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Research Article

Boundary Stabilization of Underground Multirobot System with Virtual String Constraints

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The manipulation of multiple physical connected objects can be described by virtual connected string system. We proposed a Lyapunov-based stabilization control approach for virtual connected string with midway discontinue vertical force. The system is with Riemannian boundary. We use the backstepping method to transform the system into a stable system with Dirichlet boundary. In this way, we get the controller to make sure the closed-loop system converge to the zero point exponentially. Then, we construct a Lyapunov function to analyze the stability for the closed-loop system. We also show the system is well-posedness. Also, we use the active disturbance rejection control (ADRC) to reject the disturbance when disturbance is present. Some numerical stimulations show that the control law is the effective.

1. Introduction

Multirobot transportation system has been one of the most potential research fields in recent years, and the majority of these studies are focused on multiple robots manipulating a single object, while the manipulated object in this paper is a class of multiple physical connected objects (MPO). The manipulation of MPO has been widely used both in daily life and industrial application. As an example, flood shield is used to obstruct flood disaster. These shields are connected with each other, and each shield is immobilized by fixed oblique support, flexible oblique support, or vertical support, which are represented in Figures 1(a)–1(c), respectively. Retaining wall is another typical application of MPO, i.e., retaining mesh, which is supported by multiple linked rods, as shown in Figure 1(d). One more abstracted application, a multiple link model-based human body is held by two-arm service robot, as shown in Figure 1(e). In this application, human beings are abstracted as free-ended multiple connected rigid bodies, and these rigid bodies are manipulated by two arms, respectively. Similar to above applications, the pushing manipulation problem of mobile support robot (MSR, also known as hydraulic support or roof support) on underground fully mechanized coal mining face is a such kind of system as well. Since the manipulated object, i.e., scraper conveyor, is composed of multiple middle troughs (MTs), we simplified scraper conveyor as multiple connected objects with physical connections. As shown in Figures 1(f) and 1(g), both the physical and simulated workplace of multiple MTs manipulated by MSRs on ground is represented, respectively. With simplification, as shown in Figure 1(h), MPO is replaced by external virtual string. Then, the simulated external constraint using virtual connected string is presented, as shown in Figure 1(i).

Due to loadable characteristics as well as its mobility through pushing and advancing, the manipulation problem of MSRs is one of the most complicated problems on underground fully mechanized coal mining face. Based on our previous research on the underground multirobot system [1, 2], we found the main difficulty is modeling the external constraints. Considering the commonalities of above applications, a reasonable assumption is proposed by replacing MPO with virtual connected string constraints, and the main problem is boundary stabilization of virtual connected string

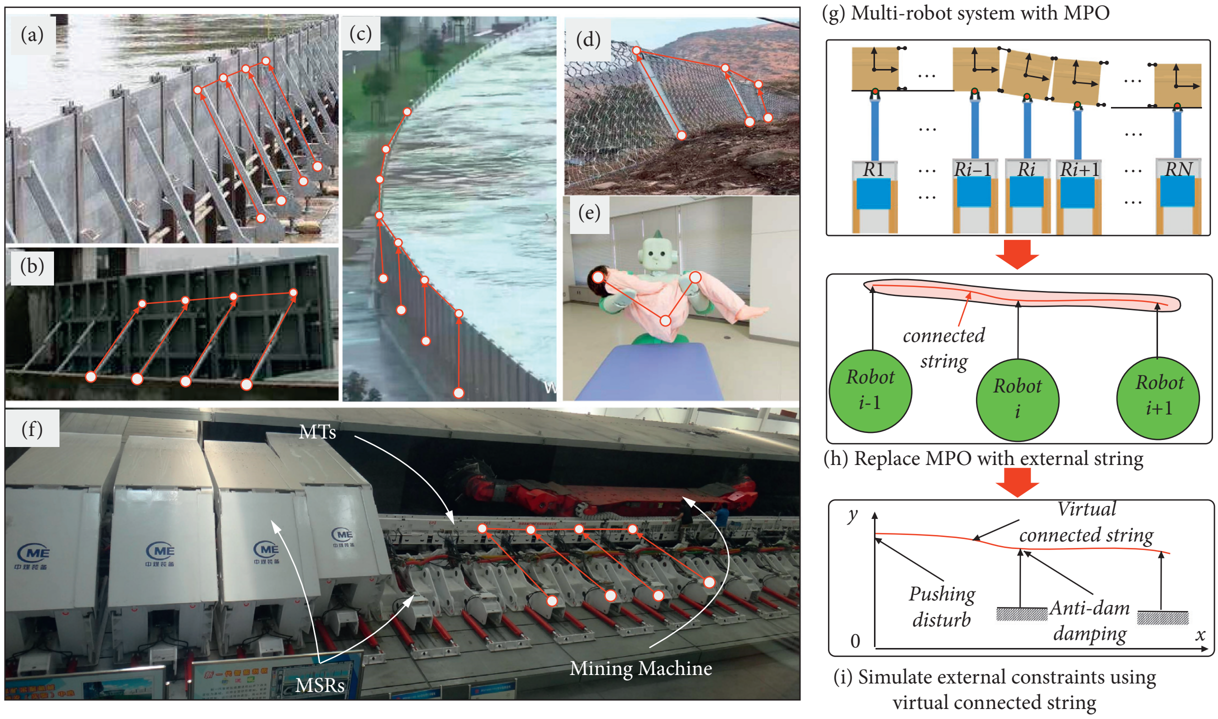

Figure 1: Widely applied scenarios of MPO. (a) Flood shield immobilized by multiple fixed oblique supports. (b) Flood shield immobilized by multiple flexible oblique supports. (c) Flood shield immobilized by multiple vertical supports. (d) Retaining wall supported by multiple bars. (e) Multiple link model-based person held by the two-arm service robot. (f) The simulated workplace of multiple MTs manipulated by MSRs on the ground. The scraper conveyor is composed of many middle troughs (MTs) and each MT is connected with other. MSRs push corresponding MT forward. (g) The diagram of the multirobot system with MPO. (h) Generalized system architecture of the multirobot system, whose manipulating object is replaced with virtual external string. (i) Simulated external constraints through virtual connected string, and both the disturbance and antidamping are presented.

system. Thus, one contribution of this paper is replacing the manipulated MPO with a virtual connected string. Then, the problem is converted to the system stabilization of the connected string system considering pushing disturbance and antidamping.

The study for the virtual connected string equations has drawn significant attention in recent years. It can be applied widely in engineering field. For example, aerial cable, traffic problem, and transmission lines can be described by connected string systems [3, 4]. In [5], the stability of strings with various connected feedbacks is studied by spectral analysis. In [2], the control of a connected string converter modeled from electrical power is studied. In [6], the symmetry is considered for the system of strings. Other literature relevant can be found in [7–9] and the references therein.

Although there are some works on the stability of connected strings, there is still no result on the Lyapunov stability analysis for connected strings. It needs some mathematical techniques to find the suitable Lyapunov function for connected strings. Moreover, these models are without disturbance. However, disturbance is usually present in practical applications [10–12]. So, we are also concerned with the disturbance for the model. Sliding mode control and active disturbance rejection control (ADRC) are usually used to reject the disturbance [13–19]. ADRC is used in this paper. ADRC shows its advantages because it can not only estimate the disturbance but also counteract the disturbance (see [20–24]).

We consider a connected string system, which is connected at point $x = 1$ in $[0, 2]$:

$$\begin{cases} \chi_{tt}(x,t) = \chi_{xx}(x,t), & x \in (0,1)\cup(1,2), t > 0, \\ \chi(1^-,t) = \chi(1^+,t), & t \geq 0, \\ \chi_x(1^-,t) - \chi_x(1^+,t) = q\chi_t(1,t), & t \geq 0, \\ \chi_x(0,t) = U(t) + r(t), & t \geq 0, \\ \chi_x(2,t) = 0, & t \geq 0, \end{cases} \tag{1}$$

where χ is the state, $U(t)$ is the input, $q>0, q\neq 2$, and r is the disturbance; we also assume $|r(t)|\leq M, |\dot{r}(t)|\leq M, \forall t>0$. The connected term is the discontinuity of vertical force component; it is antistable.

We introduce a new variable $\varphi(x,t)=[\varphi_1(x,t), \varphi_2(x,t)]^T$, where

$$\varphi_1(x,t)=\chi(x,t), \varphi_2(x,t)=\chi(2-x,t), x\in[0,1], t\geq 0. \tag{2}$$

Then, system (1) becomes

$$\begin{cases} \varphi_{tt}(x,t)=\varphi_{xx}(x,t), & 0<x<1, t>0, \\ \varphi_1(1,t)=\varphi_2(1,t), & t\geq 0, \\ \varphi_{1x}(1,t)+\varphi_{2x}(1,t)=q\varphi_{1t}(1,t), & t\geq 0, \\ \varphi_{1x}(0,t)=U(t)+r(t), & t\geq 0, \\ \varphi_{2x}(0,t)=0, & t\geq 0. \end{cases} \tag{3}$$

It can be seen that the string is unstable because there is an antidamping in the term of discontinuity of the vertical force. Also, the boundary is Riemannian, which gives rise to difficulty to make it converge to zero point. The contribution is compensating the unstable term as well to make sure that it can converge to zero. We used the backstepping transformation (see [25–27]) to deal with the unstable term, and we are the first to give the Lyapunov function for connected strings.

The paper is organized as follows. Section 2 designs the controller. Section 3 shows MATLAB simulations, and Section 4 gives the conclusion remarks.

2. The Control Design

An invertible backstepping transformation is introduced by

$$\begin{aligned} \theta_1(x,t) = {} & \varphi_1(x,t)+\frac{q(q+c)}{4-q^2}[\varphi_1(x,t)+\varphi_2(x,t)]-\frac{q(q+c)}{4-q^2}[\varphi_1(1,t)+\varphi_2(1,t)] \\ & +\frac{2(q+c)}{4-q^2}\int_x^1[\varphi_{1t}(y,t)+\varphi_{2t}(y,t)]\mathrm{d}y, \end{aligned} \tag{4}$$

$$\theta_2(x,t)=\varphi_2(x,t), \tag{5}$$

where $q>0, q\neq 2$ and $c>0, c\neq 2$; its inverse transformation is

$$\begin{aligned} \varphi_1(x,t) = {} & \theta_1(x,t)+\frac{q(q+c)}{4-c^2}[\theta_1(x,t)+\theta_2(x,t)]-\frac{q(q+c)}{4-c^2}[\theta_1(1,t)+\theta_2(1,t)] \\ & +\frac{2(q+c)}{4-c^2}\int_x^1[\theta_{1t}(y,t)+\theta_{2t}(y,t)]\mathrm{d}y, \\ \varphi_2(x,t) = {} & \theta_2(x,t). \end{aligned} \tag{6}$$

By transformation (4) and (5), system (3) is going to be

$$\begin{cases} \theta_{tt}(x,t)=\theta_{xx}(x,t), & 0<x<1, \quad t>0, \\ \theta_1(1,t)=\theta_2(1,t), & t\geq 0, \\ \theta_{1x}(1,t)+\theta_{2x}(1,t)=-c\theta_{1t}(1,t), & t\geq 0, \\ \theta_{1x}(0,t)=\dfrac{4-c^2}{4+qc}(U(t)+r(t))-\dfrac{2(q+c)}{4+qc}(\theta_{1t}(0,t)+\theta_{2t}(0,t)), & t\geq 0, \\ \theta_{2x}(0,t)=0, & t\geq 0. \end{cases} \tag{7}$$

By the transformation, the connected antidamping is moved to the boundary. We give a controller:

$$U_1(t) = \frac{2(q+c)}{4-c^2}\left[\theta_{1t}(0,t) + \theta_{2t}(0,t)\right] + k\frac{4+qc}{4-c^2}\theta_1(0,t), k>0, \tag{8}$$

to deal with the antidamping term at point $x = 0$. Under the control of

$$U(t) = U_0(t) + U_1(t), \tag{9}$$

where $U_0(t)$ is a new control to be designed to contract the disturbance. System (7) becomes

$$\begin{cases} \theta_{tt}(x,t) = \theta_{xx}(x,t), & 0<x<1, t>0, \\ \theta_1(1,t) = \theta_2(1,t), & t\geq 0, \\ \theta_{1x}(1,t) + \theta_{2x}(1,t) = -c\theta_{1t}(1,t), & t\geq 0, \\ \theta_{1x}(0,t) = \dfrac{4-c^2}{4+qc}\left(U_0(t) + r(t)\right) + k\theta_1(0,t), & t\geq 0, \\ \theta_{2x}(0,t) = 0, & t\geq 0. \end{cases} \tag{10}$$

We consider system (10) in the Hilbert space

$$\mathscr{H} = \left\{(u_1,v_1,u_2,v_2) \in \left(H^1(0,1)\times L^2(0,1)\right)^2 | u_1(1) = u_2(1)\right\}. \tag{11}$$

With the inner product induced norm, for $\forall (u_1,v_1,u_2,v_2) \in \mathscr{H}$,

$$\|(u_1,v_1,u_2,v_2)\| = \int_0^1\left(|u_1'(x)|^2 + |v_1(x)|^2 + |u_2'(x)|^2 + |v_2(x)|^2\right)\mathrm{d}x + k|u_1(0)|^2. \tag{12}$$

Introduce an unbounded linear operator $\mathscr{A}\colon D(\mathscr{A})(\subset \mathscr{H}) \longrightarrow \mathscr{H}$:

$$\begin{cases} \mathscr{A}Z = \left(v_1, u_1'', v_2, u_2''\right), \forall Z = \left(u_1,v_1,u_2,v_2\right) \in D(\mathscr{A}), \\ D(\mathscr{A}) = \left\{Z \in H \cap \left(H^2(0,1)\times H^1(0,1)\right)^2 \middle| u_1'(1) + u_2'(1) = -cv_1(1), \right. \\ u_1'(0) = ku_1(0), \\ u_2'(0) = 0, \\ v_1(1) = v_2(1), \end{cases} \tag{13}$$

and then, system (10) can be written in the evolution equation form in $\mathscr{H}$:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}Z(t) = \mathscr{A}\left(\theta_1(\cdot,t), \theta_{1t}(\cdot,t), \theta_2(\cdot,t), \theta_{2t}(\cdot,t)\right) - \dfrac{4-c^2}{4+qc}B\left(U_0(t) + r(t)\right) \\ Z(\cdot,0) = \left(\theta_1(\cdot,0), \theta_{1t}(\cdot,0), \theta_2(\cdot,0), \theta_{2t}(\cdot,0)\right), \end{cases} \tag{14}$$

where $Z(t) = (\theta_1(\cdot,t), \theta_{1t}(\cdot,t), \theta_2(\cdot,t), \theta_{2t}(\cdot,t))$ and $\mathscr{B} = (0, \delta(x), 0, 0)$. We get the following two lemmas.

Lemma 1. *$\mathscr{A}$ generates an exponentially stable C_0-semigroup.*

Proof. We will first demonstrate that $\mathscr{A}$ generates a C_0-semigroup. A direct computation gives

$$\begin{aligned}
&\mathrm{Re}\mathscr{A}\left(u_1, v_1, u_2, v_2\right)^T, \left(u_1, v_1, u_2, v_2\right)^T \\
&= \int_0^1 \left(v_1'(x,t)u_1''(x,t) + u_1''(x,t)v_1(x,t) + v_2'(x,t)u_2''(x,t) + u_2''(x,t)v_2(x,t)\right)\mathrm{d}x + kv_1(0)u_1(0) \\
&= -c\left|v_1(1)\right|^2 \le 0.
\end{aligned} \tag{15}$$

So, $\mathscr{A}$ is dissipative.

For any given $(\phi_1, \varphi_1, \phi_2, \varphi_2) \in \mathscr{H}$, solve $\mathscr{A}(u_1, v_1, u_2, v_2) = (\phi_1, \varphi_1, \phi_2, \varphi_2)$, to obtain

$$\begin{cases} v_1(x) = \phi_1(x), \\ v_2(x) = \phi_2(x), \end{cases} \tag{16}$$

which gives

$$\begin{cases}
u_1(x) = (kx+1)u_1(0) + \displaystyle\int_0^x \varphi_1(\eta)(x-\eta)d\eta\ , \\
u_2(x) = u_2(0) + \displaystyle\int_0^x \varphi_2(\eta)(x-\eta)d\eta, \\
u_1(0) = -\dfrac{1}{k}\left[\displaystyle\int_0^1 (\varphi_1(\eta) + \varphi_2(\eta))d\eta + cv_1(1)\right], \\
u_2(0) = (k+1)u_1(0) + \displaystyle\int_0^1 \varphi_1(\eta)(1-\eta)d\eta - \int_0^1 \varphi_2(\eta)(1-\eta)d\eta\ .
\end{cases} \tag{17}$$

So, $(u_1, v_1, u_2, v_2) \in \mathscr{H}$ is unique. Hence, $\mathscr{A}^{-1}$ exists, and from the Sobolev embedding theorem, it is also compact on $\mathscr{H}$. Using Lumer–Phillips theorem (see [28]), $\mathscr{A}$ generates a C_0-semigroup of contractions $e^{\mathscr{A}t}$.

Next, we show that $\mathscr{A}$ is exponentially stable. We consider the evolution equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}Z(t) = \mathscr{A}Z(t), Z(t) = \left(\theta_1(\cdot,t), \theta_{1t}(\cdot,t), \theta_2(\cdot,t), \theta_{2t}(\cdot,t)\right), \tag{18}$$

which equals to the following PDE:

$$\begin{cases}
\theta_{tt}(x,t) = \theta_{xx}(x,t), & 0 < x < 1, t > 0, \\
\theta_1(1,t) = \theta_2(1,t), & t \ge 0, \\
\theta_{1x}(1,t) + \theta_{2x}(1,t) = -c\theta_{1t}(1,t) & t \ge 0, \\
\theta_{1x}(0,t) = k\theta_1(0,t), & t \ge 0, \\
\theta_{2x}(0,t) = 0 & t \ge 0.
\end{cases} \tag{19}$$

Let $V(t)$ be a function given by

$$\begin{aligned}
V(t) = &\frac{1}{2}\int_0^1 \left(\theta_{1x}^2(x,t) + \theta_{1t}^2(x,t) + \theta_{2x}^2(x,t) + \theta_{2t}^2(x,t)\right)\mathrm{d}x + \frac{1}{2}k\theta_1^2(0,t) \\
&+ \delta\int_0^1 \left[\theta_{1x}(x,t)\theta_{1t}(x,t) + \theta_{2x}(x,t)\theta_{2t}(x,t)\right]\mathrm{d}x \\
&+ \delta\int_0^1 \left[\theta_{1t}(x,t)\theta_{2x}(x,t) + \theta_{2t}(x,t)\theta_{1x}(x,t)\right]\mathrm{d}x,
\end{aligned} \tag{20}$$

where $\delta > 0$ and δ is sufficiently small.

From Cauchy–Schwartz and Young's inequalities, it is found that, for $\delta \le \min\{1/3, k/3\}$, there exists $m_1, m_2 > 0$ so that $m_1\|Z(t)\|^2 \le V(t) \le m_2\|Z(t)\|^2$, where

$$\begin{aligned} m_1 &= \min\left\{\frac{1-3\delta}{2}, \frac{k-3\delta}{2}\right\},\\ m_2 &= \max\left\{\frac{1+3\delta}{2}, \frac{k+3\delta}{2}\right\},\\ \|Z(t)\|^2 &= \left\|\theta_{1x}\right\|^2 + \left\|\theta_{1t}\right\|^2 + \left\|\theta_1(0)\right\|^2 + \left\|\theta_{2x}\right\|^2 + \left\|\theta_{2t}\right\|^2. \end{aligned} \tag{21}$$

Hence, $V(t)$ is a proper Lyapunov function for (23). By taking the deviation of $V(t)$ in t, we have

$$\begin{aligned} \dot V(t) = &\int_0^1 [\theta_{1x}(x,t)\theta_{1xt}(x,t) + \theta_{1t}(x,t)\theta_{1tt}(x,t) + \theta_{2x}(x,t)\theta_{2xt}(x,t) + \theta_{2t}(x,t)\theta_{2tt}(x,t)]\mathrm{d}x\\ &+ k\theta_1(0,t)\theta_{1t}(0,t) + \delta\int_0^1 [\theta_{1xt}(x,t)\theta_{1t}(x,t) + \theta_{1x}(x,t)\theta_{1tt}(x,t) + \theta_{2xt}(x,t)\theta_{2t}(x,t) + \theta_{2x}(x,t)\theta_{2tt}(x,t)]\mathrm{d}x\\ &+ \delta\int_0^1 [\theta_{1tt}(x,t)\theta_{2x}(x,t) + \theta_{1t}(x,t)\theta_{2xt}(x,t) + \theta_{2tt}(x,t)\theta_{1x}(x,t) + \theta_{2t}(x,t)\theta_{1xt}(x,t)]\mathrm{d}x,\\ = &\int_0^1 [\theta_{1x}(x,t)\theta_{1xt}(x,t) + \theta_{1t}(x,t)\theta_{1xx}(x,t) + \theta_{2x}(x,t)\theta_{2xt}(x,t) + \theta_{2t}(x,t)\theta_{2xx}(x,t)]\mathrm{d}x\\ &+ k\theta_1(0,t)\theta_{1t}(0,t) + \delta\int_0^1 [\theta_{1xt}(x,t)\theta_{1t}(x,t) + \theta_{1x}(x,t)\theta_{1xx}(x,t) + \theta_{2xt}(x,t)\theta_{2t}(x,t) + \theta_{2x}(x,t)\theta_{2xx}(x,t)]\mathrm{d}x\\ &+ \delta\int_0^1 [\theta_{1xx}(x,t)\theta_{2x}(x,t) + \theta_{1t}(x,t)\theta_{2xt}(x,t) + \theta_{2xx}(x,t)\theta_{1x}(x,t) + \theta_{2t}(x,t)\theta_{1xt}(x,t)]\mathrm{d}x,\\ = &\,\theta_{1t}(1,t)\theta_{1x}(1,t) - \theta_{1t}(0,t)\theta_{1x}(0,t) + \theta_{2t}(1,t)\theta_{2x}(1,t) - \theta_{2t}(0,t)\theta_{2x}(0,t)\\ &+ k\theta_1(0,t)\theta_{1t}(0,t) - \frac{\delta}{2}\int_0^1 \left[\theta_{1x}^2(x,t) + \theta_{1t}^2(x,t) + \theta_{2x}^2(x,t) + \theta_{2t}^2(x,t)\right]\mathrm{d}x\\ &+ \frac{\delta}{2}\left[\theta_{1t}^2(1,t) + \theta_{1x}^2(1,t) + \theta_{2t}^2(1,t) + \theta_{2x}^2(1,t)\right] - \frac{\delta}{2}\left[\theta_{1x}^2(0,t) + \theta_{1t}^2(0,t) + \theta_{2x}^2(0,t) + \theta_{2t}^2(0,t)\right]\\ &+ \delta[\theta_{1x}(1,t)\theta_{2x}(1,t) - \theta_{1x}(0,t)\theta_{2x}(0,t) + \theta_{1t}(1,t)\theta_{2t}(1,t) - \theta_{1t}(0,t)\theta_{2t}(0,t)],\\ = &\left(\left(2 + \frac{c^2}{2}\right)\delta - c\right)\theta_{1t}^2(1,t) - \frac{\delta}{2}\int_0^1 \left(\theta_{1x}^2(x,t) + \theta_{1t}^2(x,t) + \theta_{2x}^2(x,t) + \theta_{2t}^2(x,t)\right)\mathrm{d}x\\ &- \frac{\delta}{2}\left(k^2\theta_1^2(0,t) + \left(\theta_{1t}(0,t) + \theta_{2t}(0,t)\right)^2\right). \end{aligned} \tag{22}$$

Let $\delta < \min\{1/3, (k/3)4c/4 + c^2\}$ and $\alpha = \max\{1, k^2\}\delta/2m_1$, and we get $\dot V(t) \le -\alpha V(t)$, which gives the result. ☐

Lemma 2. *$\mathscr{B}$ is admissible to $e^{\mathscr{A}t}$.*

Proof. By direct computation, we obtain

$$\begin{cases} \mathscr{A}^*(u_1, v_1, u_2, v_2) = (-v_1, -u_1'', -v_2, -u_2''),\\ D(\mathscr{A}^*) = \left\{(u_1, v_1, u_2, v_2) \in H \cap \left(\mathscr{H}^2(0,1) \times \mathscr{H}^1(0,1)\right)^2 | u_1'(1) + u_2'(1) = cv_1(1), u_1'(0) = ku_1(0), u_2'(0) = 0, v_1 = v_2\right\}. \end{cases} \tag{23}$$

The dual system of (10) is

$$\begin{cases} \theta_{tt}^*(x,t) = \theta_{xx}^*(x,t), & 0 < x\langle 1, t\rangle 0, \\ \theta_1^*(1,t) = \theta_2^*(1,t), & t \geq 0, \\ \theta_{1x}^*(1,t) + \theta_{2x}^*(1,t) = c\theta_{1t}^*(1,t), & t \geq 0, \\ \theta_{1x}^*(0,t) = k\theta_1^*(0,t), & t \geq 0, \\ \theta_{2x}^*(0,t) = 0, & t \geq 0, \\ y(t) = -\dfrac{4-c^2}{4+qc}\theta_{1t}^*(0,t). \end{cases} \tag{24}$$

$\mathscr{A}$ generates a C_0-semigroup, so $\mathscr{A}*$ generates a C_0-semigroup.

The energy of system (25) is

$$E(t) = \frac{1}{2}\int_0^1 \left(\theta_{1x}^*(x,t)^2 + \theta_{1t}^*(x,t)^2 + \theta_{2x}^*(x,t)^2 + \theta_{2t}^*(x,t)^2\right)\mathrm{d}x + k\theta_1^*(0,t)^2. \tag{25}$$

Then, $E(t) \leq E(0)$. Let $\rho(t) = \int_0^1 (x-1)\theta_{1x}^*(x,t)\theta_{1t}^*(x,t) + x\theta_{2x}^*(x,t)\theta_{2t}^*(x,t)\mathrm{d}x$. Differentiate $\rho(t)$ with respect to t to yiel

$$\dot{\rho}(t) = \frac{1}{2}\left[\theta_{1x}^{*2}(0,t) + \theta_{1t}^{*2}(0,t) + \theta_{2x}^{*2}(1,t) + \theta_{2t}^{*2}(1,t)\right]\mathrm{d}x - E(t) + \frac{1}{2}k\theta_1^{*2}(0,t). \tag{26}$$

We, thus, have

$$\int_0^T |y(t)|^2\mathrm{d}t = \left(\frac{4-c^2}{4+qc}\right)^2 \int_0^T \left|\theta_{1t}^*(0,t)\right|^2\mathrm{d}t \leq 2T\left(\frac{(4-c^2)}{4+qc}\right)^2 E(0). \tag{27}$$

Similar to the computation of A^{-1}

$$\mathscr{A}^{*-1}\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} (1+kx)a - \int_0^x v_1(\eta)(x-\eta)\mathrm{d}\eta \\ -u_1(x) \\ b - \int_1^x v_2(\eta)(x-\eta)\mathrm{d}\eta \\ -u_2(x) \end{pmatrix}, \quad \forall \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} \in \mathscr{H}, \tag{28}$$

where

$$a = \frac{1}{k}\left[\int_0^1 (\varphi_1(\eta) + \varphi_2(\eta))\mathrm{d}\eta - cv_1(1)\right], \quad b = (k+1)a - \int_0^1 \varphi_1(\eta)(1-\eta)d\eta + \int_0^1 \varphi_2(\eta)(1-\eta)\mathrm{d}\eta, \tag{29}$$

$$\mathscr{B}^*\mathscr{A}^{*-1}\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = -u_1(0).$$

Therefore, on $\mathscr{H}\mathscr{B}^*\mathscr{A}^{*-1}$ is bounded. Combined with (26), we show that $\mathscr{B}$ is admissible to $e^{\mathscr{A}t}$ (see [29, 30]).

From the two lemmas above, the evolution equation (14) has a unique weak solution in $\mathscr{H}$. Therefore, for $\forall(\theta_1(.,t), \theta_{1t}(.,t), \theta_2(.,t), \theta_{2t}(.,t))^T \in \mathscr{H}, U_0 \in L^2_{loc}(0,\infty)$, system (10) has a unique weak solution $(\theta_1(x,t), \theta_{1t}(x,t), \theta_2(x,t), \theta_{2t}(x,t))^T \in \mathscr{H}$, $\forall(u_1, v_1, u_2, v_2)^T \in D(\mathscr{A}^*)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \begin{pmatrix} \theta_1 \\ \theta_{1t} \\ \theta_2 \\ \theta_{2t} \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \theta_1 \\ \theta_{1t} \\ \theta_2 \\ \theta_{2t} \end{pmatrix}, \mathscr{A}^*\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} \right\rangle - g_1(0)\frac{4-c^2}{4+qc}\left[U_0(t) + r(t)\right], \tag{30}$$

that is,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left[\theta_{1x}(x,t)u_1'(x)+\theta_{1t}(x,t)v_1(x)+\theta_{2x}(x,t)u_2'(x)+\theta_{2t}(x,t)v_2(x)\right]\mathrm{d}x\\
&=\int_0^1\left[-\theta_{1x}(x,t)v_1'(x)-\theta_{1t}(x,t)u_1^2(x)-\theta_{2x}(x,t)v_2'(x)-\theta_{2t}(x,t)u_2^2(x)\right]\mathrm{d}x\\
&\quad-v_1(0)\frac{4-c^2}{4+qc}\left[U_0(t)+r(t)\right].
\end{aligned}\tag{31}$$

Next, we will reject the disturbance d. An observer is chosen as

$$\begin{aligned}
y_1(t)&=\int_0^1\left(\theta_{1t}(x,t)+\frac{1}{2}cx\theta_{1x}(x,t)+\theta_{2t}(x,t)+\frac{1}{2}cx\theta_{2x}(x,t)\right)\mathrm{d}x,\ t\geq 0,\\
y_2(t)&=\int_0^1\left(\theta_{1t}(x,t)+\theta_{2t}(x,t)\right)\mathrm{d}x,\ t\geq 0.
\end{aligned}\tag{32}$$

From equality (31), we have

$$\dot{y}_1(t)=-\frac{c}{2}y_2(t)-\frac{4-c^2}{4+qc}\left(U_0+r(t)\right).\tag{33}$$

We choose the high gain estimator to be (see [23])

$$\begin{cases}
\dot{\hat{y}}(t)=-\dfrac{4-c^2}{4+qc}\left(U_0(t)+\hat{r}(t)\right)-\dfrac{c}{2}y_2(t)-\dfrac{1}{\varepsilon}\left(\hat{y}(t)-y_1(t)\right),\\[2ex]
\dot{\hat{r}}(t)=\dfrac{1}{\varepsilon^2}\dfrac{4+qc}{4-c^2}\left(\hat{y}(t)-y_1(t)\right),
\end{cases}\tag{34}$$

where $\varepsilon>0$ is the design small parameter and $\hat{r}$ is the approximation of disturbance r. Let

$$\tilde{y}(t)=\hat{y}(t)-y_1(t),\tilde{r}(t)=\hat{r}(t)-r(t),\tag{35}$$

be the errors. Then, $\tilde{y}$ and $\tilde{r}$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\tilde{y}(t)\\ \tilde{d}(t)\end{pmatrix}=\begin{pmatrix}-\dfrac{1}{\varepsilon} & -\dfrac{4-c^2}{4+qc)}\\[2ex] \dfrac{1}{\varepsilon^2}\dfrac{4+qc}{4-c^2} & 0\end{pmatrix}\begin{pmatrix}\tilde{y}(t)\\ \tilde{d}(t)\end{pmatrix}+\begin{pmatrix}0\\ -1\end{pmatrix}\dot{d}(t)=A\begin{pmatrix}\tilde{y}(t)\\ \tilde{d}(t)\end{pmatrix}+B\dot{d}(t).\tag{36}$$

The collocated feedback controller of (10) is designed to be

$$U_0(t)=-sat\left(\hat{r}(t)\right),\tag{37}$$

where $sat(x)=\begin{cases}M, x\geq M+1,\\ x, x\in(-M-1,M+1),\\ -M, x\leq -M-1.\end{cases}$ with controller $U_0(t)=U_0(t)+U_1(t)$; the closed-loop system of (10) is

$$\begin{cases} \theta_{tt}(x,t=\theta_{xx}(x,t)), & 0<x<1, t>0,\\ \theta_1(1,t)=\theta_2(1,t), & t\geq 0,\\ \theta_{1x}(1,t)+\theta_{2x}(1,t)=c\theta_{1t}(1,t), & t\geq 0,\\ \theta_{1x}(0,t)=\dfrac{4-c^2}{4+qc}(-sat(\tilde{r}(t)+r(t))+r(t))+k\theta_1(0,t), & t\geq 0,\\ \theta_{2x}(0,t)=0, & t\geq 0,\\ \hat{y}(t)=-\dfrac{4-c^2}{4+qc}(-sat(\hat{r}(t))+\hat{r}(t))-\dfrac{c}{2}y_2(t)-\dfrac{1}{\varepsilon}(\hat{y}(t)-y_1(t)), & t\geq 0,\\ \hat{r}(t)=\dfrac{1}{\varepsilon^2}\dfrac{4-qc}{4+c^2}(\hat{y}(t)-y(t)), & t\geq 0. \end{cases} \tag{38}$$

We get the following theorem concerning the stability when disturbance in involved. □

Theorem 1. *If $|r|$ and $|\dot{r}|$ are bounded measurable, then, for any initial value $\theta(\cdot,t)\in\mathscr{H}$, there exists a unique solution $(\theta,\theta_t)\in C(0,\infty;\mathscr{H})$ for the closed-loop system (39) of (7):*

$$\begin{cases} \theta_{tt}(x,t=\theta_{xx}(x,t)), & 0<x<1, t>0,\\ \theta_1(1,t)=\theta_2(1,t), & t\geq 0,\\ \theta_{1x}(1,t)+\theta_{2x}(1,t)=c\theta_{1t}(1,t), & t\geq 0,\\ \theta_{1x}(0,t)=\dfrac{4-c^2}{4+qc}(-sat(\tilde{r}(t)+r(t))+r(t))+k\theta_1(0,t), & t\geq 0,\\ \theta_{2x}(0,t)=0, & t\geq 0,\\ \tilde{y}(t)=-\dfrac{1}{\varepsilon}\tilde{y}(t)-\dfrac{4-c^2}{4+qc}\tilde{r}(t), & t\geq 0,\\ \tilde{r}(t)=\dfrac{1}{\varepsilon^2}\dfrac{4-qc}{4+c^2}\tilde{y}(t)-r(t), & t\geq 0,\\ y_1(t)=\displaystyle\int_0^1\left(\theta_{1t}(x,t)+\frac{1}{2}cx\theta_{1x}(x,t)+\theta_{2t}(x,t)+\frac{1}{2}cx\theta_{2x}(x,t)\right)\mathrm{d}x, & t\geq 0,\\ y_2(t)=\displaystyle\int_0^1(\theta_{1t}(x,t)+\theta_{2t}(x,t))\mathrm{d}x, & t\geq 0. \end{cases} \tag{39}$$

Moreover, the solution of (39) converges to any arbitrary given neighborhood of zero when $t\longrightarrow\infty$ and $\varepsilon\longrightarrow 0$.

For the proof of Theorem 1, we omit the proof, since it can be easily proved followed by [19, 24].

3. Numerical Simulations

In this section, we compute the state for the closed-loop system (39) by the finite difference method [19]. Here,

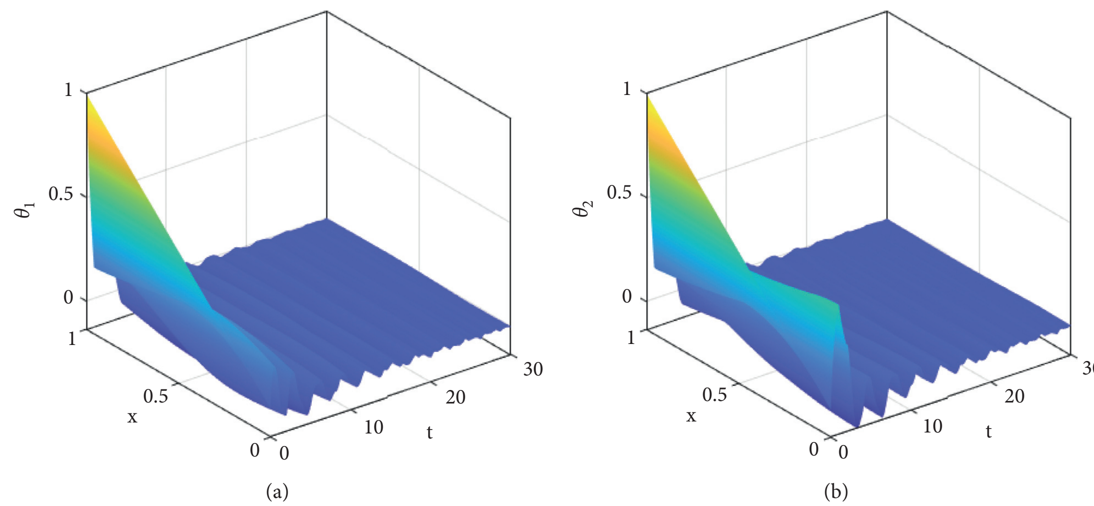

FIGURE 2: (a) The displacement of θ_1; (b) the displacement of θ_2.

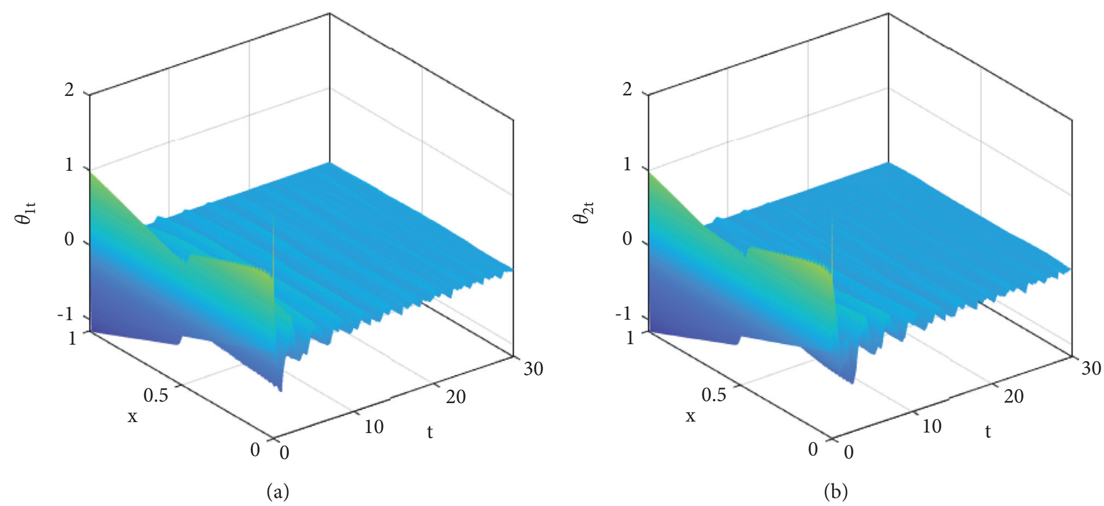

FIGURE 3: (a) The velocity of θ_{1t}; (b) the velocity of θ_{2t}.

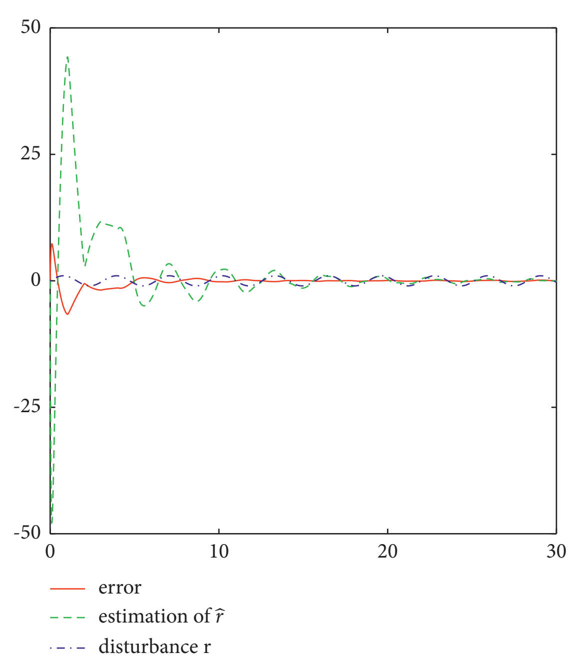

FIGURE 4: The displacement of r, $\hat{r}$, and $\tilde{r}$.

$q = 5, c = 1.5, k = 5$, and $d = 3\sin(2t)$. Figure 2 shows that the displacement of θ_1 and θ_2 converges to zero. Figure 3 shows the velocity of θ_{1t} and θ_{2t} tend to zero. Figure 4 shows the estimation of $\hat{r}$ fits well with the value of r. Thus, the control law is effective.

4. Conclusion

A virtual connected string system with midway discontinue vertical force model is proposed for the underground multirobot system, and the Lyapunov-based stabilization control approach for virtual connected string is studied in the paper. The system is with Riemannian boundary. The backstepping is used to move the antistable term to the boundary thus to give the controller and Lyapunov approach is used to analyze the stability for the closed-loop system when the disturbance is absent. Then, ADRC is used to estimate and reject the disturbance. We can study the n stings with discontinue vertical force in the further research.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

Lingling Su and Lin Zhang conceived and designed the experiments; Xianhua Zheng and Shang Feng performed the experiments; Lin Zhang and Lingling Su built the pushing dynamics; Shang Feng and Yongshi Song analyzed the data; Lingling Su wrote the paper. Shang Feng translated and re-edited the manuscript.

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