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Research Article

Further Results on Total Edge-Vertex Domination

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Total edge-vertex domination is a new total domination-type parameter. In this paper, the author shows that determining the total edge-vertex domination number in bipartite planar graphs is NP-complete. Also, the author obtains a structural relation between total domination number and total edge-vertex domination number and characterizes the trees whose total edge-vertex domination number is equal to total domination number.

1. Introduction

Graphs are indispensable tools in every phase of human daily life and in many disciplines where mathematics permeates [1–3]. Graph theory also has its own invariants. The concept of domination is one of the main parameters in graph theory. The domination is used in the solution of many problems and analysis of events in the historical process of human life. It is also used in solving network problems. Until now, many types of domination have been defined. A study on the total variations of two of these important domination types is conducted here. Their relationship with each other is examined in this study.

A graph is usually demonstrated by the symbol $G = (V, E)$ in which vertices are shown with V and edges are shown with E. Here, we can express two important definitions as follows: the set $N_G(v) = \{u|uv \in E\}$ is the open neighborhood of a vertex v in a graph G. The set $N_G[v] = N_G(v) \cup \{v\}$ is the closed neighborhood of v.

The symbol $d_G(v)$ indicates the degree of a vertex $v \in V$ and it is the number of vertices adjacent to v. A vertex with degree one is named a last vertex. If a vertex has degree at least two, it is called an internal vertex. A vertex adjacent to a last vertex is called the remote. There are two types of the remote vertices. Provided that a remote vertex is adjacent to a last vertex, it is called a weak remote and provided that a remote vertex is adjacent to at least two last vertices, it is called a strong remote. An edge of the graph G which is incident to a last vertex is called an end edge. The edge adjacent to an end edge (dissimilar from an end edge) is called a remote edge. The distance $d(u, v)$ between any two vertices u and v is the minimum of the lengths of paths between u and v. The diameter of a graph is defined as $\max\{d(u, v)\}$. The eccentricity is specified as the maximum distance of one vertex from other vertex. If a tree has a even diameter, the middle vertex of the tree is called a central vertex. A path and a tree are indicated by the symbols P_n and T, respectively. The diameter of a tree T is indicated by $\text{diam}(T)$. Provided that there is an edge $e \in E(T)$ such that e is an edge containing v as an endpoint, any vertex v of the tree is adjacent to a P_n.

Let B be a subset of vertices set of the graph G. The subset B is a dominating set (simply, DS), provided that every vertex in G either is an element of B or is adjacent to at least one vertex in B. The symbol $\gamma(G)$ indicates the domination number of a graph G and $\gamma(G)$ is equal to the minimum cardinality of a DS in G. Similarly, a subset $B \subseteq V$ is a total dominating set (simply, TDS), provided that every vertex of V has a neighbor in B. The symbol $\gamma_t(G)$ indicates the total domination number of a graph G and $\gamma_t(G)$ is equal to the minimum cardinality of a TDS in G [4–7]. We refer to the book [8] for more details about domination theory.

An edge e ev-dominates a vertex v provided that e is incident to v or e is incident to a vertex which is adjacent to v. A subset $B \subseteq E$ is an edge-vertex dominating set (simply, EVDS) of G, provided that every vertex of a graph G is

ev-dominated by at least one edge in B. The minimum cardinality of an EVDS is the ev-domination number and denoted by $\gamma_{ev}(G)$. Peters presented the concept of edge-vertex domination [9]. Lewis made studies that made additional contributions about it [10]. These two studies [11, 12] on the edge-vertex domination number of trees also attract attention.

A vertex v ve-dominates an edge e which is incident to v and any edge which is adjacent to e. A set $B \subseteq V$ is a ve-dominating set (simply, VEDS) provided that all edges of a graph G are ve-dominated by at least one vertex in B [9]. The symbol $\gamma_{ve}(G)$ indicates ve-domination number of a graph G and $\gamma_{ve}(G)$ is equal to the minimum cardinality of a VEDS in G.

In [13], Boutrig and Chellali presented the total vertex-edge domination. A subset $B \subseteq V$ is a total vertex-edge dominating set (simply, TVEDS) of G, provided that B is a VEDS and every vertex in B has a neighbor in B [13]. The symbol $\gamma_{ve}^t(G)$ indicates the total ve-domination number of a graph G and $\gamma_{ve}^t(G)$ is equal to the minimum cardinality of a TVEDS.

The total edge-vertex domination was presented in [14]. A subset $B \subseteq E$ is a total edge-vertex dominating set (simply, TEVDS) of G, provided that B is an EVDS and every edge in B shares an endpoint with another edge in B. The symbol $\gamma_{ev}^t(G)$ indicates the total ev-domination number of a graph G and $\gamma_{ev}^t(G)$ is equal to the minimum cardinality of a TEVDS. In [14], the authors show that determining the total ev-domination number of bipartite graphs is NP-complete. In this paper, we extend this result to bipartite planar graphs. Moreover, we obtain a relation between total domination number and total ev-domination number and characterize the trees that provide this property.

2. Preliminaries

First of all, let us start by stating some basic results that we will use in this study.

Observation 1 Every remote vertex of G is contained in every TDS of G.

Observation 2 (See [11]). A $\gamma_{ev}(G)$-set can be determined such that it does not contain any end edge if the diameter of every connected graph G is at least three.

Observation 3 (See [14]). A $\gamma_{ev}^t(T)$-set can be determined such that it does not contain any end edge if the diameter of every tree T is at least four.

Observation 4 (See [14]). Every remote edge is included by a minimum total edge-vertex dominating set if the diameter of every tree is at least four.

Since the graphs with only one edge have no total edge-vertex dominating set, we will not take into account graphs like these ones, by the definition of total domination.

Observation 5 (See [14]). For a nontrivial tree T, $\gamma_{ev}(T) \leq \gamma_{ev}^t(T)$.

Lemma 1 (See [14]). *For a nontrivial tree T, $\gamma_{ev}^t(T) \leq \gamma_t(T)$.*

3. Complexity Conclusion

The NP-completeness conclusion for the total ev-domination issue in bipartite graphs is here constructed. As stated below, it is shown that TOTAL EV-DOM is NP-complete for bipartite planar graphs and VERTEX COVER is used for this process.

3.1. Vertex Cover

3.1.1. Instance. A graph $G = (V, E)$ and an integer $k \leq |V|$.

3.1.2. Question. Is there a subset $V' \subseteq V$ such that $|V'| \leq k$ and, for each edge $uv \in E$, at least one of u and v belongs to V' (i.e., a subset of the vertices of the graph which dominates all of its edges)?

Theorem 1. *For bipartite planar graphs, problem TOTAL EV-DOM is NP-Complete.*

Proof. Figure 1(a) illustrates this proof. Consider an instance graph G of Vertex Cover on a planar graph, a known $\mathcal{NP}$-complete problem [15]. A subdivision graph S_1 of G has $V(S_1) = V(G) \cup E(G)$ and $E(S_1) = \{(u, uv), (uv, v) | uv \in E(G)\}$, i.e., S_1 is obtained from G by replacing its edges by paths of size three. Consider a graph H acquired off the subdivision graph of G by suffixing, for every $V(G)$-vertex j in S_1 a path of size five with vertices r_j, s_j, u_j, v_j, w_j and add an edge between u_j, the center of the P_5, and j. A subdivision S_1 of G is a bipartite graph, since we double the size of any cycle of G. Moreover, if G was planar, S_1 remains planar with the same planar representation and the addition of the P_5 with center adjacent to each $V(G)$-vertex of S_1 clearly maintain the graph both bipartite and planar.

We claim that G is of a vertex cover set V' with $|V'| \leq k$ if and only if H is of a total edge-vertex dominating set D with $|D| \leq 2|V(G)| + k$.

Let V' be a vertex cover set of G with size k. Consider the set S of edges of H composed by the edges $\{s_i u_i, u_i v_i | i \in V(G)\} \cup \{iu_i | i \in V\}$. S is composed of triples of edges connected to a central vertex u_i. Clearly, all P_5's vertices and all $V(G)$-vertices of H are dominated by the edges $\{s_i u_i, u_i v_i | i \in V\}$. Moreover, since S is a vertex cover set of G, all $E(G)$-vertices of H are dominated by the edges $\{iu_i | i \in V\}$. Therefore, S is a TEVDS set with size $2|V(G)| + k$.

Conversely, consider a TEVDS of H with size $2|V(G)| + k$. We need the edges $\{s_i u_i, u_i v_i | v \in V(G)\}$ in S to dominate the vertices r_i and w_i for every $i \in V(G)$. Consequently, S has $2|V(G)|$ such edges and another k edges remain. Provided that there exists an edge between a $V(G)$-vertex p and $E(G)$-vertex pq of H, since all $V(G)$-vertices are dominated by the already $2|V(G)|$ chosen edges, we can pick another total edge-vertex dominating set S' of same size by replacing this edge by the edge pu_p. Now the set of edges $S'/\{s_iu_i, u_iv_i|v \in V(G)\}$ has size k with edges of the type pu_p. Since S' is a total edge-vertex dominating set, these edges dominate all $E(G)$-vertices of H and, therefore, a set $V' = \{p|pu_p \in S'\}$ is a vertex cover set of G with size k. □

Note that we could also relate the domination (and edge domination) problem of a graph G with total ev-dom of a variation of the graph H, if we consider the upper and lower vertices of H as vertices (edges) and there exists an edge among two vertices of H provided that the upper vertex dominates the lower vertex, as described in Figure 1(b) and 1(c)). The same reasoning also applies for an ev-edge dominating set and ve-dominating set, just consider the edges between the upper and lower part of the graph if the upper element dominates the lower element. □

4. Total ev-Dominating Number and Total Dominating Number

Note that the edges of a minimum TEVDS of a graph G do not form cycles on the graph. Otherwise, we could remove one of these edges of the cycle and we would still get a TEVDS of smaller size. Therefore, the edges of a minimum TEVDS of G form a forest subgraph of G.

Assume that S is a minimum TEVDS and let $T_1, \ldots, T_p$ be the p trees of the forest given by the edges of S. We have that the $|\cup_{1\leq i\leq p}E(T_i)| + p$ vertices of these p trees are a TDS of G. Therefore, $\gamma_t(G) \leq \gamma_{ev}^t(G) + p$, i.e., $\gamma_{ev}^t(G) \geq \gamma_t(G) - p$.

Consider a minimum TDS, which is shown by B, of G with size $\gamma_t(G)$. Let $T_1, T_2, \ldots, T_q$ be a forest between vertices of B. If all trees have three or more vertices, then the edges of these trees are a TEVDS, which is shown by S, of G. This holds due to the definition of "total". Otherwise, we need to add an additional edge to the trees with two vertices to it be a TEVDS. For each tree T_i with $|V(T_i)| = 2$ vertices in B, we include two edges in S. And for each tree T_i with $|V(T_i)| \geq 3$ vertices in B, we include $|V(T_i)| - 1$ edges of the tree in S. Therefore, $\gamma_{ev}^t(G) \leq \gamma_t(G)$ and the upper bound is tight only when all trees $T_1, T_2, \ldots, T_q$ have two vertices. Otherwise, for each tree with three or more vertices, we increase the gap between $\gamma_{ev}^t(G)$ and $\gamma_t(G)$ by one unit. I.e., if there are x of such trees with size three or more, then $\gamma_{ev}^t(G) + x \leq \gamma_t(G)$.

Consider now a minimum TEVDS of G. It would be seen as the minimum value of $|S| - u$ among whole the total dominating sets S of G (each TDS has a value u given with the maximum number of trees with three vertices or more connecting the vertices of S). In the worst case scenario when $u = 0$, all trees connecting the vertices of S have two vertices and we have $\gamma_{ev}^t(G) = |S|$. Otherwise, each tree with three or more vertices connecting the vertices of S increase the value of u in one unit. Since u is upper bounded by $|S|/3$ (each unit of u correspond a tree with at least three vertices of S), we have $\min\{2|S|/3\} \leq \min\{|S| - u\} = \gamma_{ev}^t(G)$. And since $\gamma_t(G) \leq |S|$, we have $2\gamma_t(G)/3 \leq \gamma_{ev}^t(G)$.

Therefore, $2\gamma_t(G)/3 \leq \gamma_{ev}^t(G) \leq \gamma_t(G)$. This information is interesting for the following reasons. Since it is hard to approximate $\gamma_t(G)$ [16], it will be also hard to approximate the value of $\gamma_{ev}^t(G)$.

Now, we qualify whole trees on the equality $\gamma_t(T) = \gamma_{ev}^t(T)$. We will use a family $\mathscr{F}$ of trees $T = T_q$. Suppose that $T_1 = \{P_3, P_4, P_7, P_8, P_{12}\}$ and for a $q \in \mathbb{Z}^+$, T_{q+1} is a tree repetitively acquired over T_q per one of the three processes mentioned below:

Process O_1: suffix a vertex by joining it to any remote vertex of T_q or an internal vertex adjacent to a remote vertex.

Process O_2: suffix a path P_3 by joining one of its last vertices to a vertex of T_q which is adjacent to a path P_3.

Process O_3: suffix a path P_4 by joining one of its last vertices to an internal vertex of T_q for which is not a remote vertex.

We can give an example of the implementation of these processes as shown in Figure 2.

There is an exceptional situation in the family $\mathscr{F}$ because of the structure of the tree P_4. P_4 is the only path which has two adjacent remote vertices and has not an internal vertex other than these remote vertices. Thus, if a path P_4 is attached by joining one of its last vertices to a remote vertex of $R' = T_1 = P_4$, it is found as $\gamma_t(T) = \gamma_{ev}^t(T)$ and $T \in \mathscr{F}$. The two examples of trees for the exceptional situation are depicted in Figure 3.

Theorem 2. *Let T be a tree. If $T \in \mathscr{F}$, then $\gamma_t(T) = \gamma_{ev}^t(T)$.*

Proof. The induction is operated over the number q of processes used to build the tree $T = T_{q+1}$. Provided that $T = T_1 \in \{P_3, P_4, P_7, P_8, P_{12}\}$, then $\gamma_t(P_3) = 2 = \gamma_{ev}^t(P_3)$, $\gamma_t(P_4) = 2 = \gamma_{ev}^t(P_4)$, $\gamma_t(P_7) = 4 = \gamma_{ev}^t(P_7)$, $\gamma_t(P_8) = 4 = \gamma_{ev}^t(P_8)$, and $\gamma_t(P_{12}) = 6 = \gamma_{ev}^t(P_{12})$. Then, suppose that the equality is achieved provided each $R' = T_q$ that is a component of $\mathscr{F}$ acquired through $q - 1$ processes.

First suppose that $T = T_{q+1}$ is gotten over R' through Process O_1. Assume that a is a vertex suffixed to a remote vertex or an internal vertex h of R' which is adjacent to a remote vertex. Let B' is a TDS of R'. B' is also a TDS of T by Observation 1. Thus, we have $\gamma_t(T) \leq \gamma_t(R')$. Moreover, we know that $\gamma_{ev}^t(R') \leq \gamma_{ev}^t(T)$. So, $\gamma_t(T) \leq \gamma_t(R') = \gamma_{ev}^t(R') \leq \gamma_{ev}^t(T)$. By Lemma 1 $\gamma_{ev}^t(T) \leq \gamma_t(T)$. As a result, $\gamma_{ev}^t(T) = \gamma_t(T)$.

Suppose that $T = T_{q+1}$ is gotten over R' per Process O_2. Now we assume that Process O_2 is applied a vertex h that is adjacent to a path P_3 xyz. Let abc is suffixed by joining a to h. Let B' be a $\gamma_t(R')$-set. z and c are dominated. Then, the remote vertices b, y and the neighbors of these remote vertices a, x are included by a $\gamma_t(T)$-set. Consequently, $B' \cup \{a, b\}$ is a $\gamma_t(T)$-set and $\gamma_t(T) \leq \gamma_t(R') + 2$. Also, to ev-dominate the vertices c and z, the edges ab, xy and the neighbors of these edges ha, hx are included by a $\gamma_{ev}^t(T)$-set. In this way, provided that B is a $\gamma_{ev}^t(T)$-set, $B/\{ha, ab\}$ is a $\gamma_{ev}^t(R')$-set. We get that $\gamma_{ev}^t(T) \geq \gamma_{ev}^t(R') + 2$. Consequently, $\gamma_t(T) \leq \gamma_t(R') + 2 = \gamma_{ev}^t(R') + 2 \leq \gamma_{ev}^t(T) - 2 + 2 = \gamma_{ev}^t(T)$. From the result and Lemma 1, we obtain that $\gamma_{ev}^t(T) = \gamma_t(T)$.

Suppose that $T = T_{q+1}$ is acquired over R' per Process O_3. We assume a path P_4 $abcl$ is suffixed by joining a to an

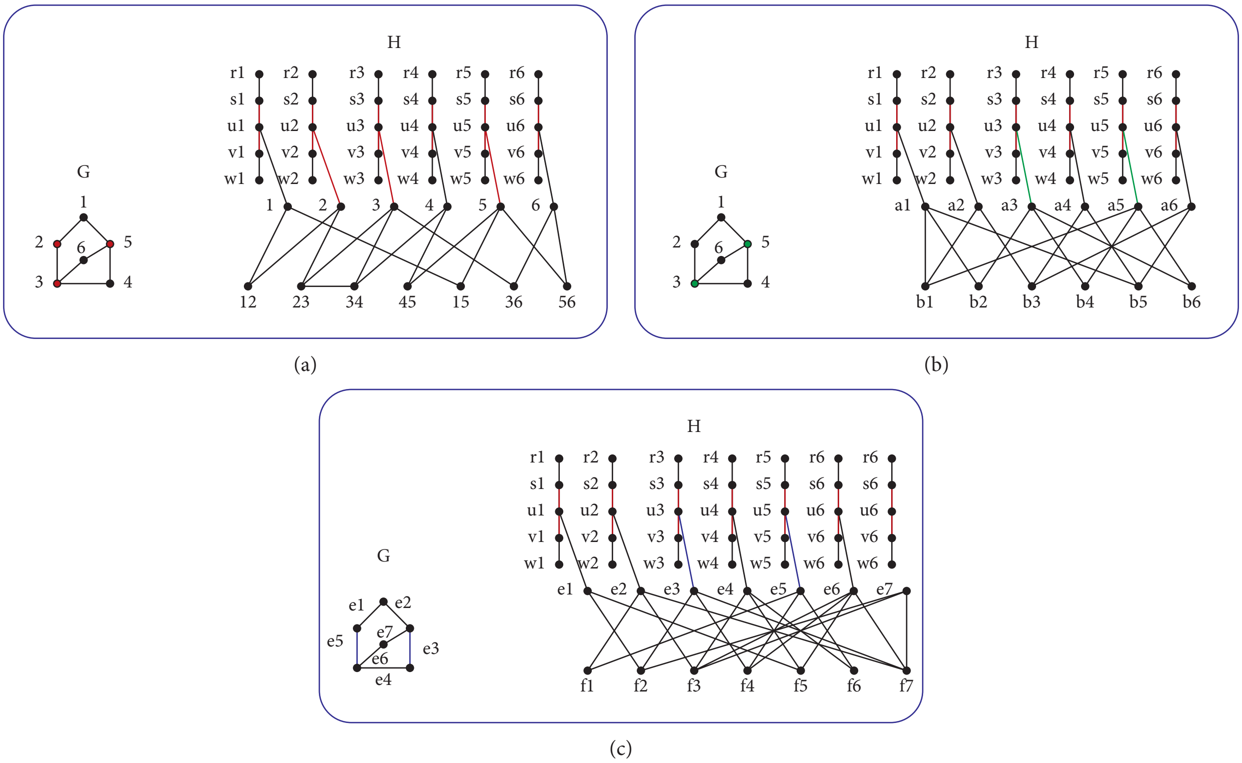

Figure 1: Planar graphs used for Theorem 1.

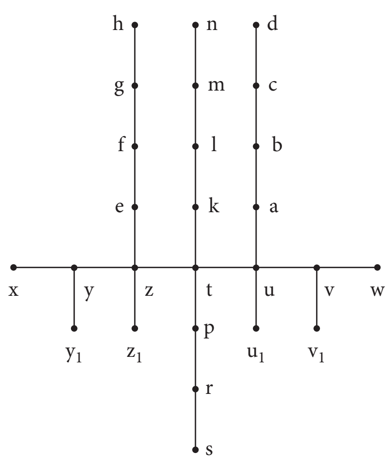

Figure 2: A tree $T \in \mathcal{F}$ which is obtained from T_1: $xyztuvw$, by adding the vertices y_1, z_1, u_1, v_1 with Process O_1, by adding the path P_3: prs, with Process O_2 and by adding the paths P_4: $abc\,d$, $klmn$, $ef\,gh$, with Process O_3 such that$\gamma_t(T) = \gamma_{ev}^t(T) = 12$.

internal vertex x of $T = T_q$ that is not a last vertex. Consider that B' be a $\gamma_t(R')$-set. To dominate l, the remote vertex c and its neighbor b are included by a $\gamma_t(T)$-set. So, $B' \cup \{b, c\}$ is a $\gamma_t(T)$-set and $\gamma_t(T) \leq \gamma_t(R') + 2$. Also, to ev-dominate the vertex l, the edge bc and its neighbor ab are included by a $\gamma_{ev}^t(T)$-set. So, provided that B is a $\gamma_{ev}^t(T)$-set, $B \setminus \{ab, bc\}$ is a $\gamma_{ev}^t(R')$-set. We obtain that $\gamma_{ev}^t(T) \geq \gamma_{ev}^t(R') + 2$.

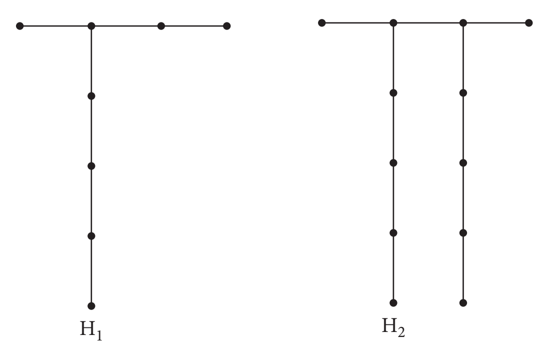

Figure 3: The two exceptional trees H_1, H_2 obtained from $T_1 = P_4$ such that $\gamma_t(H_1) = \gamma_{ev}^t(H_1) = 4$ and $\gamma_t(H_2) = \gamma_{ev}^t(H_2) = 6$.

Consequently, $\gamma_t(T) \leq \gamma_t(R') + 2 = \gamma_{ev}^t(R') + 2 \leq \gamma_{ev}^t(T) - 2 + 2 = \gamma_{ev}^t(T)$. From Lemma 1, we obtain that $\gamma_{ev}^t(T) = \gamma_t(T)$. □

Theorem 3. *Let T be a tree. If $\gamma_{ev}^t(T) = \gamma_t(T)$, then $T \in \mathcal{F}$.*

Proof. There is no TEVDS in a graph that possesses a single edge. Thus, we consider trees with at least two edges. If $\operatorname{diam}(T) = 2$, then T is a star. If $T = P_3$, then $T \in \mathcal{F}$. If T is a star different from P_3 then it can be obtained from $T = T_1$ by Process O_1. Therefore, $T \in \mathcal{F}$. Now, assume that $\operatorname{diam}(T)$ is at least three and the result is true for each tree $R' = T_q$ with order $4 \leq n' < n$.

First assume some remote vertex of T, for example x, is strong. Let y be a last vertex adjacent to x and $R' = T - y$. Let B' be a $\gamma_{ev}^t(R')$-set. By Observation 3, a $\gamma_{ev}^t(T)$-set can be determined such that it does not contain any end edge. So, B' is a $\gamma_{ev}^t(T)$-set and $\gamma_{ev}^t(T) \leq \gamma_{ev}^t(R')$. Consequently, $\gamma_t(R') \leq \gamma_t(T)$. We have $\gamma_t(R') \leq \gamma_t(T) = \gamma_{ev}^t(T) \leq \gamma_{ev}^t(R')$. From Lemma 1, $\gamma_{ev}^t(R') \leq \gamma_t(R')$. Thus, we obtain that $\gamma_{ev}^t(R') = \gamma_t(R')$. According to the inductive hypothesis, we obtain that $R' \in \mathcal{F}$ and T can be obtained from R' by Process O_1. Thus, from now on, we assume that every remote vertex of T is weak.

Now, we root T at a vertex r of maximum eccentricity $\operatorname{diam}(T)$. Consider that t is a last vertex at maximum distance from r, v is parent of t, h is parent of v, g is parent of h, provided that $\operatorname{diam}(T) \geq 5$, l is parent of g and provided that $\operatorname{diam}(T) \geq 6$, e is parent of l in the rooted tree. The symbol T_x indicates the subtree induced by a vertex x and its descendants in the rooted tree T. Assume that B is a $\gamma_{ev}^t(T)$-set. B' is a $\gamma_{ev}^t(R')$-set. Likewise, let S be a $\gamma_t(T)$-set, S' is a $\gamma_t(R')$-set. □

Case 1. We observed the situations over v. Let $d_T(v)$ be equal to 2. Due to the choice of the diametrical path, h is of a remote vertex else v or h has a last vertex.

Subcase 1.1. Assume that some child of h is a last vertex and it is indicated by x. As a result, it is found that h is a remote vertex. Let $R' = T - x$. From Observation 3, there exists a $\gamma_{ev}^t(T)$-set that includes no end edge. Thus, the vertex x is ev-dominated by hv and $\gamma_{ev}^t(T) \leq \gamma_{ev}^t(R')$. Furthermore, the vertices h and v are included by every $\gamma_t(T)$-set according to Observation 1. Thus, $\gamma_t(R') \leq \gamma_t(T)$. Therefore, $\gamma_t(R') \leq \gamma_t(T) = \gamma_{ev}^t(T) \leq \gamma_{ev}^t(R')$ and by Lemma 1, $\gamma_{ev}^t(R') \leq \gamma_t(R')$. Consequently, we obtain that $\gamma_{ev}^t(R') = \gamma_t(R')$. According to the inductive hypothesis, we obtain that $R' \in \mathcal{F}$ and T can be obtained over R' by Process O_1.

Subcase 1.2. Suppose that within the children of h there exists a remote vertex x, else v. Assume that $R' = T - T_v$. By Observation 4 the edges hv, hx are included by a minimum TEVDS of T. Thus, $B' \cup \{hv\}$ is a $\gamma_{ev}^t(T)$-set. In this case, B' is not a TEVDS but the edge hx has not a neighbor in B' for this situation. In order to attain the totality of B', the edge gh has to be included by B'. So, $(B/\{hv\}) \cup \{gh\}$ is a $\gamma_{ev}^t(R')$-set and $\gamma_{ev}^t(R') = \gamma_{ev}^t(T)$. Moreover, the vertices h, v, x are included by S. Then, $S/\{v\}$ is a $\gamma_t(R')$-set, we obtain that $\gamma_t(T) \geq \gamma_t(R') + 1$. Consequently, $\gamma_t(R') \leq \gamma_t(T) - 1 = \gamma_{ev}^t(T) - 1 = \gamma_{ev}^t(R') - 1 < \gamma_{ev}^t(R')$.

Case 2. Let $d_T(h)$ be equal to 2. Due to the choice of the diametrical path, g is of a child else h, e.g., x such that the distance of g to the farthest vertex of T_x is one or two or three.

Subcase 2.1. Suppose that some child of g is a last vertex and it is shown by x. Consider that $R' = T - x$. From Observation 3, a $\gamma_{ev}^t(T)$-set can be determined such that it does not contain any end edge. Thus, the vertex x is ev-dominated by lg and $\gamma_{ev}^t(T) \leq \gamma_{ev}^t(R')$. Furthermore, to dominate x, the vertex g is included by every $\gamma_t(T)$-set. So, $S' = S - \{g\}$ is a $\gamma_t(R')$-set, $\gamma_t(R') \leq \gamma_t(T) - 1$. Therefore, $\gamma_t(R') \leq \gamma_t(T) - 1 = \gamma_{ev}^t(T) - 1 \leq \gamma_{ev}^t(R') - 1 < \gamma_{ev}^t(R')$.

Subcase 2.2. A remote vertex x can be determined within the children of g, where T_x: xy. Consider that $R' = T - T_g$. To ev-dominate the vertices y, t the edges gx, gh and hv are included by a minimal TEVDS of T. So, $B' \cup \{gx, gh, hv\}$ is a $\gamma_{ev}^t(T)$-set. $\gamma_{ev}^t(T) \leq \gamma_{ev}^t(R\prime) + 3$. Moreover, the remote vertices x, v and their neighbors g, h are included by S. So, $S/\{x, g, h, v\}$ is a $\gamma_t(R')$-set, we obtain that $\gamma_t(T) \geq \gamma_t(R') + 4$. Therefore, $\gamma_t(R') \leq \gamma_t(T) - 4 = \gamma_{ev}^t(T) - 4 = \gamma_{ev}^t(R') + 3 - 4 < \gamma_{ev}^t(R')$.

Subcase 2.3. A vertex x can be determined within the children of g, where T_x: xyz. Consider that $R' = T - T_h$. So, $B' \cup \{gh, hv\}$ is a $\gamma_{ev}^t(T)$-set. $\gamma_{ev}^t(T) \leq \gamma_{ev}^t(R') + 2$. Also, $S/\{h, v\}$ is a $\gamma_t(R')$-set and $\gamma_t(T) \geq \gamma_t(R') + 2$. Then, $\gamma_t(R') \leq \gamma_t(T) - 2 = \gamma_{ev}^t(T) - 2 \leq \gamma_{ev}^t(R') + 2 - 2 = \gamma_{ev}^t(R')$. From Lemma 1, we obtain that $\gamma_t(R') = \gamma_{ev}^t(R')$ and according to the inductive hypothesis, $R' \in \mathcal{F}$ and T is acquired off R' per Process O_2.

Case 3. Let $d_T(g)$ be equal to 2. Due to the choice of the diametrical path, l is of a child else g, e.g., x such that the distance of l to the farthest vertex of T_x is one or two or three or four.

Subcase 3.1. Consider that some child of l is a last vertex. It is shown by x. Consider that $R' = T - T_g$. In order to ev-dominate the vertices x and t, the edges lg, gh, hv are included by a $\gamma_{ev}^t(T)$-set. We obtain that $B' \cup \{\text{lg}, gh, hv\}$ is a $\gamma_{ev}^t(T)$-set and $\gamma_{ev}^t(T) \leq \gamma_{ev}^t(R') + 3$. Furthermore, to dominate x and t, the remote vertices l, v and their neighbors g, h are included by every $\gamma_t(R')$-set. So, $S' = S - \{l, g, h, v\}$ is a $\gamma_t(R')$-set. $\gamma_t(R') \leq \gamma_t(T) - 4$. Therefore, $\gamma_t(R') \leq \gamma_t(T) - 4 = \gamma_{ev}^t(T) - 4 \leq \gamma_{ev}^t(R') + 3 - 4 < \gamma_{ev}^t(R')$.

If $R' = P_4$, then there exists an exceptional situation in here. Since P_4 do not contain internal vertices out of remote vertices, after a path P_4 is attached to $R' = P_4$ we obtain that $\gamma_t(T) = \gamma_{ev}^t(T) = 4$ and $T \in \mathcal{F}$.

Subcase 3.2. A remote vertex x can be determined among the children of l, where T_x: xy. Consider that $R' = T - T_g$. To ev-dominate the vertices y, t, the edges lx, hv and their neighbors el, gh are included by a minimal TEVDS of T. Then, $B' \cup \{gh, hv\}$ is a $\gamma_{ev}^t(T)$-set and $\gamma_{ev}^t(T) \leq \gamma_{ev}^t(R') + 2$. Hence, to dominate the last vertices y, t, the remote vertices x, v and their neighbors l, h are included by S. Consequently, $S/\{h, v\}$ is a $\gamma_t(R')$-set, we obtain $\gamma_t(T) \geq \gamma_t(R') + 2$. So, $\gamma_t(R') \leq \gamma_t(T) - 2 = \gamma_{ev}^t(T) - 2 = \gamma_{ev}^t(R') + 2 - 2 \leq \gamma_{ev}^t(R')$. According to the inductive hypothesis, we obtain that $R' \in \mathcal{F}$ and T can be obtained from R' by Process O_3.

Subcase 3.3. A vertex x can be determined within the children of l, where T_x: xyz. Consider that $R' = T - T_g$. To ev-dominate the vertices z, t, the remote edges xy, hv and their neighbors lx, gh are included by a minimal TEVDS of

T. So $B' \cup \{gh, hv\}$ is a $\gamma_{ev}^t(T)$-set and $\gamma_{ev}^t(T) \le \gamma_{ev}^t(R') + 2$. To dominate the last vertices z, t, the remote vertices y, v and their neighbors x, h are included by S. Consequently, $S/\{h, v\}$ is a $\gamma_t(R')$-set. We obtain $\gamma_t(T) \ge \gamma_t(R') + 2$. Then, $\gamma_t(R') \le \gamma_t(T) - 2 = \gamma_{ev}^t(T) - 2 = \gamma_{ev}^t(R') + 2 - 2 \le \gamma_{ev}^t(R')$. Per Lemma 1, we obtain that $\gamma_{ev}^t(R') = \gamma_t(R')$ and according to the inductive hypothesis, we obtain that $R' \in \mathcal{F}$ and T can be obtained from R' by Process O_3.

Subcase 3.4. A vertex x can be determined among the children of l, where T_x: $xyzs$. Consider that $R' = T - T_g$. To ev-dominate the vertices s, t, the remote edges yz, hv and their neighbors xy, gh are included by a minimal TEVDS of T. So $B' \cup \{gh, hv\}$ is a $\gamma_{ev}^t(T)$-set and $\gamma_{ev}^t(T) \le \gamma_{ev}^t(R') + 2$. To dominate the last vertices s, t, the remote vertices z, v and their neighbors y, h are included by S. Consequently, $S/\{h, v\}$ is a $\gamma_t(R')$-set. We obtain $\gamma_t(T) \ge \gamma_t(R') + 2$. Hence, $\gamma_t(R') \le \gamma_t(T) - 2 = \gamma_{ev}^t(T) - 2 = \gamma_{ev}^t(R') + 2 - 2 \le \gamma_{ev}^t(R')$. From Lemma 1, we obtain that $\gamma_{ev}^t(R') = \gamma_t(R')$ and according to the inductive hypothesis, we obtain that $R' \in \mathcal{F}$ and T can be obtained from R' by Process O_3.

Case 4. Let $d_T(l)$ be equal to 2. We obtain that $T = P_6 \notin \mathcal{F}$ provided that $d_T(e) = 1$. Now assume that $d_T(e) \ge 3$. Let $R' = T - T_l$. In this case, $B' \cup \{gh, hv\}$ is a $\gamma_{ev}^t(T)$-set. We have $\gamma_{ev}^t(T) \le \gamma_{ev}^t(R') + 2$. On the other hand, g, h, v are contained in a TDS of T. Finally, $S' = S/\{g, h, v\}$ is a $\gamma_t(R')$-set. $\gamma_t(R') + 3 \le \gamma_t(T)$. Thus, $\gamma_t(R') \le \gamma_t(T) - 3 = \gamma_{ev}^t(T) - 3 \le \gamma_{ev}^t(R') + 2 - 3 < \gamma_{ev}^t(R')$.

Data Availability

In this paper, complete data are included.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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