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Research Article

Stability and Bifurcation Analysis of a Fractional-Order Food Chain Model with Two Time Delays

Hao Qi and Wencai Zhao

College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China

Correspondence should be addressed to Wencai Zhao; zhaowencai@sdust.edu.cn

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In this study, the stability and bifurcation problems of a fractional food chain system with two kinds of delays are studied. Firstly, the nonnegative, bounded, and unique properties of the solutions of the system are proved. The asymptotic stability of the equilibrium points of the system is discussed. Furthermore, the global asymptotic stability of the positive equilibrium point is deduced by using Lyapunov function method. Secondly, the system takes two kinds of time delays as bifurcation parameters and calculates the critical values of Hopf bifurcation accurately. The results show that Hopf bifurcation can advance with increasing fractional order and another delay. In conclusion, numerical simulation verifies and illustrates the theoretical results.

1. Introduction

In the ecosystem, no species exists in isolation. The different populations are all related to each other. Predator relationship, competition relationship, reciprocity relationship, and parasitic relationship are the main population relationships. In these major relationships, predator-prey relationship is universal in nature and is of great significance to complex ecosystems. It is precisely because of the important application background and practical value of the predation system that the food chain system has been researched extensively by many scholars [1–5].

In nature, the phenomenon of time delay is exhibited universally in biological population. The phenomenon of time delay is mainly caused by many factors such as gestation, maturation, and food digestion of population. The phenomenon of time delay signifies that the related properties of the system are related to not only the present state but also the previous period. Aiello and Freedman studied a single population system with a time delay and stage structure [6]. Beddington et al. [7] proved that time delay could affect the stability of the dynamical model. Gazi et al. [8] researched the influence of harvest and discrete time delay on the prey-predator populations and obtained the discrete time-delay length required to remain the stability of the system. Jana et al. [9] analyzed the time-delay predator-prey system including prey shelter and demonstrated the global asymptotic stability of the system. Yan et al. [10] considered the predator-prey model with delayed reaction diffusion and analyzed the global asymptotic stability of the positive equilibrium point of the model. Vinoth et al. [11] put forward a delayed prey-predator system with additive Allee effect, and the local asymptotic stability of the model at equilibrium point was studied. Numerous studies have shown that a population system with time delay could exhibit more complex nonlinear dynamic behaviors. Therefore, time delay has a profound impact on the stability behavior of biological systems.

Differential equation theory has been widely used in automation system, aerospace technology, information engineering, and so on. In these practical applications, the system usually has some parameters. If the parameters of the system change, the topological structure of the phase diagram in phase space also changes; then, the phenomenon is called bifurcation [12–14]. Hopf bifurcation theory has become a classical tool to research the generation and extinction periodic solutions of small amplitude differential equations. When a parameter passes a marginal value, the equilibrium point will lose stability and a periodic solution will appear [15–19]. Deka et al. [15] proposed and analyzed a one-predator and two-prey system with a general Gauss type, and the stability and direction of the Hopf bifurcation were proved by regarding the mortality of the predator as the bifurcation argument. In [16], a predator-prey model with discrete time delay of habitat complexity and sanctuary for prey was proposed and the occurrence criterion of Hopf bifurcation was obtained by taking the time lag as argument. In [20], Guo et al. established a food chain system with a couple of time lags and Holling II type functions:

$$\begin{cases} x'(t) = x(t)\left(r_1 - a_1 x(t) - \dfrac{y(t-\tau_1)}{1+mx(t)}\right), \\ y'(t) = y(t)\left(\dfrac{a_2 x(t-\tau_2)}{1+mx(t)} - r_2 - a_3 z(t-\tau_1)\right), \\ z'(t) = z(t)\left(a_4 y(t-\tau_2) - r_3\right). \end{cases} \tag{1}$$

Among them, the biological significance of each parameter of system (1) is well illustrated in Table 1.

At the same time, the existence of the positive equilibrium point was proved, and the occurrence criterion of Hopf bifurcation was obtained by taking the time lag as the parameter.

Fractional-order calculus is a method that rises recently. It is a method that extends the ordinary integral calculus to nonintegral calculus [21–27]. So far, fractional calculus has been applied to many domains, such as neural network [28], medicine [29], finance system [30], and safety communication [31]. A great deal of studies have proved that the fractional dynamical system is to a higher degree suitable to biological systems because the fractional differential is connected with the entire time zone, while the integer differential is only related to a particular moment. Because biological systems generally have the characteristics of heredity and memory, so more and more scholars believe that the method of fractional calculus can better characterize the behavior of biological system. At present, some scholars have spread the classical integer-order differential systems to the fractional-order differential systems [32–37]. Rihan et al. [32] studied a fractional-order food chain model with time delay as well as infection in predators; sufficient criterion for asymptotic stability of the stable condition of the model was established. Huang et al. [36] discovered that the bifurcation dynamics of the model could be resultfully controlled as long as other parameters of the system are determined, and the extended feedback delay or fractional order is carefully adjusted.

Based on the above discussion, model (1) is extended in this study to obtain the following fractional-order food chain model:

Table 1: Biological significance of symbols.

Symbols	Biological significance
$x(t)$	The density of prey population at time t
$y(t)$	The density of the primary predator population at time t
$z(t)$	The density of the top predator population at time t
r_1	The intrinsic growth rate of prey population
r_2	The death rate of the primary predator population
r_3	The death rate of the top predator population
a_1	The internal competition rate of the prey population
a_2	The nutrient conversion rate from prey to primary predator
a_3	The rate of capture by top predators on primary predators
a_4	The digestibility of top predators to primary predators
m	The semisaturation of predator
τ_1	The capture time
τ_2	The maturity time

$$\begin{cases} D^\theta x(t) = x(t)\left(r_1 - a_1 x(t) - \dfrac{y(t-\tau_1)}{1+mx(t)}\right), \\ D^\theta y(t) = y(t)\left(\dfrac{a_2 x(t-\tau_2)}{1+mx(t)} - r_2 - a_3 z(t-\tau_1)\right), \\ D^\theta z(t) = z(t)\left(a_4 y(t-\tau_2) - r_3\right), \end{cases} \tag{2}$$

where $0 < \theta \leq 1$; the biological significance of each variable and parameter of model (2) is the same as that of model (1). The initial conditions are $x(t) = \zeta_1(t) \geq 0, y(t) = \zeta_2(t) \geq 0, z(t) = \zeta_3(t) \geq 0$, and $t \in [-\max(\tau_1, \tau_2), 0]$. The model is established on the sense of Caputo derivative.

The rest of the study is organized as follows. Several definitions as well as lemmas are addressed in Section 2. In Section 3, the corresponding nondelay system of (2) is discussed. The Hopf bifurcation of system (2) is studied in Section 4. Some numerical simulations are presented in Section 5. Conclusions are drawn in the end.

2. Preliminaries

For the theoretical derivation, we first give the relevant definitions and lemmas of Caputo calculus.

Definition 1 (see [21]). The fractional integral of order θ for a function $f(t)$ is defined as

$$I^\theta f(t) = \frac{1}{\Gamma(\theta)} \int_{t_0}^{t} (t-s)^{\theta-1} f(s) \mathrm{d}s, \tag{3}$$

where $t \geq t_0$, $\theta > 0$, and $\Gamma(\cdot)$ is the Gamma function, $\Gamma(\theta) = \int_0^\infty t^{\theta-1} e^{-t} \mathrm{d}t$.

Definition 2 (see [21]). The Caputo fractional derivative of order θ for a function $f(t)$ is defined as

$$D^\theta f(t) = \frac{1}{\Gamma(n-\theta)} \int_{t_0}^{t} \frac{f^{(n)}(s)}{(t-s)^{\theta-n+1}} \mathrm{d}s, \tag{4}$$

where n is a positive integer, $n-1<\theta\leq n$, and $t\geq t_0$. So, specifically, if $0<\theta<1$,

$$D^{\theta}f(t)=\frac{1}{\Gamma(1-\theta)}\int_{t_0}^{t}\frac{f'(s)}{(t-s)^{\theta}}\mathrm{d}s. \tag{5}$$

Lemma 1 (see [22]). *Define $w\in C^{\theta}([t_0,T],R)$. Suppose that there exist $t_1\in(t_0,T]$, such that $w(t_1)=0$ and $w(t)>0$ for $t_0\leq t<t_1$; then,*

$$D^{\theta}w(t_1)<0. \tag{6}$$

Lemma 2 (see [21]). *Define $\theta>0$, $n-1<\theta\leq n$. Suppose $m(t)$ is n times continuous differentiable function and $D^{\theta}m(t)$ is piecewise continuous on $[t_0,\infty)$; we have*

$$L\{D^{\theta}m(t)\}=s^{\theta}\Upsilon(s)-\sum_{k=0}^{n-1}s^{\theta-k-1}m^{(k)}(t_0), \tag{7}$$

where $\Upsilon(s)=L\{m(t)\}$.

Lemma 3 (see [38]). *Assume $\mathbb{M}$ represents the complex plane, for $\forall a>0$ and $b>0$, and $Q\in\mathbb{M}$; then,*

$$L\{t^{b-1}E_{a,b}(Qt^{a})\}=\frac{s^{a-b}}{s^{a}-Q}, \tag{8}$$

for $\mathbb{R}(s)>|Q|^{1/a}$, $\mathbb{R}(s)$ signifies the real part of the complex number s, and $E_{a,b}$ is the following Mittag–Leffler function described by

$$E_{a,b}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(an+b)}. \tag{9}$$

Lemma 4 (see [24]). *Consider the system:*

$$D^{\theta}X(t)=\varrho(t,X),\quad t_0>0, \tag{10}$$

with initial condition $X(t_0)=X_{t_0}$, where $\theta\in(0,1]$ and ϱ: $[t_0,\infty)\times\eta\longrightarrow R^{n}$, $\eta\subseteq R^{n}$, if $\varrho(t,X)$ meets the local Lipschitz criteria with respect to $X\in R^{n}$:

$$\|\varrho(t,X)-\varrho(t,\overline{X})\|\leq\delta\|X-\overline{X}\|, \tag{11}$$

so it has a unique solution of (10) on $[t_0,\infty)$, where

$$\left\|X(u_1,u_2,\ldots,u_n)-\overline{X}(v_1,v_2,\ldots,v_n)\right\|=\sum_{i=1}^{n}|u_i-v_i|,\quad u_i,v_i\in R. \tag{12}$$

Lemma 5 (see [23]). *Define $0<\theta<1$, $X(t)\in R^{n}$, and f: $R^{n}\longrightarrow R^{n}$; think about the following nonlinear fractional system of the same order:*

$$D^{\theta}X(t)=f(X(t)),\quad X(0)=X_0. \tag{13}$$

If the eigenvalues λ_i $(i=1,\ldots,n)$ of the Jacobian matrix corresponding to the equilibrium point of the system meet the following criterion $|\arg(\lambda_i)|>(\theta\pi/2)$, $i=1,2,\ldots n$, then system (13) is asymptotically stable.

Lemma 6 (see [26]). *Assume $l(t)\in R_{+}$ is a continuous and differentiable function. For $\forall t\geq t_0$, $D^{\theta}[l(t)-l^{*}-l^{*}\ln(l(t)/l^{*})]\leq(1-(l^{*}/l(t)))D^{\theta}l(t)$, $l^{*}\in R_{+}$, $\forall\theta\in(0,1)$.*

Lemma 7 (see [23]). *Think about the under n-dimensional linear fractional-order time-delay system:*

$$\begin{cases}D^{\theta_1}x_1(t)=b_{11}x_1(t-\tau_{11})+b_{12}x_2(t-\tau_{12})+\cdots+b_{1n}x_n(t-\tau_{1n}),\\ D^{\theta_2}x_2(t)=b_{21}x_1(t-\tau_{21})+b_{22}x_2(t-\tau_{22})+\cdots+b_{2n}x_n(t-\tau_{2n}),\\ \vdots\\ D^{\theta_n}x_n(t)=b_{n1}x_1(t-\tau_{n1})+b_{n2}x_2(t-\tau_{n2})+\cdots+b_{nn}x_n(t-\tau_{nn}).\end{cases} \tag{14}$$

Among them, $\forall\theta_i\in(0,1)$, and the initial conditions $x_i(t)=\xi_i(t)$ are provided for $-\max_{1\leq i,j\leq n}\tau_{ij}=-\tau_{\max}\leq t\leq 0$, $i=1,2,\ldots,n$. It is defined as

$$\Lambda(s)=\begin{bmatrix}s^{\theta_1}-b_{11}e^{-s\tau_{11}} & -b_{12}e^{-s\tau_{12}} & \cdots & -b_{1n}e^{-s\tau_{1n}}\\ -b_{21}e^{-s\tau_{21}} & s^{\theta_2}-b_{22}e^{-s\tau_{22}} & \cdots & -b_{2n}e^{-s\tau_{2n}}\\ \vdots & \vdots & \ddots & \vdots\\ -b_{n1}e^{-s\tau_{n1}} & -b_{n2}e^{-s\tau_{n2}} & \cdots & s^{\theta_n}-b_{nn}e^{-s\tau_{nn}}\end{bmatrix}. \tag{15}$$

If all roots of $\det(\Lambda(s)) = 0$ have negative real parts, so the zero solution of system (14) is Lyapunov globally asymptotically stable.

3. Analysis of the Nondelayed Model

First, we research the delay-free system of (2):

$$\begin{cases} D^{\theta}x(t) = x(t)\left(r_1 - a_1x(t) - \dfrac{y(t)}{1+mx(t)}\right), \\ D^{\theta}y(t) = y(t)\left(\dfrac{a_2x(t)}{1+mx(t)} - r_2 - a_3z(t)\right), \\ D^{\theta}z(t) = z(t)\left(a_4y(t) - r_3\right). \end{cases} \tag{16}$$

The nonnegativity and boundedness, existence, and uniqueness of solutions about systems (2) and (16) are discussed in Sections 3.1 and 3.2. The local stability of the equilibrium points of system (2) is discussed, and the global asymptotic stability of the positive equilibrium point of system (2) is demonstrated in Section 3.3.

3.1. Nonnegativity and Boundedness of Solutions. Think about the biological implications of reality, it is significant to analyze the nonnegativity of the system. To prove the following theorem, let R^+ denote the collection of entire positive real numbers containing 0, $\eta_+ = \{(x,y,z) \in \eta\colon\ x,y,z \in R^+\}$.

Theorem 1. *The solutions about system (16) from η_+ are nonnegative and uniformly bounded.*

Proof. When $t = 0$, then $x(0) > 0$; we desire to obtain the solution $x(t)$ from η_+ is nonnegative, i.e., $x(t) \geq 0$, for $\forall t \geq 0$. Suppose it exhibits a constant $t' > 0$, $x(t') < 0$; according to $x(t)$ which is a continuous function, there exists $t'' \in (0,t')$ and $x(t'') = 0$. Define $t_1 = \min\{t'' \in (0,t') | x(t'') = 0\}$; then, when $t = t_1 > 0$, from system (16), one obtains $D^{\theta}x(t)|_{t=t_1} = x(t)(r_1 - a_1x(t) - (y(t)/(1+mx(t)))) = 0$. However, according to the definition of t_1, $x(0) > 0$ and $x(t_1) = 0$; moreover, $x(t) > 0$, $t \in [0,t_1)$; by Lemma 1, we have $D^{\theta}x(t_1) < 0$. Hence, we derive a contradiction; therefore, $x(t) \geq 0$, $\forall t \in [0,\infty)$. Likewise, we can demonstrate $y(t), z(t) \geq 0$, $\forall t \in [0,\infty)$.

For boundedness, we think about the following function:

$$W(x(t),y(t),z(t)) = x(t) + \frac{1}{a_2}y(t) + \frac{a_3}{a_2a_4}z(t). \tag{17}$$

According to system (16), one has

$$\begin{aligned} D^{\theta}W(t) &= r_1x(t) - a_1x^2(t) - \frac{1}{a_2}r_2y(t) - \frac{a_3r_3}{a_2a_4}z(t), \\ D^{\theta}W(t) + \hat{k}W(t) &= -a_1\left(x(t) - \frac{r_1+\hat{k}}{2a_1}\right)^2 + \frac{\left(r_1+\hat{k}\right)^2}{4a_1} - \frac{r_2-\hat{k}}{a_2}y(t) - \frac{a_3}{a_2a_4}\left(r_3-\hat{k}\right)z(t) \\ &\leq -a_1\left(x(t) - \frac{r_1+\hat{k}}{2a_1}\right)^2 + \frac{\left(r_1+\hat{k}\right)^2}{4a_1} \\ &\leq \frac{\left(r_1+\hat{k}\right)^2}{4a_1} = \mathbb{A}, \end{aligned} \tag{18}$$

where $\hat{k} = \min\{r_2, r_3\} > 0$. Therefore,

$$D^{\theta}W(t) + \hat{k}W(t) \leq \mathbb{A}. \tag{19}$$

By Lemma 2, making Laplace transform of both sides of (19), we obtain

$$s^{\theta}\Upsilon(s) - s^{\theta-1}W(0) + \hat{k}\Upsilon(s) \leq \frac{\mathbb{A}}{s}, \tag{20}$$

where $\Upsilon(s) = L\{W(t)\}$. From this, we can obtain

$$\Upsilon(s) \leq \frac{s^{\theta-1}W(0)}{s^{\theta}+\hat{k}} + \frac{\mathbb{A}}{s\left(s^{\theta}+\hat{k}\right)}. \tag{21}$$

Making inverse Laplace transform of (21), then

$$W(t) \leq W(0)L^{-1}\left(\frac{s^{\theta-1}}{s^{\theta}+\hat{k}}\right) + \mathbb{A}L^{-1}\left(\frac{s^{\theta-(\theta+1)}}{s^{\theta}+\hat{k}}\right). \tag{22}$$

By Lemma 3, one has

$$W(t) \leq W(0)E_{\theta,1}\left\{-\hat{k}t^{\theta}\right\} + \mathbb{A}t^{\theta}E_{\theta,\theta+1}\left\{-\hat{k}t^{\theta}\right\}. \tag{23}$$

According to

$$E_{\vartheta,\iota}(z) = zE_{\vartheta,\vartheta+\iota}(z) + \frac{1}{\Gamma(\iota)}, \tag{24}$$

so we have

$$E_{\theta,1}\left(-\hat{k}t^{\theta}\right) = \left(-\hat{k}t^{\theta}\right)E_{\theta,\theta+1}\left(-\hat{k}t^{\theta}\right) + \frac{1}{\Gamma(1)},$$
$$t^{\theta}E_{\theta,\theta+1}\left(-\hat{k}t^{\theta}\right) = -\frac{1}{\hat{k}}\left(E_{\theta,1}\left(-\hat{k}t^{\theta}\right) - 1\right). \tag{25}$$

Hence,

$$W(t) \le \left\{W(0) - \frac{\mathbb{A}}{\hat{k}}\right\}E_{\theta,1}\left(-\hat{k}t^{\theta}\right) + \frac{\mathbb{A}}{\hat{k}}, \tag{26}$$

where, if $t \longrightarrow \infty$, we have $E_{\theta,1}(-\hat{k}t^{\theta}) \longrightarrow 0$.

Furthermore, the set $\mathbb{D}$ attracts all the solutions of system (16), where

$$\mathbb{D} = \left\{(x, y, z) \in \eta_+ | x(t) + \frac{1}{a_2}y(t) + \frac{a_3}{a_2a_4}z(t) \le \frac{1}{\hat{k}}\frac{\left(r_1 + \hat{k}\right)^2}{4a_1} + \epsilon, \quad \epsilon > 0\right\}. \tag{27}$$

□

3.2. Existence and Uniqueness of Solutions

Theorem 2. *System (16) only exhibits a solution $X(t) = (x(t), y(t), z(t)) \in \eta_+$ for any given initial value $X(t_0) = (x_{t_0}, y_{t_0}, z_{t_0}) \in \eta_+$.*

Proof. According to Theorem 1, the solutions of system (16) from η_+ are nonnegative and uniformly bounded; then, there exists a constant P, such that $\max\{x(t), y(t), z(t)\} \le P$. Define a mapping $Q(X) = (Q_1(X), Q_2(X), Q_3(X))$, in which

$$Q_1(X) = x(t)\left(r_1 - a_1x(t) - \frac{y(t)}{1 + mx(t)}\right),$$
$$Q_2(X) = y(t)\left(\frac{a_2x(t)}{1 + mx(t)} - r_2 - a_3z(t)\right), \tag{28}$$
$$Q_3(X) = z(t)\left(a_4y(t) - r_3\right).$$

Let X and $\overline{X}$ be any two solutions to system (16); we can derive

$$\begin{aligned}
&\|Q(X) - Q(\overline{X})\| \\
&= \left|Q_1(X) - Q_1(\overline{X})\right| + \left|Q_2(X) - Q_2(\overline{X})\right| + \left|Q_3(X) - Q_3(\overline{X})\right| \\
&= \left|x\left(r_1 - a_1x - \frac{y}{1 + mx}\right) - \overline{x}\left(r_1 - a_1\overline{x} - \frac{\overline{y}}{1 + m\overline{x}}\right)\right| + \left|y\left(\frac{a_2x}{1 + mx} - r_2 - a_3z\right) - \overline{y}\left(\frac{a_2\overline{x}}{1 + m\overline{x}} - r_2 - a_3\overline{z}\right)\right| \\
&\quad + \left|z(a_4y - r_3) - \overline{z}(a_4\overline{y} - r_3)\right| \\
&\le r_1|x - \overline{x}| + a_1\left|x^2 - \overline{x}^2\right| + \left|\frac{xy - \overline{x}\overline{y} + mxy\overline{x} - m\overline{x}\overline{y}x}{(1 + mx)(1 + m\overline{x})}\right| + r_2|y - \overline{y}| + a_3|yz - \overline{y}\overline{z}| \\
&\quad + a_2\left|\frac{xy - \overline{x}\overline{y} + mxy\overline{x} - m\overline{x}\overline{y}x}{(1 + mx)(1 + m\overline{x})}\right| + a_4|yz - \overline{y}\overline{z}| + r_3|z - \overline{z}| \\
&\le r_1|x - \overline{x}| + 2a_1P|x - \overline{x}| + P|y - \overline{y}| + P|x - \overline{x}| + mP^2|y - \overline{y}| + mP^2|x - \overline{x}| \\
&\quad + mP^2|x - \overline{x}| + r_2|y - \overline{y}| + a_3P|z - \overline{z}| + a_3P|y - \overline{y}| \\
&\quad + a_2\left(P|y - \overline{y}| + P|x - \overline{x}| + mP^2|y - \overline{y}| + mP^2|x - \overline{x}| + mP^2|x - \overline{x}|\right) \\
&\quad + a_4P|z - \overline{z}| + a_4P|y - \overline{y}| + r_3|z - \overline{z}| \\
&= \left(r_1 + 2a_1P + P + 2mP^2 + a_2P + 2a_2mP^2\right)|x - \overline{x}| \\
&\quad + \left(P + mP^2 + r_2 + a_3P + a_2P + a_2mP^2 + a_4P\right)|y - \overline{y}| \\
&\quad + (a_3P + a_4P + r_3)|z - \overline{z}| \\
&\le \delta\|X - \overline{X}\|,
\end{aligned} \tag{29}$$

where $\delta = \max\{M_1, M_2, M_3\}$ and $M_1 = r_1 + 2a_1 P + P + 2mP^2 + a_2 P + 2a_2 mP^2$, $M_2 = P + mP^2 + r_2 + a_3 P + a_2 P + a_2 mP^2 + a_4 P$, and $M_3 = a_3 P + a_4 P + r_3$. Hence, $Q(X)$ meets the Lipschitz criteria about X. By Lemma 4, it has only a solution $X(t)$ of system (16) with initial value $X(t_0) = (x_{t_0}, y_{t_0}, z_{t_0})$. □

3.3. Stability Analysis of Balance Point. For analyzing the possible equilibrium of system (16), we first present the following assumptions:

(i) $[\mathbb{H}_1]$: $r_1(a_2 - mr_2) - a_1 r_2 > 0$.
(ii) $[\mathbb{H}_2]$: $a_4 r_1 - r_3 > 0$ and $a_2 x^* - r_2(1 + mx^*) > 0$, in which x^* is the positive root of the following equation:

$$-ma_1 a_4 x^2 + (mr_1 a_4 - a_1 a_4)x + a_4 r_1 - r_3 = 0. \quad (30)$$

We can find the following four biologically feasible equilibrium points:

(1) $E_0 = (0, 0, 0)$
(2) $E_1 = ((r_1/a_1), 0, 0)$
(3) $E_2 = (\breve{x}, \breve{y}, 0)$; it exhibits if the condition $[\mathbb{H}_1]$ is true, where $\breve{x} = (r_2/(a_2 - mr_2))$ and $\breve{y} = (a_2[r_1(a_2 - mr_2) - a_1 r_2]/(a_2 - mr_2)^2)$
(4) $E_3 = (x^*, y^*, z^*)$; it exhibits if the condition $[\mathbb{H}_2]$ is true, where $y^* = (r_3/a_4)$ and $z^* = (a_2 x^* - r_2(1 + mx^*))/(1 + mx^*)a_3$

The Jacobian matrix about system (16) at arbitrary point (x, y, z) is as follows

$$J = \begin{bmatrix} r_1 - 2a_1 x - \dfrac{y}{(1+mx)^2} & -\dfrac{x}{1+mx} & 0 \\ \dfrac{a_2 y}{(1+mx)^2} & \dfrac{a_2 x}{1+mx} - r_2 - a_3 z & -a_3 y \\ 0 & a_4 z & a_4 y - r_3 \end{bmatrix}. \quad (31)$$

Theorem 3. *The trivial equilibrium E_0 of system (16) is unstable.*

Proof. The Jacobian matrix at $E_0 = (0, 0, 0)$ is as follows:

$$J(E_0) = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & -r_2 & 0 \\ 0 & 0 & -r_3 \end{bmatrix}. \quad (32)$$

Let $\lambda = s^\theta$; the characteristic equation of (32) is

$$(\lambda - r_1)(\lambda + r_2)(\lambda + r_3) = 0. \quad (33)$$

So, the roots of the characteristic equation are $\lambda_1 = r_1, \lambda_2 = -r_2$, and $\lambda_3 = -r_3$. Therefore,

$$\begin{aligned} |\arg(\lambda_1)| &= 0 < \frac{\theta\pi}{2}, \\ |\arg(\lambda_2)| &= |\arg(\lambda_3)| = \pi > \frac{\theta\pi}{2}. \end{aligned} \quad (34)$$

Consequently, by Lemma 5, the trivial equilibrium E_0 of system (16) is unstable. □

Theorem 4. *If $(a_2 r_1/(a_1 + mr_1)) - r_2 < 0$ is met, the boundary equilibrium E_1 of system (16) is locally asymptotically stable.*

Proof. The corresponding Jacobian matrix at $E_1 = ((r_1/a_1), 0, 0)$ is shown below:

$$J(E_1) = \begin{bmatrix} -r_1 & -\dfrac{r_1}{a_1 + mr_1} & 0 \\ 0 & \dfrac{a_2 r_1}{a_1 + mr_1} - r_2 & 0 \\ 0 & 0 & -r_3 \end{bmatrix}. \quad (35)$$

At this point, the characteristic equation corresponding to (35) is

$$(\lambda + r_1)(\lambda + r_3)\left(\lambda - \left(\frac{a_2 r_1}{a_1 + mr_1} - r_2\right)\right) = 0. \quad (36)$$

So, the roots of the characteristic equation are $\lambda_1 = -r_1, \lambda_2 = (a_2 r_1/(a_1 + mr_1)) - r_2, \lambda_3 = -r_3$. Owing to the assumptions,

$$|\arg(\lambda_1)| = |\arg(\lambda_2)| = |\arg(\lambda_3)| = \pi > \frac{\theta\pi}{2}. \quad (37)$$

Consequently, E_1 of system (16) is locally asymptotically stable by Lemma 5. □

Theorem 5. *In the case of $[\mathbb{H}_1]$, if one of the following conditions is met,*

(1) $r_1(a_2 - mr_2) - a_1 r_2 = (a_1 a_2/m) < (r_3(a_2 - mr_2)^2/a_2 a_4)$
(2) $r_1(a_2 - mr_2) - a_1 r_2 < \min\{(a_1 a_2/m), (r_3(a_2 - mr_2)^2/a_2 a_4)\}$
(3) $(a_1 a_2/m) < r_1(a_2 - mr_2) - a_1 r_2 < (r_3(a_2 - mr_2)^2/a_2 a_4)$ and $(p_1/2\sqrt{p_2}) < \cos(\theta\pi/2)$, where p_1 and p_2 are given in the following proof; then, the boundary equilibrium $E_2 = (\breve{x}, \breve{y}, 0)$ of system (16) is locally asymptotically stable

Proof. In the case of $[\mathbb{H}_1]$, the boundary equilibrium $E_2 = (\breve{x}, \breve{y}, 0)$ of system exhibits. The Jacobian matrix at $E_2 = (\breve{x}, \breve{y}, 0)$ is as follows:

$$J(E_2) = \begin{bmatrix} \frac{m\check{x}\check{y}}{(1+m\check{x})^2} - a_1\check{x} & -\frac{\check{x}}{1+m\check{x}} & 0 \\ \frac{a_2\check{y}}{(1+m\check{x})^2} & 0 & -a_3\check{y} \\ 0 & 0 & a_4\check{y} - r_3 \end{bmatrix}. \quad (38)$$

The characteristic equation of (38) is

$$(\lambda^2 - p_1\lambda + p_2)(\lambda - p_3) = 0, \quad (39)$$

where $p_1 = (m\check{x}\check{y}/(1+m\check{x})^2) - a_1\check{x}$, $p_2 = (a_2\check{x}\check{y}/(1+m\check{x})^3)$, and $p_3 = a_4\check{y} - r_3$.

(1) If $r_1(a_2 - mr_2) - a_1r_2 = (a_1a_2/m) < (r_3(a_2 - mr_2)^2/a_2a_4)$, we can find that $p_3 < 0$ and $p_1 = 0$; all characteristic roots of equation (39) are $\lambda_1 = \sqrt{p_2}i, \lambda_2 = -\sqrt{p_2}i$, and $\lambda_3 = p_3$; therefore,

$$\begin{aligned} &|\arg(\lambda_1)| = |\arg(\lambda_2)| = \frac{\pi}{2} > \frac{\theta\pi}{2}, \\ &|\arg(\lambda_3)| = \pi > \frac{\theta\pi}{2}. \end{aligned} \quad (40)$$

Hence, by Lemma 5, the equilibrium E_2 about system (16) is locally asymptotically stable.

(2) If $r_1(a_2 - mr_2) - a_1r_2 < \min\{(a_1a_2/m), (r_3(a_2 - mr_2)^2/a_2a_4)\}$, we can find that $p_3 < 0$ and $p_1 < 0$. Obviously,

$$\begin{aligned} &|\arg(\lambda_1)| > \frac{\theta\pi}{2}, \\ &|\arg(\lambda_2)| > \frac{\theta\pi}{2}, \\ &|\arg(\lambda_3)| = \pi > \frac{\theta\pi}{2}. \end{aligned} \quad (41)$$

Accordingly, by Lemma 5, the equilibrium E_2 about system (16) is locally asymptotically stable.

(3) Using the given conditions, we can obtain all characteristic roots of equation (39) are $\lambda_1 = (p_1/2) + (\sqrt{4p_2 - p_1^2}/2)i, \lambda_2 = (p_1/2) - (\sqrt{4p_2 - p_1^2}/2)i$, and $\lambda_3 = p_3$. Owing to $(p_1/2\sqrt{p_2}) < \cos(\theta\pi/2)$, therefore,

$$\begin{aligned} &|\arg(\lambda_1)| > \frac{\theta\pi}{2}, \\ &|\arg(\lambda_2)| > \frac{\theta\pi}{2}, \\ &|\arg(\lambda_3)| = \pi > \frac{\theta\pi}{2}. \end{aligned} \quad (42)$$

As a result, by Lemma 5, the equilibrium E_2 about system (16) is locally asymptotically stable. □

Theorem 6. *If* $[\mathbb{H}_3]$: $a_1 > (my^*/(1+mx^*))$ *is satisfied, then the positive equilibrium* $E_3 = (x^*, y^*, z^*)$ *about system (16) is globally asymptotically stable.*

Proof. The Jacobian matrix at $E_3 = (x^*, y^*, z^*)$ is as follows:

$$J(E_3) = \begin{bmatrix} \frac{mx^*y^*}{(1+mx^*)^2} - a_1x^* & -\frac{x^*}{1+mx^*} & 0 \\ \frac{a_2y^*}{(1+mx^*)^2} & 0 & -a_3y^* \\ 0 & a_4z^* & 0 \end{bmatrix}. \quad (43)$$

The characteristic equation of (43) is

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \quad (44)$$

where $A_1 = -((mx^*y^*/(1+mx^*)^2) - a_1x^*)$, $A_2 = (a_2x^*y^*/(1+mx^*)^3) + a_3a_4y^*z^*$, and $A_3 = -a_3a_4y^*z^*((mx^*y^*/(1+mx^*)^2) - a_1x^*)$. Based on our assumptions, we have

$$\begin{aligned} &A_1 > 0, \\ &\begin{vmatrix} A_1 & A_3 \\ 1 & A_2 \end{vmatrix} > 0, \\ &A_3 > 0. \end{aligned} \quad (45)$$

By the Routh–Hurwitz criterion, all roots of (44) are negative real parts; therefore,

$$\begin{aligned} &|\arg(\lambda_1)| > \frac{\theta\pi}{2}, \\ &|\arg(\lambda_2)| > \frac{\theta\pi}{2}, \\ &|\arg(\lambda_3)| > \frac{\theta\pi}{2}. \end{aligned} \quad (46)$$

As a result, by Lemma 5, the equilibrium E_3 about system (16) is locally asymptotically stable.

Let us consider the Lyapunov function:

$$V(x, y, z) = \left(x - x^* - x^* \ln \frac{x}{x^*}\right) + \frac{1 + mx^*}{a_2}\left(y - y^* - y^* \ln \frac{y}{y^*}\right) + \frac{a_3(1 + mx^*)}{a_2 a_4}\left(z - z^* - z^* \ln \frac{z}{z^*}\right). \quad (47)$$

Obviously, $V(x, y, z) > 0$ for any $x, y, z > 0$, except for the positive equilibrium $E_3 = (x^*, y^*, z^*)$.

By Lemma 6, we have

$$\begin{aligned}
D^\theta V(x, y, z) \le{}& \left(1 - \frac{x^*}{x}\right) D^\theta x(t) + \frac{1 + mx^*}{a_2}\left(1 - \frac{y^*}{y}\right) D^\theta y(t) \\
&+ \frac{a_3(1 + mx^*)}{a_2 a_4}\left(1 - \frac{z^*}{z}\right) D^\theta z(t) \\
={}& (x - x^*)\left(r_1 - a_1 x - \frac{y}{1 + mx}\right) + \frac{1 + mx^*}{a_2}(y - y^*)\left(\frac{a_2 x}{1 + mx} - r_2 - a_3 z\right) \\
&+ \frac{a_3(1 + mx^*)}{a_2 a_4}(z - z^*)(a_4 y - r_3) \\
={}& (x - x^*)\left(a_1 x^* - a_1 x - \frac{y}{1 + mx} + \frac{y^*}{1 + mx}\right) \\
&+ \frac{1 + mx^*}{a_2}(y - y^*)\left(\frac{a_2 x}{1 + mx} - \frac{a_2 x^*}{1 + mx^*} - a_3 z + a_3 z^*\right) \\
&+ \frac{a_3(1 + mx^*)}{a_2 a_4}(y - y^*) a_4 (y - y^*) \\
={}& -a_1 (x - x^*)^2 + \frac{m y^* (x - x^*)^2}{(1 + mx)(1 + mx^*)}.
\end{aligned} \quad (48)$$

Since $a_1 > (my^*/(1 + mx^*))$, then we have $D^\theta V(x, y, z) \le 0$. Thus, E_3 is globally asymptotically stable. □

4. Analysis of the Delayed Model

The conditions for nonnegativity boundedness, existence, and uniqueness derived for system (16) also apply to system (2). Systems (2) and (16) have identical equilibrium points. Due to the impact of time lags τ_1 and τ_2, the stability of system (2) needs to be rediscussed. Next, the stability and branch of system (2) are studied by selecting τ_1 and τ_2 as key parameters, and the critical bifurcation value is discussed precisely.

4.1. The Bifurcation of System (2) Caused by Delay τ_1. In the following analysis, we focus on time delay τ_1 as the bifurcation parameter of system (2) and obtain the critical value of Hopf bifurcation of the system.

Making transformation, $P_1(t) = x(t) - x^*$, $P_2(t) = y(t) - y^*$, and $P_3(t) = z(t) - z^*$. In consequence, system (2) is able to be transformed into

$$\begin{cases} D^{\theta}P_1(t) = (P_1(t)+x^*)\left(r_1 - a_1(P_1(t)+x^*) - \dfrac{P_2(t-\tau_1)+y^*}{1+m(P_1(t)+x^*)}\right), \\ D^{\theta}P_2(t) = (P_2(t)+y^*)\left(\dfrac{a_2(P_1(t-\tau_2)t+nx^*)}{1+m(P_1(t)+x^*)} - r_2 - a_3(P_3(t-\tau_1)+z^*)\right), \\ D^{\theta}P_3(t) = (P_3(t)+z^*)(a_4(P_2(t-\tau_2)+y^*)-r_3). \end{cases} \tag{49}$$

The linearized scheme from system (49) results in

$$\begin{cases} D^{\theta}P_1(t) = b_{11}P_1(t) + b_{12}P_2(t-\tau_1), \\ D^{\theta}P_2(t) = b_{21}P_1(t) + b_{22}P_2(t) + b_{23}P_1(t-\tau_2) + b_{24}P_3(t-\tau_1), \\ D^{\theta}P_3(t) = b_{31}P_3(t) + b_{32}P_2(t-\tau_2), \end{cases} \tag{50}$$

where

$$\begin{aligned} b_{11} &= r_1 - 2a_1x^* - \frac{y^*}{(1+mx^*)^2}, \\ b_{12} &= -\frac{x^*}{1+mx^*}, \\ b_{21} &= -\frac{a_2mx^*y^*}{(1+mx^*)^2}, \\ b_{22} &= \frac{a_2x^*}{1+mx^*} - r_2 - a_3z^*, \\ b_{23} &= \frac{a_2y^*}{1+mx^*}, \\ b_{24} &= -a_3y^*, \\ b_{31} &= a_4y^* - r_3, \\ b_{32} &= a_4z^*. \end{aligned} \tag{51}$$

The characteristic equation of system (50) is as shown below:

$$U_1(s) + U_2(s)e^{-s\tau_1} = 0, \tag{52}$$

where

$$\begin{aligned} U_1(s) &= s^{3\theta} + (-b_{31}-b_{22}-b_{11})s^{2\theta} + (b_{22}b_{31}+b_{11}b_{31}+b_{11}b_{22})s^{\theta} - b_{11}b_{22}b_{31}, \\ U_2(s) &= b_{12}b_{21}b_{31} - b_{21}b_{12}s^{\theta} + \left(b_{12}b_{23}b_{31} + b_{11}b_{32}b_{24} - b_{12}b_{23}s^{\theta} - b_{32}b_{24}s^{\theta}\right)e^{-s\tau_2}. \end{aligned} \tag{53}$$

The real and imaginary parts of $U_k(s)$ $(k=1,2)$ are represented by U_k^r and U_k^i. Suppose s is a purely imaginary root of (52), where $s = \omega_1(\cos(\pi/2) + i\sin(\pi/2))$ $(\omega_1 > 0)$; it follows from (52) that

$$\begin{cases} U_2^r\cos\omega_1\tau_1 + U_2^i\sin\omega_1\tau_1 = -U_1^r, \\ U_2^i\cos\omega_1\tau_1 - U_2^r\sin\omega_1\tau_1 = -U_1^i. \end{cases} \tag{54}$$

In view of (54), we derive that

$$\begin{cases} \cos\omega_1\tau_1 = -\dfrac{h_1(\omega_1)}{h_3(\omega_1)}, \\ \sin\omega_1\tau_1 = -\dfrac{h_2(\omega_1)}{h_3(\omega_1)}, \end{cases} \tag{55}$$

where $h_1(\omega_1) = U_1^rU_2^r + U_1^iU_2^i$, $h_2(\omega_1) = U_1^rU_2^i - U_2^rU_1^i$, and $h_3(\omega_1) = (U_2^r)^2 + (U_2^i)^2$. It is apparent from (55) that

$$h_1^2(\omega_1) + h_2^2(\omega_1) - h_3^2(\omega_1) = 0. \tag{56}$$

In terms of $\cos \omega_1\tau_1 = -(h_1(\omega_1)/h_3(\omega_1))$, we obtain

$$\tau_{10}^k = \frac{1}{\omega_1}\left[\arccos\left(-\frac{h_1(\omega_1)}{h_3(\omega_1)}\right) + 2k\pi\right], \quad k = 0, 1, 2 \ldots . \tag{57}$$

Suppose the equation of (56) has a positive real root ω_{10}; we make

$$\tau_{10} = \min\left\{\tau_{10}^k, \quad k = 0, 1, 2 \ldots\right\}, \tag{58}$$

where τ_{10}^k is provided by (57).

If $\tau_1 = 0$, then (52) becomes

$$\varsigma_1(s) + \varsigma_2(s)e^{-s\tau_2} = 0, \tag{59}$$

where

$$\begin{aligned}\varsigma_1(s) &= s^{3\theta} + (-b_{31} - b_{22} - b_{11})s^{2\theta} + (b_{22}b_{31} + b_{11}b_{31} + b_{11}b_{22} - b_{12}b_{21})s^{\theta} + b_{12}b_{21}b_{31} - b_{11}b_{22}b_{31},\\ \varsigma_2(s) &= (-b_{12}b_{23} - b_{32}b_{24})s^{\theta} + b_{12}b_{23}b_{31} + b_{11}b_{32}b_{24}.\end{aligned} \tag{60}$$

Suppose that ς_k^r and ς_k^i represent the real and imaginary parts of $\varsigma_k(s)$ $(k = 1, 2)$, s is a purely imaginary root of (59), and $s = \overline{\omega}_1(\cos(\pi/2) + i\sin(\pi/2))$ $(\overline{\omega}_1 > 0)$; we can get that

$$\begin{cases}\varsigma_2^r \cos\overline{\omega}_1\tau_2 + \varsigma_2^i \sin\overline{\omega}_1\tau_2 = -\varsigma_1^r,\\ \varsigma_2^i \cos\overline{\omega}_1\tau_2 - \varsigma_2^r \sin\overline{\omega}_1\tau_2 = -\varsigma_1^i.\end{cases} \tag{61}$$

Based on (61), we have

$$\begin{cases}\cos\overline{\omega}_1\tau_2 = -\dfrac{k_1(\overline{\omega}_1)}{k_3(\overline{\omega}_1)},\\[2ex] \sin\overline{\omega}_1\tau_2 = -\dfrac{k_2(\overline{\omega}_1)}{k_3(\overline{\omega}_1)},\end{cases} \tag{62}$$

where $k_1(\overline{\omega}_1) = \varsigma_1^r\varsigma_2^r + \varsigma_1^i\varsigma_2^i$, $k_2(\overline{\omega}_1) = \varsigma_1^r\varsigma_2^i - \varsigma_2^r\varsigma_1^i$, and $k_3(\overline{\omega}_1) = (\varsigma_2^r)^2 + (\varsigma_2^i)^2$. It is apparent from (62) that

$$k_1^2(\overline{\omega}_1) + k_2^2(\overline{\omega}_1) - k_3^2(\overline{\omega}_1) = 0. \tag{63}$$

In the light of $\cos\overline{\omega}_1\tau_2 = -(k_1(\overline{\omega}_1)/k_3(\overline{\omega}_1))$, we obtain

$$\overline{\tau}_{20}^k = \frac{1}{\overline{\omega}_1}\left[\arccos\left(-\frac{k_1(\overline{\omega}_1)}{k_3(\overline{\omega}_1)}\right) + 2k\pi\right], \quad k = 0, 1, 2 \ldots . \tag{64}$$

Suppose the equation of (63) has a positive real root, we make

$$\overline{\tau}_{20} = \min\left\{\overline{\tau}_{20}^k, \quad k = 0, 1, 2 \ldots\right\}, \tag{65}$$

where $\overline{\tau}_{20}^k$ is provided by (64).

Remark 1. If equation (56) has no positive roots, then the system does not have bifurcation points. On the contrary, if equation (56) has more than one positive root, we take the minimum of all the roots. As mentioned above, $\tau_{10} = \min\{\tau_{10}^k, k = 0, 1, 2 \ldots\}$. Similarly, $\overline{\tau}_{20}^k$ is obtained this way.

In order to better search for the criterion of the occurrence for bifurcation, the following hypotheses are helpful and essential: $[\mathbb{H}_4]$: $((\widehat{E}_1\widehat{F}_1 + \widehat{E}_2\widehat{F}_2)/(\widehat{F}_1^2 + \widehat{F}_2^2)) > 0$, where $\widehat{E}_1$, $\widehat{E}_2$, $\widehat{F}_1$, and $\widehat{F}_2$ are described in the following.

Lemma 8. *Let $s(\tau_1) = \xi(\tau_1) + i\omega_1(\tau_1)$ be the root of (17) near $\tau_1 = \tau_{1j}$ meeting $\xi(\tau_{1j}) = 0$ and $\omega_1(\tau_{1j}) = \omega_{10}$, so the following transversality criteria are true:*

$$\mathrm{Re}\left[\frac{ds}{d\tau_1}\right]\Big|_{(\omega_1=\omega_{10},\tau_1=\tau_{10})} > 0, \tag{66}$$

where $\omega_{10}$and τ_{10} are the critical frequency and the bifurcation point individually.

Proof. After differentiating equation (52) about τ_1, we have

$$U_1'(s)\frac{ds}{d\tau_1} + U_2'(s)\frac{ds}{d\tau_1}e^{-s\tau_1} + U_2(s)e^{-s\tau_1}\left(-\tau_1\frac{ds}{d\tau_1} - s\right) = 0. \tag{67}$$

So, we can obtain

$$\frac{ds}{d\tau_1} = \frac{\widehat{E}(s)}{\widehat{F}(s)}, \tag{68}$$

where

$$\begin{aligned}\widehat{E}(s) &= sU_2(s)e^{-s\tau_1},\\ \widehat{F}(s) &= U_1'(s) + [U_2'(s) - \tau_1U_2(s)]e^{-s\tau_1}.\end{aligned} \tag{69}$$

Let $\widehat{E}_1$ and $\widehat{E}_2$ be the real and imaginary parts of $\widehat{E}(s)$ individually. $\widehat{F}_1$ and $\widehat{F}_2$ be the real and imaginary parts of $\widehat{F}(s)$ severally. After several algebraic calculation, we get from (68) that

$$\mathrm{Re}\left[\frac{ds}{d\tau_1}\right]\Big|_{(\omega_1=\omega_{10},\tau_1=\tau_{10})} = \frac{\widehat{E}_1\widehat{F}_1 + \widehat{E}_2\widehat{F}_2}{\widehat{F}_1^2 + \widehat{F}_2^2}, \tag{70}$$

where

$$\begin{aligned}
\widehat{E}_1 &= \omega_{10}\left(U_2^r \sin \omega_{10}\tau_{10} - U_2^i \cos \omega_{10}\tau_{10}\right),\\
\widehat{E}_2 &= \omega_{10}\left(U_2^r \cos \omega_{10}\tau_{10} + U_2^i \sin \omega_{10}\tau_{10}\right),\\
\widehat{F}_1 &= (U_1')^r + \left((U_2')^r - \tau_{10}U_2^r\right)\cos \omega_{10}\tau_{10} + \left((U_2')^i - \tau_{10}U_2^i\right)\sin \omega_{10}\tau_{10},\\
\widehat{F}_2 &= (U_1')^i + \left((U_2')^i - \tau_{10}U_2^i\right)\cos \omega_{10}\tau_{10} - \left((U_2')^r - \tau_{10}U_2^r\right)\sin \omega_{10}\tau_{10}.
\end{aligned} \tag{71}$$

As a result, suppose $[\mathbb{H}_4]$ implies that the transversality criteria are true. That is the proof of Lemma 8. □

With the support of Lemmas 7 and 8, the under theorem can be derived.

Theorem 7. *In the case of* $[\mathbb{H}_2]$, $[\mathbb{H}_4]$, *and* $\tau_2 \in [0, \overline{\tau}_{20})$, *we have the following results:*

(1) The positive equilibrium E_3 *of system (2) is asymptotically stable when* $\tau_1 \in [0, \tau_{10})$.

(2) System (2) exhibits a Hopf bifurcation at E_3 *when* $\tau_1 = \tau_{10}$, *i.e., it has a branch of periodic solution bifurcating from* E_3 *near* $\tau_1 = \tau_{10}$

4.2. The Bifurcation of System (2) Caused by Delay τ_2. In the following discussion, time delay τ_2 is taken as the bifurcation parameter of system (2), and the Hopf bifurcation criterion of the system is obtained through theoretical analysis.

The characteristic equation about system (2) is available:

$$V_1(s) + V_2(s)e^{-s\tau_2} = 0, \tag{72}$$

where

$$\begin{aligned}
V_1(s) &= s^{3\theta} + (-b_{31} - b_{22} - b_{11})s^{2\theta} + (b_{22}b_{31} + b_{11}b_{31} + b_{11}b_{22})s^{\theta}\\
&\quad - b_{11}b_{22}b_{31} + \left(b_{12}b_{21}b_{31} - b_{21}b_{12}s^{\theta}\right)e^{-s\tau_1},\\
V_2(s) &= \left(b_{12}b_{23}b_{31} + b_{11}b_{32}b_{24} - b_{12}b_{23}s^{\theta} - b_{32}b_{24}s^{\theta}\right)e^{-s\tau_1}.
\end{aligned} \tag{73}$$

The real and imaginary parts of $V_k(s)$ $(k = 1, 2)$ are represented by V_k^r and V_k^i. Suppose s is a purely imaginary root of (72), where $s = \omega_2(\cos(\pi/2) + i\sin(\pi/2))$ $(\omega_2 > 0)$, and we have

$$\begin{cases} V_2^r \cos \omega_2\tau_2 + V_2^i \sin \omega_2\tau_2 = -V_1^r,\\ V_2^i \cos \omega_2\tau_2 - V_2^r \sin \omega_2\tau_2 = -V_1^i. \end{cases} \tag{74}$$

With the help of (74), we have

$$\begin{cases} \cos \omega_2\tau_2 = -\dfrac{l_1(\omega_2)}{l_3(\omega_2)},\\[2ex] \sin \omega_2\tau_2 = -\dfrac{l_2(\omega_2)}{l_3(\omega_2)}, \end{cases} \tag{75}$$

where $l_1(\omega_2) = V_1^rV_2^r + V_1^iV_2^i$, $l_2(\omega_2) = V_1^rV_2^i - V_2^rV_1^i$, and $l_3(\omega_2) = (V_2^r)^2 + (V_2^i)^2$. From (75), one has

$$l_1^2(\omega_2) + l_2^2(\omega_2) - l_3^2(\omega_2) = 0. \tag{76}$$

In terms of $\cos \omega_2\tau_2 = -(l_1(\omega_2)/l_3(\omega_2))$, we obtain

$$\tau_{20}^k = \frac{1}{\omega_2}\left[\arccos\left(-\frac{l_1(\omega_2)}{l_3(\omega_2)}\right) + 2k\pi\right], \quad k = 0, 1, 2 \ldots. \tag{77}$$

Support the equation of (76) has a positive real root ω_{20}, and we make

$$\tau_{20} = \min\left\{\tau_{20}^k, \quad k = 0, 1, 2 \ldots\right\}, \tag{78}$$

where τ_{20}^k is provided by (77).

Once eliminating τ_2 from (72), then

$$v_1(s) + v_2(s)e^{-s\tau_1} = 0, \tag{79}$$

where

$$\begin{aligned}
v_1(s) &= s^{3\theta} + (-b_{31} - b_{22} - b_{11})s^{2\theta} + (b_{22}b_{31} + b_{11}b_{31} + b_{11}b_{22})s^{\theta} - b_{11}b_{22}b_{31},\\
v_2(s) &= b_{12}b_{21}b_{31} + b_{12}b_{23}b_{31} + b_{11}b_{32}b_{24} + (-b_{21}b_{12} - b_{12}b_{23} - b_{32}b_{24})s^{\theta}.
\end{aligned} \tag{80}$$

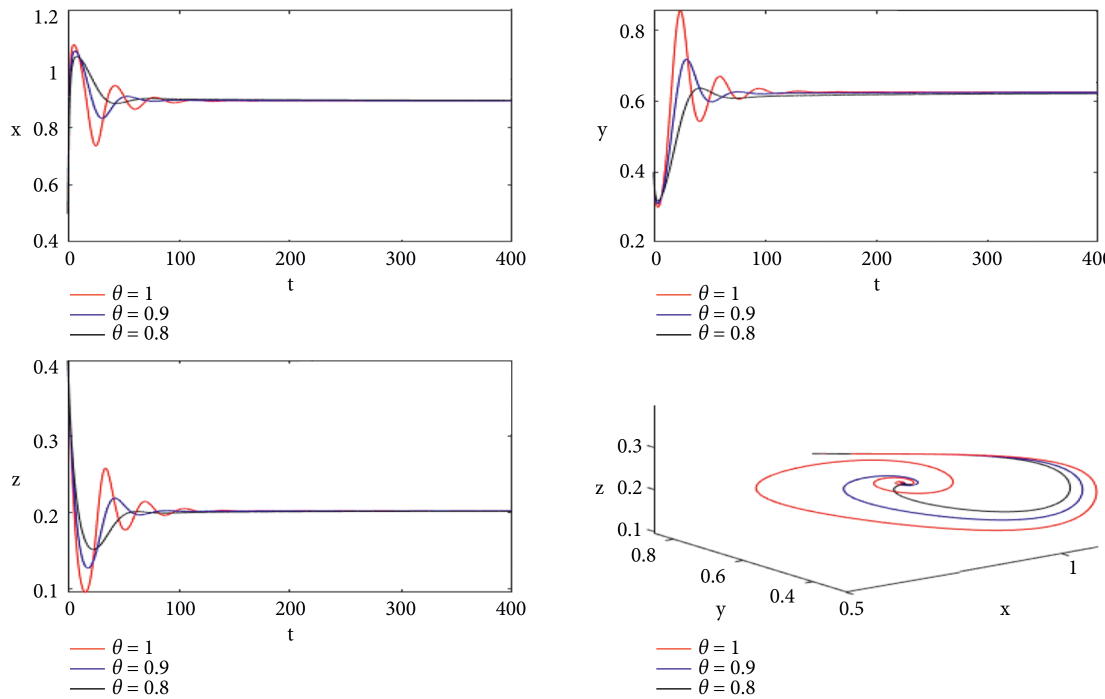

FIGURE 1: Case 1: the trajectories and phase diagrams of system (92) over time at $\tau_1 = \tau_2 = 0$.

Labels v_k^r and v_k^i represent the real and imaginary parts of $v_k(s)$ $(k = 1, 2)$. Suppose s is a purely imaginary root of (79), where $s = \overline{\omega}_2(\cos(\pi/2) + i\sin(\pi/2))$ $(\overline{\omega}_2 > 0)$, and we can derive from (79) that

$$\begin{cases} v_2^r \cos\overline{\omega}_2\tau_1 + v_2^i \sin\overline{\omega}_2\tau_1 = -v_1^r, \\ v_2^i \cos\overline{\omega}_2\tau_1 - v_2^r \sin\overline{\omega}_2\tau_1 = -v_1^i. \end{cases} \tag{81}$$

By means of (81), we have

$$\begin{cases} \cos\overline{\omega}_2\tau_1 = -\dfrac{q_1(\overline{\omega}_2)}{q_3(\overline{\omega}_2)}, \\ \\ \sin\overline{\omega}_2\tau_1 = -\dfrac{q_2(\overline{\omega}_2)}{q_3(\overline{\omega}_2)}, \end{cases} \tag{82}$$

where $q_1(\overline{\omega}_2) = v_1^r v_2^r + v_1^i v_2^i$, $q_2(\overline{\omega}_2) = v_1^r v_2^i - v_2^r v_1^i$, and $q_3(\overline{\omega}_2) = (v_2^r)^2 + (v_2^i)^2$. It is obtained from (82) that

$$q_1^2(\overline{\omega}_2) + q_2^2(\overline{\omega}_2) - q_3^2(\overline{\omega}_2) = 0. \tag{83}$$

Because of $\cos\overline{\omega}_2\tau_1 = -(q_1(\overline{\omega}_2)/q_3(\overline{\omega}_2))$, we obtain

$$\overline{\tau}_{10}^k = \frac{1}{\overline{\omega}_2}\left[\arccos\left(-\frac{q_1(\overline{\omega}_2)}{q_3(\overline{\omega}_2)}\right) + 2k\pi\right], \quad k = 0, 1, 2 \ldots. \tag{84}$$

Suppose (83) has a positive real root; we make

$$\overline{\tau}_{10} = \min\left\{\overline{\tau}_{10}^k, k = 0, 1, 2 \ldots\right\}, \quad k = 0, 1, 2 \ldots, \tag{85}$$

where $\overline{\tau}_{10}^k$ is provided by (84).

In order to better search for the criterion of the occurrence for bifurcation, the following hypotheses are available and essential: $[\mathbb{H}_5]$: $((\aleph_1\mathfrak{I}_1 + \aleph_2\mathfrak{I}_2)/(\mathfrak{I}_1^2 + \mathfrak{I}_2^2)) > 0$, where $\aleph_1$, $\aleph_2$, $\mathfrak{I}_1$, and $\mathfrak{I}_2$ are described in the following.

Lemma 9. *Let $s(\tau_2) = \xi(\tau_2) + i\omega_2(\tau_2)$ be the root of (29) near $\tau_2 = \tau_{2j}$ meeting $\xi(\tau_{2j}) = 0$ and $\omega_2(\tau_{2j}) = \omega_{20}$, so the following transversality criteria are true:*

$$\operatorname{Re}\left[\frac{ds}{d\tau_2}\right]\Big|_{(\omega_2=\omega_{20},\tau_2=\tau_{20})} > 0, \tag{86}$$

where ω_{20} and τ_{20} are the critical frequency and the bifurcation point individually.

Proof. Differentiating equation (72) with respect to τ_2, we have

$$V_1'(s)\frac{ds}{d\tau_2} + V_2'(s)\frac{ds}{d\tau_2}e^{-s\tau_2} + V_2(s)e^{-s\tau_2}\left(-\tau_2\frac{ds}{d\tau_2} - s\right) = 0. \tag{87}$$

So, we can obtain

$$\frac{ds}{d\tau_2} = \frac{\aleph(s)}{\mathfrak{I}(s)}, \tag{88}$$

where

$$\begin{aligned} \aleph(s) &= sV_2(s)e^{-s\tau_2}, \\ \mathfrak{I}(s) &= V_1'(s) + \left[V_2'(s) - \tau_2 V_2(s)\right]e^{-s\tau_2}. \end{aligned} \tag{89}$$

Define $\aleph_1$ and $\aleph_2$ be the real and imaginary parts of $\aleph(s)$ individually. $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be the real and imaginary parts of $\mathfrak{I}(s)$ individually. After several algebraic calculations, we receive from (88) that

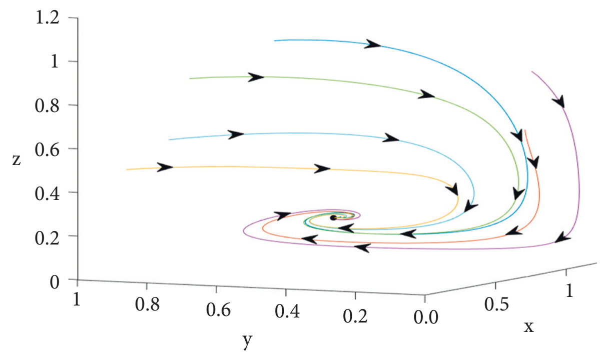

FIGURE 2: Case 1: the phase diagrams for system (92) at $\tau_1 = \tau_2 = 0$ and $\theta = 0.9$.

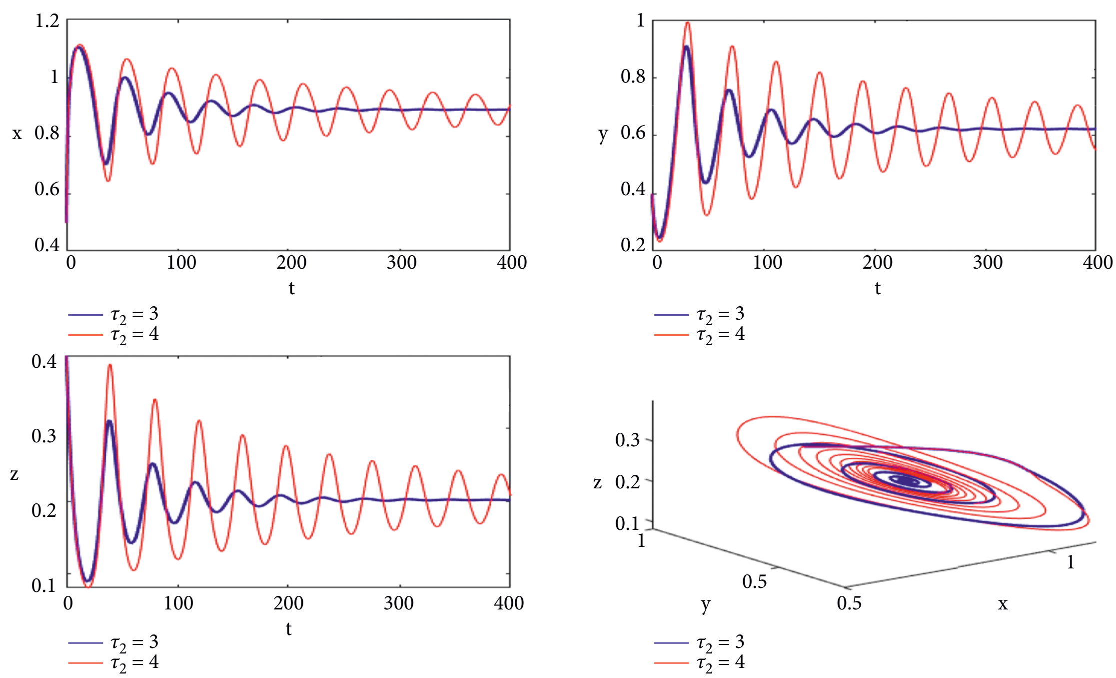

FIGURE 3: Case 2: the trajectories and phase diagrams of system (92) over time at $\tau_1 = 0$.

$$\operatorname{Re}\left[\frac{ds}{d\tau_2}\right]\Big|_{(\omega_2=\omega_{20},\tau_2=\tau_{20})} = \frac{\aleph_1\mathfrak{I}_1 + \aleph_2\mathfrak{I}_2}{\mathfrak{I}_1^2 + \mathfrak{I}_2^2}, \tag{90}$$

where

$$\begin{aligned}
\aleph_1 &= \omega_{20}\left(V_2^r \sin \omega_{20}\tau_{20} - V_2^i \cos \omega_{20}\tau_{20}\right),\\
\aleph_2 &= \omega_{20}\left(V_2^r \cos \omega_{20}\tau_{20} + V_2^i \sin \omega_{20}\tau_{20}\right),\\
\mathfrak{I}_1 &= (V_1')^r + \left((V_2')^r - \tau_{20}V_2^r\right)\cos \omega_{20}\tau_{20} + \left((V_2')^i - \tau_{20}V_2^i\right)\sin \omega_{20}\tau_{20},\\
\mathfrak{I}_2 &= (V_1')^i + \left((V_2')^i - \tau_{20}V_2^i\right)\cos \omega_{20}\tau_{20} - \left((V_2')^r - \tau_{20}V_2^r\right)\sin \omega_{20}\tau_{20}.
\end{aligned} \tag{91}$$

As a result, suppose $[\mathbb{H}_5]$ implies the transversality criteria are true. That is the proof of Lemma 9. □

With the support of Lemmas 7 and 8, the under theorem can be derived.

Theorem 8. *In the case of* $[\mathbb{H}_2]$, $[\mathbb{H}_5]$, *and* $\tau_1 \in [0, \overline{\tau}_{10})$, *we have the following results:*

(1) The equilibrium E_3 *of system (2) is asymptotically stable when* $\tau_2 \in [0, \tau_{20})$

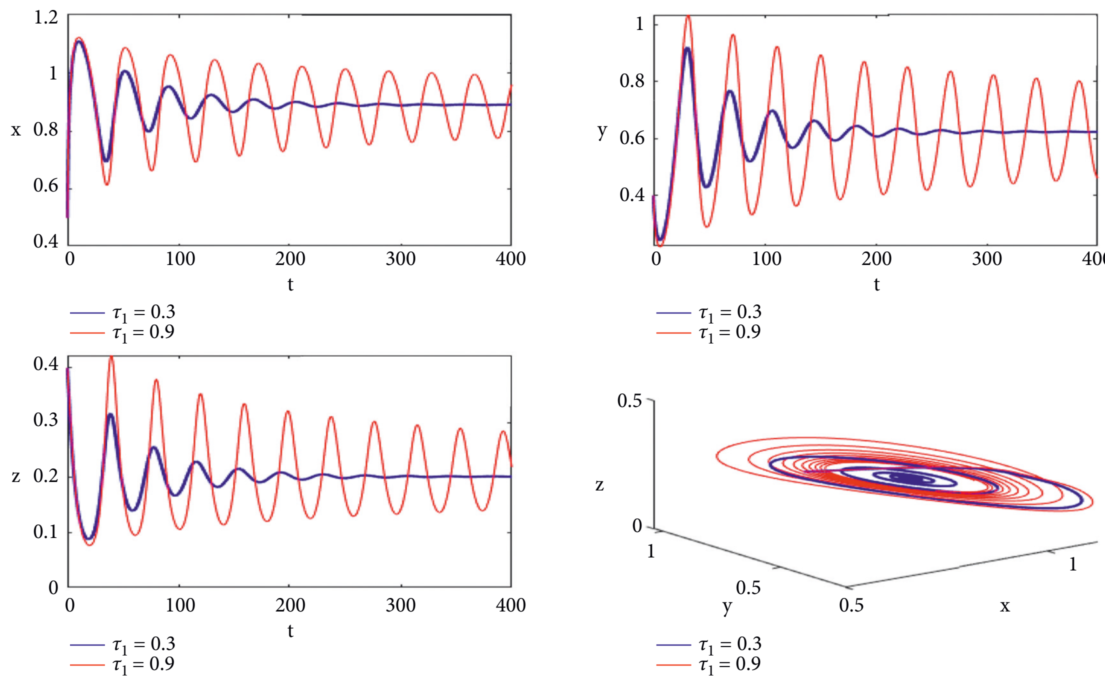

FIGURE 4: Case 2: the trajectories and phase diagrams of system (92) over time at $\tau_2 = 2.5$.

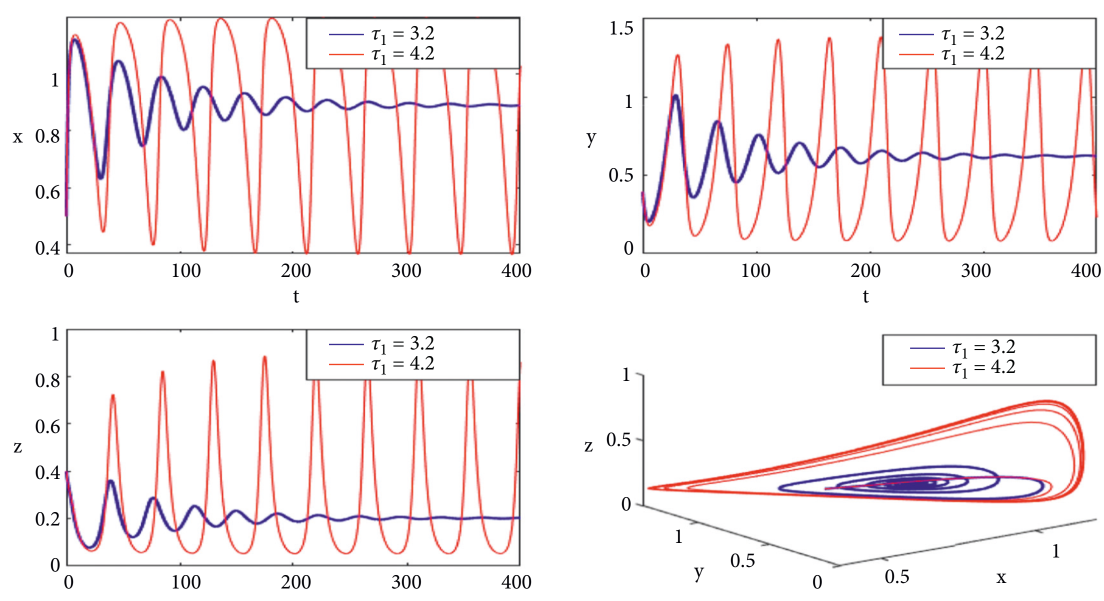

FIGURE 5: Case 3: the trajectories and phase diagrams of system (92) over time at $\tau_2 = 0$.

(2) System (2) occurs a Hopf bifurcation at E_3 when $\tau_2 = \tau_{20}$, i.e., it has a branch of periodic solution bifurcating from E_3 near $\tau_2 = \tau_{20}$

Remark 2. In the previous work, many authors discussed Hopf bifurcations for fractional-order systems with single delay [39, 40], but in this study, we study Hopf bifurcations for fractional-order systems with two delays, which is of great significance for the discussion of Hopf bifurcations for systems with multiple delays.

Remark 3. In fact, the fractional-order system has a wider stability region than the integer order system. In other words, the fractional-order number will affect the stability of the system, taking the fractional-order number as the bifurcation parameter will also cause Hopf bifurcation.

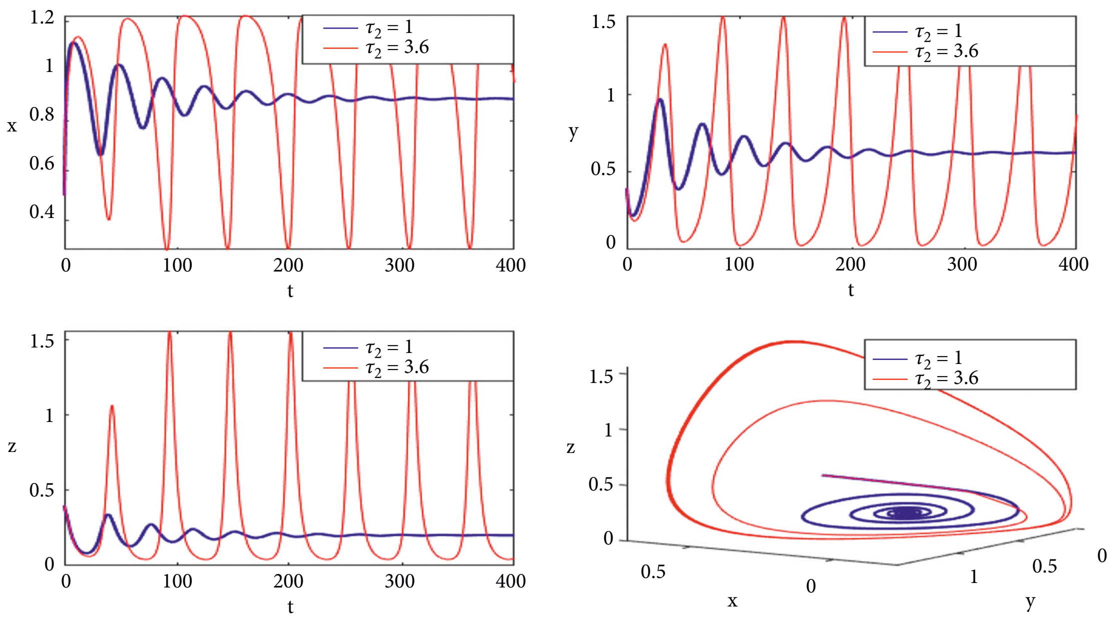

FIGURE 6: Case 3: the trajectories and phase diagrams of system (92) over time at $\tau_1 = 2$.

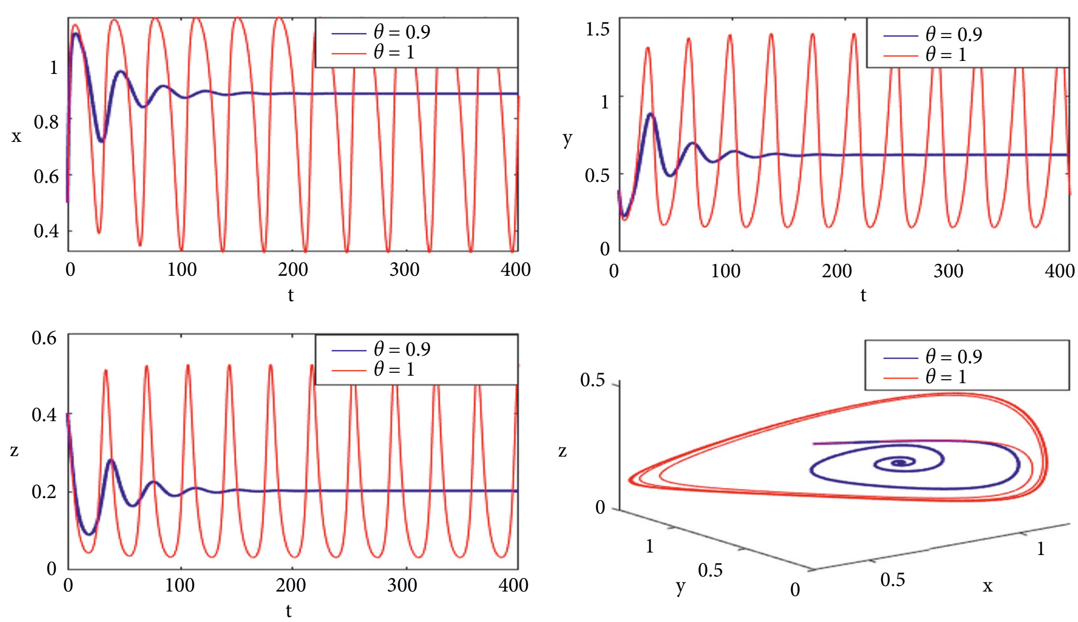

FIGURE 7: Case 4: the trajectories and phase diagrams of system (92) at $\tau_1 = 2$ and $\tau_2 = 0.4$.

5. Numerical Results

Now, we verify our theoretical consequences by numerical simulations. For practical reasons, we only analyze the positive equilibrium E_3, rather than E_0, E_1, or E_2. We set $r_1 = 1.5$, $r_2 = 0.125$, $r_3 = 0.25$, $a_1 = 1.2$, $a_2 = 0.4$, $a_3 = 0.6$, $a_4 = 0.4$, and $m = 0.5$; then, system (2) can be transformed into

$$\begin{cases} D^{\theta}x(t) = x(t)\left(1.5 - 1.2x(t) - \dfrac{y(t-\tau_1)}{1+0.5x(t)}\right), \\ D^{\theta}y(t) = y(t)\left(\dfrac{0.4x(t-\tau_2)}{1+0.5x(t)} - 0.125 - 0.6z(t-\tau_1)\right), \\ D^{\theta}z(t) = z(t)\left(0.4y(t-\tau_2) - 0.25\right). \end{cases} \tag{92}$$

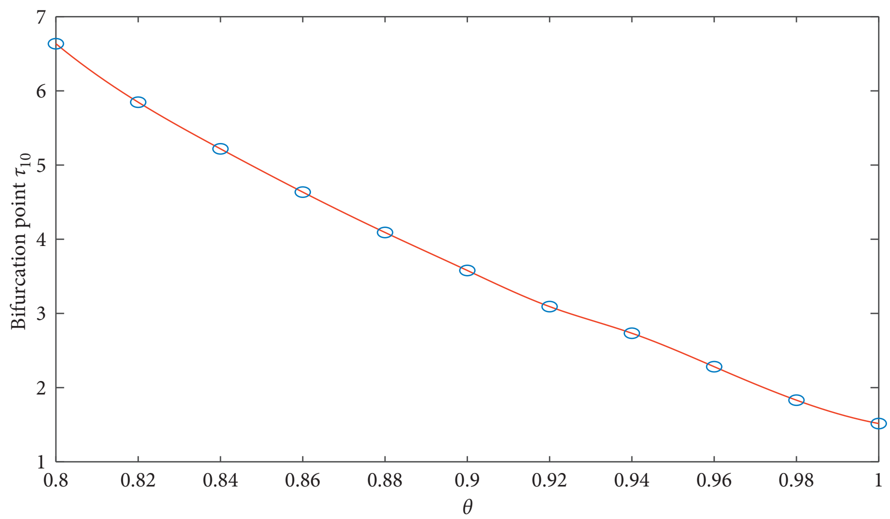

Figure 8: Case 4: effect of θ on bifurcation point τ_{10}.

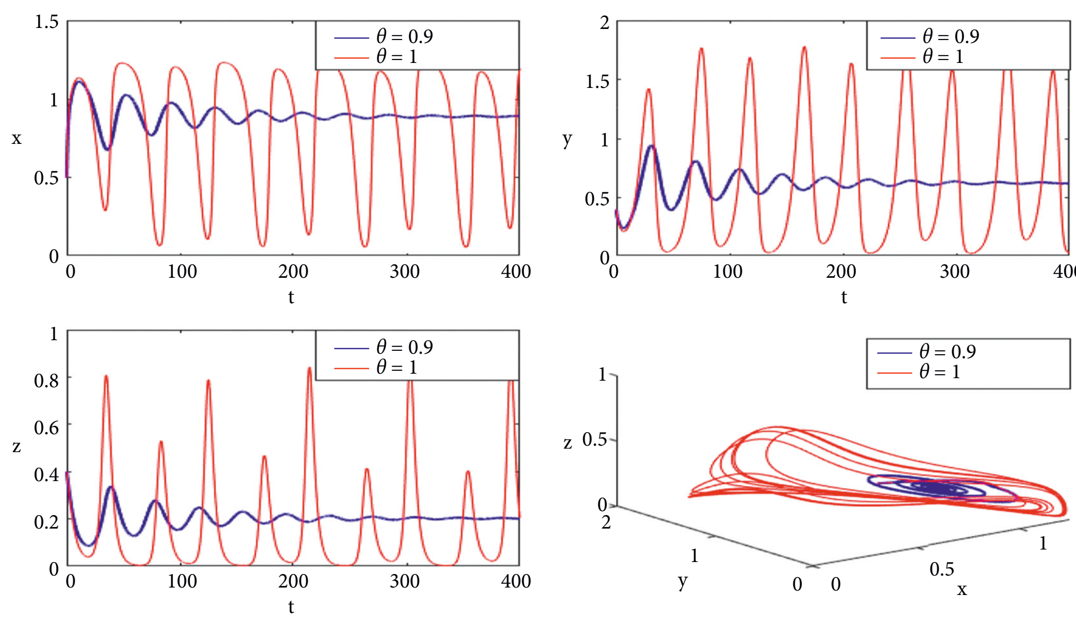

Figure 9: Case 5: the trajectories and phase diagrams of system (92) at $\tau_1 = 0.2$ and $\tau_2 = 3.3$.

We can easily verify that the system only has a positive equilibrium, $E_3 = (0.8895, 0.6250, 0.2021)$.

Case 1. We set $\tau_1 = \tau_2 = 0, \theta = 0.8, 0.9, 1$, and then, we can verify that the system meets the condition of Theorem 6; Figure 1 indicates the positive equilibrium E_3 of system (92) is stable. Particularly, if $\theta = 0.9$, Figure 2 shows that E_3 of system (92) is globally asymptotically stable.

To discuss the bifurcation points about system (92), let us define $\theta = 0.9$.

Case 2. We fix τ_2 and choose τ_1 as the branch parameter to consider the bifurcation of system (92). If $\tau_1 = 0$, we can figure out that $\overline{\tau}_{20} = 3.6581$. In Figure 3, it implies system (92) is asymptotically stable when $\tau_1 = 0$ and $\tau_2 = 3$, and we can clearly find that the system is not stable and Hopf bifurcation appears when $\tau_1 = 0$ and $\tau_2 = 4$. Then, we can calculate that $\tau_{10} = 0.8649$ when $\tau_2 = 2.5$. Based on Theorem 7, system (92) is asymptotically stable if $\tau_1 = 0.3\,(< 0.8649)$ and $\tau_2 = 2.5$. However, if we increase τ_1 from 0.3 to $0.9\,(> 0.8649)$, system (92) is unstable and Hopf bifurcation occurs, see Figure 4.

Case 3. We fix τ_1 and choose τ_2 as the branch parameter to consider the bifurcation of system (92). When $\tau_2 = 0$, we can calculate that $\overline{\tau}_{10} = 4.0646$. In Figure 5, it implies system (92) is asymptotically stable when $\tau_1 = 3.2$ and $\tau_2 = 0$, and we can clearly find that system (92) is not stable and Hopf bifurcation appears when $\tau_1 = 4.2$ and $\tau_2 = 0$. Then, we can get that $\tau_{20} = 3.5566$ when $\tau_1 = 2$. By Theorem

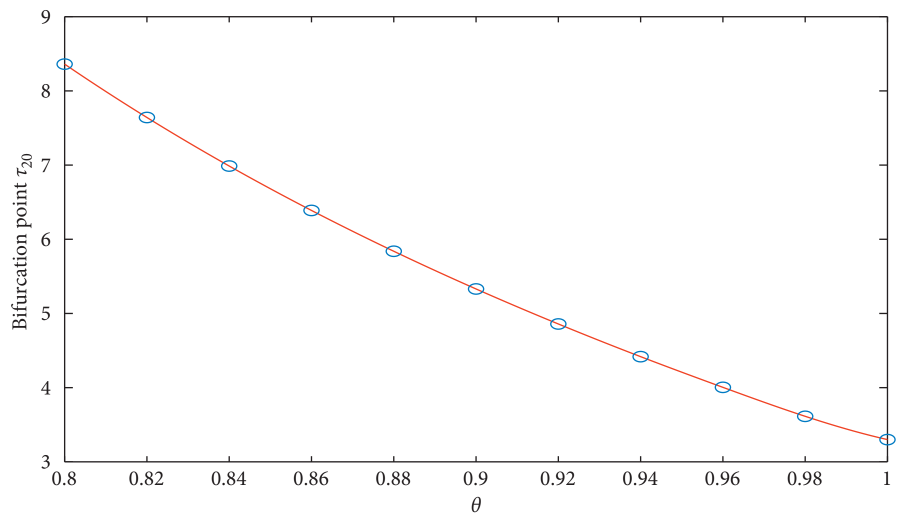

FIGURE 10: Case 5: effect of θ on bifurcation point τ_{20}.

8, system (92) is asymptotically stable when $\tau_1 = 2$ and $\tau_2 = 1\,(< 3.5566)$. However, if we increase τ_2 from 1 to $3.6\,(> 3.5566)$, system (92) is unstable and Hopf bifurcation occurs, see Figure 6.

In order to study the impact of fractional order on bifurcation points, we make the following simulation results.

Case 4. We choose $\tau_2 = 0.4$. When $\theta = 1$, we can calculate $\tau_{10} = 1.5156$. At this time, we take $\tau_1 = 2\,(> 1.5156)$, and Hopf bifurcation appears in system (92). When $\theta = 0.9$, we can calculate $\tau_{10} = 3.5790$; then, we take $\tau_1 = 2\,(< 3.5790)$; system (92) is asymptotically stable. Look at Figure 7. Figure 8 shows the impact of fractional order on τ_1.

Case 5. Similarly, we take $\tau_1 = 0.2$. When $\theta = 1$, we can obtain $\tau_{20} = 3.2996$. At this time, we take $\tau_2 = 3.3\,(> 3.2996)$, and Hopf bifurcation appears in system (92). When $\theta = 0.9$, we can get $\tau_{20} = 5.3302$, and we take $\tau_2 = 3.3\,(< 5.3302)$; system (92) is asymptotically stable (see Figure 9). Figure 10 shows the impact of fractional order on τ_2.

6. Conclusion

In this study, a fractional-order food chain system involving two time delays has been presented. Nonnegative, bounded, existence, and uniqueness about the solution of the system have been proved. For nondelay system, we have discussed the local stability of the system equilibrium point and proved the globally asymptotically stability of the positive equilibrium point by constructing Lyapunov functions. By using time delays as parameters to discuss the Hopf bifurcation, which has showed that when the delay exceeds the critical value, the Hopf bifurcation will appear in the system, that is to say, the system will change from stable to unstable and a periodic solution will appear. In particular, the periodic oscillation behavior of the system could be suppressed by fractional order, which has indicated that the fractional-order system has a larger range of stability region than the integer-order system.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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