considered CBS model are further general and the particular values for the associated constants some exact solutions are available in the literature which are abundant novel and not found in the previous research.

4. The Graphical Representations and Physical Description

In this part, we will demonstrate the 3-dimensional and contour structure for the obtained solitons of the models using the Wolfram Mathematica program to visualize the internal mechanism of them. For simplicity, some figures of the gained solitons are depicted and some are skipped here.

4.1. The Graphical Representations. Here, we have demonstrated the diverse nature of attained solitons graphically of the considered nonlinear evolution equations for dissimilar values of the integrated parameters inside appropriate interval but the values of traveling wave velocity ω varies. The influence of wave velocity is exposed in the following shapes.

The profiles of the wave solution (11) for diverse traveling wave velocity ω are shown in Figures 1–3.

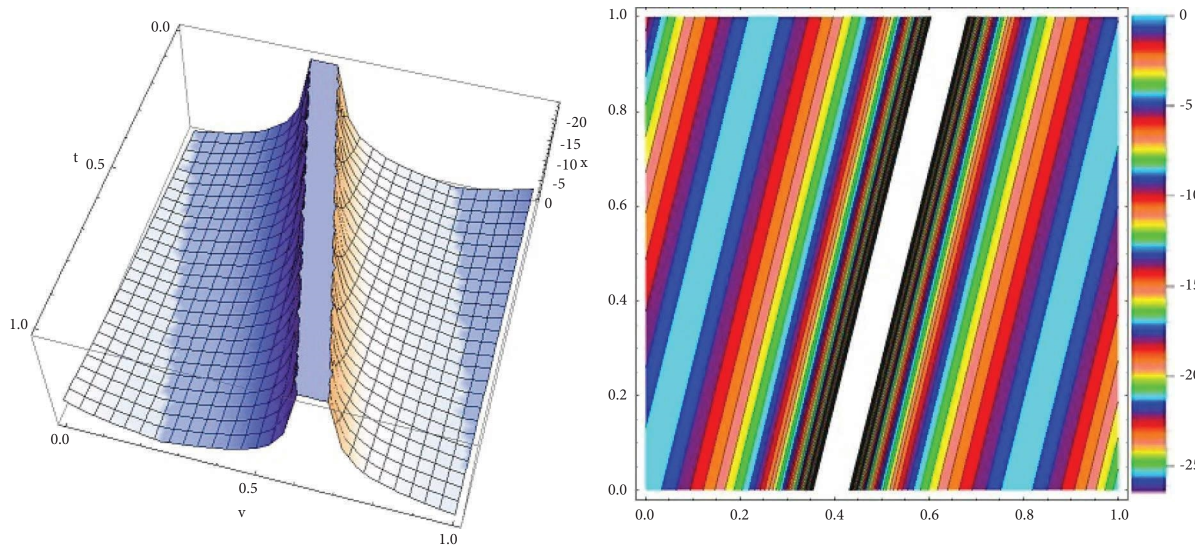

Figure 1: Sketched the singular bell shape soliton from (11) when $\omega = 5.0$.

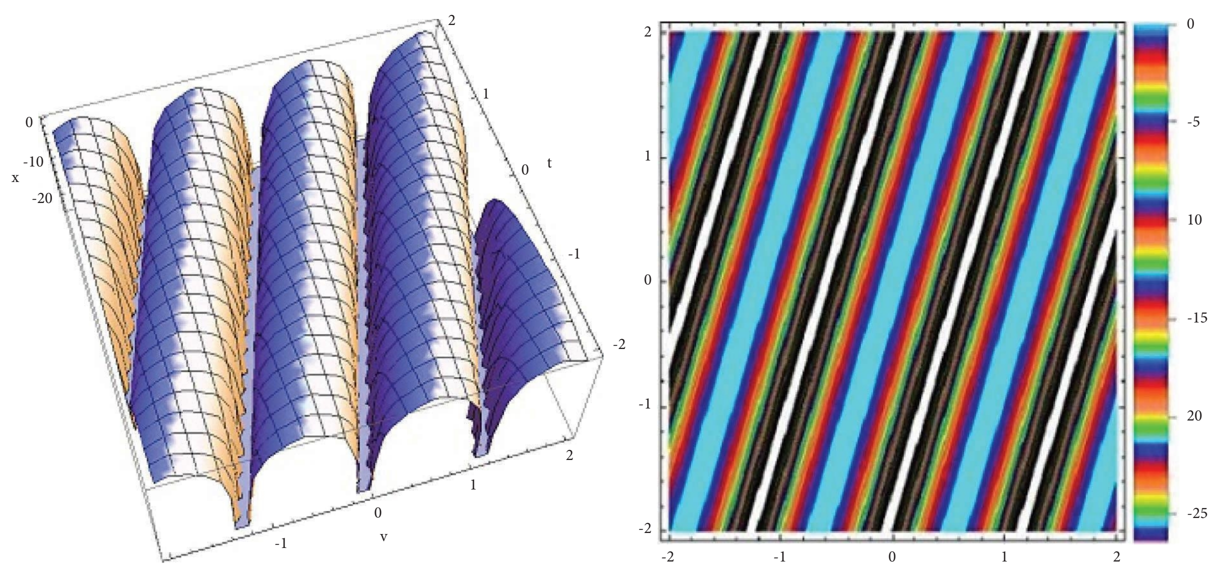

Figure 2: Sketched the singular periodic soliton from (11) when $\omega = 3.0$.

The above-given figures of (11) represent singular bell shape soliton and this shape is depicted for the value of the traveling wave velocity $\omega = 5.0$ in the interval $0 \le x, t \le 1$ and the contour graph is depicted at $t = 0$ for the same velocity and shown in Figure 1. But for the velocity decreasing, i.e., $\omega = 3.0$, the solution (11) displays the singular periodic soliton in the interval $-2 \le x, t \le 2$. The contour graph of this solution for the velocity $\omega = 3.0$ is depicted at $t = 0$ and portrayed in Figure 2. Again when the traveling wave velocity is gradually tends to zero i.e., $\omega = 0.8$, the compacton type soliton of (11) is demonstrated inside the range $-2 \le x, t \le 2$. The contour graph of this solution for the velocity $\omega = 0.8$ is depicted at $t = 0$ and sketched in Figure 3.

The profiles of the wave solution (13) for different traveling wave velocities ω are depicted in Figures 4–7.

It is observed from the above-given analysis, the background of the soliton (13) is a spike for the traveling wave velocity $\omega = 44.53$ portrayed surrounded by interval $-1 \le x, t \le 1$. The contour graph of this soliton for the velocity $\omega = 44.53$ is depicted at $t = 0$ and displayed in Figure 4. But, when the wave velocity is decreased i.e., $\omega = 3.43$, the solution function (13) represents singular bell shape soliton within $-2 \le x, t \le 2$ and for this velocity the contour shape of the solution is depicted at $t = 0$ and presented in Figure 5. When the wave velocity is further decreased i.e., $\omega = 2.25$, the solution function (13) represents bell shape soliton inside range $-2 \le x, t \le 2$ and also the contour graph is depicted at $t = 0$ (for same velocity) in Figure 6. Again, if the wave velocity gradually tends to zero i.e., $\omega = 0.25$, the solution (13) represents the singular

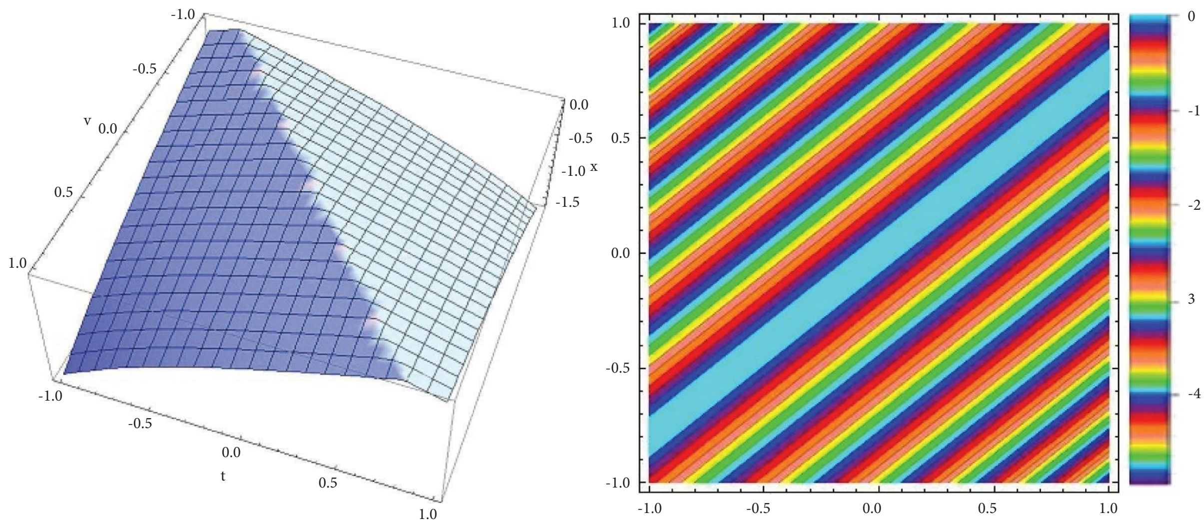

Figure 3: Sketched the compacton type soliton from (11) when $\omega = 0.8$.

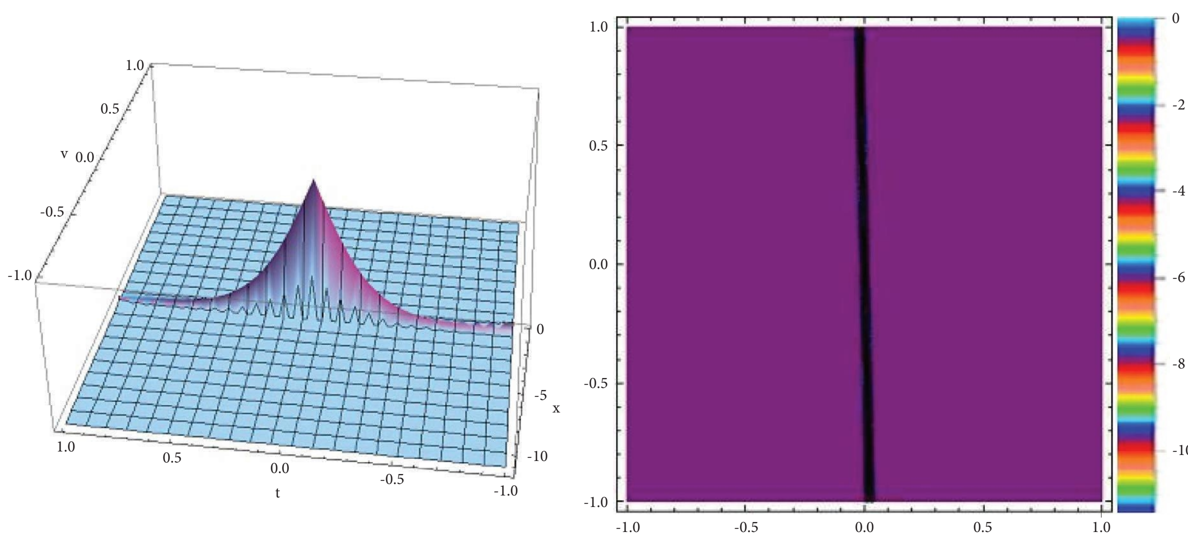

Figure 4: Design3D and contour shape of (13) for $\omega = 44.53$.

periodic soliton in the period $-10 \leq x, t \leq 10$ and the contour design of this solution for the same velocity is drawn at $t = 0$ and shown in Figure 7.

The profiles of the traveling wave solution (34) for diverse wave velocity ω are given as follows.

The soliton (34) represents the singular kink soliton for the wave velocity $\omega = 3.2$ portrayed within the interval $-1 \leq x, t \leq 1$. The graph of the contour of the soliton for the wave velocity $\omega = 3.2$ is depicted at $t = 0$ and reported in Figure 8. When the traveling wave velocity is $\omega = 3$, the solution (34) represents a singular periodic surrounded by $-2 \leq x, t \leq 2$ and its contour graph of the soliton is designated in Figure 9 for the same velocity $\omega = 3$. Also, when the wave velocity is $\omega = 1$, the solution (34) also represents singular periodic in $-8 \leq x, t \leq 8$ and in this case contour shape is depicted in Figure 10 at $t = 0$.

The profiles of the wave solution (36) for different traveling wave velocity ω are given as follows.

The solution (36) represents kink type soliton for the traveling wave velocity $\omega = 12$ shown inside the range $-2 \leq x, t \leq 2$. The shape (contour) of the soliton is depicted at $t = 0$ for the velocity $\omega = 12$ and documented in Figure 11. But, for decreasing wave velocity, i.e., $\omega = 2.25$, the solution (36) represents the flat kink shape soliton in the space $-2 \leq x, t \leq 2$. The graph of the solution (contour) for the wave velocity $\omega = 2.25$ is depicted at $t = 0$ and displayed in Figure 12. Again, when the wave velocity is gradually tends to zero i.e., $\omega = 0.44$, the solution (36) represents general soliton within the space $-2 \leq x, t \leq 8$. The contour figure for the traveling wave velocity $\omega = 0.44$ of solution is represented at $t = 0$ and indicated in Figure 13.

The profiles of the wave solution (44) for various traveling wave velocity ω are given as follows.

When the traveling wave velocity is $\omega = 0.84$, the solution (44) stand for singular periodic soliton contained by the space $-8 \leq x, t \leq 8$. The contour graph of this solution

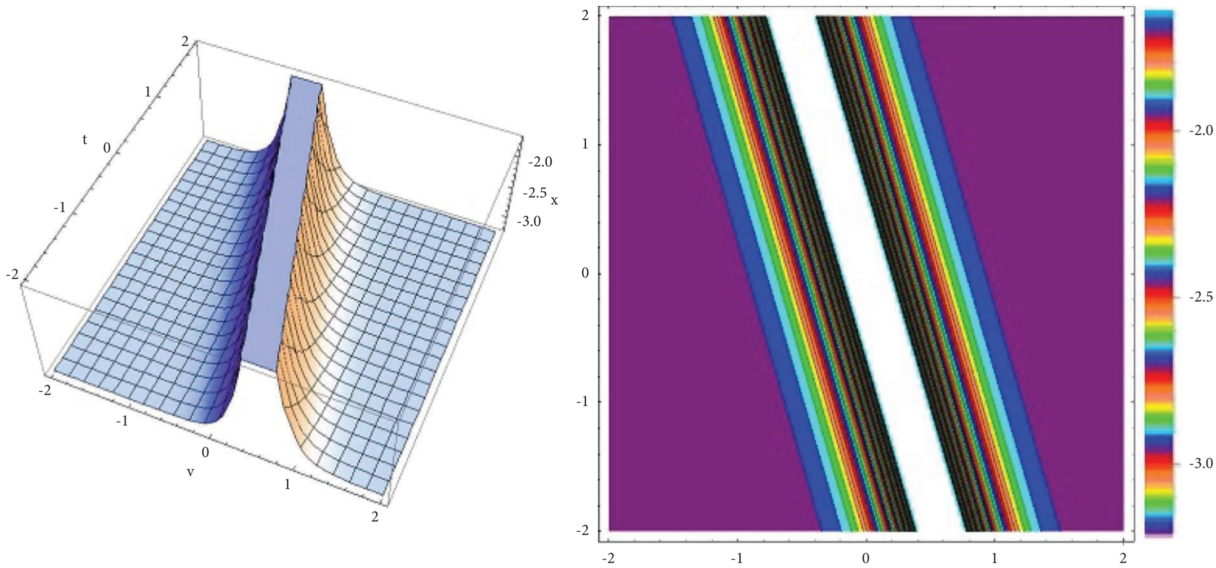

Figure 5: Singular bell shape soliton drawn from (13) for. $\omega = 3.43$.

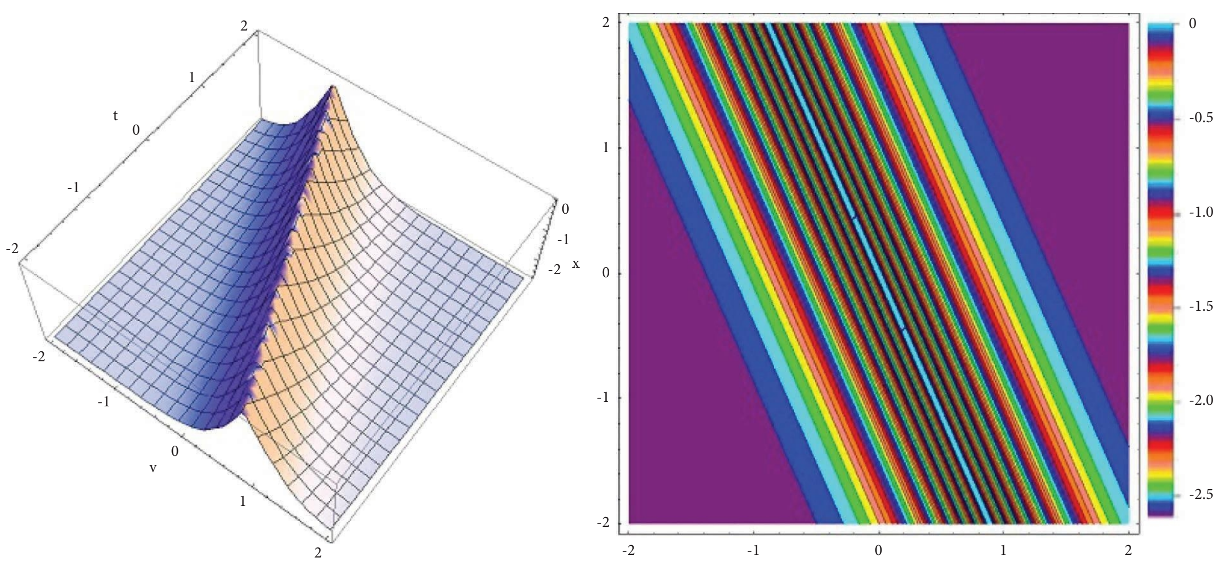

Figure 6: Sharp bell shape soliton drawn from (13) for $\omega = 2.25$.

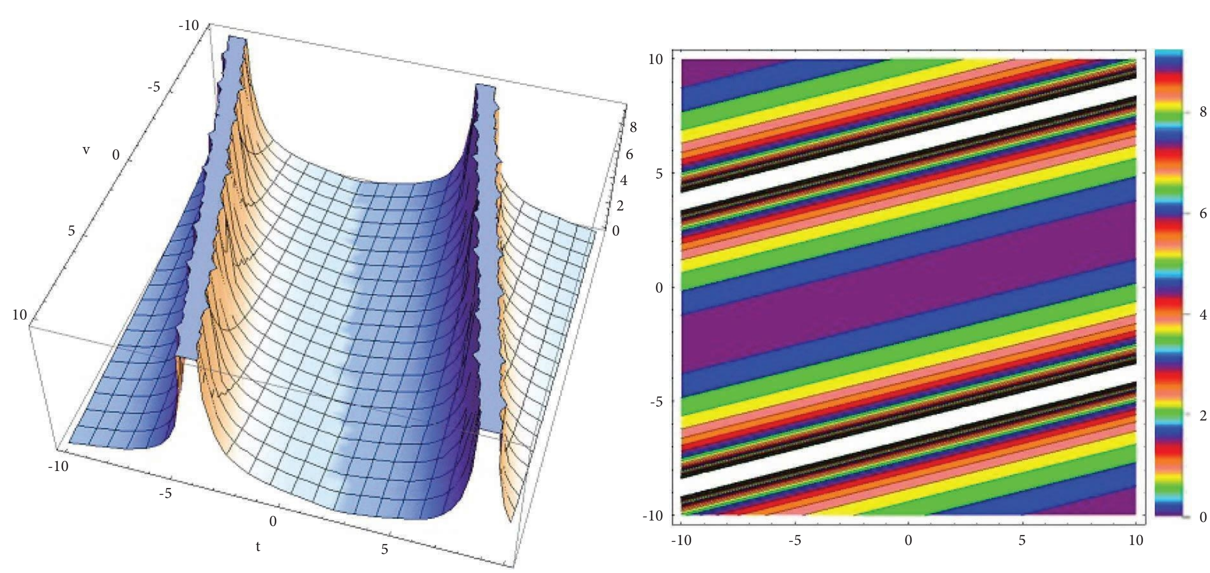

Figure 7: Singular periodic soliton sketched from (13) for $\omega = 0.25$.

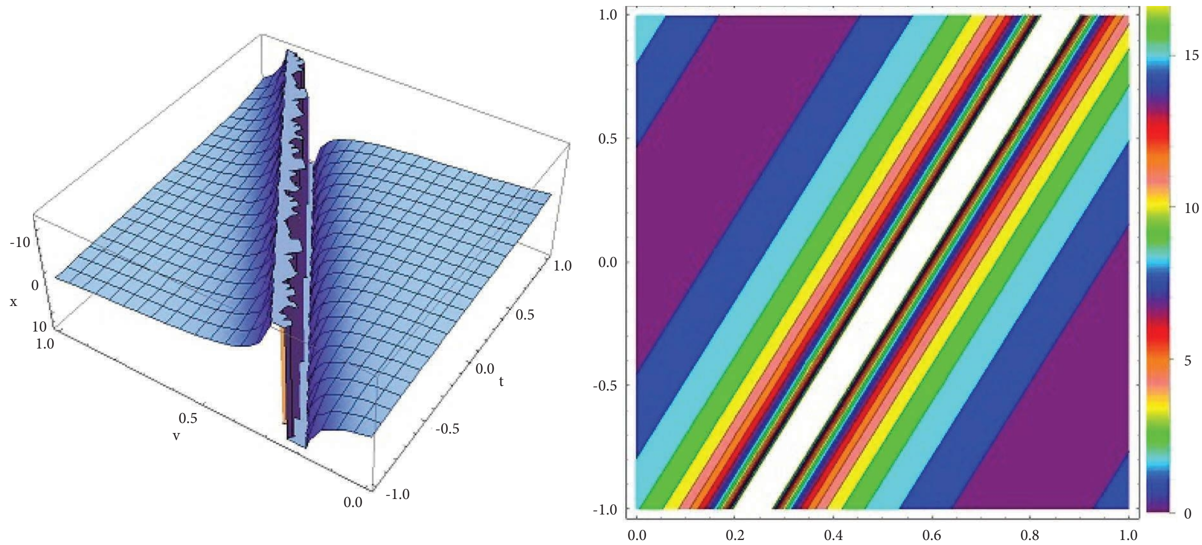

Figure 8: Exact periodic soliton depicted from (34) for $\omega = 3.2$.

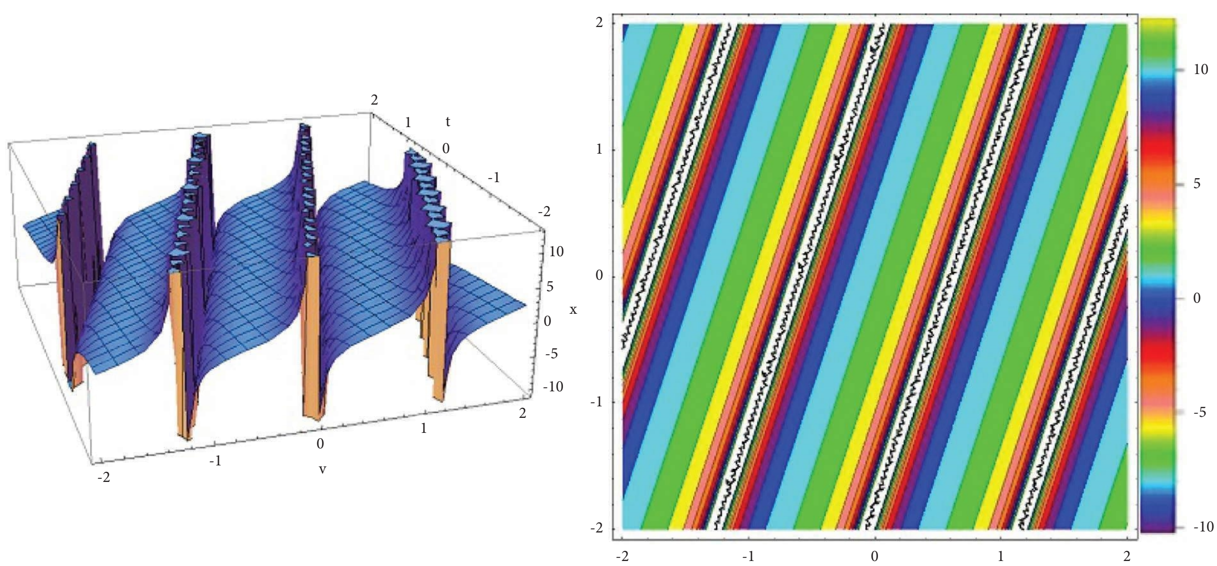

Figure 9: Multiple singular periodic soliton depicted from (34) for $\omega = 3$.

for the wave velocity $\omega = 0.84$ is depicted at $t = 0$ and portrayed in Figure 14. If the wave velocity gradually tends to zero i.e., $\omega = 0.004$, the solution (44) represents a concave parabolic wave contained by the range $-8 \leq x, t \leq 8$. The graph of this soliton (contour) for the wave velocity $\omega = 0.004$ is depicted at $t = 0$ and documented in Figure 15.

The profiles of the wave solution (47) for various traveling wave velocity ω are given as follows.

The solution (47) signifies singular soliton for the traveling wave velocity $\omega = 10$ described in the range $-2 \leq x, t \leq 2$. The figure of this solution (contour) for the wave velocity $\omega = 10$ is depicted at $t = 0$ and displayed in Figure 16. When the wave velocity decreases i.e., $\omega = 4$, the solution (47) represents a singular kink type soliton within the period $-2 \leq x, t \leq 2$. The contour design of this soliton for the traveling wave velocity $\omega = 4$ is depicted at $t = 0$ and specified in Figure 17. Besides, when the wave velocity gradually decreases $(i, e\, \omega = 2)$, the solution (47) represents a singular anti-bell soliton within $-8 \leq x, t \leq 8$ and contour graph of this solution for the same velocity is depicted at $t = 0$ and specified in Figure 18.

4.2. The Physical Significance of the Established Wave Solutions. The Presented contour and 3D graphical depiction of the established solutions of contemplated models might be practical to explicate the internal contrivances of the phenomena related with the considered models. For particular values of the traveling wave velocity, we obtain the diverse shape of solitons and outlined in the graphical representation section. Furthermore, all of the attained solutions exit numerous traveling wave solutions which are of key significance in elucidating several physical circumstances which are useful in the field of wave

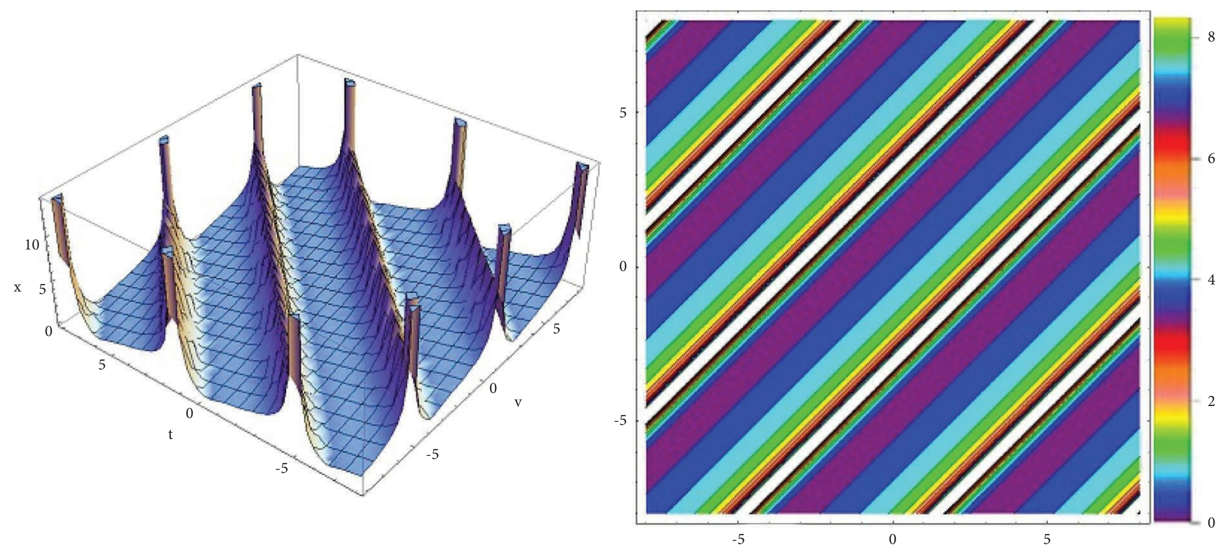

Figure 10: Singular periodic soliton depicted from (34) for $\omega = 1$.

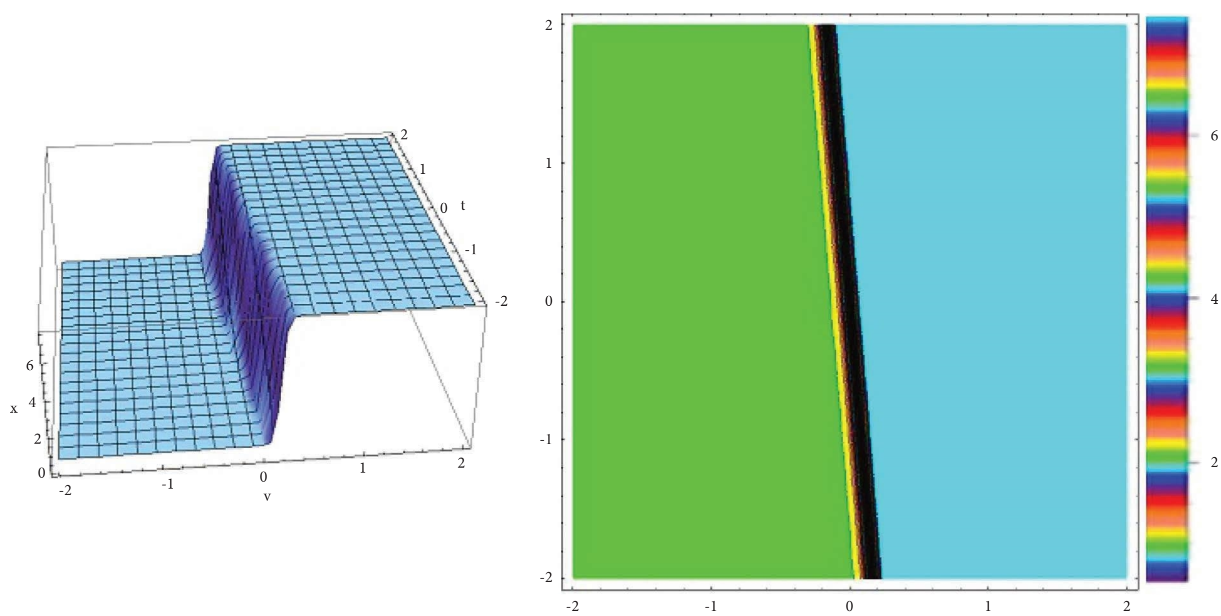

Figure 11: Kink soliton drawing from (36) for $\omega = 12$.

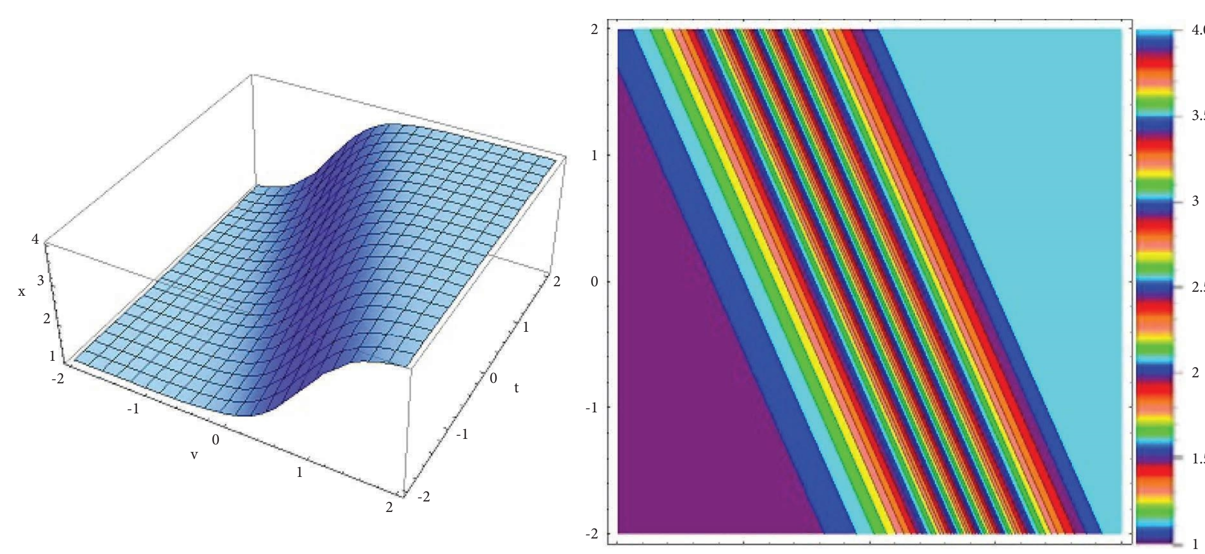

Figure 12: Flat kink shape soliton drawing from (36) for $\omega = 2.25$.

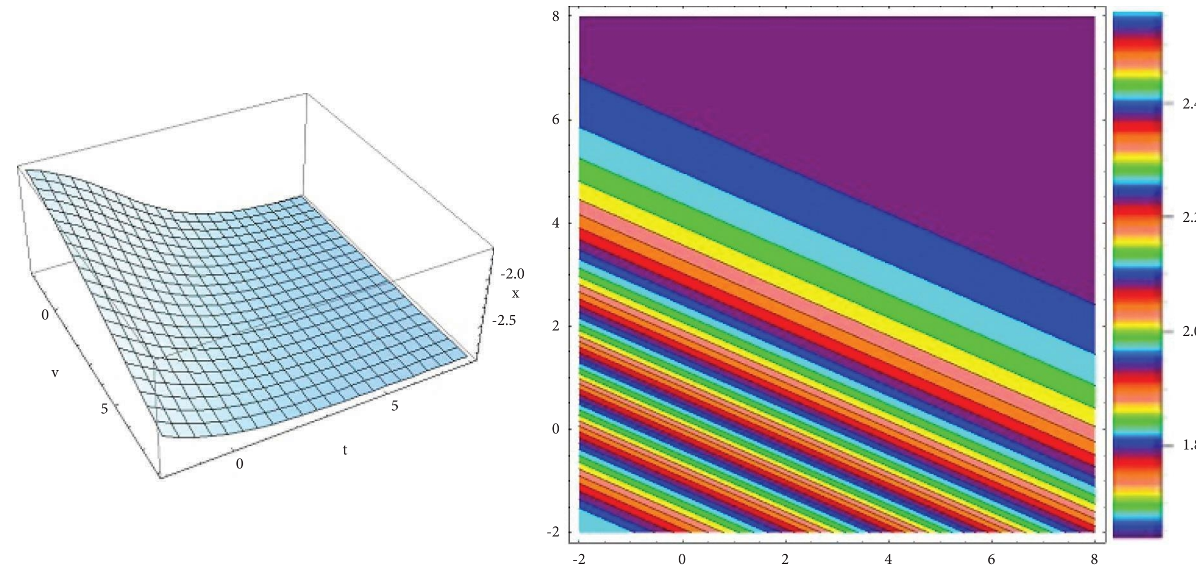

Figure 13: 3D and contour plot soliton drawing from (36) for $\omega = 0.44$.

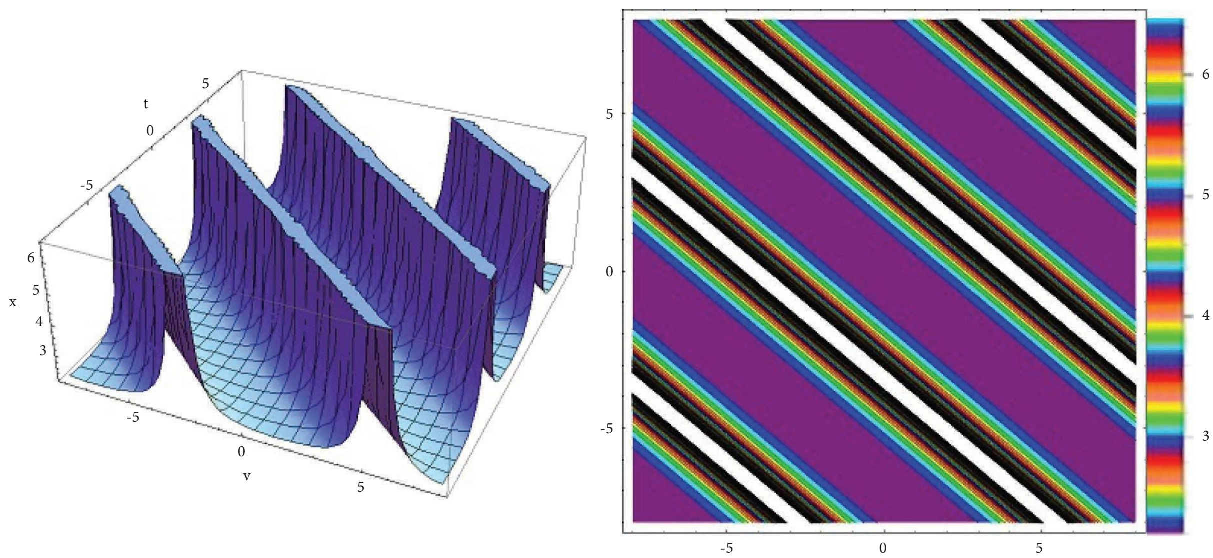

Figure 14: Singular periodic soliton sketching from (44) for $\omega = 0.84$.

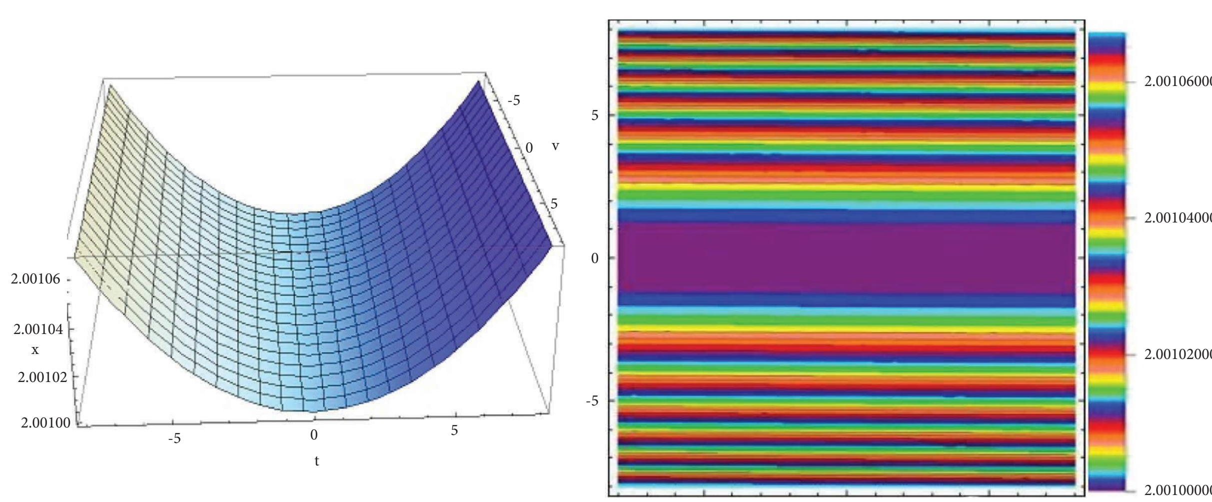

Figure 15: Concave parabolic wave soliton of (44) for $\omega = 0.004$.

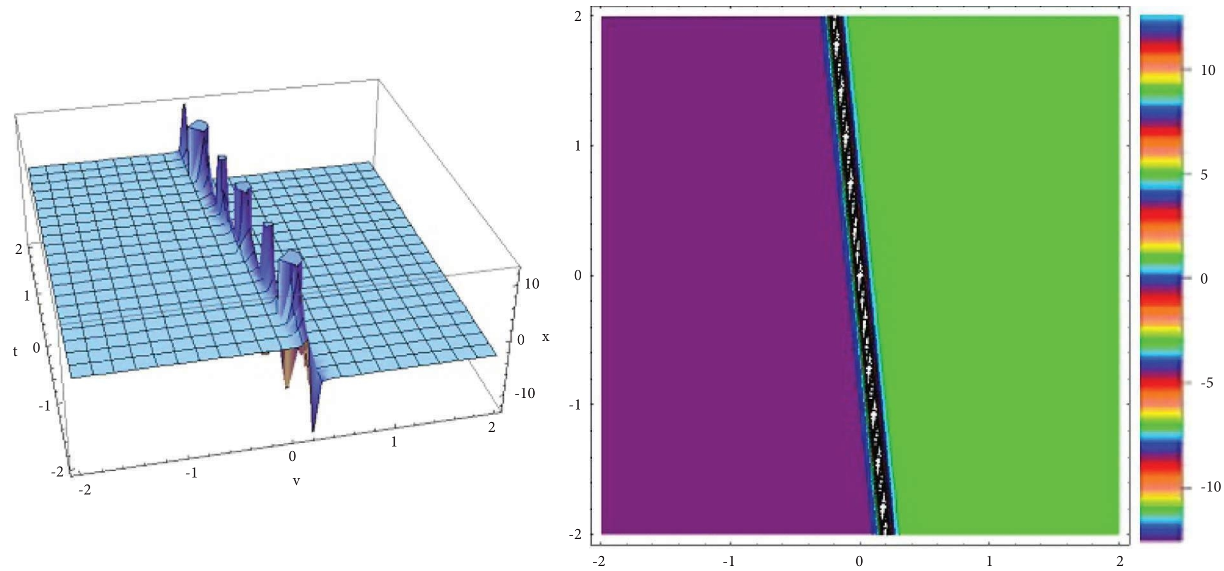

Figure 16: Desing singular soliton of (47) for = 10.

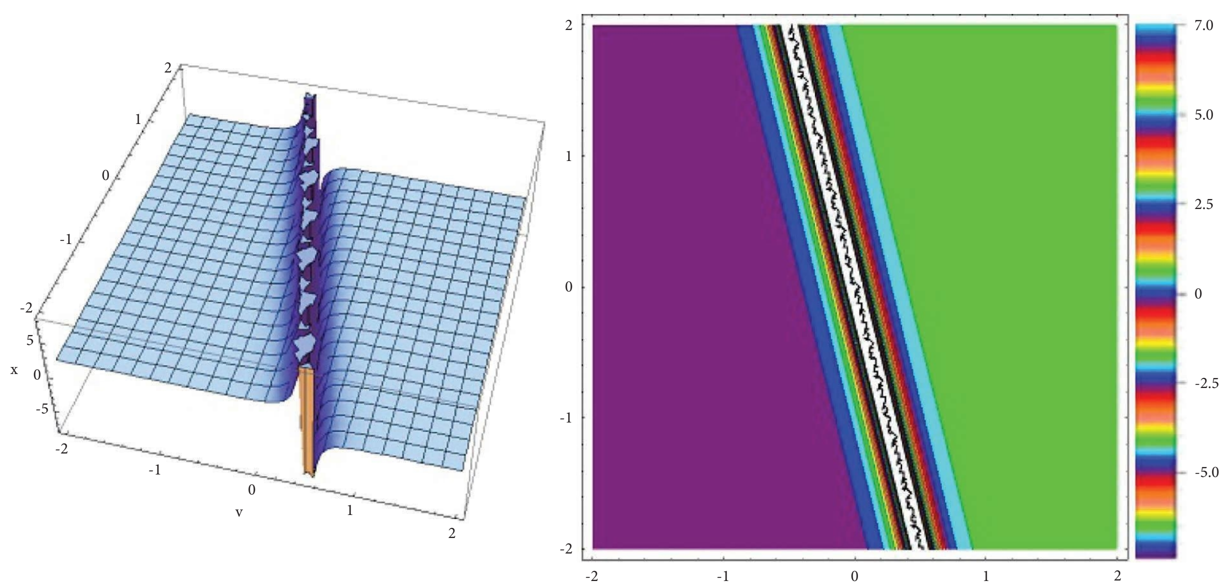

Figure 17: Desing of singular soliton of (47) for = 4.

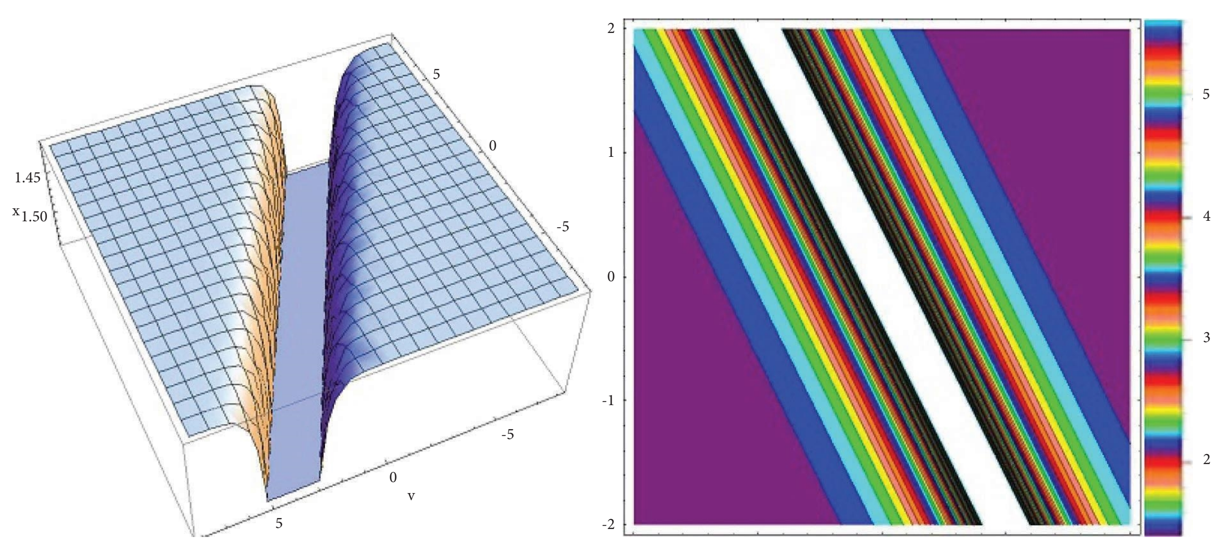

Figure 18: Singular bell type soliton sketching from (47) for = 2.

Table 1: Comparison of the results of the modified Zakharov–Kuznetsov (mZK) model.

Solutions obtained in by the (G'/G)-expansion method	Solutions obtained in this article
1. If $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, the solution (4.13) becomes: $u_{1,2}(\xi) = \pm \sqrt{3}i\lambda \tan h((\lambda/2)\xi) \pm \sqrt{3}i$. 2. If $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, the solution (4.14) becomes: $u_{3,4}(\xi) = \pm \sqrt{3}\lambda \tan((i\lambda/2)\xi) \pm \sqrt{3}i$	1. If $m = 1, 2\sqrt{-\ln} = \lambda, \psi = \xi$, $v(\psi) = u_{1,2}(\xi)$, the solution (23) becomes: $u_{1,2}(\xi) = \pm \sqrt{3}i\lambda \tan h((\lambda/2)\xi) \pm \sqrt{3}i$. 2. If $m = 1, 2l = i\lambda, \psi = \xi$, $v(\psi) = u_{3,4}(\xi)$, the solution (29) turns into: $u_{3,4}(\xi) = \pm \sqrt{3}\lambda \tan((i\lambda/2)\xi) \pm \sqrt{3}i$.

Table 2: Comparison of the results of the Calogero–Bogoyavlenskii–Schiff (CBS) model.

Solutions obtained in by the modified tanh-function method	Solutions obtained in this article
1. $u_1(x,t) = a_0 + 2 \tan h(x + z - 4t)$.	1. If $\sqrt{(m^2 - 4l^2)} = 2$, $m = a_0$, $\xi = x + z - 4t$, $v(\xi) = u_1(x,t)$, the solution (36) becomes $u_1(x,t) = a_0 + 2 \tan h(x + z - 4t)$.
2. $u_2(x,t) = a_0 + \tan h(x + z - t) + \sqrt{-1/\sigma}\sqrt{\sigma(1 - y^2)}$.	2. If $\sqrt{(m^2 + 4l^2)} = 1$, $m = a_0 + \sqrt{-1/\sigma}\sqrt{\sigma(1 - y^2)}$, $(\xi/2) = x + z - t$, $v(\xi) = u_2(x,t)$, the solution (40) turns into $u_2(x,t) = a_0 + \tan h(x + z - t) + \sqrt{-1/\sigma}\sqrt{\sigma(1 - y^2)}$
3. $u_3(x,t) = a_0 + \tan h(x + z - t) - \sqrt{-1/\sigma}\sqrt{\sigma(1 - y^2)}$.	3. If $\sqrt{(m^2 - 4l^2)} = 1, m = a_0 - \sqrt{-1/\sigma}\sqrt{\sigma(1 - y^2)}, ((\sqrt{(4l^2 - m^2)}/2)\xi) = x + z - t$, $v(\xi) = u_3(x,t)$, the solution (44) becomes $u_3(x,t) = a_0 + \tan h(x + z - t) - \sqrt{-1/\sigma}\sqrt{\sigma(1 - y^2)}$

mechanics such as wave behaviors of deep oceans, in the two-dimensional discrete electrical lattice, plasma physics, nonlinear optics, and more. We mention that all presented solutions insert valuable insights into associated studies on soliton solutions and are supplementary general in wave nature. We also study that, the solutions hold different natures of recognized profiles of solitary wave solutions as for instance, kink, singular kink, periodic and singular periodic, spike, compacton, bell shaped, and singular bell shaped.

In the above-given graphical representation, the attained solutions of the models disclose their different dynamical characteristics for different wave patterns in their corresponding media, show their different behavior such as the bell shaped wave indicates that the considered models are typically continuous or smooth, asymptotically approach to zero and commonly symmetric. The compacton wave specifies that the solutions of the models are stable and carry finite wavelength. And, the kink wave rises or falls from one asymptotic state to another and it approaches a constant at infinity. Furthermore, the periodic wave indicates that the wave family involves stable parts and the singular periodic wave shows that the wave family involves unstable parts.

5. Comparison of the Attained Solutions

In this part, we compare the exact traveling wave solutions of the modified Zakharov–Kuznetsov (mZK) model by executing the AE method with those solutions attained by the (G'/G)-expansion method. Also, we compare the solutions of the $(2+1)$-dimensional Calogero–Bogoyavlenskii–Schiff (CBS) model with those solutions obtained by the modified tanh-function method. It is important to observe that the obtained solutions are compatible, straightforward, and further general. The attained solutions might be helpful to examine the physical significance for the considered models.

5.1. The Modified Zakharov–Kuznetsov (mZK) Model. Bekir [45] investigate the mZK model and established only two solutions (see Appendix A) by executing the (G'/G)-expansion method. We identify that some of our obtained solutions are identical to Bekir's solutions and some are different. In Table 1, we compare the obtained solutions by the two methods [45].

The AE method delivered nineteen different solutions. From Table 1, we notice that the solutions $u_{1,2}$ and $u_{3,4}$ obtained by the (G'/G)-expansion method and the solutions (23) and (29) obtained by the AE method are identical respectively. The solution $u_{5,6}$ (see Appendix A) obtained by the (G'/G)-expansion method is a trivial solution and does not carry any physical significance. Therefore, the solution $u_{5,6}$ is not considered. In addition, we found some other solutions, namely, (11)–(23), and (24)–(28) which are not found in [45]. Therefore, by comparing the solutions obtained by the two methods, it might be concluded, that we attain further solutions which are general and compatible.

5.2. The Calogero–Bogoyavlenskii–Schiff (CBS) Model. By executing the modified tanh-function method, Taha et al. [46] obtained only three solutions (see Appendix B) for the $(2+1)$-dimensional CBS model. We identify that some of the attained solutions are identical to the Taha et al. solutions and some are different. In Table 2, we compare the solutions examined by the two methods [46].

By applying the AE method, we determine nineteen fresh solutions. From Table 2, we see that the solutions u_1, u_2, u_3 obtained by the modified tanh-function method and the solutions (36), (40), and (44) obtained by the AE method are identical, respectively. In addition, we attain other sixteen solutions (34), (35), (37)–(39), (41)–(43), (45)–(52)which are not found by Taha et al. [46]. Therefore, from the comparison of the solutions obtained by the two methods, it might be concluded that we have attained further general, functional solutions than Taha et al. [46] solutions.

6. Conclusion

The auxiliary equation approach has effectively been implemented to the modified Zakharov–Kuznetsov and the $(2+1)$-dimensional Calogero–Bogoyavlenskii–Schiff models in this article. The established wave solutions are further generic than the reachable results in the literature and for distinct values of the associated parameters standard solutions are originated and a few existing solutions are restored. In this study, it is established diverse solutions and the established traveling wave solutions are constructed through trigonometric, exponential, and hyperbolic functions and their integration. The accuracy of the investigated solutions have been verified by placing them into their original models and found accurate. From the analysis, it is perceived that the auxiliary equation approach can be used to study the further nonlinear evolution models that frequently occur in physics, mathematical physics, engineering, and other scientific real-time application fields. The ascertained solutions have revealed that the auxiliary equation method is an effective algorithm, powerful, further generalized, and can be utilized to arrange the wave velocity. In this study, we examine the exact solutions of the nonlinear mZK and CBS equations but it might assert that the approach can be implemented for other NLEEs. Therefore, in future, we can investigate the exact solutions of other NLEEs, such as the Calogero–Degasperis (CD) model, the Zakharov–Kuznetsov–Benjamin–Bona–Mahony (ZK-BBM) model, the $(3+1)$-dimensional Boiti–Leon–Manna–Pempinelli equation and some others by means of the auxiliary equation method.

Appendix

A. Bekir [45] Solutions

The solution of Bekir [45], investigated by the (G'/G)-expansion method for the modified Zakharov–Kuznetsov (mZK) model are scheduled as follows:

$$u_{1,2}(\xi) = \pm\sqrt{3}i\lambda \tan h\left(\frac{\lambda}{2}\xi\right) \pm \sqrt{3}i, \tag{A.1}$$

$$u_{3,4}(\xi) = \pm\sqrt{3}\lambda \tan\left(\frac{i\lambda}{2}\xi\right) \pm \sqrt{3}i, \tag{A.2}$$

$$u_{5,6}(x,t) = \pm\frac{\pm\sqrt{3}iC_2}{C_1 + C_2 x}. \tag{A.3}$$

B. Taha et al. [46] Solutions

The solution of Taha et al. [46], examined by the modified tanh-function method for the Calogero–Bogoyavlenskii–Schiff (CBS) model are arranged as follows:

$$u_1(x,t) = a_0 + 2\tan h(x+z-4t), \tag{B.1}$$

$$u_2(x,t) = a_0 + \tan h(x+z-t) + \sqrt{\frac{-1}{\sigma}}\sqrt{\sigma(1-y^2)}, \tag{B.2}$$

$$u_3(x,t) = a_0 + \tan h(x+z-t) - \sqrt{\frac{-1}{\sigma}}\sqrt{\sigma(1-y^2)}. \tag{B.3}$$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors would like to express their sincere thanks to the anonymous referees for their valuable comments and suggestions.

References

[1] M. M. El-Borai, W. G. El-Sayed, and R. M. Al-Masroub, "Exact solutions for time fractional coupled Whitham-Broer-Kaup equations via exp-function method," *International Research Journal of Engineering and Technology*, vol. 2, no. 6, pp. 307–315, 2015.
[2] M. N. Islam and M. A. Akbar, "New exact wave solutions to the space-time fractional-coupled Burgers equations and the space-time fractional foam drainage equation," *Cogent Physics*, vol. 5, no. 1, Article ID 1422957, 2018.
[3] M. N. Islam and M. A. Akbar, "Closed form solutions to the coupled space-time fractional evolution equations in mathematical physics through analytical method," *Journal of Mechanics of Continua and Mathematical Sciences*, vol. 13, no. 2, pp. 1–23, 2018.
[4] M. A. Akbar and N. H. M. Ali, "The alternative (G'/G)-expansion method and its applications to nonlinear partial differential equations," *International Journal of the Physical Sciences*, vol. 6, no. 35, pp. 7910–7920, 2014.
[5] M. N. Islam and M. A. Akbar, "Closed form wave solutions to the time fractional Boussinesq-type and the time fractional Zakharov-Kuznetsov equations," *The Journal of the National Science Foundation of Sri Lanka*, vol. 47, no. 2, pp. 149–160, 2019.
[6] Y. Liu, J. Roberts and Y. Yan, A note on finite difference methods for nonlinear fractional differential equations with non-uniform meshes," *International Journal of Computer Mathematics*, vol. 95, no. 6-7, pp. 1151–1169, 2017.
[7] E. Hm Zahran and M. Ma Khater, "Modified extended tanh-function method and its applications to the Bogoyavlenskii equation," *Applied Mathematical Modelling*, vol. 40, no. 3, pp. 1769–1775, 2016.
[8] B. Zheng and Q. Feng, "The Jacobi elliptic equation method for solving fractional partial differential equations," *Abstract*

and Applied Analysis, vol. 2014, Article ID 249071, 9 pages, 2014.

[9] H. M. Baskonus, "Complex soliton solutions to the gilson-pickering model," *Axioms*, vol. 8, no. 1, p. 18, 2019.

[10] M. S. Osman, D. Lu, M. M. A. Khater, and R. A. M. Attia, Complex wave structures for abundant solutions related to the complex Ginzburg-Landau model," *Optik*, vol. 192, Article ID 162927, 2019.

[11] M. N. Islam, M. Asaduzzaman and M. S. Ali, Exact wave solutions to the simplified modified Camassa-Holm equation in mathematical physics," *AIMS Math*, vol. 5, no. 1, pp. 26–41, 2019.

[12] O. A. Ilhan, M. N. Islam, and M. A. Akbar, "Construction of functional closed form wave solutions to the ZKBBM equation and the Schrodinger equation," *Iranian Journal of Science and Technology-Transactions of Mechanical Engineering is a Journal*, vol. 14, 2020.

[13] N. A. Kudryashov, "First integrals and solutions of the traveling wave reduction for the Triki-Biswas equation," *Optik*, vol. 185, pp. 275–281, 2019.

[14] A. M. Wazwaz and L. Kaur, "Optical solitons and Peregrine solitons for nonlinear Schrodinger equation by variational iteration method," *Optik*, vol. 179, pp. 804–809, 2019.

[15] K. Hosseini and R. Ansari, "New exact solutions of nonlinear conformable time-fractional Boussinesq equations using the modified Kudryashov method," *Waves in Random and Complex Media*, vol. 27, no. 4, pp. 628–636, 2017.

[16] J. G. Liu, M. Eslami, H. Rezazadeh, and M. Mirzazadeh, "The dynamical behavior of mixed type lump solutions on the $(3+1)$-dimensional generalized Kadomtsev-Petviashvili-Boussinesq equation," *International Journal of Nonlinear Sciences and Numerical Stimulation*, vol. 21, no. 7-8, pp. 661–665, 2020.

[17] A. M. Wazwaz, "A two-mode burgers equation of weak shock waves in a fluid: multiple kink solutions and other exact solutions," *International Journal of Algorithms, Computing and Mathematics*, vol. 3, no. 4, pp. 3977–3985, 2017.

[18] H. Bulut, H. M. Baskonus and Y. Pandir, The modified trial equation method for fractional wave equation and time fractional generalized Burgers equation," *Abstract and Applied Analysis*, vol. 2013, Article ID 636802, 8 pages, 2013.

[19] J. F. Alzaidy, "Fractional sub-equation method and its applications to the space–time fractional differential equations in mathematical physics," *British Journal of Mathematics & Computer Science*, vol. 3, no. 2, pp. 153–163, 2013.

[20] M. S. Hashemi, "Invariant subspaces admitted by fractional differential equations with conformable derivatives," *Chaos, Solitons & Fractals*, vol. 107, pp. 161–169, 2018.

[21] A. Yusuf, M. Inc, and D. Baleanu, "Optical solitons with M-truncated and beta derivatives in nonlinear optics," *Frontiers in Physics*, vol. 7, no. 126, 2019.

[22] M. Inc, A. I. Aliyu, A. Yusuf, and D. Baleanu, "Optical solitons for Biswas-Milovic Model in nonlinear optics by Sine-Gordon equation method," *Optik*, vol. 157, pp. 267–274, 2018.

[23] W. Gao, H. Rezazadeh, Z. Pinar, H. M. Baskonus, S. Sarwar, and G. Yel, Novel explicit solutions for the nonlinear Zoomeron equation by using newly extended direct algebraic technique," *Optical and Quantum Electronics*, vol. 52, no. 1, p. 52, 2020.

[24] M. A. Akbar, N. H. M. Ali and T. Tanjim, Outset of multiple soliton solutions to the nonlinear Schrodinger equation and the coupled Burgers equation," *Journal of Physics Communications*, vol. 3, no. 9, Article ID 095013, 2019.

[25] H. Rezazadeh, A. Korkmaz, M. Eslami, and S. M. Mirhosseini-Alizamini, "A large family of optical solutions to Kundu-Eckhaus model by a new auxiliary equation method," *Optical and Quantum Electronics*, vol. 51, no. 3, p. 84, 2019.

[26] A. Ismail, "Exact solution for fractional DDEs via auxiliary equation method coupled with the fractional complex transform," *Mathematical Methods in the Applied Sciences*, vol. 39, no. 18, pp. 5619–5625, 2016.

[27] M. Al-Amin, M. N. Islam, and M. A. Akbar, "Adequate wide-ranging closed-form wave solutions to a nonlinear biological model," *Partial Differential Equations in Applied Mathematics*, vol. 4, no. 4, Article ID 100042, 2021.

[28] S. Munro and E. J. Parkes, "The derivation of a modified Zakharov-Kuznetsov equation and the stability of its solutions," *Journal of Plasma Physics*, vol. 62, no. 3, Article ID S0022377899007874, 1999.

[29] S. Munro and E. J. Parkes, "Stability of solitary-wave solutions to a modified Zakharov-Kuznetsov equation," *Journal of Plasma Physics*, vol. 64, no. 4, pp. 411–426, 2000.

[30] M. Eslami, B. F. Vajargah, and M. Mirzazadeh, "Exact solutions of modified Zakharov-Kuznetsov equation by the homogeneous balance method," *Ain Shams Engineering Journal*, vol. 5, pp. 221–225, 2014.

[31] O. I. Bogoyavlenskiĭ, "Breaking solitons. II," *Math. USSR Izv*, vol. 35, no. 1, pp. 245–248, 1990.

[32] D. E. Baldwin and W. Hereman, "A symbolic algorithm for computing recursion operators of nonlinear partial differential equations," *International Journal of Computer Mathematics*, vol. 87, no. 5, pp. 1094–1119, 2010.

[33] A. M. Wazwaz, "Multiple-soliton solutions for the Calogero-Bogoyavlenskii-Schiff-Jimbo-Miwa and YTSF equations," *Applied Mathematics and Computation*, vol. 203, no. 2, pp. 592–597, 2008.

[34] D. j. Zhang, J. Ji, and S. l. Zhao, "Soliton scattering with amplitude changes of a negative order AKNS equation," *Physica D: Nonlinear Phenomena*, vol. 238, no. 23-24, pp. 2361–2367, 2009.

[35] F. Tascan, A. Bekir and M. Koparan, Travelling wave solutions of nonlinear evolution equations by using the first integral methodfirst integral method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 5, pp. 1810–1815, 2009.

[36] A. M. Wazwaz, "The extended tanh method for the Zakharov-Kuznetsov (ZK) equation, the modified ZK equation, and its generalized formsfied ZK equation and its generalized forms," *Communications in Nonlinear Science and Numerical Simulation*, vol. 13, no. 6, pp. 1039–1047, 2008.

[37] X. Zhou, W. Shan, Z. Niu, and P. Xiao, Y. Wang, Lie symmetry analysis and some exact solutions for modified Zakharov-Kuznetsov equationfied Zakharov-Kuznetsov equation," *Modern Physics Letters B*, vol. 32, no. 31, Article ID 1850383, 2018.

[38] S. T. Mohyud-Din, M. A. Noor, and K. I. Noor, "Exp-function method for traveling wave solutions of modified Zakharov-Kuznetsov equation," *Journal of King Saud University Science*, vol. 22, no. 4, pp. 213–216, 2010.

[39] X. Wang, S. A. Javed, A. Majeed, M. Kamran, and M. Abbas, "Investigation of exact solutions of nonlinear evolution equations using unified method," *Mathematics*, vol. 10, no. 16, p. 2996, 2022.

[40] M. O. Al-Amr, "Exact solutions of the generalized $(2+1)$-dimensional nonlinear evolution equations via the modified simple equation method," *Computers & Mathematics with Applications*, vol. 69, pp. 390–397, 2015.

[41] M. T. Darvishi, N. Maliheh, and N. Mohammad, "New application of EHTA for the generalized $(2+1)$-dimensional nonlinear evolution equations," *International Journal of Mathematics and Computer Science*, vol. 6, no. 3, 2010.

[42] H. Naher and F. A. Abdullah, "The improved (G'/G)-expansion method for the $(2+1)$-Dimensional modified Zakharov-Kuznetsov equation," *Journal of Applied Mathematics*, vol. 438928, p. 20, 2012.

[43] S. R. Islam, A. Akbulut, and S. M. Y. Arafat, "Exact solutions of the different dimensional CBS equations in mathematical physics," *Partial Differential Equations in Applied Mathematics*, vol. 5, Article ID 100320, 2022.

[44] L. Han, S. Bilige, X. Wang, M. Li, and R. Zhang, "Rational wave solutions and dynamics properties of the generalized $(2+1)$-dimensional Calogero-Bogoyavlenskii-Schiff equation by using bilinear method," *Advances in Mathematical Physics*, vol. 2021, Article ID 295547, 10 pages, 2021.

[45] A. Bekir, "Application of the (G'/G)-expansion method for nonlinear evolution equations," *Physics Letters A*, vol. 372, no. 19, pp. 3400–3406, 2008.

[46] W. M. Taha, A. H. Kadhim, R. A. Hameed, and M. S. M. Noorani, "New modified tanh-function method for nonlinear evolution equations," *Tikrit Journal of Pure Science*, vol. 21, no. 6, pp. 187–190, 2016.