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Research Article

Optimal Weak Type Estimates for the P-ADIC Hardy Type Operators on Higher-Dimensional Product Spaces

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In this paper, we introduce the fractional p-adic Hardy operators and its conjugate operators and obtain its optimal weak type estimates on the p-adic Lebesgue product spaces.

1. Introduction

In recent years, p-adic analysis has been widely used in quantum mechanics, the probability theory, and the dynamical systems [1, 2]. Meanwhile, there is an increasing attention in pseudo-differential equations, wavelet theory, and harmonic analysis (see [3–8]).

For a prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers, a nonzero rational number x is represented as $x = p^{\gamma} m/n$, where γ is an integer and the integers m, n are not divisible by p. Then, the norm is defined as $|x|_p = p^{-\gamma}$, and it is easy to see that the norm satisfies the following properties:

(i) $|x|_p \geq 0, \forall x \in \mathbb{Q}_p, |x|_p = 0 \Leftrightarrow x = 0$

(ii) $|xy|_p = |x|_p |y|_p, \forall x,\ y \in \mathbb{Q}_p$

(iii) $|x+y|_p \leq \max\{|x|_p, |y|_p\}, \forall x,\ y \in \mathbb{Q}_p$, in the case when $|x|_p \neq |y|_p$, we have $|x+y|_p = \max\{|x|_p, |y|_p\}$

It is well known that $\mathbb{Q}_p$ is a typical model of non-Archimedean local fields. From the standard p-adic analysis, any $x \in \mathbb{Q}_p/\{0\}$ can be uniquely represented as a canonical form

$$x = p^{\gamma} \sum_{k=0}^{\infty} \alpha_k p^k, \tag{1}$$

where $\alpha_k, \gamma \in \mathbb{Z}, \alpha_0 \neq 0 \leq \alpha_k < p$, note that the series (1) converges with respect to the norm $|x|_p$ because one has $|p^{\gamma} \alpha_k p^k|_p = p^{-\gamma-k}$. The space $\mathbb{Q}_p^n$ consists of elements $x = (x_1, x_2, \ldots, x_n)$, where $x_i \in \mathbb{Q}_p$, $i = 1, 2, \ldots, n$. The norm in this space is

$$|x|_p := \max_{1\leq i\leq n}\{|x|_{ip}\}, \quad x \in \mathbb{Q}_p^n. \tag{2}$$

The symbols $B_{\gamma}(a)$ and $S_{\gamma}(a)$ represent, respectively, the ball and the sphere with center at $a \in \mathbb{Q}_p^n$ and radius p^{γ}, defined by

$$B_{\gamma}(a) = \{x \in \mathbb{Q}_p^n\colon |x-a|_p \leq p^{\gamma}\}, S_{\gamma}(a) = \{x \in \mathbb{Q}_p^n\colon |x-a|_p = p^{\gamma}\}. \tag{3}$$

It is clear that $S_{\gamma}(a) = B_{\gamma}(a)/B_{\gamma-1}(a)$, and we set $B_{\gamma}(0) = B_{\gamma}$ and $S_{\gamma}(0) = S_{\gamma}$.

As $\mathbb{Q}_p^n$ is a locally compact commutative group with respect to addition, there exists a Harr measure dx on $\mathbb{Q}_p^n$, which is unique up to a positive constant factor and is translation invariant, that is, $d(x+a) = dx$. We normalize the measure dx such that

$$\int_{B_0(0)} \mathrm{d}x = |B_0(0)|_H = 1, \tag{4}$$

where $|B|_H$ denotes the Harr measure of a measure subset B of $\mathbb{Q}_p^n$. By simple calculation, we can obtain that

$$|B_{\gamma}(a)|_H = p^{\gamma n}, \ |S_{\gamma}(a)|_H = p^{\gamma n}(1-p^{-n}). \tag{5}$$

The classical Hardy operator

$$\mathscr{H} f(x) := \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t, \quad x > 0 \tag{6}$$

was introduced by Hardy in [9], and a celebrated integral inequality states that

$$\|\mathscr{H} f\|_{L^q(\mathbb{R}^+)} \le \frac{q}{q-1}\|f\|_{L^q(\mathbb{R}^+)}, \quad 1 < q < \infty. \tag{7}$$

It was also shown that the constant factor $q/(q-1)$ is optimal, knowing its importance in analysis.

Faris in [10] and Christ and Grafakos in [11] proposed an extension of (1) and its adjoint to the n-dimensional Euclidian spaces $\mathbb{R}^n$ of which the equivalent forms are

$$Hf(x) = \frac{1}{|x|^n}\int_{|y|\le|x|} f(y)\,\mathrm{d}y, H^* f(x) = \int_{|y|>|x|}\frac{f(y)}{|y|^n}\mathrm{d}y, \quad \frac{x \in \mathbb{R}^n}{\{0\}}. \tag{8}$$

The norm of H and H^* on $L^q(\mathbb{R}^n)$ was evaluated and found to be equal to that of the classical Hardy operator. For more details about the boundedness of the Hardy operator and its adjoint, we included some references [12–14].

On the other hand, the fractional Hardy operator and its adjoint are obtained by merely interchanging $|\cdot|^n$ with $|\cdot|^{n+\alpha}\,(0 \le \alpha < n)$ in (8). The weak and strong type optimal bounds for the fractional Hardy and adjoint Hardy operator have also spotlighted many researchers in the past, see [15–18].

The n-dimensional fractional p-adic Hardy and adjoint Hardy operator are defined and studied in [19], which, for $f \in L_{\mathrm{loc}}(\mathbb{Q}_p^n)$ and $0 \le \beta < \infty$, are given as

$$\begin{aligned} H_\beta^p f(x) &= \frac{1}{|x|_p^{n-\beta}}\int_{|y|_p\le|x|_p} f(y)\,\mathrm{d}y, \quad H_\beta^{p,*} f(x) \\ &= \int_{|y|_p\ge|x|_p}\frac{f(y)}{|y|_p^{n-\beta}}\mathrm{d}y, \quad \frac{x \in \mathbb{Q}_p^n}{\{0\}}. \end{aligned} \tag{9}$$

when $\beta = 0$, the fractional p-adic Hardy and adjoint Hardy operator reduces to p-adic Hardy and adjoint Hardy operator. Some other papers showing the boundedness of p-adic Hardy-type operators are included [19–26].

In 2020, Li et al. [27] introduced the definition of the fractional Hardy operator on higher-dimensional product spaces as follows:

$$H_{\beta_1,\ldots\beta_m} f(x) := \left(\prod_{i=1}^m \frac{1}{\left|B\left(0,|x_i|\right)\right|^{1-\beta_i/n_i}}\right)\int_{|y_1|<|x_1|}\cdots\int_{|y_m|<|x_m|} f(y_1,\ldots,y_m)\,\mathrm{d}y_1\cdots\mathrm{d}y_m, \tag{10}$$

where f be a nonnegative integrable function on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_m}$, $x_i \in \mathbb{R}^{n_i}$, $m \in \mathbb{N}, 0 \le \beta_i < n_i$, $i = 1,\ldots,m$, $x = (x_1,x_2,\ldots,x_m) \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_m}$ with $\prod_{i=1}^m |x_i| \ne 0$. Furthermore, the corresponding operator norm on the weak Lebesgue product spaces was obtained.

Next, we will introduce the definition of the fractional Hardy operator on higher-dimensional p-adic product spaces and obtain sharp weak bounds.

Definition 1. Let $m, n_i \in \mathbb{N}$, $x_i \in \mathbb{Q}_p^{n_i}$, $0 \le \beta_i < n_i$, $i = 1,\ldots,m$, and f be a nonnegative integrable function on $\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\times\cdots\times\mathbb{Q}_p^{n_m}$. Define the fractional p-adic Hardy operator on higher-dimensional product spaces by

$$H^p_{\beta_1,\ldots\beta_m} f(x) = 9\left(\prod_{i=1}^m \frac{1}{\left|B\left(0,|x_i|_p\right)\right|_H^{1-\beta_i/n_i}}\right)00\int_{|y_1|_p<|x_1|_p}\cdots\int_{|y_m|_p<|x_m|_p} f(y_1,\cdots y_m)\,\mathrm{d}y_1\cdots\mathrm{d}y_m, \tag{11}$$

where $x = (x_1,x_2,\ldots,x_m) \in \mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\times\cdots\times\mathbb{Q}_p^{n_m}$ with $\prod_{i=1}^m |x_i|_p \ne 0$.

In 2020, Wang et al. [28] gave the definition of fractional conjugate Hardy operator on higher-dimensional product spaces as follows:

$$H^{*}_{\beta_1,\ldots\beta m}f(x) := \int_{|y_1|>|x_1|}\cdots\int_{|y_m|>|x_m|}\left(\prod_{i=1}^{m}\frac{f(y_1,\ldots,y_m)}{\left|B(0,|y_i|)\right|^{1-\beta_i/n_i}}\right)\mathrm{d}y_1\ldots\mathrm{d}y_m, \tag{12}$$

where f be a nonnegative integrable function on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_m}$, $x_i\in\mathbb{R}^{n_i}$, $m\in\mathbb{N}$, $0\le\beta_i<n_i, i=1,\ldots,m$, $x=(x_1,x_2,\ldots,x_m)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_m}$ with $\prod_{i=1}^{m}|x_i|\neq 0$, and they also got the corresponding operator norm on the weak Lebesgue product spaces.

Next, we will give a higher-dimensional version of the fractional p-adic conjugate Hardy operator and obtain sharp weak bounds.

Definition 2. Let $m\in\mathbb{N}, n_i\in\mathbb{N}, x_i\in\mathbb{Q}_p^{n_i}$, $0\le\beta_i<n_i, i=1,\ldots,m$, and f be a nonnegative integrable function on $\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\times\cdots\times\mathbb{Q}_p^{n_m}$. Define the fractional p-adic conjugate Hardy operator on higher-dimensional product spaces by

$$H^{p,*}_{\beta_1,\ldots\beta m}f(x) = \int_{|y_1|_p>|x_1|_p}\cdots\int_{|y_m|_p>|x_m|_p}\left(\prod_{i=1}^{m}\frac{f(y_1,\ldots,y_m)}{\left|B\left(0,|y_i|_p\right)\right|_H^{1-\beta_i/n_i}}\right)\mathrm{d}y_1\ldots\mathrm{d}y_m, \tag{13}$$

where $x=(x_1,x_2,\ldots,x_m)\in\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\times\cdots\times\mathbb{Q}_p^{n_m}$ with $\prod_{i=1}^{m}|x_i|\neq 0$.

In this article, we will obtain sharp weak bounds for the fractional p-adic Hardy operators and its conjugate operators on the p-adic Lebesgue product spaces. Our method of proving the main results involves a frequent use of the following formula:

$$\int_{\mathbb{Q}_p^n}f(x)\,\mathrm{d}x = \sum_{\gamma\in\mathbb{Z}}\int_{S_\gamma}f(x)\,\mathrm{d}x. \tag{14}$$

2. Sharp Weak Bounds for Fractional Hardy Operators

This section considers the problem of obtaining optimal weak bounds for $H^p_{\beta_1,\ldots\beta m}$ and our results as follows.

Theorem 1. *Set $0<\beta_i<n_i$, let $Q=(n_1/(n_1-\beta_1),\ldots,n_m/(n_m-\beta_m))$, $i=1,\ldots,m$. If $f\in L^1(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\times\cdots\times\mathbb{Q}_p^{n_m})$, then we have*

$$\left\|H^p_{\beta_1,\ldots,\beta_m}f\right\|_{wL^Q\left(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\times\cdots\times\mathbb{Q}_p^{n_m}\right)}\le 1\cdot\|f\|_{L^1\left(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\times\cdots\times\mathbb{Q}_p^{n_m}\right)}. \tag{15}$$

Furthermore,

$$\left\|H^p_{\beta_1,\cdots\beta m}\right\|_{L^1\left(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\times\cdots\times\mathbb{Q}_p^{n_m}\right)\longrightarrow wL^Q\left(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\times\cdots\times\mathbb{Q}_p^{n_m}\right)} = 1. \tag{16}$$

To obtain the desired result, we need the following lemma.

Lemma 1. *Suppose that $0\le\beta<n$, if $f\in L^1(\mathbb{Q}_p^n)$, then for any $\lambda>0$,*

$$\left\|H^p_\beta f\right\|_{L^{n/n-\beta,\infty}\left(\mathbb{Q}_p^n\right)}\le 1\cdot\|f\|_{L^1\left(\mathbb{Q}_p^n\right)}. \tag{17}$$

Moreover,

$$\left\|H^p_\beta\right\|_{L^1\left(\mathbb{Q}_p^n\right)\longrightarrow L^{n/n-\beta,\infty}\left(\mathbb{Q}_p^n\right)} = 1. \tag{18}$$

Proof. Since

$$\begin{aligned}\left|H^p_\beta f(x)\right| &\le \frac{1}{|x|_p^{n-\beta}}\int_{|y|_p\le|x|_p}|f(y)|\,\mathrm{d}y\\ &\le |x|_p^{-(n-\beta)}\|f\|_{L^1\left(\mathbb{Q}_p^n\right)}.\end{aligned} \tag{19}$$

Next, we let $L=\|f\|_{L^1(\mathbb{Q}_p^n)}$, then, for any $\lambda>0$, we get

$$\left\{x\in\mathbb{Q}_p^n:\ \left|H^p_\beta f(x)\right|>\lambda\right\}\subset\left\{x\in\mathbb{Q}_p^n:\ |x|_p<(L/\lambda)^{1/n-\beta}\right\}. \tag{20}$$

Thus,

$$\begin{aligned}\left\|H^p_\beta f\right\|_{L^{n/n-\beta,\infty}\left(\mathbb{Q}_p^n\right)} &= \sup_{\lambda>0}\lambda\left(\int_{\mathbb{Q}_p^n}\chi_{\left\{x\in\mathbb{Q}_p^n:\ \left|H^p_\beta f(x)\right|>\lambda\right\}}(x)\,\mathrm{d}x\right)^{n-\beta/n}\\ &\le \sup_{\lambda>0}\lambda\left(\int_{\mathbb{Q}_p^n}\chi_{\left\{x\in\mathbb{Q}_p^n:\ |x|_p<(L/\lambda)^{1/(n-\beta)}\right\}}(x)\,\mathrm{d}x\right)^{n-\beta/n}\\ &= \sup_{\lambda>0}\lambda\left(\int_{|x|_p<(L/\lambda)^{1/(n-\beta)}}\mathrm{d}x\right)^{n-\beta/n}\\ &= \sup_{\lambda>0}\lambda\left(\sum_{j=-\infty}^{\log_p(L/\lambda)^{1(n-\beta)}}\int_{S_j}\mathrm{d}x\right)^{n-\beta/n}\\ &= \left(1-p^{-n}\right)^{(n-\beta)/n}\sup_{\lambda>0}\lambda\left(\sum_{j=-\infty}^{\log_p(L/\lambda)^{1/(n-\beta)}}p^{jn}\,\mathrm{d}x\right)^{n-\beta/n}\\ &= \sup_{\lambda>0}\lambda\left(\frac{L}{\lambda}\right) = \|f\|_{L^1\left(\mathbb{Q}_p^n\right)}.\end{aligned} \tag{21}$$

On the other hand, we let $f_0(x) = \chi_{\left\{x\in\mathbb{Q}_p^n:\ |x|_p\le 1\right\}}(x)$, then

$$\|f_0\|_{L^1\left(\mathbb{Q}_p^n\right)} = \int_{\mathbb{Q}_p^n}\chi_{\left\{x\in\mathbb{Q}_p^n:\ |x|_p\le 1\right\}}(x)\,\mathrm{d}x = 1. \tag{22}$$

Also,

$$\begin{aligned}\left|H_\beta^p f_0(x)\right| &= \frac{1}{|x|_p^{n-\beta}}\int_{|y|_p\le|x|_p} f_0(y)dy\\ &= \frac{1}{|x|_p^{n-\beta}}\int_{|y|_p\le|x|_p}\chi_{\left\{x\in\mathbb{Q}_p^n:\ |x|_p\le 1\right\}}\mathrm{d}y\\ &= \frac{1}{|x|_p^{n-\beta}}\begin{cases}\int_{|y|_p\le|x|_p}\mathrm{d}y, & |x|_p\le 1,\\ \int_{|y|_p<1}\mathrm{d}y, & |x|_p>1.\end{cases}\\ &= \begin{cases}|x|_p^{\beta}, & |x|_p\le 1,\\ |x|_p^{\beta-n}, & |x|_p>1.\end{cases}\end{aligned} \tag{23}$$

Now,

$$\left\{x\in\mathbb{Q}_p^n:\ \left|H_\beta^p f_0(x)\right|>\lambda\right\} = \left\{|x|_p\le 1:\ |x|_p^{\beta}>\lambda\right\}\cup\left\{|x|_p>1:\ |x|_p^{\beta-n}>\lambda\right\}. \tag{24}$$

Since $0<\beta<n$, when $\lambda\ge 1$, then

$$\left\{x\in\mathbb{Q}_p^n:\ \left|H_\beta^p f_0(x)\right|>\lambda\right\} = \varnothing. \tag{25}$$

Also, when $0<\lambda<1$, then

$$\left\{x\in\mathbb{Q}_p^n:\ \left|H_\beta^p f_0(x)\right|>\lambda\right\} = \left\{x\in\mathbb{Q}_p^n:\ (\lambda)^{1/\beta}<|x|_p<\left(\frac{1}{\lambda}\right)^{1/(n-\beta)}\right\}. \tag{26}$$

Therefore,

$$\begin{aligned}\left\|H_\beta^p f_0\right\|_{L^{n/(n-\beta,\infty)}\left(\mathbb{Q}_p^n\right)} &= \sup_{0<\lambda<1}\lambda\left(\int_{\mathbb{Q}_p^n}\chi_{\left\{x\in\mathbb{Q}_p^n:\ \lambda^{1/\beta}<|x|_p<(1/\lambda)^{1/(\beta-n)}\right\}}(x)dx\right)^{n-\beta/n}\\ &= \sup_{0<\lambda<1}\lambda\left(\int_{\lambda^{1/\beta}<|x|_p<(1/\lambda)^{1/(n-\beta)}}\mathrm{d}x\right)^{n-\beta/n}\\ &= \sup_{0<\lambda<1}\lambda\left(\sum_{j=\log_p^{\lambda^{1/\beta}}}^{\log_p(1/\lambda)^{1/(n-\beta)}}\int_{S_j}\mathrm{d}x\right)^{n-\beta/n}\\ &= \left(1-p^{-n}\right)^{(n-\beta)/n}\sup_{0<\lambda<1}\lambda\left(\frac{p^{\left(\log_p^{\lambda^{1/\beta}}+1\right)n}-p^{\left(\log_p^{\lambda^{1/(\beta-n)}}+1\right)n}}{1-p^n}\right)^{n-\beta/n}\\ &= \left(1-p^{-n}\right)^{(n-\beta)/n}\sup_{0<\lambda<1}\left(\frac{1-\lambda^{n/\beta}\lambda^{n/(n-\beta)}}{1-p^{-n}}\right)^{n-\beta/n}\\ &= \|f_0\|_{L^1\left(\mathbb{Q}_p^n\right)}.\end{aligned} \tag{27}$$

Thus, as above, we get

$$\left\|H_\beta^p\right\|_{L^1\left(\mathbb{Q}_p^n\right)\longrightarrow L^{n/n-\beta,\infty}\left(\mathbb{Q}_p^n\right)} = 1. \tag{28}$$

Now let us prove Theorem 1. □

Proof. Without loss of generality, we consider only the situation when $m=2$, and then, the case $m\ge 3$ is just a repetition of the case $m=2$. For $m=2$, the operator H_{β_1,β_2}^p can be written as

$$\left(H^{p}_{\beta_1,\beta_2}f\right)(x_1,x_2)=\frac{1}{\left|B\left(0,|x_1|_p\right)\right|_H^{1-\beta_1/n_1}}\frac{1}{\left|B\left(0,|x_2|_p\right)\right|_H^{1-\beta_2/n_2}}\int_{|y_1|_p\le|x_1|_p}\int_{|y_2|_p\le|x_2|_p}f(y_1,y_2)\,dy_1dy_2. \tag{29}$$

When $f\in L^1(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2})$, we get

$$\frac{1}{\left|B\left(0,|x_2|_p\right)\right|_H^{1-\beta_2/n_2}}\int_{|y_2|_p\le|x_2|_p}f(\cdot,y_2)\,dy_2\in L^1\left(\mathbb{Q}_p^{n_1}\right),\quad\text{for all } x_2\in\mathbb{Q}_p^{n_2}. \tag{30}$$

Then by Lemma 1,

$$\begin{aligned}\left\|\left(H^{p}_{\beta_1,\beta_2}f\right)(\cdot,x_2)\right\|_{L^{n_1/n_1-\beta_1,\infty}\left(\mathbb{Q}_p^{n_1}\right)}&=\sup_{\lambda_1>0}\lambda_1\left|\left\{x_1:\left(H^{p}_{\beta_1,\beta_2}\right)(x_1,x_2)>\lambda_1\right\}\right|^{n_1-\beta_1/n_1}\\&\le 1\cdot\left\|\frac{1}{\left|B\left(0,|x_2|_p\right)\right|_H^{1-\beta_2/n_2}}\int_{|y_2|_p<|x_2|_p}f(\cdot,y_2)\,dy_2\right\|_{L^1\left(\mathbb{Q}_p^{n_1}\right)}.\end{aligned} \tag{31}$$

Using Fubini theorem, we obtain that

$$\begin{aligned}&\left\|\frac{1}{|B(0,|x_2|_p)|_H^{1-\beta_2/n_2}}\int_{|y_2|_p<|x_2|_p}f(\cdot,y_2)\,dy_2\right\|_{L^1\left(\mathbb{Q}_p^{n_1}\right)}\\&=\frac{1}{|B(0,|x_2|_p)|_H^{1-\beta_2/n_2}}\int_{\mathbb{Q}_p^{n_1}}\left|\int_{|y_2|_p<|x_2|_p}f(y_1,y_2)\,dy_2\right|dy_1\\&\le\frac{1}{|B(0,|x_2|_p)|_H^{1-\beta_2/n_2}}\int_{|y_2|_p<|x_2|_p}\int_{\mathbb{Q}_p^{n_1}}|f(y_1,y_2)|\,dy_1\,dy_2.\end{aligned} \tag{32}$$

Obviously, $\int_{\mathbb{Q}_p^{n_1}}|f(y_1,y_2)|\,dy_1\in L^1(\mathbb{Q}_p^{n_2})$, if $f\in L^1(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2})$. Then, applying the lemma again, we get

$$\begin{aligned}&\left\|\frac{1}{|B(0,|x_2|_p)|_H^{1-\beta_2/n_2}}\int_{|y_2|_p\le|x_2|_p}\left(\int_{\mathbb{Q}_p^{n_1}}|f(y_1,y_2)|\,dy_1\right)dy_2\right\|_{L^{n_2/n_2-\beta_2,\infty}\left(\mathbb{Q}_p^{n_2}\right)}\\&=\sup_{\lambda_2>0}\lambda_2\left|\left\{x_2:\frac{1}{|B(0,|x_2|_p)|_H^{1-\beta_2/n_2}}\int_{|y_2|_p\le|x_2|_p}\left(\int_{\mathbb{Q}_p^{n_1}}|f(y_1,y_2)|\,dy_1\right)dy_2>\lambda_2\right\}\right|^{n_2-\beta_2/n_2}\\&\le 1\cdot\int_{\mathbb{Q}_p^{n_2}}\int_{\mathbb{Q}_p^{n_1}}|f(y_1,y_2)|\,dy_1\,dy_2\\&=\|f\|_{L^1\left(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\right)}.\end{aligned} \tag{33}$$

Combining (31)–(33), we get

$$\left\|H^{p}_{\beta_1,\beta_2}f\right\|_{wL^{n_1/n_1-\beta_1,n_2/n_2-\beta_2}\left(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\right)}\le 1\cdot\|f\|_{L^1\left(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\right)}, \tag{34}$$

for all $f\in L^1(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2})$.

Conversely, to prove that the constant 1 is optimal, we took

$$f_0(x) = \chi_{\left\{x\in\mathbb{Q}_p^n:\, |x|_p \le 1\right\}}(x). \tag{35}$$

And choosing $F(x_1, x_2) = f_0(x_1) f_0(x_2)$, where $x_1 \in \mathbb{Q}_p^{n_1}, x_2 \in \mathbb{Q}_p^{n_2}$, we get from the definition of $H^p_{\beta_1,\beta_2}$ that

$$\begin{aligned} H^p_{\beta_1,\beta_2} F(x_1, x_2) &= \left(\prod_{i=1}^{2} \frac{1}{\left|\left(B0, |x_i|_p\right)\right|_H^{1-\beta_i/n_i}} \right) \int_{|y_1|_p<|x_1|_p} \int_{|y_2|_p<|x_2|_p} f_0(y_1) f_0(y_2)\, \mathrm{d}y_1 \mathrm{d}y_2 \\ &= H^p_{\beta_1} f_0(x_1) H^p_{\beta_2} f_0(x_2). \end{aligned} \tag{36}$$

Also,

$$\begin{aligned} \left|H^p_{\beta_1} f_0(x_1)\right| &= \frac{1}{\left|B\left(0, |x_1|_p\right)\right|_H^{1-\beta_1/n_1}} \int_{|y_1|_p \le |x_1|_p} f_0(y_1)\, \mathrm{d}y \\ &= \frac{1}{|x_1|_p^{n_1-\beta_1}} \begin{cases} \int_{|y_1|_p \le |x_1|_p} \mathrm{d}y, & |x_1|_p < 1, \\ \int_{|y_1|_p < 1} \mathrm{d}y, & |x_1|_p \ge 1, \end{cases} \\ &= \begin{cases} |x_1|_p^{\beta_1} & |x_1|_p < 1, \\ |x_1|_p^{\beta_1-n_1}, & |x_1|_p \ge 1. \end{cases} \end{aligned} \tag{37}$$

We now that let $L = H^p_{\beta_2} f_0(x_2)$, for $0 < \lambda_1 < L$, then combining both the cases, we get

$$\begin{aligned} &\left|\left\{x_1 \in \mathbb{Q}_p^{n_1}:\, \left|H^p_{\beta_1,\beta_2} F(x_1, x_2)\right| > \lambda_1\right\}\right| \\ &= \left|\left\{|x_1|_p < 1:\, |x_1|_p^{\beta_1} L > \lambda_1\right\} + \left\{|x_1|_p \ge 1:\, |x_1|_p^{\beta_1-n_1} L > \lambda_1\right\}\right| \\ &= \left|\left\{x_1 \in \mathbb{Q}_p^{n_1}:\, \left(\frac{\lambda_1}{L}\right)^{1/\beta_1} < |x_1|_p < 1\right\} + \left\{x_1 \in \mathbb{Q}_p^{n_1}:\, 1 \le |x_1|_p < \left(\frac{L}{\lambda_1}\right)^{1/n_1-\beta_1}\right\}\right| \\ &= \left|\left\{x_1 \in \mathbb{Q}_p^{n_1}:\, \left(\frac{\lambda_1}{L}\right)^{1/\beta_1} < |x_1|_p < \left(\frac{L}{\lambda_1}\right)^{1/n_1-\beta_1}\right\}\right|. \end{aligned} \tag{38}$$

Therefore, we have

$$
\begin{aligned}
&\left\|H^{p}_{\beta_1,\beta_2}F(x_1,x_2)\right\|_{L^{\frac{n_1}{n_1-\beta_1},\infty}\left(\mathbb{Q}_p^{n_1}\right)}\\
&= \sup_{0<\lambda_1<L}\lambda_1\left(\int_{\mathbb{Q}_p^{n_1}}\chi_{\left\{(\lambda_1/L)^{1/\beta_1}<|x_1|_p<(L/\lambda_1)^{1(n_1-\beta_1)}\right\}}(x_1)dx_1\right)^{n_1-\beta_1/n_1}\\
&= \sup_{0<\lambda_1<L}\lambda_1\left(\int_{(\lambda_1/L)^{1/\beta_1}<|x_1|_p<(L/\lambda_1)^{1/(n_1-\beta_1)}}dx_1\right)^{n_1-\beta_1/n_1}\\
&= \sup_{0<\lambda_1<L}\lambda_1\left(\sum_{j=-\log_p^{(\lambda_1/L)^{\beta_1}}+1}^{\log_p^{(L/\lambda_1)^{1/(n_1-\beta_1)}}}\int_{S_j}dx_1\right)^{n_1-\beta_1/n_1}\\
&= \left(1-p^{-n_1}\right)^{(n_1-\beta_1)/n_1}\sup_{0<\lambda_1<L}\lambda_1\left(\sum_{j=-\log_p^{(\lambda_1/L)^{\beta_1}}+1}^{\log_p^{(L/\lambda_1)^{1/(n_1-\beta_1)}}}p^{jn_1}\right)^{n_1-\beta_1/n_1}\\
&= \left(1-p^{-n_1}\right)^{(n_1-\beta_1)/n_1}\sup_{0<\lambda_1<L}\lambda_1\left(\frac{p^{\left(-\log_p^{(\lambda_1/L)^{\beta_1}}+1\right)n_1}-p^{\left(\log_p^{(L/\lambda_1)^{1/(n_1-\beta_1)}}+1\right)n_1}}{1-p^{n_1}}\right)^{n_1-\beta_1/n_1}\\
&= \left(1-p^{-n_1}\right)^{(n_1-\beta_1)/n_1}\sup_{0<\lambda_1<L}\lambda_1\left(\frac{(\lambda_1/L)^{n_1/\beta_1}-(L/\lambda_1)^{n_1/(n_1-\beta_1)}}{p^{-n_1}-1}\right)^{n_1-\beta_1/n_1}\\
&= \left(1-p^{-n_1}\right)^{(n_1-\beta_1)/n_1}\sup_{0<\lambda_1<L}\lambda_1\left(\frac{L^{n_1/(n_1-\beta_1)}-(\lambda_1/L)^{n_1/\beta_1}(\lambda_1)^{n_1/(n_1-\beta_1)}}{1-p^{-n_1}}\right)^{n_1-\beta_1/n_1}\\
&= 1\cdot L.
\end{aligned}
\tag{39}
$$

For $0<\lambda_2<1$, we also divide x_2 into two cases $|x_2|_p<1$ and $|x_2|_p\geq 1$. As above, we get

$$\sup_{0<\lambda_2<1}\lambda_2\left|\left\{x_2\in\mathbb{Q}_p^{n_2}:\ 1\cdot H^{p}_{\beta_2}f_0(x_2)>\lambda_2\right\}\right|^{(n_2-\beta_2)/n_2}=1\times 1. \tag{40}$$

Since $\|F\|_{L^1(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2})}=1\times 1$, by combining (37) with (38),

$$\left\|H^{p}_{\beta_1,\beta_2}F\right\|_{wL^{n_1/n_1-\beta_1,n_2/n_2-\beta_2}\left(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\right)}=1\cdot\|F\|_{L^1\left(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\right)}. \tag{41}$$

This completes the proof. □

3. Sharp Weak Bounds for Fractional Conjugate Hardy Operators

Likewise, this section contains the results having sharp weak bounds for fractional p-adic conjugate Hardy operators, and our results are as follows.

Theorem 2. *Set* $0<\beta_i<n_i$, *let* $Q=n_1/(n_1-\beta_1),\dots,n_m/(n_m-\beta_m)$, $i=1,\dots,m$. *If* $f\in L^1(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\times\cdots\times\mathbb{Q}_p^{n_m})$, *then we have*

$$\left\|H^{p,*}_{\beta_1,\dots,\beta_m}f\right\|_{wL^Q\left(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\times\cdots\times\mathbb{Q}_p^{n_m}\right)}\leq 1\cdot\|f\|_{L^1\left(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\times\cdots\times\mathbb{Q}_p^{n_m}\right)}. \tag{42}$$

Furthermore,

$$\left\|H^{p,*}_{\beta_1,\cdots\beta m}\right\|_{L^1\left(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\times\cdots\times\mathbb{Q}_p^{n_m}\right)\longrightarrow wL^Q\left(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\times\cdots\times\mathbb{Q}_p^{n_m}\right)}=1. \tag{43}$$

In order to prove our theorem, we need the following lemma.

Lemma 2. *Suppose that* $0\leq\beta<n$, *if* $f\in L^1(\mathbb{Q}_p^n)$, *then for any* $\lambda>0$,

$$\left\|H^{p,*}_{\beta}f\right\|_{L^{n/n-\beta,\infty}\left(\mathbb{Q}_p^n\right)}\leq 1\cdot\|f\|_{L^1\left(\mathbb{Q}_p^n\right)}. \tag{44}$$

Moreover,

$$\left\|H^{p,*}_{\beta}\right\|_{L^1\left(\mathbb{Q}_p^n\right)\longrightarrow L^{n/n-\beta,\infty}\left(\mathbb{Q}_p^n\right)}=1. \tag{45}$$

The proof of this result is almost the same as Lemma 1; here, we omit the proof details. Next, we give the proof of Theorem 2.

Proof. Without loss of generality, we only discuss the case $m = 2$, and then, the case $m \geq 3$ is just a repetition of the case $m = 2$. When $m = 2$, the operator $H^{p,*}_{\beta_1,\beta_2}$ can be written as

$$\left(H^{p,*}_{\beta_1,\beta_2} f\right)(x_1, x_2) = \int_{|y_1|_p > |x_1|_p} \int_{|y_2|_p > |x_2|_p} \frac{f(y_1, y_2)}{\left|B\left(0, |y_1|_p\right)\right|_H^{1-\beta_1/n_1} \left|B\left(0, |y_2|_p\right)\right|_H^{1-\beta_2/n_2}} \mathrm{d}y_1 \mathrm{d}y_2. \tag{46}$$

Using Lemma 2 and Fubini theorem, it implies that

$$\begin{aligned}
&\left\|\left(H^{p,*}_{\beta_1,\beta_2} f\right)(\cdot, x_2)\right\|_{L^{n_1/(n_1-\beta_1),\infty}\left(\mathbb{Q}_p^{n_1}\right)} \\
&= \sup_{\lambda_1 > 0} \lambda_1 \left|\left\{x_1 : \left(H^{p,*}_{\beta_1,\beta_2} f\right)(X_1, x_2) > \lambda_1\right\}\right|^{n_1/(n_1-\beta_1)} \\
&\leq 1 \cdot \left\| \int_{|y_2|_p > |x_2|_p} \frac{f(\cdot, y_2)}{\left|B\left(0, |y_2|_p\right)\right|^{1-\beta_2/n_2}} \right\|_{L^1\left(\mathbb{Q}_p^{n_1}\right)} \\
&\leq 1 \cdot \int_{|y_2|_p > |x_2|_p} \frac{1}{\left|B\left(0, |y_p|\right)\right|^{1-\beta_2/n_2}} \int_{\mathbb{Q}_p^{n_1}} f(y_1, y_2) \,\mathrm{d}y_1 \,\mathrm{d}y_2.
\end{aligned} \tag{47}$$

We conclude that

$$\begin{aligned}
&\left\| \int_{|y_2|_p > |x_2|_p} \frac{1}{\left|B\left(0, |y_p|\right)\right|^{1-\beta_2/n_2}} \int_{\mathbb{Q}_p^{n_1}} f(y_1, y_2) \,\mathrm{d}y_1 \,\mathrm{d}y_2 \right\|_{L^{n_2/(n_2-\beta_2),\infty}\left(\mathbb{Q}_p^{n_2}\right)} \\
&= \sup_{\lambda_2 > 0} \lambda_2 \left| \left\{ x_2 : \int_{|y_2|_p > |x_2|_p} \frac{1}{\left|B\left(0, |y_p|\right)\right|^{1-\beta_2/n_2}} \int_{\mathbb{Q}_p^{n_1}} f(y_1, y_2) \,dy_1 dy_2 > \lambda_2 \right\} \right|^{n_2/n_2-\beta_2} \\
&\leq 1 \cdot \int_{\mathbb{Q}_p^{n_2}} \int_{\mathbb{Q}_p^{n_1}} f(y_1, y_2) \,\mathrm{d}y_1 \mathrm{d}y_2 \\
&= 1 \cdot \|f\|_{\mathbb{Q}_p^{n_1} \times \mathbb{Q}_p^{n_2}}.
\end{aligned} \tag{48}$$

Consequently, combining (45) and (46), we get

$$\left\|H^{p,*}_{\beta_1,\beta_2} f\right\|_{wL^{n_1/(n_1-\beta_1),n_2/(n_2-\beta_2)}\left(\mathbb{Q}_p^{n_1} \times \mathbb{Q}_p^{n_2}\right)} \leq 1 \cdot \|f\|_{L^1\left(\mathbb{Q}_p^{n_1} \times \mathbb{Q}_p^{n_2}\right)}. \tag{49}$$

On the other hand, for any $0 < \varepsilon < 1$, we took

$$f_\varepsilon(x) = \begin{cases} |x|_p^{-(\beta+n)/\varepsilon}, & |x|_p \geq 1, \\ 0, & |x|_p < 1. \end{cases} \tag{50}$$

Let $F(x_1, x_2) = f_{\epsilon_1}(x_1) f_{\epsilon_2}(x_2)$, where $x_1 \in \mathbb{Q}_p^{n_1}$, $x_2 \in \mathbb{Q}_p^{n_2}$, then

$$\begin{aligned}
\|F\|_{L^1\left(\mathbb{Q}_p^{n_1} \times \mathbb{Q}_p^{n_2}\right)} &= \|f\|_{\epsilon_1 L^1\left(\mathbb{Q}_p^{n_1}\right)} \|f\|_{\epsilon_2 L^1\left(\mathbb{Q}_p^{n_2}\right)} \\
&= \frac{1 - p^{-n_1}}{p^{((\beta_1+n_1)/\epsilon_1)-n_1} - 1} \frac{1 - p^{-n_2}}{p^{((\beta_2+n_2)/\epsilon_2)-n_2} - 1}.
\end{aligned} \tag{51}$$

We have

$$\begin{aligned}
H^{p,*}_{\beta_i} f_{\epsilon_i}(x_i) &= \int_{|y_i|_p > |x_i|_p} |y_i|_p^{-(\beta_i+n_i)/\varepsilon_i - (n_i-\beta_i)} \chi_{\{|y_i|_p \geq 1\}}(y_i) \,\mathrm{d}y_i \\
&= \begin{cases} \dfrac{(1 - p^{-n_i}) |x_i|^{\beta_i - (\beta_i+n_i)/\varepsilon_i}}{p^{(\beta_i+n_i)/\varepsilon_i - \beta_i} - 1}, & |x_i|_p \geq 1, \\ \dfrac{(1 - p^{-n_i})}{p^{(\beta_i+n_i)/\varepsilon_i - \beta_i} - 1}, & |x_i|_p < 1. \end{cases}
\end{aligned} \tag{52}$$

Set $C_{\epsilon_i} = (1 - p^{-n_i})/p^{(\beta_i+n_i)/\epsilon_i-\beta_i} - 1$ and $M = (C_{\epsilon_1} H_{\beta_2}^{p,*} f_2(x_2)/\lambda_1)^{1/((\beta_1+n_1)/\epsilon_1-\beta_1)}$, we obtain that

$$
\begin{aligned}
&\left|\left\{x_1 \in \mathbb{Q}_p^{n_1}: |H_{\beta_1,\beta_2}^{p,*} F(x_1,x_2)| > \lambda_1\right\}\right| \\
&= \left|\left\{|x_1|_p < 1: C_{\epsilon_1} H_{\beta_2}^{p,*} f_2(x_2) > \lambda_1\right\}\right| + \left|\left\{|x_1|_p \geq 1: C_{\epsilon_1} |x_1|^{\beta_1-(\beta_1+n_1)/\epsilon_1} H_{\beta_2}^{p,*} f_2(x_2) > \lambda_1\right\}\right| \\
&= \left|\left\{x_1 \in \mathbb{Q}_p^{n_1}: |x_1|_p^{(\beta_1+n_1)/\epsilon_1-\beta_1} < C_{\epsilon_1} H_{\beta_2}^{p,*} f_2(x_2)/\lambda_1\right\}\right|.
\end{aligned} \tag{53}
$$

Notice that when $\lambda_i > C_{\epsilon_i}, \{x_i \in \mathbb{Q}_p^{n_1} : |H_{\beta_i}^{p,*} f_i(x_i)| > \lambda_i\} = \varnothing$, if ϵ_i is small enough, C_{ϵ_i} tends to zero; therefore, when ε_i is small enough, we get

$$
\begin{aligned}
I_0 &:= \sup_{0<\lambda_1 < H_{\beta_2}^{p,*} f_2(x_2)} \lambda_1 \left|\left\{x_1 \in \mathbb{Q}_p^{n_1}: |H_{\beta_1,\beta_2}^{p,*}| > \lambda_1\right\}\right|^{n_1/(n_1-\beta_1)} \\
&= \sup_{0<\lambda_1 < H_{\beta_2}^{p,*} f_2(x_2)} \lambda_1 \left(\int_{|x_1|_p^{(\beta_1+n_1)/\varepsilon_1-\beta_1} < C_{\varepsilon_1} H_{\beta_2}^{p,*} f_2(x_2)/\lambda_1} dx_1 \right)^{n_1/(n_1-\beta_1)} \\
&= \sup_{0<\lambda_1 < H_{\beta_2}^{p,*} f_2(x_2)} \lambda_1 \left(\sum_{j=-\infty}^{\log_p^M} \int_{S_j} dx_1 \right)^{n_1/(n_1-\beta_1)} \\
&= (1 - p^{-n_1})^{n_1/(n_1-\beta_1)} \sup_{0<\lambda_1 < H_{\beta_2}^{p,*} f_2(x_2)} \lambda_1 \left(\sum_{j=-\infty}^{\log_p^M} p^{jn_1} \right)^{n_1/(n_1-\beta_1)} \\
&= \sup_{0<\lambda_1 < H_{\beta_2}^{p,*} f_2(x_2)} \lambda_1 (M^{n_1})^{n_1/(n_1-\beta_1)} \\
&= \sup_{0<\lambda_1 < H_{\beta_2}^{p,*} f_2(x_2)} \lambda_1 \left(\left(C_{\varepsilon_1} H_{\beta_2}^{p,*} f_2(x_2)/\lambda_1 \right)^{1/(((\beta_1+n_1)/\varepsilon_1-\beta_1)n_1)} \right)^{n_1/(n_1-\beta_1)} \\
&= \left(C_{\varepsilon_1} H_{\beta_2}^{p,*} f_2(x_2) \right)^{1/((n_1-\beta_1)((\beta_1+n_1)/\varepsilon_1-\beta_1))} \\
&\quad \times \sup_{0<\lambda_1 < \min\left\{H_{\beta_2}^{p,*} f_2(x_2), C_{\varepsilon_1}\right\}} \lambda_1^{1-1/(n_1-\beta_1)((\beta_1+n_1)/\varepsilon_1-\beta_1)} \\
&= C_{\varepsilon_1} \left(H_{\beta_2}^{p,*} f_2(x_2) \right)^{1/((n_1-\beta_1)((\beta_1+n_1)/\varepsilon_1-\beta_1))} \\
&= \left(H_{\beta_2}^{p,*} f_2(x_2) \right)^{1/((n_1-\beta_1)((\beta_1+n_1)/\varepsilon_1-\beta_1))} \left[\frac{p^{(\beta_1+n_1)/\varepsilon_1-n_1} - 1}{p^{(\beta_1+n_1)/\varepsilon_1-\beta_1} - 1} \|f_{\varepsilon_1}\|_{L^1(\mathbb{Q}_p^{n_1})} \right].
\end{aligned} \tag{54}
$$

Using the same method for x_2, we obtain that

$$\begin{aligned}
&\sup_{\lambda_2>0}\lambda_2\left|\left\{x_2\in\mathbb{Q}_p^{n_2}:\ I_0>\lambda_2\right\}\right|^{n_2/(n_2-\beta_2)}\\
&=\left(C_{\varepsilon_2}\left[\frac{p^{(\beta_1+n_1)/\varepsilon_1-n_1}-1}{p^{(\beta_1+n_1)/\varepsilon_1-\beta_1}-1}\left\|f_{\varepsilon_1}\right\|_{L^1(\mathbb{Q}_p^{n_1})}\right]^{((n_1-\beta_1)/(n_2-\beta_2))\left(\left(p^{(\beta_1+n_1)/\varepsilon_1-n_1}-1\right)/\left(p^{(\beta_1+n_1)/\varepsilon_1-\beta_1}-1\right)\right)}\right)\\
&\quad\times\sup_{0<\lambda_2<C_{\varepsilon_2}}\lambda_2^{1-((n_1-\beta_1)/(n_2-\beta_2))\left(\left(p^{(\beta_1+n_1)/\varepsilon_1-n_1}-1\right)/\left(p^{(\beta_1+n_1)/\varepsilon_1-\beta_1}-1\right)\right)}\\
&=C_{\varepsilon_2}\left(\left[\frac{p^{(\beta_1+n_1)/\varepsilon_1-n_1}-1}{p^{(\beta_1+n_1)/\varepsilon_1-\beta_1}-1}\left\|f_{\varepsilon_1}\right\|_{L^1(\mathbb{Q}_p^{n_1})}\right]^{((n_1-\beta_1)/(n_2-\beta_2))\left(\left(p^{(\beta_1+n_1)/\varepsilon_1-n_1}-1\right)/\left(p^{(\beta_1+n_1)/\varepsilon_1-\beta_1}-1\right)\right)}\right)\\
&=\left(\left[\frac{p^{(\beta_1+n_1)/\varepsilon_1-n_1}-1}{p^{(\beta_1+n_1)/\varepsilon_1-\beta_1}-1}\left\|f_{\varepsilon_1}\right\|_{L^1(\mathbb{Q}_p^{n_1})}\right]^{((n_1-\beta_1)/(n_2-\beta_2))\left(\left(p^{(\beta_1+n_1)/\varepsilon_1-n_1}-1\right)/\left(p^{(\beta_1+n_1)/\varepsilon_1-\beta_1}-1\right)\right)}\right)\\
&\quad\times\left(\left[\frac{p^{(\beta_2+n_2)/\varepsilon_2-n_2}-1}{p^{(\beta_2+n_2)/\varepsilon_2-\beta_2}-1}\left\|f_{\varepsilon_2}\right\|_{L^1(\mathbb{Q}_p^{n_2})}\right]\right)
\end{aligned}\tag{55}$$

Let $\varepsilon_1 \longrightarrow 0^+$ and $\varepsilon_2 \longrightarrow 0^+$, it implies that

$$\left\|H_{\beta_1,\beta_2}^{p,*}F\right\|_{wL^{(n_1/(n_1-\beta_1)),(n_2/(n_2-\beta_2))}\left(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\right)}\geq 1\cdot\|F\|_{L^1\left(\mathbb{Q}_p^{n_1}\times\mathbb{Q}_p^{n_2}\right)}. \tag{56}$$

This finishes the proof of Theorem 2. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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