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Research Article

A Generalized Definition of Fuzzy Subrings

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In this study, under the condition that L is a completely distributive lattice, a generalized definition of fuzzy subrings is introduced. By means of four kinds of cut sets of fuzzy subset, the equivalent characterization of L-fuzzy subring measures are presented. The properties of L-fuzzy subring measures under these two kinds of product operations are further studied. In addition, an L-fuzzy convexity is directly induced by L-fuzzy subring measure, and it is pointed out that ring homomorphism can be regarded as L-fuzzy convex preserving mapping and L-fuzzy convex-to-convex mapping. Next, we give the definition and related properties of the measure of L-fuzzy quotient ring and give a new characterization of L-fuzzy quotient ring when the measure of L-fuzzy quotient ring is 1.

1. Introduction

Since Rosenfeld [1] introduced the concept of fuzzy subgroup, fuzzy algebra has developed rapidly. Liu et al. [2, 3] defined the concept of fuzzy subring. Malik and Mordeson discussed the direct sum operation properties of L-fuzzy subrings [4]. Shi [5] proposed the concept of fuzzy subgroup measure for the first time. Shi and Xin [6] generalized fuzzy subgroup measure to L-fuzzy subgroup measure. Li and Shi introduced the notion of L-fuzzy convex sublattice measure and induced an L-fuzzy convex structure [7]. Han and Shi [8] defined L-convex fuzzy ideal measure and further studied its induced L-fuzzy convex structure. The notions of M-fuzzifying convex structure and (L, M)-fuzzy convex structure are introduced by Shi and Xiu in [9, 10]. Actually, fuzzy convexity exists in many mathematical research areas, such as fuzzy vector spaces, fuzzy groups, fuzzy lattices, and fuzzy topologies (see [7, 8, 11–24]).

Inspired by this, the definition of L-fuzzy subring measure was proposed in this study; its equivalent characterizations were given by using four kinds of cut sets, and the relevant properties were discussed. In addition, an L-fuzzy subring measure can induce an L-fuzzy convex structure, and a ring homomorphism between two rings can exactly be regarded as an L-fuzzy convex-to-convex mapping and an L-fuzzy convex preserving mapping. And when the subring A of ring R is fuzzy and the ideal N is crisp, we can also obtain the concept of L-fuzzy quotient ring measure. On this basis, we use the representation theorem to give the equivalent characterization of fuzzy quotient ring when the measure of L-fuzzy quotient ring is 1. At the same time, we study the relationship between L-fuzzy subring measure and L-fuzzy quotient ring measure. It is proposed that the ring homomorphism from ring to quotient ring can be regarded as L-fuzzy convex-to-convex mapping and L-fuzzy convex preserving mapping.

2. Preliminaries

Throughout this study, L is a completely distributive lattice, and the smallest element and the largest element in L are denoted by $\bot$ and $\top$, respectively.

An element a in L is called a prime element if $a \geqslant b \wedge c$ implies $a \geqslant b$ or $a \geqslant c$. a in L is called co-prime if $a \leqslant b \vee c$ implies $a \leqslant b$ or $a \leqslant c$ [25]. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $J(L)$.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$, if and only if, for every subset $D \subseteq L$, the relation $b \leqslant \sup D$ always implies the existence of $d \in D$ with

$a \leqslant d$ [26]. $\{a \in L | a \prec b\}$ is called the greatest minimal family of b in the sense of [27], denoted by $\beta(b)$; moreover, define the binary relation $\triangleleft$ in L as follows: for $a, b \in L$, $b \triangleleft a$ if and only if, for every subset $E \subseteq L$, the relation $b \geqslant \inf E$ always implies the existence of $e \in E$ with $a \geqslant e$ [28]. $\{a \in L | b \triangleleft a\}$ is called the greatest maximal family of a, denoted by $\alpha(b)$.

In a completely distributive lattice L, α is an $\wedge$ - $\cup$ map, β is a union-preserving map, and there exist $\alpha(b)$ and $\beta(b)$ for each $b \in L$ such that $b = \vee\beta(b) = \wedge\alpha(b)$ (see [29]).

For a complete distributive lattice L, $A \in L^X$ and $a \in L$, we can define

$$\begin{aligned} A_{[a]} &= \{x \in X | A(x) \geqslant a\}, \\ A^{(a)} &= \{x \in X | A(x) \nleq a\}, \\ A_{(a)} &= \{x \in X | a \in \beta(A(x))\}, \\ A^{[a]} &= \{x \in X | a \notin \alpha(A(x))\}. \end{aligned} \tag{1}$$

Theorem 1 (see [27, 30]). *For each L-fuzzy set $A \in L^X$, the following conditions are true:*

(1) $A = \vee_{a \in L}(a \wedge A_{[a]}) = \vee_{a \in M(L)}(a \wedge A_{[a]}) = \vee_{a \in L}(a \wedge A_{(a)}) = \vee_{a \in M(L)}(a \wedge A_{(a)})$

(2) $A = \wedge_{a \in L}(a \vee A_{[a]}) = \wedge_{a \in P(L)}(a \vee A_{[a]}) = \wedge_{a \in L}(a \vee A_{(a)}) = \wedge_{a \in P(L)}(a \vee A_{(a)})$

Since a completely distributive lattice is a complete Heyting algebra, there exists a binary operation $\longmapsto$ in L. Explicitly the implication is given by

$$a \mapsto b = \vee\{c \in L | a \wedge c \leqslant b\}. \tag{2}$$

We list some properties of the implication operation in the following lemma.

Lemma 1 (see [6]). *Let L be a completely distributive lattice and let $\longmapsto$ be the implication operator corresponding to $\wedge$. Then, for all $a, b, c \in L$, $\{a_i\}_{i \in I} \subseteq L$, the following statements hold:*

(1) $\top \longmapsto a = a$

(2) $c \leqslant a \longmapsto b \Leftrightarrow a \wedge c \leqslant b$

(3) $a \longmapsto b = \top \Leftrightarrow a \leqslant b$

(4) $a \longmapsto (\wedge_{i \in I} a_i) = \wedge_{i \in I}(a \longmapsto a_i)$, hence, $a \longmapsto b \leqslant a \longmapsto c$ whenever $b \leqslant c$

(5) $(\vee_{i \in I} a_i) \longmapsto b = \wedge_{i \in I}(a_i \longmapsto b)$, hence, $b \longmapsto c \leqslant a \longmapsto c$ whenever $a \leqslant b$

(6) $(a \longmapsto c) \wedge (c \longmapsto b) \leqslant a \longmapsto b$

(7) $(a \longmapsto b) \wedge (c \longmapsto d) \leqslant a \wedge c \longmapsto b \wedge d$

Lemma 2 (see [6]). *Let $f: R \longrightarrow R'$ be a set mapping. Let μ and η be two L-fuzzy subsets in R and R', respectively. Then, we have*

(i) $f_L^{\longrightarrow}(f_L^{\longleftarrow}(\eta)) = \eta$ if f is surjective

(ii) $f_L^{\longleftarrow}(f_L^{\longrightarrow}(\mu)) = \mu$ if f is injective, where $f_L^{\longrightarrow}: L^R \longrightarrow L^{R'}$ and $f_L^{\longleftarrow}: L^{R'} \longrightarrow L^R$ are defined by $f_L^{\longrightarrow}(\mu)(r') = \vee\{\mu(r) | f(r) = r'\}$, $f_L^{\longleftarrow}(\eta) = \eta \circ f$

Definition 1 (see [10]). A mapping $\mathscr{C}: L^X \longrightarrow M$ is called an (L, M)-fuzzy convexity on X if it satisfies the following conditions:

(LMC1) $\mathscr{C}(\chi_{\varnothing}) = \mathscr{C}(\chi_X) = \top_M$

(LMC2) If $\{A_i | i \in \Omega\} \subseteq L^X$ is nonempty, then $\wedge_{i \in \Omega}\mathscr{C}(A_i) \leqslant \mathscr{C}(\wedge_{i \in \Omega} A_i)$

(LMC3) If $\{A_i | i \in \Omega\} \subseteq L^X$ is nonempty and totally ordered, then $\wedge_{i \in \Omega}\mathscr{C}(A_i) \leqslant \mathscr{C}(\vee_{i \in \Omega} A_i)$

The pair $(X, \mathscr{C})$ is called an (L, M)-fuzzy convex space. An (L, L)-fuzzy convex space is called an L-fuzzy convex space for short.

Definition 2 (see [10]). Let $(X, \mathscr{C})$ and $(Y, \mathscr{D})$ be (L, M)-fuzzy convexity spaces. A mapping $f: X \longrightarrow Y$ is called

(1) An (L, M)-fuzzy convex preserving mapping provided $\mathscr{D}(\lambda) \leqslant \mathscr{C}(f_L^{\longleftarrow}(\lambda))$ for all $\lambda \in L^Y$

(2) An (L, M)-fuzzy convex-to-convex mapping provided $\mathscr{C}(\lambda) \leqslant \mathscr{D}(f_L^{\longrightarrow}(\lambda))$ for all $\lambda \in L^S$

(3) A mapping $f: X \longrightarrow Y$ is called an (L, M)-fuzzy isomorphism provided f is bijective, (L, M)-fuzzy convex preserving and (L, M)-fuzzy convex-to-convex

An (L, L)-fuzzy convex preserving mapping is called an L-fuzzy convex preserving mapping, an (L, L)-fuzzy convex-to-convex mapping is called an L-fuzzy convex-to-convex mapping, and an (L, L)-fuzzy isomorphism is called an L-fuzzy isomorphism for short.

In [4], Malik and Mordeson introduced the following two operations.

Definition 3 (see [4]). Let A and B be two L-fuzzy subsets of a ring R. Define the L-fuzzy subset AB of R by $\forall x \in R$:

$$(AB)(x) = \bigvee_{n \in \mathbb{N}, x = \sum_{i=1}^{n} y_i z_i} \left(\bigwedge_{i=1}^{n} (A(y_i) \wedge B(z_i)) \right). \tag{3}$$

Definition 4 (see [4]). Let $\{R_i | i \in \Omega\}$ be a collection of rings, and let A_i be an L-fuzzy subset of X_i, $\forall i \in \Omega$. Define the Cartesian product $\prod_{i \in \Omega} A_i$ of $\{A_i | i \in \Omega\}$ by $\forall x_i \in R_i$:

$$\left(\prod_{i \in \Omega} A_i \right)(x) = \bigwedge_{i \in \Omega} A_i(x_i). \tag{4}$$

Theorem 2 (see [31]). *Let N be an ideal of ring R, A be an fuzzy subring of R, and A^N; then, A/N is an fuzzy subring of R/N.*

Definition 5 (see [32]). Let A be an L-fuzzy subring of R and N be an subring of ring R, definition ${}^{A}Z_N(x \circ \mathrm{N}) = \vee_{n \in N} A(x \circ \mathrm{n})$. Then, ${}^{A}Z_N$ is an L-fuzzy subring of R/N, called L-fuzzy quotient ring of A/N.

3. A Generalized definition of Fuzzy Subrings

In this section, firstly, the definition of L-fuzzy subring measure is given, and the equivalent characterizations are carried out with the help of four kinds of cut sets, and the related properties and their proofs are given, inspired by the concept of L-fuzzy subgroup measure in [5]. It is natural to introduce the following concept.

Definition 6. Let μ be an L-fuzzy subset in a ring R. Then, the L-fuzzy subring measure $\mathscr{R}(\mu)$ of μ is defined as

$$\mathscr{R}(\mu) = \bigwedge_{x,y\in R}[(\mu(x)\wedge\mu(y)) \longmapsto (\mu(xy)\wedge\mu(x-y))]. \quad (5)$$

We also say that μ is an L-fuzzy subring of R with respect to measure $\mathscr{R}(\mu)$.

Example 1. Let Z_3 be the remaining class ring of module 3 and specify the operations on Z_3 such as regular addition and multiplication. Define $A, B, C \in L^{Z_3}$ by

$$A(z) = \begin{cases} 0.8, & z = 0, \\ 0.3, & z = 1, \\ 0.4, & z = 2, \end{cases}$$
$$B(z) = \begin{cases} 0, & z = 0, \\ 1, & z = 1, \\ 0.4, & z = 2, \end{cases} \quad (6)$$
$$C(z) = \begin{cases} 0, & z = 0, \\ 0, & z = 1, \\ 0, & z = 2. \end{cases}$$

We can get $\mathscr{R}(A) = 0.3, \mathscr{R}(B) = 0$, and $\mathscr{R}(C) = 1$.

(1) In fact, $A(x)\wedge A(y) \longmapsto A(x-y)\wedge A(xy) =$
$$\begin{cases} 0.8 \longmapsto 0.8 = 1, & \text{if } x = 0, y = 0, \\ 0.3 \longmapsto 0.4 = 1, & \text{if } x = 0, y = 1, \\ 0.4 \longmapsto 0.3 = 0.3, & \text{if } x = 0, y = 2, \\ 0.3 \longmapsto 0.3 = 1, & \text{if } x = 1, y = 0, \\ 0.3 \longmapsto 0.3 = 1, & \text{if } x = 1, y = 1, \\ 0.3 \longmapsto 0.4 = 1, & \text{if } x = 1, y = 2, \\ 0.4 \longmapsto 0.4 = 1, & \text{if } x = 2, y = 0, \\ 0.3 \longmapsto 0.3 = 1, & \text{if } x = 2, y = 1, \\ 0.4 \longmapsto 0.3 = 0.3, & \text{if } x = 2, y = 2. \end{cases}$$
So, we can obtain $\mathscr{R}(A) = 1 \wedge 0.3 = 0.3$.

(2) $B(x)\wedge B(y) \longmapsto B(x-y)\wedge B(xy) =$
$$\begin{cases} 0 \longmapsto 0 = 1, & \text{if } x = 0, y = 0, \\ 0 \longmapsto 0 = 1, & \text{if } x = 0, y = 1, \\ 0 \longmapsto 0 = 1, & \text{if } x = 0, y = 2, \\ 0 \longmapsto 0 = 1, & \text{if } x = 1, y = 0, \\ 1 \longmapsto 0 = 0, & \text{if } x = 1, y = 1,. \\ 0.4 \longmapsto 0.4 = 1, & \text{if } x = 1, y = 2, \\ 0 \longmapsto 0 = 1, & \text{if } x = 2, y = 0, \\ 0.4 \longmapsto 0.4 = 1, & \text{if } x = 2, y = 1, \\ 0.4 \longmapsto 0 = 0, & \text{if } x = 2, y = 2. \end{cases}$$
So, we can obtain $\mathscr{R}(B) = 1 \wedge 0 = 0$.

(3) Obviously, $\mathscr{R}(C) = 1$.

Theorem 3. *Let μ be an L-fuzzy subset in a ring R. Then,*

$$\mathscr{R}(\mu) = \bigwedge_{x,y\in R}((\mu(x)\wedge\mu(y)) \longmapsto \mu(xy)) \wedge \bigwedge_{x,y\in R}((\mu(x)\wedge\mu(y)) \longmapsto \mu(x+y)) \wedge \bigwedge_{y\in R}(\mu(y) \longmapsto \mu(-y)). \quad (7)$$

Proof. From Definition 6, we can obtain that $\forall x, y \in R$,

$$\begin{aligned} &\mathscr{R}(\mu)\wedge\mu(x)\wedge\mu(y) \leqslant \mu(xy), \\ &\mathscr{R}(\mu)\wedge\mu(x)\wedge\mu(y) \leqslant \mu(x-y). \end{aligned} \quad (8)$$

In particular, we have that

$$\begin{aligned} &\mathscr{R}(\mu)\wedge\mu(y) \leqslant (0), \\ &\mathscr{R}(\mu)\wedge\mu(y) \leqslant \mathscr{R}(\mu)\wedge\mu(0)\wedge\mu(y) \leqslant \mu(-y), \\ &\mathscr{R}(\mu)\wedge\mu(y) = \mathscr{R}(\mu)\wedge\mu(-y), \\ &\mathscr{R}(\mu)\wedge\mu(y)\wedge\mu(y) = \mathscr{R}(\mu)\wedge\mu(x)\wedge\mu(-y). \end{aligned} \quad (9)$$

This shows that

$$\begin{aligned} &\mathscr{R}(\mu) \leqslant \bigwedge_{x,y\in R}(\mu(x)\wedge\mu(y) \longmapsto \mu(xy)), \\ &\mathscr{R}(\mu) \leqslant \bigwedge_{x,y\in R}(\mu(x)\wedge\mu(y) \longmapsto \mu(x+y)) \wedge \bigwedge_{y\in R}(\mu(y) \longmapsto \mu(-y)). \end{aligned} \quad (10)$$

So,

$$\mathscr{R}(\mu) \leqslant \bigwedge_{x,y\in R}((\mu(x)\wedge\mu(y)) \longmapsto \mu(xy)) \wedge \bigwedge_{x,y\in R}((\mu(x)\wedge\mu(y)) \longmapsto \mu(x+y)) \wedge \bigwedge_{y\in R}(\mu(y) \longmapsto \mu(-y)). \quad (11)$$

Analogously, we can prove

$$\mathscr{R}(\mu) \geqslant \bigwedge_{x,y\in R}((\mu(x)\wedge\mu(y)) \longmapsto \mu(xy)) \wedge \bigwedge_{x,y\in R}((\mu(x)\wedge\mu(y)) \longmapsto \mu(x+y)) \wedge \bigwedge_{y\in R}(\mu(y) \longmapsto \mu(-y)). \quad (12)$$

The following lemma is obvious. □

Lemma 3. *Let μ be an L-fuzzy subset in a ring R. Then, $\mathscr{R}(\mu) \geqslant a$ if and only if, for any $x, y \in R$,*

$$\begin{aligned} &\mu(x)\wedge\mu(y)\wedge a \leqslant \mu(xy), \\ &\mu(x)\wedge\mu(y)\wedge a \leqslant \mu(x-y). \end{aligned} \quad (13)$$

The next theorem presents some equivalent descriptions of L-fuzzy subring measure.

Theorem 4. *Let μ be an L-fuzzy set in a ring R. Then,*

(1) $\mathscr{R}(\mu) = \vee\{a \in L | \mu(x)\wedge\mu(y)\wedge a \leqslant \mu(xy), \mu(x)\wedge\mu(y)\wedge a \leqslant \mu(x-y), \forall x, y \in R\}$

(2) $\mathscr{R}(\mu) = \vee\{a \in L | \forall b \leqslant a,\ \mu_{[b]} \text{ is a subring of } R\}$
(3) $\mathscr{R}(\mu) = \vee\{a \in L | \forall b \notin \alpha(a),\ \mu^{[b]} \text{ is a subring of } R\}$
(4) $\mathscr{R}(\mu) = \vee\{a \in L | \forall b \in P(L), b \ngeq a, \mu^{(b)} \text{ is a subring of } R\}$
(5) $\mathscr{R}(\mu) = \vee\{a \in L | \forall b \in \beta(a), \mu_{(b)} \text{ is a subring of } R\}$ *if* $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ *for any* $a, b \in L$

Proof

(1)⇔(2). Suppose that $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy)$ and $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x - y)$, for any $x, y \in R$. Then, for any $b \leqslant a$ and for any $x, y \in \mu_{[b]}$, we have

$$\begin{aligned} \mu(xy) &\geqslant \mu(x) \wedge \mu(y) \wedge a \geqslant b, \\ \mu(x - y) &\geqslant \mu(x) \wedge \mu(y) \wedge a \geqslant b, \end{aligned} \tag{14}$$

this shows $xy \in \mu_{[b]}, x - y \in \mu_{[b]}$. Therefore, $\mu_{[b]}$ is a subring of R. Hence,

$$\begin{aligned} \mathscr{R}(\mu) = \vee\{a \in L | &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy), \\ &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x - y), \forall x, y \in R\} \\ &\leqslant \vee\{a \in L | \forall b \leqslant a, \mu_{[b]} \text{ is a subring of } R\}. \end{aligned} \tag{15}$$

Conversely, assume that $a \in L$ and $\forall b \leqslant a$, $\mu_{[b]}$ is a subring of R. For any $x, y \in R$, let $b = \mu(x) \wedge \mu(y) \wedge a$. Then, $b \leqslant a$ and $x, y \in \mu_{[b]}$; thus, $xy,\ x - y \in \mu_{[b]}$, i.e.,

$$\begin{aligned} \mu(xy) &\geqslant b = \mu(x) \wedge \mu(y) \wedge a, \\ \mu(x - y) &\geqslant b = \mu(x) \wedge \mu(y) \wedge a. \end{aligned} \tag{16}$$

This means that

$$\begin{aligned} \mathscr{R}(\mu) = \vee\{a \in L | &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy), \\ &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x - y), \forall x, y \in R\} \\ &\geqslant \vee\{a \in L | \forall b \leqslant a, \mu_{[b]} \text{ is a subring of } R\}. \end{aligned} \tag{17}$$

So, (1)⇔(2) is clearly established.

(1)⇔(3) Suppose that $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy)$ and $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x - y)$, for any $x, y \in R$. Then, for any $b \notin \alpha(a)$ and $x, y \in \mu^{[b]}$, we have

$$b \notin \alpha(\mu(x)) \cup \alpha(\mu(y)) \cup \alpha(a) = \alpha(\mu(x) \wedge \mu(y) \wedge \mu(a)). \tag{18}$$

By

$$\begin{aligned} \mu(x) \wedge \mu(y) \wedge a &\leqslant \mu(xy), \\ \mu(x) \wedge \mu(y) \wedge a &\leqslant \mu(x - y), \end{aligned} \tag{19}$$

we know

$$\begin{aligned} \alpha(\mu(xy)) &\subseteq \alpha(\mu(x) \wedge \mu(y) \wedge a), \\ \alpha(\mu(x - y)) &\subseteq \alpha(\mu(x) \wedge \mu(y) \wedge a). \end{aligned} \tag{20}$$

Hence, $b \notin \alpha(\mu(xy))$ and $b \notin \alpha(\mu(x - y))$, i.e. $xy, x - y \in \mu^{[b]}$.

This means that $\mu^{[b]}$ is a subring of R and $a \in \{a \in L | \forall b \notin \alpha(a), \mu^{[b]} \text{ is a subring of } R\}$.

This shows that

$$\begin{aligned} \mathscr{R}(\mu) = \vee\{a \in L | &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy), \\ &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x - y), \forall x, y \in R\} \\ &\leqslant \vee\{a \in L | \forall b \notin \alpha(a), \mu^{[b]} \text{is a subring of } R\}. \end{aligned} \tag{21}$$

Conversely, assume that

$$a \in \{a \in L | \forall b \notin \alpha(a), \mu^{[b]} \text{ is a subring of } R\}. \tag{22}$$

Now, we prove, for any $x, y \in R$,

$$\begin{aligned} \mu(x) \wedge \mu(y) \wedge a &\leqslant \mu(xy), \\ \mu(x) \wedge \mu(y) \wedge a &\leqslant \mu(x - y). \end{aligned} \tag{23}$$

Suppose that $b \notin \alpha(\mu(x) \wedge \mu(y) \wedge a)$. By

$$\alpha(\mu(x) \wedge \mu(y) \wedge a) = \alpha(\mu(x)) \cup \alpha(\mu(x)) \cup \alpha(a), \tag{24}$$

we know that $b \notin \alpha(a)$ and $x, y \in \mu^{[b]}$. Since $\mu^{[b]}$ is a subring of R, it holds that $xy, x - y \in \mu^{[b]}$, i.e., $b \notin \alpha(\mu(xy))$ and $b \notin \alpha(\mu(x - y))$.

This shows

$$\begin{aligned} \mu(x) \wedge \mu(y) \wedge a &\leqslant \mu(xy), \\ \mu(x) \wedge \mu(y) \wedge a &\leqslant \mu(x - y). \end{aligned} \tag{25}$$

It is proved that

$$\begin{aligned} \mathscr{R}(\mu) = \vee\{a \in L | &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy), \\ &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x - y), \forall x, y \in R\} \\ &\geqslant \vee\{a \in L | \forall b \notin \alpha(a), \mu^{[b]} \text{ is a subring of } R\}. \end{aligned} \tag{26}$$

So, (1)⇔(3) is clearly established;

(1)⇔(4) Suppose that $a \in L$, and for any $x, y \in R$,

$$\begin{aligned} \mu(x) \wedge \mu(y) \wedge a &\leqslant \mu(xy), \\ \mu(x) \wedge \mu(y) \wedge a &\leqslant \mu(x - y). \end{aligned} \tag{27}$$

Let $b \in P(L), b \ngeq a$, and $x, y \in \mu^{(b)}$. Now, we prove $xy, x - y \in \mu^{(b)}$. If $xy, x - y \notin \mu^{(b)}$, i.e., $\mu(xy) \leqslant b$ and $\mu(x - y) \leqslant b$, then

$$\begin{aligned} \mu(x) \wedge \mu(y) \wedge a &\leqslant \mu(xy) \\ \mu(x) \wedge \mu(y) \wedge a &\leqslant \mu(x - y) \leqslant b. \end{aligned} \tag{28}$$

By $b \in P(L)$ and $x, y \in \mu^{(b)}$, we have $a \leqslant b$, which contradicts $b \ngeq a$. Hence, $xy, x - y \in \mu^{(b)}$. This shows that $\mu^{(b)}$ is a subring of R.

Therefore,

$$\begin{aligned} \mathscr{R}(\mu) = \vee\{a \in L | &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy), \\ &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x - y), \forall x, y \in R\} \\ &\leqslant \vee\{a \in L | \forall b \in P(L), b \ngeq a, \mu^{(b)} \text{ is a subring of } R\}. \end{aligned} \tag{29}$$

Conversely, assume that

$$a \in \left\{a \in L | \forall b \in P(L), b \ngeq a, \mu^{(b)} \text{ is a subring of } R\right\}. \tag{30}$$

Now, we prove that, for any $x, y \in R$,

$$\begin{aligned} &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy), \\ &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y). \end{aligned} \tag{31}$$

Let $b \in P(L)$ and $\mu(x) \wedge \mu(y) \wedge a \nleq b$. Then, $\mu(x) \nleq b$, $\mu(y) \nleq b$, and $a \nleq b$, i.e., $x, y \in \mu^{(b)}$. Since $\mu^{(b)}$ is a subring of R, it holds that $xy \in \mu^{(b)}, x - y \in \mu^{(b)}$, i.e., $\mu(xy) \nleq b$, $\mu(x-y) \nleq b$.

This shows that

$$\begin{aligned} &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy), \\ &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y). \end{aligned} \tag{32}$$

Therefore,

$$\begin{aligned} \mathscr{R}(\mu) = \vee\{a \in L | &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy), \\ &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y), \forall x, y \in R\} \\ &\geqslant \vee\left\{a \in L | \forall b \in P(L), b \ngeq a, \mu^{(b)} \text{ is a subring of } R\right\}. \end{aligned} \tag{33}$$

So, (1)⇔(4) is clearly established.

(1)⇔(5) Suppose that

$$\begin{aligned} a \in \{a \in L | &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy), \mu(x) \wedge \mu(y) \\ &\wedge a \leqslant \mu(x-y), \quad \forall x, y \in R\}. \end{aligned} \tag{34}$$

Then, for any $b \in \beta(a)$ and for any $x, y \in \mu_{(b)}$, it holds that

$$\begin{aligned} &b \in \beta(\mu(x)) \cap \beta(\mu(y)) \cap \beta(a) = \beta(\mu(x) \wedge \mu(y) \wedge a) \subseteq \beta(\mu(xy)), \\ &b \in \beta(\mu(x)) \cap \beta(\mu(y)) \cap \beta(a) = \beta(\mu(x) \wedge \mu(y) \wedge a) \subseteq \beta(\mu(x-y)), \end{aligned} \tag{35}$$

i.e., $xy, x - y \in \mu_{(b)}$. This shows that $\mu_{(b)}$ is a subring of R. This means that

$$\begin{aligned} \mathscr{R}(\mu) = \vee\{a \in L | &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy), \\ &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y), \quad \forall x, y \in R\} \\ &\leqslant \vee\left\{a \in L | \forall b \in \beta(a), \mu_{(b)} \text{ is a subring of } R\right\}. \end{aligned} \tag{36}$$

Conversely, assume that

$$a \in \left\{a \in L | \forall b \in \beta(a), \mu_{(b)} \text{ is a subring of } R\right\}. \tag{37}$$

Now, we prove that, for any $x, y \in R$,

$$\begin{aligned} &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy), \\ &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y). \end{aligned} \tag{38}$$

Let $b \in \beta(\mu(x) \wedge \mu(y) \wedge a)$. By

$$\beta(\mu(x) \wedge \mu(y) \wedge a) = \beta(\mu(x)) \cap \beta(\mu(y)) \cap \beta(a), \tag{39}$$

we know that $x, y \in \mu_{(b)}$ and $b \in \beta(a)$. Since $\mu_{(b)}$ is a subring of R, it holds that $xy, x - y \in \mu_{(b)}$, i.e., $b \in \beta(\mu(xy))$ and $b \in \beta(\mu(x-y))$.

This shows that

$$\begin{aligned} &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy), \\ &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y). \end{aligned} \tag{40}$$

Therefore,

$$\begin{aligned} \mathscr{R}(\mu) = \vee\{a \in L | &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy), \\ &\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y), \quad \forall x, y \in R\} \\ &\geqslant \left\{a \in L | \forall b \in \beta(a), \mu_{(b)} \text{ is a subring of } R\right\}. \end{aligned} \tag{41}$$

So, (1)⇔(5) is clearly established.

In conclusion, (1)⇔(2)⇔(3)⇔(4)⇔(5). □

4. The Relation between *L*-Fuzzy Subring Measure and *L*-Fuzzy Convexity

In this section, we will investigate the relation between L-fuzzy subring measure and L-fuzzy convexity. We will prove that a ring homomorphism is exactly an L-fuzzy convex preserving mapping and an L-fuzzy convex-to-convex mapping.

For each $\mu \in L^R$, $\mathscr{R}(\mu)$ can be naturally considered as a mapping $\mathscr{R}: L^R \longrightarrow L$ defined by $\mu \longmapsto \mathscr{R}(\mu)$.

The following theorem shows that $\mathscr{R}$ is exactly an L-fuzzy convexity on R.

Theorem 5. *Let R be a ring. Then, the mapping $\mathscr{R}: L^R \longrightarrow L$ defined by $\mu \longmapsto \mathscr{R}(\mu)$ is an L-fuzzy convexity on R, which is called the L-fuzzy convexity induced by L-fuzzy subring measure on R.*

Proof. (LMC1) It is straightforward that

$$\mathscr{R}(\chi_{\varnothing}) = \mathscr{R}(\chi_R) = \top. \tag{42}$$

(LMC2) Let $\{\mu_i\}_{i\in\Omega}$ be a family of L-fuzzy subsets in a ring R. Now, we prove that

$$\mathscr{R}\left(\bigwedge_{i\in\Omega} \mu_i\right) \geqslant \bigwedge_{i\in\Omega} \mathscr{R}(\mu_i). \tag{43}$$

Suppose that $a \leqslant \wedge_{i\in\Omega} \mathscr{R}(\mu_i)$, Then, $\forall x, y \in R$,

$$\begin{aligned} &\mu_i(x) \wedge \mu_i(y) \wedge a \leqslant \mu_i(xy), \\ &\mu_i(x) \wedge \mu_i(y) \wedge a \leqslant \mu_i(x-y). \end{aligned} \tag{44}$$

Hence,

$$\begin{aligned} &\bigwedge_{i\in\Omega} \mu_i(x) \wedge \bigwedge_{i\in\Omega} \mu_i(y) \wedge a \leqslant \bigwedge_{i\in\Omega} \mu_i(xy), \\ &\bigwedge_{i\in\Omega} \mu_i(x) \wedge \bigwedge_{i\in\Omega} \mu_i(y) \wedge a \leqslant \bigwedge_{i\in\Omega} \mu_i(x-y). \end{aligned} \tag{45}$$

This shows $a \leqslant \mathscr{R}(\wedge_{i\in\Omega} \mu_i)$. So, we can obtain $\mathscr{R}(\wedge_{i\in\Omega} \mu_i) \geqslant \wedge_{i\in\Omega} \mathscr{R}(\mu_i)$.

(LMC3) Let $\{\mu_i | i \in \Omega\} \subseteq L^R$ be nonempty and totally ordered. In order to prove $\wedge_{i\in\Omega} \mathscr{R}(\mu_i) \leqslant \mathscr{R}(\vee_{i\in\Omega} \mu_i)$, it needs to show that $a \leqslant \mathscr{R}(\vee_{i\in\Omega} \mu_i)$ for any $a \leqslant \wedge_{i\in\Omega} \mathscr{R}(\mu_i)$.

By Lemma 3, for all $i \in \Omega$, $x, y, \in R$, we have

$$\begin{aligned}&\mu_i(x) \wedge \mu_i(y) \wedge a \leqslant \mu_i(xy),\\&\mu_i(x) \wedge \mu_i(y) \wedge a \leqslant \mu_i(x - y).\end{aligned} \tag{46}$$

Let $b \in J(L)$ such that

$$b \prec \left(\bigvee_{i\in\Omega} \mu_i(x)\right) \wedge \left(\bigvee_{i\in\Omega} \mu_i(y)\right) \wedge a. \tag{47}$$

Then, we have

$$\begin{aligned}&b \prec \bigvee_{i\in\Omega} \mu_i(x),\\&b \prec \bigvee_{i\in\Omega} \mu_i(y), \qquad b \leqslant a.\end{aligned} \tag{48}$$

Hence, there exists some $i, j \in \Omega$ such that $b \leqslant \mu_i(x), b \leqslant \mu_j(y), b \leqslant a$. Since $\{\mu_i | i \in \Omega\}$ is totally ordered, we assume $\mu_j \leqslant \mu_i$; it follows that $b \leqslant \mu_i(x) \wedge \mu_i(y) \wedge a$.

By

$$\begin{aligned}&\mu_i(x) \wedge \mu_i(y) \wedge a \leqslant \mu_i(xy),\\&\mu_i(x) \wedge \mu_i(y) \wedge a \leqslant \mu_i(x - y),\end{aligned} \tag{49}$$

we obtain $b \leqslant \mu_i(xy)$ and $b \leqslant \mu_i(x - y)$. Hence, $b \leqslant \vee_{i\in\Omega}\mu_i(xy)$ and $b \leqslant \vee_{i\in\Omega}\mu(x - y)$.

From the arbitrariness of b, we have

$$\begin{aligned}&\left(\bigvee_{i\in\Omega} \mu_i(x)\right) \wedge \left(\bigvee_{i\in\Omega} \mu_i(y)\right) \wedge a \leqslant \bigvee_{i\in\Omega} \mu_i(xy),\\&\left(\bigvee_{i\in\Omega} \mu_i(x)\right) \wedge \left(\bigvee_{i\in\Omega} \mu_i(y)\right) \wedge a \leqslant \bigvee_{i\in\Omega} \mu_i(x - y),\end{aligned} \tag{50}$$

Combining Lemma 3, we have $a \leqslant \mathscr{R}(\vee_{i\in\Omega}\mu_i)$. By the arbitrariness of a, we obtain

$$\bigwedge_{i\in\Omega} \mathscr{R}(\mu_i) \leqslant \mathscr{R}\left(\bigvee_{i\in\Omega} \mu_i\right). \tag{51}$$

Therefore, $\mathscr{R}$ is an L-fuzzy convexity on R.

Now, we consider the L-fuzzy subring measures of homomorphic image and preimage of L-fuzzy subsets.

Theorem 6. *Let $f: R \longrightarrow R'$ be a ring homomorphism, and $\mu \in L^R$ and $\eta \in L^{R'}$. Then,*

(1) $\mathscr{R}(f_L^{\rightarrow}(\mu)) \geqslant \mathscr{R}(\mu)$, and if f is injective, then $\mathscr{R}(f_L^{\rightarrow}(\mu)) = \mathscr{R}(\mu)$

(2) $\mathscr{R}(f_L^{\leftarrow}(\eta)) \geqslant \mathscr{R}(\eta)$, and if f is surjective, then $\mathscr{R}(f_L^{\leftarrow}(\eta)) = \mathscr{R}(\eta)$

Proof

(1) can be proved from Theorem 4 and the following fact:

$$\begin{aligned}\mathscr{R}(f_L^{\rightarrow}(\mu)) &= \vee\left\{a \in L | \forall b \in P(L), b \ngeq a, (f_L^{\rightarrow}(\mu))^{(b)} \text{ is a subring of } R\right\}\\&= \vee\left\{a \in L | \forall b \in P(L), b \ngeq a, f_L^{\rightarrow}\left(\mu^{(b)}\right) \text{ is a subring of } R\right\}\\&\geqslant \vee\left\{a \in L | \forall b \in P(L), b \ngeq a, \mu^{(b)} \text{ is a subring of } R\right\}\\&= \mathscr{R}(\mu).\end{aligned} \tag{52}$$

If f is injective, the above $\geq$ can be replaced by $=$. Hence, $\mathscr{R}(\mu) = \mathscr{R}(f_L^{\rightarrow}(\mu))$.

(2) can be similarly obtained by Theorem 5:

$$\begin{aligned}\mathscr{R}(f_L^{\leftarrow}(\eta)) &= \vee\left\{a \in L | \forall b \leqslant a, (f_L^{\leftarrow}(\eta))_{[b]} \text{ is a subring of } R\right\}\\&= \vee\left\{a \in L | \forall b \leqslant a, f_L^{\leftarrow}\left(\eta_{[b]}\right) \text{ is a subring of } R\right\}\\&\geqslant \vee\left\{a \in L | \forall b \leqslant a, \eta_{[b]} \text{ is a subring of } R\right\}\\&= \mathscr{R}(\mu).\end{aligned} \tag{53}$$

If f is surjective, the above $\geq$ can be replaced by $=$. Thus, we can obtain that $\mathscr{R}(\eta) = \mathscr{R}(f_L^{\leftarrow}(\eta))$.

By Theorem 6, we obtain the following theorem.

Theorem 7. *Let $f: R \longrightarrow R'$ be ring homomorphism. Let $\mathscr{R}_R$ and $\mathscr{R}_{R'}$ be the L-fuzzy convexities induced by L-fuzzy subring measures on R and R', respectively. Then, $f: (R, \mathscr{R}_R) \longrightarrow (R', \mathscr{R}_{R'})$ is an L-fuzzy convex preserving mapping and an L-fuzzy convex-to-convex mapping.*

5. The Operations of L-Fuzzy Subrings

In this section, we shall discuss some operation properties of L-fuzzy subring measures. In a ring R, given two L-fuzzy sets A, B, and AB is defined in Definition 3. Now, we present its representations by means of cut sets.

Theorem 8. *Let R be a ring and $A, B \in L^R$. Then, the following conditions are true.*

(1) $\forall a \in L$, $(AB)_{(a)} \subseteq A_{(a)}B_{(a)} \subseteq A_{[a]}B_{[a]} \subseteq (AB)_{[a]}$

(2) $\forall a \in L$, $(AB)^{(a)} \subseteq A^{(a)}B^{(a)} \subseteq A^{[a]}B^{[a]} \subseteq (AB)^{[a]}$; in particular, if $a \in P(L)$, then $(AB)^{(a)} = A^{(a)}B^{(a)}$

(3) $AB = \vee_{a\in L}\left\{a \wedge (A_{[a]}B_{[a]})\right\} = \vee_{a\in M(L)}\left\{a \wedge (A_{[a]}B_{[a]})\right\}$

(4) $AB=\vee_{a\in L}\{a\wedge(A_{(a)}B_{(a)})\}=\vee_{a\in M(L)}\{a\wedge(A_{(a)}B_{(a)})\}$

(5) $AB=\wedge_{a\in L}\{a\vee(A^{[a]}B^{[a]})\}=\wedge_{a\in P(L)}\{a\vee(A^{[a]}B^{[a]})\}$

(6) $AB=\wedge_{a\in L}\{a\vee(A^{(a)}B^{(a)})\}=\wedge_{a\in P(L)}\{a\vee(A^{(a)}B^{(a)})\}$

Proof

(1) $\forall a\in L$; first, we prove that $(AB)_{(a)}\subseteq A_{(a)}B_{(a)}$. By

$$\begin{aligned}x\in(AB)_{(a)}\Rightarrow a\in\beta(AB(x))&=\beta\left(\bigvee_{n\in\mathbb{N},x=\sum_{i=1}^{n}y_iz_i}\left(\bigwedge_{i=1}^{n}(A(y_i)\wedge B(z_i))\right)\right)\\&=\bigcup_{n\in\mathbb{N},x=\sum_{i=1}^{n}y_iz_i}\beta\left(\bigwedge_{i=1}^{n}(A(y_i)\wedge B(z_i))\right)\\&\subseteq\bigcup_{n\in\mathbb{N},x=\sum_{i=1}^{n}y_iz_i}\bigcap_{i=1}^{n}(\beta(A(y_i))\cap\beta(B(z_i))).\end{aligned}\tag{54}$$

We know that there are $y_i,z_i\in R$ and $n\in\mathbb{N}$ such that $x=\sum_{i=1}^{n}y_iz_i$, $a\in\beta(A(y_i))$, and $a\in\beta(B(z_i))$, that is, $y_i\in A_{(a)},z_i\in B_{(a)}$, so $x=\sum_{i=1}^{n}y_iz_i\in A_{(a)}B_{(a)}$. This shows $(AB)_{(a)}\subseteq A_{(a)}B_{(a)}$.

It is obvious that $A_{(a)}B_{(a)}\subseteq A_{[a]}B_{[a]}$.

Now, we prove $A_{[a]}B_{[a]}\subseteq(AB)_{[a]}$. Suppose that $x\in A_{[a]}B_{[a]}$.

Then, by

$$A_{[a]}B_{[a]}=\left\{x\mid\exists n\in\mathbb{N},\ y_i\in A_{[a]}\text{ and }z_i\in B_{[a]}\text{ such that }x=\sum_{i=1}^{n}y_iz_i\right\},\tag{55}$$

we have

$$\begin{aligned}(AB)(x)&=\bigvee_{n\in\mathbb{N},x=\sum_{i=1}^{n}y_iz_i}\left\{\bigwedge_{i=1}^{n}(A(y_i)\wedge B(z_i))\right\}\\&=\bigvee_{n\in\mathbb{N},x=\sum_{i=1}^{n}y_iz_i}\left\{\bigwedge_{i=1}^{n}(a\wedge a)\right\}\geqslant a.\end{aligned}\tag{56}$$

So, we obtain $x\in(AB)_{[a]}$. It is proved that $A_{[a]}B_{[a]}\subseteq(AB)_{[a]}$.

(2) $(AB)^{(a)}\subseteq A^{(a)}B^{(a)}$ can be proved from the following implications.

$$\begin{aligned}x\in(AB)^{(a)}&\Rightarrow(AB)(x)\nleq a\\&\Rightarrow\bigvee_{n\in\mathbb{N},x=\sum_{i=1}^{n}y_iz_i}\left(\bigwedge_{i=1}^{n}(A(y_i)\wedge B(z_i))\right)\nleq a\\&\Rightarrow\exists n\in\mathbb{N},y_i,z_i\in R\text{ such that }x=\sum_{i=1}^{n}y_iz_i,\quad A(y_i)\nleq a,B(z_i)\nleq a\\&\Rightarrow\exists n\in\mathbb{N},y_i,z_i\in R\text{ such that }x=\sum_{i=1}^{n}y_iz_i,\quad y_i\in A^{(a)},z_i\in B^{(a)}\\&\Rightarrow x\in A^{(a)}B^{(a)}.\end{aligned}\tag{57}$$

In particular, if $a \in P(L)$, then the inverse of the above implications are true. In this case, $(AB)^{(a)} = A^{(a)}B^{(a)}$.

It is obvious that $A^{(a)}B^{(a)} \subseteq A^{[a]}B^{[a]}$. Next, we prove that $A^{[a]}B^{[a]} \subseteq (AB)^{[a]}$. Suppose that $x \notin (AB)^{[a]}$. Then, $a \in \alpha((AB)(x))$. By

$$\begin{aligned}
a \in \alpha((AB)(x)) &= \alpha\left(\bigvee_{n\in\mathbb{N},x=\sum_{i=1}^{n} y_i z_i}\left(\bigwedge_{i=1}^{n}(A(y_i)\wedge B(z_i))\right)\right)\\
&\subseteq \alpha\left(\bigwedge_{i=1}^{n}(A(y_i)\wedge B(z_i))\right)\subseteq \bigcup_{i=1}^{n}(\alpha(A(y_i))\cup\alpha(B(z_i)))\\
&\Rightarrow \exists i \in \mathbb{N}, \text{ such that when } x=\sum_{i=1}^{n} y_i z_i, \quad a\in\alpha(A(y_i)) \text{or } a\in\alpha(B(z_i))\\
&\Rightarrow \exists i \in \mathbb{N}, \text{ such that when } x=\sum_{i=1}^{n} y_i z_i, \quad y_i\notin A^{[a]} \text{ or } z_i \notin B^{[a]}\\
&\Rightarrow x\notin A^{[a]}B^{[a]},
\end{aligned} \tag{58}$$

we know that $A^{(a)}B^{(a)} \subseteq A^{[a]}B^{[a]}$.

From (1), (2), and Theorem 1 we can obtain (3), (4), (5), and (6).

Next, two theorems are generalizations of the results in [4].

Theorem 9. *Let A, B be two L-fuzzy subsets of a ring R. Then, $\mathscr{R}(AB) \geq \mathscr{R}(A)\wedge\mathscr{R}(B)$.*

Proof. By Theorem 4 (4), we can obtain the following fact:

$$\begin{aligned}
\mathscr{R}(AB) &= \vee\left\{a\in L|\forall b\in P(L), b\ngeq a, (AB)^{(b)} \text{ is a subring of } R\right\}\\
&= \vee\left\{a\in L|\forall b\in P(L), b\ngeq a, A^{(b)}B^{(b)} \text{ is a subring of } R\right\}\text{by Theorem 8 (2)}\\
&\geqslant \left(\vee\left\{a\in L|\forall b\in P(L), b\ngeq a, A^{(b)} \text{ is a subring of } R\right\}\right)\\
&\wedge\left(\vee\left\{a\in L|\forall b\in P(L), b\ngeq a, B^{(b)} \text{ is a subring of } R\right\}\right)\\
&= \mathscr{R}(A)\wedge\mathscr{R}(B).
\end{aligned} \tag{59}$$

Theorem 10. *Let $\{R_i|i\in\Omega\}$ be a collection of rings and A_i an L-fuzzy subset of R_i, $\forall i\in\Omega$. Then, $\mathscr{R}(\prod_{i\in\Omega}A_i) \geqslant \wedge_{i\in\Omega}\mathscr{R}(A_i)$.*

Proof. It is easy to check that $\prod_{i\in\Omega}A_i = \wedge_{i\in\Omega}P_i^{-1}(A_i)$, where P_i is the projection from $\prod_{i\in\Omega}R_i$ to R_i. From (LMC2) in the proof of Theorem 5, we can obtain

$$\mathscr{R}\left(\prod_{i\in\Omega}A_i\right) = \mathscr{R}\left(\bigwedge_{i\in\Omega}P_i^{-1}(A_i)\right) \geqslant \bigwedge_{i\in\Omega}\mathscr{R}\left(P_i^{-1}(A_i)\right). \tag{60}$$

Since $P_i\colon \prod_{i\in\Omega}R_i \longrightarrow R_i$ is a ring homomorphism, by means of Theorem 5, we have $\mathscr{R}(P_i^{-1}(A_i)) \geqslant \mathscr{R}(A_i)$. Thus, we obtain

$$\mathscr{R}\left(\prod_{i\in\Omega}A_i\right) \geqslant \bigwedge_{i\in\Omega}\mathscr{R}(A_i). \tag{61}$$

From Theorem 6, we can obtain the following corollary.

Corollary 1. *Let $\{R_i|i\in\Omega\}$ be a collection of rings, and let $\mathscr{R}_\Pi$ and $\mathscr{R}_i$ are, respectively, L-fuzzy convexity induced by L-fuzzy subring measures of $\prod_{i\in\Omega}R_i$, R_i. Then, $P_i\colon (\prod_{i\in\Omega}R_i, \mathscr{R}_\Pi) \longrightarrow (R_i, \mathscr{R}_i)$ is an L-fuzzy convex preserving mapping and an L-fuzzy convex-to-convex mapping.*

6. A Generalized definition of Fuzzy Quotient Rings

In this section, firstly, we can get the definition of L-fuzzy quotient ring measure from the given definition of L-fuzzy subring measure; Secondly, we study the relationship between L-fuzzy subring measure and L-fuzzy quotient ring measure. Finally, it is given that the ring homomorphism from ring to quotient ring is L-fuzzy convex-to-convex mapping and L-fuzzy convex preserving mapping. When the measure of L-fuzzy quotient ring is 1, a new characterization of L-fuzzy quotient ring is given by using cut set.

Definition 7. Let A be an L-fuzzy subset in a ring R, and quotient ring $R/N = \{x + N|x \in R\}$. Then, the L-fuzzy quotient ring measure $\mathscr{R}(A/N)$ of A is defined as

$$\mathscr{R}(A/N) = \bigwedge_{x,y\in R}[(A/N(x+N)\wedge A/N(y+N))\longmapsto (A/N((x+N)(y+N))\wedge A/N((x+N)-(y+N)))]. \tag{62}$$

We also say that A/N is an L-fuzzy quotient ring of R with respect to measure $\mathscr{R}(A/N)$.

The following lemma is obvious.

Lemma 4. *Let A be an L-fuzzy subset in a ring R. Then, $\mathscr{R}(A/N)\geqslant a$ if and only if, for any $x, y \in R$,*

$$\begin{aligned} A/N(x+N)\wedge A/N(y+N)\wedge a &\leqslant A/N((x+N)(y+N)),\\ A/N(x+N)\wedge A/N(y+N)\wedge a &\leqslant A/N((x+N)-(y+N)). \end{aligned} \tag{63}$$

Next, we study the relationship between L-fuzzy subring measure and L-fuzzy quotient ring measure.

Corollary 2. *Let A be an L-fuzzy subset in a ring R, N be an ideal of R, and quotient ring $R/N = \{x + N|x \in R\}$. Then,*

$$\mathscr{R}(A)\leqslant \mathscr{R}(A/N). \tag{64}$$

Proof. Suppose that $a\leqslant \mathscr{R}(A)$. Then, $\forall x, y \in R$:

$$\begin{aligned} A(x)\wedge A(y)\wedge a &\leqslant A(xy),\\ A(x)\wedge A(y)\wedge a &\leqslant A(x-y). \end{aligned} \tag{65}$$

Hence,

$$\begin{aligned} A/N(x+N)\wedge A/N(y+N)\wedge a &\leqslant A/N((x+N)(y+N)),\\ A/N(x+N)\wedge A/N(y+N)\wedge a &\leqslant A/N((x+N)-(y+N)). \end{aligned} \tag{66}$$

This shows $a\leqslant \mathscr{R}(A/N)$. So, we can obtain

$$\mathscr{R}(A)\leqslant \mathscr{R}(A/N). \tag{67}$$

From the relationship between L-fuzzy subring measure and L-fuzzy convexity, we can get the following conclusion: R is a ring, the mapping $\mathscr{R}$ defined by $A/N\longmapsto \mathscr{R}(A/N)$: $L^R \longrightarrow L$ is a L-fuzzy convexity on R, which is called L-fuzzy convexity of L-fuzzy quotient ring measure on R. The ring homomorphisms between the other two quotient rings are L-fuzzy convex preserving mapping and L-fuzzy convex–to-convex mapping.

Finally, we show that the ring homomorphism from ring R to quotient ring R/N is L-fuzzy convex preserving mapping and L-fuzzy convex-to-convex mapping.

Theorem 11. *Let R be a ring, R/N be a quotient ring, and A be a L-fuzzy subset of R. So, $f\colon R \longrightarrow R/N\,(A \longrightarrow A/N)$ is a ring homomorphism, and this ring homomorphism is L-fuzzy convex preserving mapping and L-fuzzy convex-to-convex mapping.*

Next, when the measure of L-fuzzy quotient ring is 1, a new characterization of L-fuzzy quotient ring is given by using cut set.

Theorem 12. *Let N be the ideal of R and A be the L-fuzzy subring of R. Then, the following conditions are true:*

(1) $\forall a \in L$, $(A/N)_{(a)} \subseteq A_{(a)}/N \subseteq A_{(a)}/N \subseteq (A/N)_{[a]}$

(2) $(A/N)_{[a]} = \cap_{b\in\beta(a)}(A/N)_{[b]} = \cap_{b\in\beta(a)}(A/N)_{(b)}$

(3) $(A/N)_{(a)} = \cup_{a\in\beta(b)}(A/N)_{[b]} = \cup_{a\in\beta(b)}(A/N)_{(b)}$

(4) $(A/N) = \vee_{a\in L}\{a\wedge (A_{[a]}/N)\} = \vee_{a\in M(L)}\{a\wedge (A_{[a]}/N)\}$

(5) $(A/N) = \vee_{a\in L}\{a\wedge (A_{(a)}/N)\} = \vee_{a\in M(L)}\{a\wedge (A_{(a)}/N)\}$

Proof

(1) $\forall a \in L$; first, we prove that $(A/N)_{(a)} \subseteq A_{(a)}/N$; let $x + N \in (A/N)_{(a)}$. By

$$\begin{aligned} a \in \beta((A/N)(x+N)) &= \beta\Big(\bigvee_{n\in N} A(x+n)\Big)\\ &= \bigcup_{n\in N}\beta(A(x+n)). \end{aligned} \tag{68}$$

We know that there is $n \in N$, such that $a \in \beta(A(x+n))$, so $x + n \in A_{(a)}$; let $-n \in N$, so $x = (x+n) - n \in A_{(a)} + N$, that is, $x + N \in A_{(a)}/N$. This shows $(A/N)_{(a)} \subseteq A_{(a)}/N$.

It is obvious that $A_{(a)}/N \subseteq A_{[a]}/N$.

Now, we prove $A_{[a]}/N \subseteq (A/N)_{[a]}$; suppose that $x + N \in A_{[a]}/N$; then, by

$$A_{[a]}/N = \{x + N|x \in A_{[a]}\}, \tag{69}$$

we have $A/N(x+N)=\vee_{n\in N}A(x+n)\geqslant a$. So, we obtain $x+N\in(A/N)_{[a]}$. It is proved that $A_{[a]}/N\subseteq(A/N)_{[a]}$.

(2) $\forall b\in\beta(a)$; using the definition of cut set, we can easily obtain

$$(A/N)_{[a]}\subseteq(A/N)_{(b)}\subseteq(A/N)_{[b]}\Rightarrow(A/N)_{[a]}\subseteq\bigcap_{b\in\beta(a)}(A/N)_{(b)}\subseteq\bigcap_{b\in\beta(a)}(A/N)_{[b]}, \tag{70}$$

in addition $x+N\in\cap_{b\in\beta(a)}(A/N)_{[b]}\Rightarrow(A/N)(x+N)\geqslant\vee\{b\in L|b\in\beta(a)\}=a\Rightarrow x+N\in(A/N)_{[a]}$.

Therefore, (2) is proved. Similarly, (3) can be proved to be true.

From (1) and Theorem 1, we can get that (4) and (5) hold.

Similarly, using the other two cut sets, we can also give another form of L-fuzzy quotient ring and the corresponding conclusions.

Theorem 13. *Let N be the ideal of R and A be the L-fuzzy subring of R. Then, the following conditions are true:*

(1) $\forall a\in L$, $(A/N)^{(a)}=A^{(a)}/N\subseteq A^{[a]}/N\subseteq(A/N)^{[a]}$

(2) $(A/N)^{[a]}=\cap_{a\in\alpha(b)}(A/N)^{[b]}=\cap_{a\in\alpha(b)}(A/N)^{(b)}$

(3) $(A/N)^{(a)}=\cup_{b\in\alpha(a)}(A/N)^{[b]}=\cup_{b\in\alpha(a)}(A/N)^{(b)}$

(4) $(A/N)=\wedge_{a\in L}\{a\vee(A^{[a]}/N)\}=\wedge_{a\in P(L)}\{a\vee(A^{[a]}/N)\}$

(5) $(A/N)=\wedge_{a\in L}\{a\vee(A^{(a)}/N)\}=\wedge_{a\in P(L)}\{a\vee(A^{(a)}/N)\}$

Proof

(1) $\forall a\in L$; first, we prove that $(A/N)^{(a)}=A^{(a)}/N$. Suppose that $x+N\in(A/N)^{(a)}$; then, we have

$$\begin{aligned}x+N\in(A/N)^{(a)}&\Leftrightarrow(A/N)(x+N)\nleq a\\&\Leftrightarrow\bigvee_{n\in N}A(x+n)\nleq a\\&\Leftrightarrow\forall n\in N,A(x+n)\nleq a\\&\Leftrightarrow\forall n\in N,x+n\in A^{(a)}\\&\Leftrightarrow x+N\in A^{(a)}/N.\end{aligned} \tag{71}$$

So, we can obtain $(A/N)^{(a)}=A^{(a)}/N$.

Obviously, $A^{(a)}/N\subseteq A^{[a]}/N$.

Now, we prove $A^{[a]}/N\subseteq(A/N)^{[a]}$. Suppose that $x+N\notin(A/N)^{[a]}$, so $a\in\alpha((A/N)(x+N))$. By

$$\begin{aligned}a\in\alpha((A/N)(x+N))=\alpha\Big(\bigvee_{n\in N}A(x+n)\Big)&\subseteq\bigcap_{n\in N}(\alpha(A(x+n)))\\&\Rightarrow\forall n\in N,a\in\alpha(A(x+n))\Rightarrow\forall n\in N,x+n\notin A^{[b]}\\&\Rightarrow x+N\notin A^{[a]}/N.\end{aligned} \tag{72}$$

So, we can obtain $A^{[a]}/N\subseteq(A/N)^{[a]}$.

(2) $\forall a\in\alpha(b)$;

using the definition of cut set, we can easily obtain

$$(A/N)^{[a]}\subseteq(A/N)^{(b)}\subseteq(A/N)^{[b]}\Rightarrow(A/N)^{[a]}\subseteq\bigcap_{a\in\alpha(b)}(A/N)^{(b)}\subseteq\bigcap_{a\in\alpha(b)}(A/N)^{[b]}. \tag{73}$$

In addition,

$$\begin{aligned}x+N\in\bigcap_{a\in\alpha(b)}(A/N)^{[b]}&\Rightarrow a\notin\alpha\Big(\bigcap_{a\in\alpha(b)}(A/N)(x+N)\Big)\\&\Rightarrow a\notin\alpha((A/N)(x+N))\\&\Rightarrow x+N\in(A/N)^{[a]}.\end{aligned} \tag{74}$$

Therefore, (2) is proved. Similarly, (3) can be proved to be true.

From (1) and Theorem 1, we can get that (4) and (5) hold.

7. Conclusion

This study proposes the concept of L-subset which is subring to some extent, and the L-fuzzy subring measure is characterized by four kinds of cut sets. The properties of L-fuzzy subring measures under these two operations are also studied. Furthermore, the L-fuzzy convexity in a ring is directly induced by L-fuzzy subring measure, and some properties of this fuzzy convexity are studied. Next, we give the definition and related properties of the measure of L-fuzzy quotient ring and give a new characterization of L-fuzzy quotient ring when the measure of L-fuzzy quotient ring is 1.

Importantly, this idea can be applied to different algebraic systems, such as unitary rings, domains, and prime ideals. Therefore, fuzzy convexity can be derived from different algebraic systems. In addition, we can study the measurement of fuzzy relation and its application in information system, and the measurement of fuzzy filter is considered to provide a theoretical basis for image processing and fuzzy pattern recognition.

Data Availability

The data that support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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