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*Research Article*

# Rough Approximation Spaces via Maximal Union Neighborhoods and Ideals with a Medical Application

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One of the most popular and important tools to deal with imperfect knowledge is the rough set theory. It starts from dividing the universe to obtain blocks utilizing an equivalence relation. To make it more flexibility and expand its scope of applications, many generalized rough set models have been proposed and studied. To contribute to this area, we introduce new generalized rough set models inspired by "maximal union neighborhoods and ideals." These models are created with the aim to help decision-makers to analysis and evaluate the given data more accurately by decreasing the ambiguity regions. We confirm this aim by illustrating that the current models improve the approximations operators (lower and upper) and accuracy measures more than some existing method approaches. We point out that almost all major properties with respect to rough set model can be kept using the current models. One of the interesting obtained characterizations of the current models is preserving the monotonic property, which enables us to evaluate the vagueness in the data and enhance the confidence in the outcomes. Moreover, we compare the current approximation spaces with the help of concrete examples. Finally, we show the performance of the current models to discuss the information system of dengue fever disease and eliminate the ambiguity of the medical diagnosis, which produces an accurate decision.

## 1. Introduction

Rough set theory was introduced via Pawlak [1, 2], as one of the representative granular computing models. It offers a useful mathematical instrument to cope with ambiguity/uncertainty. Presently, rough set theory has been successfully applied to several areas such as approximate reasoning [3], machine learning [4], incomplete information systems [5], and decision analysis [6]. Pawlak [1, 2] expressed the classical rough set theory by two new sets, obtained from an equivalence relation, called lower and upper approximations. To know the vagueness size and measure the completeness of data, the concepts of boundary regions and accuracy measures were presented. While the boundary region defined as the difference between the approximation operators (upper and lower), the accuracy measure represents a number obtained from the quotient of the cardinalities of lower approximation and upper approximation. To remove the strict condition imposed in the classical rough set model, many scholars replaced the equivalence relation by similarity relation [7], tolerance relation [8], or binary relation [9, 10].

Additionally, to extend the applications of this theory, Yao [11, 12] proposed new classes of blocks (or granular computing) induced from right, left, and the intersection (union) of left and right neighborhoods with respect to any binary relation. This matter forms a revolution in rough set theory and opens a door for studying many types of neighborhood systems which are identical under an equivalence relation. Following this line, different types of neighborhood systems have been established with the aim to introduce several generalized rough set models (or approximation spaces), minimal left and minimal right neighborhoods [13], neighborhoods produced by the intersection of minimal left and right neighborhoods [14], and core neighborhoods [15]. Abu-Donia [16, 17] investigated some sorts of approximation operators using a family of binary relations instead of one binary relation. Ciucci [18] investigated different types of orthopair systems. In 2018, Dai et al. [19] familiarized the maximal right neighborhoods under a similarity relation and then applied it to present three different sorts of approximation spaces. Recently, Al-shami [20] has discussed new seven kinds of rough maximal neighborhoods and benefited to introduce new approximation spaces and rank suspected individuals of COVID-19. He proved that the accuracy measures kept the monotonic property with respect to any relation. With the desire to minimize the boundary region and maximize the accuracy values, the systems of containment neighborhoods [21] and subset neighborhoods [22] were studied and explored under arbitrary relation.

In 2013, Kandi et al. [23] made use of an abstract structure called "ideal" to define new rough paradigms called "ideal approximation spaces." This paradigm proved its efficiency to enlarge the knowledge obtained from the information system, and hence, high the degree of its completeness. Moreover, it was applied to model some uncertainty phenomena as illustrated in [24–26]. Recently, Hosny [27] has inserted the ideal structures via topological approximation spaces and showed how this technique deletes some objects from the upper approximations and/or adds new objects to the lower approximations, which means improving the approximation operators more than their counterparts induced from topological approximation spaces.

More recently, Al-shami and Hosny [28, 29] have displayed new ideal approximation spaces with respect to maximal left neighborhood under any binary relation and showed their advantages compared to the previous ones. Following this line of research, we propose new generalized rough set models inspired by "maximal union neighborhoods and ideals." By these models, we successfully decrease the ambiguity regions which helps us to evaluate the given data more accurately. The good performance of the current approach is illustrated by making comparisons with the approach introduced in [20].

This manuscript is structured as follows. Section 2 is dedicated to the basic concepts and properties on rough neighborhood systems and ideals. The goal of Section 3 is to introduce new four ideals approximation spaces and their essential characterizations are explored. In Section 4, with the help of illustrated examples, the comparisons between these models are studied as well as the comparison of the first kind of these models with its counterpart in [20] is investigated. The significance of the proposed paradigms with a medical diagnosis of dengue fever disease is demonstrated in Section 5. Finally, a summary of the obtained findings is given and upcoming works are proposed in Section 6.

## 2. Preliminaries

In this section, we briefly recall the essential of approximation spaces defined with respect to equivalence relations or arbitrary binary relations. Also, we mention the main properties of ideal structures.

*Definition 1* (see [1, 2]). Take $R$ as an equivalence relation on a universe $E$ and let $[e]_R$ be the equivalence class containing $e$. For any subset $F$ of $E$, the lower and upper approximations and accuracy measure are defined, respectively, by

$$\underline{apr}(F) = \{e \in E : [e]_R \subseteq F\}; \tag{1}$$

$$\bar{apr}(F) = \{e \in E : [e]_R \cap F \neq \phi; \text{and} \tag{2}$$

$$\text{Acc}_R(F) = \frac{|\underline{apr}(F)|}{|\bar{apr}(F)|}. \tag{3}$$

The main properties of these approximations (known as Pawlak's properties) are the following.

($\mathcal{L}_1$) $\underline{apr}(F^c) = [\bar{apr}(F)]^c$, where $F^c$ is the complement of $F$.
($\mathcal{L}_2$) $\underline{apr}(E) = E$.
($\mathcal{L}_3$) $\underline{apr}(\phi) = \phi$.
($\mathcal{L}_4$) $\underline{apr}(F) \subseteq F$.
($\mathcal{L}_5$) $\underline{apr}(F \cap G) = \underline{apr}(F) \cap \underline{apr}(G)$
($\mathcal{L}_6$) $\underline{apr}(F \cup G) \supseteq \underline{apr}(F) \cup \underline{apr}(G)$
($\mathcal{L}_7$) $F \subseteq G \Rightarrow \underline{apr}(F) \subseteq \underline{apr}(G)$.
($\mathcal{L}_8$) $\underline{apr}(\underline{apr}(F)) = \underline{apr}(F)$.
($\mathcal{L}_9$) $\bar{apr}(F) \subseteq \underline{apr}(\bar{apr}(F))$.
($\mathcal{U}_1$) $\bar{apr}(F^c) = [\underline{apr}(F)]^c$.
($\mathcal{U}_2$) $\bar{apr}(E) = E$.
($\mathcal{U}_3$) $\bar{apr}(\phi) = \phi$.
($\mathcal{U}_4$) $F \subseteq \bar{apr}(F)$.
($\mathcal{U}_5$) $\bar{apr}(F \cup G) = \bar{apr}(F) \cup \bar{apr}(G)$.
($\mathcal{U}_6$) $\bar{apr}(F \cap G) \subseteq \bar{apr}(F) \cap \bar{apr}(G)$.
($\mathcal{U}_7$) $F \subseteq G \Rightarrow \bar{apr}(F) \subseteq \bar{apr}(G)$.
($\mathcal{U}_8$) $\bar{apr}(\bar{apr}(F)) = \bar{apr}(F)$.
($\mathcal{U}_9$) $\bar{apr}(\underline{apr}(F)) \subseteq \underline{apr}(F)$.

*Definition 2* (see [20]). If the relations $R_1$ and $R_2$ are equivalence on a universe $E$ such that $R_1 \subseteq R_2$. Then, the approximations inspired by these relations have the property of monotonicity (monotonic property) if $\text{Acc}_{R_2}(F) \leq \text{Acc}_{R_1}(F)$ for each $F \subseteq E$.

*Definition 3* (see [11, 19, 20]). The next neighborhoods of an element $e$ of a finite set $E \neq \varnothing$ are defined with respect to any arbitrary binary relation $R$ as follows.

(1) The right neighborhood of $e$, symbolized by $N_r(e)$, is given by $N_r(e) = \{a \in E : (e, a) \in R\}$
(2) The left neighborhood of $e$, symbolized by $N_l(e)$, is given by $N_r(e) = \{a \in E : (a, e) \in R\}$
(3) $\mu_r(e)$ is the union of all right neighborhoods containing $e$

(4) $\mu_l(e)$ is the union of all left neighborhoods containing $e$

(5) $\mu_u(e) = \mu_r(e) \cup \mu_l(e)$

**Theorem 4** [20]. *Let $E$ be a universal set and $R_1, R_2$ be two binary relations on $E$. If $R_1 \subseteq R_2$, then, $\mu_{1u}(e) \subseteq \mu_{2u}(e)$, $\forall e \in E$.*

*Definition 5* (see [20]). Let $R$ be a binary relation on a nonempty set $E$. The lower and upper approximations, boundary region, accuracy, and roughness of a nonempty subset $F$ of $E$ are defined, respectively, by

$$R_\star(F) = \{e \in E : \mu_u(e) \subseteq F\}, \tag{4}$$

$$R^\star(F) = \{e \in E : \mu_u(e) \cap F \neq \phi\}, \tag{5}$$

$$B_R^\star(F) = R^\star(F) - R_\star(F), \tag{6}$$

$$\mathrm{Acc}_R^\star(F) = \left|\frac{R_\star(F) \cap F}{R^\star(F) \cup F}\right|, \tag{7}$$

$$\mathrm{Rough}_R^\star(F) = 1 - \mathrm{Acc}_R^\star(F). \tag{8}$$

*Definition 6* (see [30]). An ideal $\mathcal{J}$ on a set $E \neq \phi$ is nonempty collection of subsets of $E$ that is closed under finite unions and subsets; i.e., it satisfies the following conditions:

(1) $F \in \mathcal{J}$ and $H \in \mathcal{J} \Rightarrow F \cup H \in \mathcal{J}$

(2) $F \in \mathcal{J}$ and $H \subseteq F \Rightarrow H \in \mathcal{J}$

*Definition 7* (see [26]). For two ideals on a nonempty set $E$, the smallest collection generating by $\mathcal{J}_1, \mathcal{J}_2$, denoted by $\mathcal{J}_1 \vee \mathcal{J}_2$, is given by

$$\mathcal{J}_1 \vee \mathcal{J}_2 = \{G \cup F : G \in \mathcal{J}_1, F \in \mathcal{J}_2\}. \tag{9}$$

**Proposition 8** (see [26]). *If $\mathcal{J}_1, \mathcal{J}_2$ are ideals on a nonempty set $E$, and $F, H$ are subsets of $E$. Then, the collection $\mathcal{J}_1 \vee \mathcal{J}_2$ satisfies the next conditions:*

*(1) $\mathcal{J}_1 \vee \mathcal{J}_2 \neq \phi$*

*(2) $F \in \mathcal{J}_1 \vee \mathcal{J}_2, H \subseteq F \Rightarrow H \in \mathcal{J}_1 \vee \mathcal{J}_2$*

*(3) $F, H \in \mathcal{J}_1 \vee \mathcal{J}_2 \Rightarrow F \cup H \in \mathcal{J}_1 \vee \mathcal{J}_2$*

*It means that the collection $\mathcal{J}_1 \vee \mathcal{J}_2$ is an ideal on $E$.*

## 3. Some Novel Rough Set Paradigms Produced by $\mu_u(e)$-Neighborhoods and Ideals

In this section, we display four paradigms of approximation spaces induced from $\check{R} < x > \check{R}$-neighborhoods and ideals. With the aid of elucidative examples and counterexamples, we point out some directions that are invalid in the obtained findings and relationships.

### *3.1. The First Method of the Improvement of the Approximations and Accuracy Measure of Subsets*

*Definition 9.* Let $R$ and $\mathcal{J}$ be, respectively, binary relation and ideal on a nonempty set $E$. The first kind of the improvement of lower and upper approximations, boundary region, accuracy, and roughness of a nonempty subset $F$ of $E$ induced from $R$ and $\mathcal{J}$ is defined, respectively, by

$$R_\star^{\mathcal{J}}(F) = \{e \in E : \mu_u(e) \cap F^c \in \mathcal{J}\}, \tag{10}$$

$$R^{\star\mathcal{J}}(F) = \{e \in E : \mu_u(e) \cap F \notin \mathcal{J}\}, \tag{11}$$

$$\mathrm{BND}_R^{\star\mathcal{J}}(F) = R^{\star\mathcal{J}}(F) - R_\star^{\mathcal{J}}(F), \tag{12}$$

$$\mathrm{ACC}_R^{\star\mathcal{J}}(F) = \frac{\left|R_\star^{\mathcal{J}}(F) \cap F\right|}{\left|R^{\star\mathcal{J}}(F) \cup F\right|}, \tag{13}$$

$$\mathrm{Rough}_R^{\star\mathcal{J}}(F) = 1 - \mathrm{ACC}_R^{\star\mathcal{J}}(F). \tag{14}$$

**Proposition 10.** *Consider $F, H \subseteq E$ and let $\mathcal{J}, \mathcal{K}$ be ideals and $R$ be a binary relation on $E$. Then,*

*(1) $R^{\star\mathcal{J}}(\phi) = \phi$*

*(2) $F \subseteq H \Rightarrow R^{\star\mathcal{J}}(F) \subseteq R^{\star\mathcal{J}}(H)$*

*(3) $R^{\star\mathcal{J}}(F \cap H) \subseteq R^{\star\mathcal{J}}(F) \cap R^{\star\mathcal{J}}(H)$*

*(4) $R^{\star\mathcal{J}}(F \cup H) = R^{\star\mathcal{J}}(F) \cup R^{\star\mathcal{J}}(H)$*

*(5) $R^{\star\mathcal{J}}(F) = (R_\star^{\mathcal{J}}(F^c))^c$*

*(6) If $F \in \mathcal{J}$, then $R^{\star\mathcal{J}}(F) = \phi$*

*(7) If $\mathcal{J} \subseteq \mathcal{K}$, then $R^{\star\mathcal{K}}(F) \subseteq R^{\star\mathcal{J}}(F)$*

*(8) If $\mathcal{J} = P(E)$, then $R^{\star\mathcal{J}}(F) = \phi$*

*(9) $R^{\star\mathcal{J}\cap\mathcal{K}}(F) = R^{\star\mathcal{J}}(F) \cup R^{\star\mathcal{K}}(F)$*

*(10) $R^{\star\mathcal{J}\vee\mathcal{K}}(F) = R^{\star\mathcal{J}}(F) \cap R^{\star\mathcal{K}}(F)$*

*Proof.*

(1) $R^{\star\mathcal{J}}(\phi) = \{e \in E : \mu_u(e) \cap \phi \notin \mathcal{J}\}. = \phi$

(2) Let $e \in R^{\star\mathcal{J}}(F)$. Then, $\mu_u(e) \cap F \notin \mathcal{J}$. Since, $F \subseteq H$ and $\mathcal{J}$ is an ideal. Thus, $\mu_u(e) \cap H \notin \mathcal{J}$. Therefore, $e \in R^{\star\mathcal{J}}(H)$. Hence, $R^{\star\mathcal{J}}(F) \subseteq R^{\star\mathcal{J}}(H)$

(3) Immediately by part (2)

(4) $R^{\star\mathcal{J}}(F) \cup R^{\star\mathcal{J}}(H) \subseteq R^{\star\mathcal{J}}(F \cup H)$ by part (2). Let $e \in R^{\star\mathcal{J}}(F \cup H)$. Then, $\mu_u(e) \cap (F \cup H) \notin \mathcal{J}$. It follows that $((\mu_u(e) \cap F) \cup (\mu_u(e) \cap H)) \notin \mathcal{J}$. Therefore, $\mu_u(e) \cap F \notin I$ or $\mu_u(e) \cap H \notin \mathcal{J}$, that means $e \in R^{\star\mathcal{J}}(F)$ or $e \in R^{\star\mathcal{J}}(H)$. Then, $e \in R^{\star\mathcal{J}}(F) \cup R^{\star\mathcal{J}}(H)$. Thus, $R^{\star\mathcal{J}}(F \cup H) \subseteq R^{\star\mathcal{J}}(F) \cup R^{\star\mathcal{J}}(H)$. Hence, $R^{\star\mathcal{J}}(F \cup H) = R^{\star\mathcal{J}}(F) \cup R^{\star\mathcal{J}}(H)$

(5) $(R_\star^{\mathcal{J}}(F^c))^c = (\{e\in E : \mu_u(e)\cap F\in\mathcal{J}\})^c. = \{e\in E : \mu_u(e)\cap F\in\mathcal{J}\}. = R^{\star\mathcal{J}}(F)$

(6) Immediately obtains by Definition 9

(7) Let $e\in R^{\star\mathcal{K}}(F)$. Then, $\mu_u(e)\cap F\in\mathcal{K}$. Since, $\mathcal{J}\subseteq\mathcal{K}$. Thus, $\mu_u(e)\cap F\in\mathcal{J}$. Therefore, $e\in R^{\star\mathcal{J}}(F)$. Hence, $R^{\star\mathcal{K}}(F)\subseteq R^{\star\mathcal{J}}(F)$

(8) Immediately obtains by Definition 9

(9) $R^{\star\mathcal{J}\cap\mathcal{K}}(F) = \{e\in E : \mu_u(e)\cap F\in\mathcal{J}\cap\mathcal{K}\}. = \{e\in E : \mu_u(e)\cap F\in\mathcal{J}\}$ or $\{e\in E : \mu_u(e)\cap F\in\mathcal{K}\}. = \{e\in E : \mu_u(e)\cap F\in\mathcal{J}\}\cup\{e\in E : \mu_u(e)\cap F\in\mathcal{K}\}. = R^{\star\mathcal{J}}(F)\cup R^{\star\mathcal{K}}(F)$

(10) $R^{\star\mathcal{J}\vee\mathcal{K}}(F) = \{e\in E : \mu_u(e)\cap F\in\mathcal{J}\vee\mathcal{K}\}. = \{e\in E : \mu_u(e)\cap F\in\mathcal{J}\cup\mathcal{K}\}. = \{e\in E : \mu_u(e)\cap F\in\mathcal{J}\}$ and $\{e\in E : \mu_u(e)\cap F\in\mathcal{K}\}. = \{e\in E : \mu_u(e)\cap F\in\mathcal{J}\}\cap\{e\in E : \mu_u(e)\cap F\in\mathcal{K}\}. = R^{\star\mathcal{J}}(F)\cap R^{\star\mathcal{K}}(F)$

□

**Proposition 11.** *Consider $F, H\subseteq E$ and let $\mathcal{J}, \mathcal{K}$ be ideals and $R$ be a binary relation on $E$. Then,*

*(1)* $R_\star^{\mathcal{J}}(E) = E$

*(2)* $F\subseteq H \Rightarrow R_\star^{\mathcal{J}}(F)\subseteq R_\star^{\mathcal{J}}(H)$

*(3)* $R_\star^{\mathcal{J}}(F)\cup R_\star^{\mathcal{J}}(H)\subseteq R_\star^{\mathcal{J}}(F\cup H)$

*(4)* $R_\star^{\mathcal{J}}(F\cap H) = R_\star^{\mathcal{J}}(F)\cap R_\star^{\mathcal{J}}(H)$

*(5)* $R_\star^{\mathcal{J}}(F) = (R^{\star\mathcal{J}}(F^c))^c$

*(6) If* $F^c\in\mathcal{J}$, *then* $R_\star^{\mathcal{J}}(F) = E$

*(7) If* $\mathcal{J}\subseteq\mathcal{K}$, *then* $R_\star^{\mathcal{J}}(F)\subseteq R_\star^{\mathcal{K}}(F)$

*(8) If* $\mathcal{J} = P(E)$, *then* $R_\star^{\mathcal{J}}(F) = E$

*(9)* $R_\star^{\mathcal{J}\cap\mathcal{K}}(F) = R_\star^{\mathcal{J}}(F)\cap R_\star^{\mathcal{K}}(F)$

*(10)* $R_\star^{\mathcal{J}\vee\mathcal{K}}(F) = R_\star^{\mathcal{J}}(F)\cup R_\star^{\mathcal{K}}(F)$

*Proof.*

(1) $R_\star^{\mathcal{J}}(E) = \{e\in E : \mu_u(e)\cap\phi\in\mathcal{J}\}. = E$

(2) Let $e\in R_\star^{\mathcal{J}}(F)$. Then, $\mu_u(e)\cap F^c\in\mathcal{J}$. Since, $H^c\subseteq F^c$ and $\mathcal{J}$ is an ideal. Thus, $\mu_u(e)\cap H^c\in\mathcal{J}$. Therefore, $e\in R_\star^{\mathcal{J}}(H)$. Hence, $R_\star^{\mathcal{J}}(F)\subseteq R_\star^{\mathcal{J}}(H)$

(3) Immediately by part (2)

(4) $R_\star^{\mathcal{J}}(F)\cap R_\star^{\mathcal{J}}(H)\supseteq R_\star^{\mathcal{J}}(F\cap H)$ by part (2). Let $e\in R_\star^{\mathcal{J}}(F)\cap R_\star^{\mathcal{J}}(H)$. Then, $\mu_u(e)\cap F^c\in\mathcal{J}$ and $\mu_u(e)\cap H^c\in\mathcal{J}$. It follows that $(\mu_u(e)\cap(F^c\cup H^c))\in\mathcal{J}$. So, $(\mu_u(e)\cap(F\cap H)^c)\in\mathcal{J}$. Therefore, $e\in R_\star^{\mathcal{J}}(F\cap H)$. Thus, $R_\star^{\mathcal{J}}(F)\cap R_\star^{\mathcal{J}}(H)\subseteq R_\star^{\mathcal{J}}(F\cap H)$. Hence, $R_\star^{\mathcal{J}}(F)\cap R_\star^{\mathcal{J}}(H) = R_\star^{\mathcal{J}}(F\cap H)$

(5) $(R^{\star\mathcal{J}}(F^c))^c = (\{e\in E : \mu_u(e)\cap F^c\in\mathcal{J}\})^c. = \{e\in E : \mu_u(e)\cap F^c\in\mathcal{J}\}. = R_\star^{\mathcal{J}}(F)$

(6) Immediately obtains by Definition 9

(7) Let $e\in R_\star^{\mathcal{J}}(F)$. Then, $\mu_u(e)\cap F^c\in\mathcal{J}$. Since, $\mathcal{J}\subseteq\mathcal{K}$. Thus, $\mu_u(e)\cap F^c\in\mathcal{K}$. Therefore, $e\in R_\star^{\mathcal{J}}(F)$. Hence, $R_\star^{\mathcal{J}}(F)\subseteq R_\star^{\mathcal{K}}(F)$

(8) Immediately obtains by Definition 9

(9) $R_\star^{\mathcal{J}\cap\mathcal{K}}(F) = \{e\in E : \mu_u(e)\cap F^c\in\mathcal{J}\cap\mathcal{K}\}. = \{e\in E : \mu_u(e)\cap F^c\in\mathcal{J}\}$ and $\{e\in E : \mu_u(e)\cap F^c\in\mathcal{K}\}. = \{e\in E : \mu_u(e)\cap F^c\in\mathcal{J}\}\cap\{e\in E : \mu_u(e)\cap F^c\in\mathcal{K}\}. = R_\star^{\mathcal{J}}(F)\cap R_\star^{\mathcal{K}}(F)$

(10) $R_\star^{\mathcal{J}\vee\mathcal{K}}(F) = \{e\in E : \mu_u(e)\cap F^c\in\mathcal{J}\vee\mathcal{K}\}. = \{e\in E : \mu_u(e)\cap F^c\in\mathcal{J}\cup\mathcal{K}\}. = \{e\in E : \mu_u(e)\cap F^c\in\mathcal{J}\}$ or $\{e\in E : \mu_u(e)\cap F^c\in\mathcal{K}\}. = \{e\in E : \mu_u(e)\cap F^c\in\mathcal{J}\}\cup\{e\in E : \mu_u(e)\cap F^c\in\mathcal{K}\}. = R_\star^{\mathcal{J}}(F)\cup R_\star^{\mathcal{K}}(F)$

□

*Remark 12.* By the next example, we elucidate that

(1) The converse of parts 2, 6, 7, and 8 in Proposition 10 and Proposition 11 is generally incorrect

(2) The inclusion relations of part 3 in Proposition 10 and Proposition 11 are generally proper

*Example 1.*

(i) Let $E = \{e_1, e_2, e_3, e_4\}$, $\mathcal{J} = \{\phi, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}\}$ and $R = \{(e_1, e_2), (e_1, e_3), (e_2, e_3), (e_2, e_4), (e_3, e_1), (e_3, e_4)\}$ be a binary relation defined on $E$. By calculations, we obtain $\mu_u(e_1) = \{e_1, e_2, e_4\}$, $\mu_u(e_2) = \{e_1, e_2, e_3\}$, $\mu_u(e_3) = \{e_2, e_3, e_4\}$, $\mu_u(e_4) = \{e_1, e_3, e_4\}$. For part 2, take $F = \{e_1, e_2\}$ and $H = \{e_1, e_4\}$, then

(a) $R^{\star\mathcal{J}}(F) = \phi$, $R^{\star\mathcal{J}}(H) = \{e_1, e_3, e_4\}$. Therefore, $R^{\star\mathcal{J}}(F)\subseteq R^{\star\mathcal{J}}(H)$, but $F\nsubseteq H$

(b) $R_\star^{\mathcal{J}}(F) = \{e_2\}$, $R_\star^{\mathcal{J}}(H) = E$. Therefore, $R_\star^{\mathcal{J}}(F)\subseteq R_\star^{\mathcal{J}}(H)$, but $F\nsubseteq H$

(ii) Let $E = \{e_1, e_2, e_3, e_4\}$, $\mathcal{K} = \{\phi, \{e_2\}\}$, $\mathcal{J} = \{\phi, \{e_1\}\}$ and $R = \{(e_2, e_2), (e_3, e_3), (e_4, e_4)\}$ be a binary relation defined on $E$. By calculations, we obtain $\mu_u(e_1) = \phi$, $\mu_u(e_2) = \{e_2\}$, $\mu_u(e_3) = \{e_3\}$, $\mu_u(e_4) = \{e_4\}$

(1) For part 6, take

(a) $F = \{e_1, e_2\}$, then, $R^{\star\mathcal{K}}(F) = \phi$. Therefore, $R^{\star\mathcal{K}}(F) = \phi$, but $F\in\mathcal{K}$.

(b) $F = \{e_3, e_4\}$, then, $R_\star^{\mathcal{K}}(F) = E$. Therefore, $R_\star^{\mathcal{K}}(F) = E$, but $F^c \in \mathcal{K}$.

(2) For part 7, take

(a) $F = \{e_1, e_2\}$, then, $R^{\star\mathcal{J}}(F) = \{e_2\}, R^{\star\mathcal{K}}(F) = \phi$. Therefore, $R^{\star\mathcal{K}}(F) \subseteq R^{\star\mathcal{J}}(F)$, but $\mathcal{J} \nsubseteq \mathcal{K}$

(b) $F = \{e_3, e_4\}$, then, $R_\star^{\mathcal{J}}(F) = \{e_1, e_3, e_4\}, R_\star^{\mathcal{K}}(F) = E$. Therefore, $R_\star^{\mathcal{J}}(F) \subseteq R_\star^{\mathcal{K}}(F)$, but $\mathcal{J} \nsubseteq \mathcal{K}$

(3) For part 8, take

(a) $F = \{e_1, e_2\}$, then, $R^{\star\mathcal{K}}(F) = \phi$, but $\mathcal{K} \neq P(E)$

(b) $F = \{e_3, e_4\}$, then, $R_\star^{\mathcal{K}}(F) = E$, but $\mathcal{K} \neq P(E)$

(iii) Let $E = \{e_1, e_2, e_3, e_4\}, \mathcal{J} = \{\phi, \{e_4\}\}$ and $R = \Delta \cup \{(e_2, e_1), (e_3, e_1), (e_4, e_1)\}$ be a binary relation defined on $E$, (where $\Delta$ is the identity relation and equal to $(e_1, e_1), (e_2, e_2), (e_3, e_3), (e_4, e_4)\})$. By calculations, we obtain $\mu_u(e_1) = \mu_u(e_2) = \mu_u(e_3) = \mu_u(e_4) = E$. For part 3, take $F = \{e_1, e_4\}, H = \{e_2, e_3\}$. Hence

(a) $F \cap H = \phi$, then $R^{\star\mathcal{J}}(F) = R^{\star\mathcal{J}}(H) = E, R^{\star\mathcal{J}}(F \cap H) = \phi$. Therefore, $R^{\star\mathcal{J}}(F) \cap R^{\star\mathcal{J}}(H) = E \neq \phi = R^{\star\mathcal{J}}(F \cap H)$

(b) $F \cup H = E$, then $R_\star^{\mathcal{J}}(F) = R_\star^{\mathcal{J}}(H) = \phi, R^{\star\mathcal{J}}(F \cup H) = E$. Therefore, $R_\star^{\mathcal{J}}(F) \cup R_\star^{\mathcal{J}}(H) = \phi \neq E = R_\star^{\mathcal{J}}(F \cup H)$

*Remark 13.* Some Pawlak's properties are not kept by this method as we demonstrate in the following.

(i) In Example 1 (i) take

(1) $F = \{e_1, e_2\}$, then $R^{\star\mathcal{J}}(F) = \phi$. Hence, $F \nsubseteq R^{\star\mathcal{J}}(F)$

(2) $F = \{e_1, e_4\}$, then $R_\star^{\mathcal{J}}(F) = E$. Hence, $R_\star^{\mathcal{J}}(F) \nsubseteq F$

(3) $F = E$, then $R^{\star\mathcal{J}}(E) = \{e_1, e_3, e_4\}$. Hence, $R^{\star\mathcal{J}}(E) \neq E$

(4) $F = \phi$, then $R_\star^{\mathcal{J}}(\phi) = \{e_2\}$. Hence, $R_\star^{\mathcal{J}}(\phi) \neq \phi$

(ii) In Example 1 (i) take $\mathcal{J} = \{\phi, \{e_1\}\}$

(1) $F = \{e_1, e_4\}$, then $R^{\star\mathcal{J}}(F) = \{e_1, e_3, e_4\}, R^{\star\mathcal{J}}(R^{\star\mathcal{J}}(F)) = E$. Hence, $R^{\star\mathcal{J}}(F) \neq R^{\star\mathcal{J}}(R^{\star\mathcal{J}}(F))$

(2) $F = \{e_2, e_3\}$, then $R_\star^{\mathcal{J}}(F) = \{e_2\}, R_\star^{\mathcal{J}}(R_\star^{\mathcal{J}}(F)) = \phi$. Hence, $R_\star^{\mathcal{J}}(F) \neq R_\star^{\mathcal{J}}(R_\star^{\mathcal{J}}(F))$

(3) $F = \{e_1, e_2\}$, then $R^{\star\mathcal{J}}(F) = \{e_1, e_2, e_3\}$ and $R_\star^{\mathcal{J}}(R^{\star\mathcal{J}}(F)) = \{e_2\}$. Hence, $R^{\star\mathcal{J}}(F) \nsubseteq R_\star^{\mathcal{J}}(R^{\star\mathcal{J}}(F))$

(4) $F = \{e_3, e_4\}$, then $R_\star^{\mathcal{J}}(F) = \{e_4\}$ and $R^{\star\mathcal{J}}(R_\star^{\mathcal{J}}(F)) = \{e_1, e_3, e_4\}$. Hence, $R^{\star\mathcal{J}}(R_\star^{\mathcal{J}}(F)) \nsubseteq R_\star^{\mathcal{J}}(F)$

**Proposition 14.** *Consider R and $\mathcal{J}$ are, respectively, relation and ideal on $E \neq \phi$ and let $\phi \neq F \subseteq E$. Then,*

(1) $0 \leq ACC_R^{\star\mathcal{J}}(F) \leq 1$

(2) $ACC_R^{\star\mathcal{J}}(E) = 1$.

*Proof.* To prove 1, it is obvious that $R^{\star\mathcal{J}}(F) \cup F \neq \phi$ for every nonempty subset $F$ of $E$. Hence, $\phi \subseteq R_\star^{\mathcal{J}}(F) \cap F \subseteq R^{\star\mathcal{J}}(F) \cup F$. Therefore, $0 \leq |R_\star{}^{\mathcal{J}}(F) \cap F| \leq |R^{\star\mathcal{J}}(F) \cup F|$. So, $0 \leq |R_\star^{\mathcal{J}}(F) \cap F|/|R^{\star\mathcal{J}}(F) \cup F| \leq 1$. It means that, $0 \leq \mathrm{ACC}_R^{\star\mathcal{J}}(F) \leq 1$. Case 2 can be proved easily. □

**Theorem 15.** *Consider R as a relation on E and let $\mathcal{J}, \mathcal{K}$ be ideals on E such that $\mathcal{J} \subseteq \mathcal{K}$. For each $F \subseteq E$, the next properties hold.*

(1) $BND_R^{\star\mathcal{K}}(F) \subseteq BND_R^{\star\mathcal{J}}(F)$

(2) $ACC_R^{\star\mathcal{J}}(F) \leq ACC_R^{\star\mathcal{K}}(F)$

(3) $Rough_R^{\star\mathcal{K}}(F) \leq Rough_R^{\star\mathcal{J}}(F)$

*Proof.*

(1) Let $e \in \mathrm{BND}_R^{\star\mathcal{K}}(F)$. Then, $e \in R^{\star\mathcal{K}}(F) - R_\star{}^{\mathcal{K}}(F)$. So, $e \in R^{\star\mathcal{K}}(F)$ and $e \in (R_\star^{\mathcal{K}}(F))^c$. Hence, $e \in R^{\star\mathcal{J}}(F)$ and $e \in (R_\star{}^{\mathcal{J}}(F))^c$ by Propositions 10 and 11 part 7. It follows that $e \in \mathrm{BND}_R^{\star\mathcal{J}}(F)$. Therefore, $\mathrm{BND}_R^{\star\mathcal{K}}(F) \subseteq \mathrm{BND}_R^{\star\mathcal{J}}(F)$

(2) $\mathrm{ACC}_R^{\star\mathcal{J}}(F) = |R_\star{}^{\mathcal{J}}(F) \cap F/R^{\star\mathcal{J}}(F) \cup F|.\leq |R_\star{}^{\mathcal{K}}(F) \cap F/R^{\star\mathcal{K}}(F) \cup F|. = \mathrm{ACC}_R^{\star\mathcal{K}}(F)$

(3) Directly follows from 2

□

*Remark 16.* In Theorem 15, the converses of parts 1, 2, and 3 are generally false. To elucidate that, take $F = \{e_3, e_4\}$ as a subset of Example 1 (ii). Then

(1) $\mathrm{BND}_R^{\star\mathcal{K}}(F) = \phi \subseteq \phi = \mathrm{BND}_R^{\star}\mathcal{J}(F)$, but $\mathcal{J} \nsubseteq \mathcal{K}$

(2) $\mathrm{ACC}_R^{\star\mathcal{J}}(F) = 1 \leq 1 = \mathrm{ACC}_R^{\star\mathcal{K}}(F)$, but $\mathcal{J} \nsubseteq \mathcal{K}$

(3) $\mathrm{Rough}_R^{\star\mathcal{K}}(F) = 0 \leq 0 = \mathrm{Rough}_R^{\star\mathcal{J}}(F)$, but $\mathcal{J} \nsubseteq \mathcal{K}$

**Theorem 17.** *Let $\phi \neq F \subseteq E$, $\mathcal{J}$ be an ideal on E and $R_1, R_2$ be two binary relations on E. If $R_1 \subseteq R_2$, then*

(1) $R_1^{\star\mathcal{J}}(F) \subseteq R_2^{\star\mathcal{J}}(F)$

(2) $R_{2\star}^{\mathcal{J}}(F) \subseteq R_{1\star}^{\mathcal{J}}(F)$

(3) $BND_{R_1}^{\star}{}^{\mathcal{J}}(F) \subseteq BND_{R_2}^{\star}{}^{\mathcal{J}}(F)$

(4) $ACC_{R_2}^{\star}{}^{\mathcal{J}}(F) \leq ACC_{R_1}^{\star}{}^{\mathcal{J}}(F)$

(5) $Rough_{R_1}^{\star}{}^{\mathcal{J}}(F) \leq Rough_{R_2}^{\star}{}^{\mathcal{J}}(F)$

*Proof.*

(1) Let $e \in R_1^{\star\mathcal{J}}(F)$. Then, $\mu_{1u}(e) \cap F \in \mathcal{J}$. Since, $\mu_{1u}(e) \subseteq \mu_{2u}(e)$ (by Theorem 4 [20]). It follows that $\mu_{2u}(e) \cap F \in \mathcal{J}$. Thus, $e \in R_2^{\star\mathcal{J}}(F)$. Hence, $R_1^{\star\mathcal{J}}(F) \subseteq R_2^{\star\mathcal{J}}(F)$

(2) Let $e \in R_{2\star}{}^{\mathcal{J}}(F)$. Then, $\mu_{2u}(e) \cap F^c \in \mathcal{J}$. Since, $\mu_{1u}(e) \subseteq \mu_{2u}(e)$ (by Theorem 4 [20]). It follows that $\mu_{1u}(e) \cap F^c \in \mathcal{J}$. Thus, $e \in R_{1\star}{}^{\mathcal{J}}(F)$. Hence, $R_{2\star}{}^{\mathcal{J}}(F) \subseteq R_{1\star}{}^{\mathcal{J}}(F)$

(3) Let $e \in \mathrm{BND}_{R_1}^{\star\,\mathcal{J}}(F)$. Then, $e \in R_1^{\star\mathcal{J}}(F) - R_{1\star}{}^{\mathcal{J}}(F)$. So, $e \in R_1^{\star\mathcal{J}}(F)$ and $e \in (R_{1\star}{}^{\mathcal{J}}(F))^c$. Thus, $e \in R_2^{\star\mathcal{J}}(F)$ and $e \in (R_{1\star}{}^{\mathcal{J}}(F))^c$ by parts 1 and 2. Hence, $e \in \mathrm{BND}_{R_2}^{\star\,\mathcal{J}}(F)$. Therefore, $\mathrm{BND}_{R_1}^{\star\,\mathcal{J}}(F) \subseteq \mathrm{BND}_{R_2}^{\star\,\mathcal{J}}(F)$

(4) $\mathrm{ACC}_{R_2}^{\star\,\mathcal{J}}(F) = |R_{2\star}{}^{\mathcal{J}}(F) \cap F / R_2^{\star\mathcal{J}}(F) \cup F|. \leq |R_{1\star}{}^{\mathcal{J}}(F) \cap F / R_1^{\star\mathcal{J}}(F) \cup F|. = \mathrm{ACC}_{R_1}^{\star\,\mathcal{J}}(F)$

(5) Straightforward by 4

In Theorem 17, the relations of inclusion and less than are generally proper as we illustrate by the next example. □

*Example 2.* Let $E = \{e_1, e_2, e_3, e_4\}$, $\mathcal{J} = \{\phi, \{e_2\}, \{e_3\}, \{e_4\}, \{e_2, e_3\}, \{e_2, e_4\}, \{e_3, e_4\}, \{e_2, e_3, e_4\}\}$ and $R_1 = \Delta \cup \{(e_1, e_4), (e_4, e_1)\}$, $R_2 = R_1 \cup \{(e_1, e_3), (e_3, e_1)\}$ be two binary relation defined on $E$ thus $\mu_{1u}(e_1) = \mu_{1u}(e_4) = \{e_1, e_4\}$, $\mu_{1u}(e_2) = \{e_2\}$, $\mu_{1u}(e_3) = \{e_3\}$, $\mu_{2u}(e_1) = \mu_{2u}(e_3) = \mu_{2u}(e_4) = \{e_1, e_3, e_4\}$, $\mu_{2u}(e_2) = \{e_2\}$. Take

(a) $F = \{e_1, e_4\}$, then

(1) $R_1^{\star\mathcal{J}}(F) = \{e_4\} \neq \{e_1, e_3, e_4\} = R_2^{\star\mathcal{J}}(F)$

(2) $\mathrm{ACC}_{R_1}^{\star\,\mathcal{J}}(F) = 2/3 \neq 1/2 = \mathrm{ACC}_{R_2}^{\star\,\mathcal{J}}(F)$

(3) $\mathrm{Rough}_{R_2}^{\star\,\mathcal{J}}(F) = 1/2 \neq 1/3 = \mathrm{Rough}_{R_1}^{\star\,\mathcal{J}}(F)$

(4) $F = \{e_4, e_3\}$, then $R_{1\star}{}^{\mathcal{J}}(F) = \{e_2, e_3, e_4\} \neq \{e_2\} = R_{2\star}{}^{\mathcal{J}}(F)$

### *3.2. The Second Method of the Improvement of the Approximations and Accuracy Measure of Subsets*

*Definition 18.* Let $R$ and $\mathcal{J}$ be, respectively, binary relation and ideal on a nonempty set $E$. The second kind of the improvement of lower and upper approximations, boundary region, accuracy, and roughness of a nonempty subset $F$ of $E$ induced from $R$ and $\mathcal{J}$ is defined, respectively, by

$$R_{\star\star}{}^{\mathcal{J}}(F) = \{e \in F : \mu_u(e) \cap F^c \in \mathcal{J}\}, \tag{15}$$

$$R^{\star\star\mathcal{J}}(F) = F \cup R^{\star\mathcal{J}}(F), \tag{16}$$

$$\mathrm{BND}_R^{\star\star\,\mathcal{J}}(F) = R^{\star\star\mathcal{J}}(F) - R_{\star\star}{}^{\mathcal{J}}(F), \tag{17}$$

$$\mathrm{ACC}_R^{\star\star\,\mathcal{J}}(F) = \frac{|R_{\star\star}{}^{\mathcal{J}}(F)|}{|R^{\star\star\mathcal{J}}(F)|}, \tag{18}$$

$$\mathrm{Rough}_R^{\star\star\,\mathcal{J}}(F) = 1 - \mathrm{ACC}_R^{\star\star\,\mathcal{J}}(F). \tag{19}$$

**Proposition 19.** *Consider $F, H \subseteq E$ and let $\mathcal{J}, \mathcal{K}$ be ideals and $R$ be a binary relation on $E$. Then,*

*(1) $F \subseteq R^{\star\star\mathcal{J}}(F)$ equality hold if $F = \phi$ or $E$*

*(2) $F \subseteq H \Rightarrow R^{\star\star\mathcal{J}}(F) \subseteq R^{\star\star\mathcal{J}}(H)$*

*(3) $R^{\star\star\mathcal{J}}(F) \subseteq R^{\star\star\mathcal{J}}(R^{\star\star\mathcal{J}}(F))$*

*(4) $R^{\star\star\mathcal{J}}(F \cap H) \subseteq R^{\star\star\mathcal{J}}(F) \cap R^{\star\star\mathcal{J}}(H)$*

*(5) $R^{\star\star\mathcal{J}}(F \cup H) = R^{\star\star\mathcal{J}}(F) \cup R^{\star\star\mathcal{J}}(H)$*

*(6) $R^{\star\star\mathcal{J}}(F) = (R_{\star\star}{}^{\mathcal{J}}(F^c))^c$*

*(7) If $F \in \mathcal{J}$, then $R^{\star\star\mathcal{J}}(F) = F$*

*(8) If $\mathcal{J} \subseteq \mathcal{K}$, then $R^{\star\star\mathcal{K}}(F) \subseteq R^{\star\star\mathcal{J}}(F)$*

*(9) If $\mathcal{J} = P(E)$, then $R^{\star\star\mathcal{J}}(F) = F$*

*(10) $R^{\star\star\mathcal{J}\cap\mathcal{K}}(F) = R^{\star\star\mathcal{J}}(F) \cup R^{\star\star\mathcal{K}}(F)$*

*(11) $R^{\star\star\mathcal{J}\vee\mathcal{K}}(F) = R^{\star\star\mathcal{J}}(F) \cap R^{\star\star\mathcal{K}}(F)$*

*Proof.* Similar to the technique provided in Proposition 10. □

**Proposition 20.** *Consider $F, H \subseteq E$ and let $\mathcal{J}, \mathcal{K}$ be ideals and $R$ be a binary relation on $E$. Then,*

*(1) $R_{\star\star}{}^{\mathcal{J}}(F) \subseteq F$ equality hold if $F = \phi$ or $E$*

*(2) $F \subseteq H \Rightarrow R_{\star\star}{}^{\mathcal{J}}(F) \subseteq R_{\star\star}{}^{\mathcal{J}}(H)$*

*(3) $R_{\star\star}{}^{\mathcal{J}}(R_{\star\star}{}^{\mathcal{J}}(F)) \subseteq R_{\star\star}{}^{\mathcal{J}}(F)$*

*(4) $R_{\star\star}{}^{\mathcal{J}}(F) \cup R_{\star\star}{}^{\mathcal{J}}(H) \subseteq R_{\star\star}{}^{\mathcal{J}}(F \cup H)$*

*(5) $R_{\star\star}{}^{\mathcal{J}}(F \cap H) = R_{\star\star}{}^{\mathcal{J}}(F) \cap R_{\star\star}{}^{\mathcal{J}}(H)$*

*(6) $R_{\star\star}{}^{\mathcal{J}}(F) = (R^{\star\star\mathcal{J}}(F^c))^c$*

*(7) If $F^c \in \mathcal{J}$, then, $R_{\star\star}{}^{\mathcal{J}}(F) = F$*

*(8) If $\mathcal{J} \subseteq \mathcal{K}$, then, $R_{\star\star}{}^{\mathcal{J}}(F) \subseteq R_{\star\star}{}^{\mathcal{K}}(F)$*

*(9) If $\mathcal{J} = P(E)$, then, $R_{\star\star}{}^{\mathcal{J}}(F) = F$*

*(10) $R_{\star\star}{}^{\mathcal{J}\cap\mathcal{K}}(F) = R_{\star\star}{}^{\mathcal{J}}(F) \cap R_{\star\star}{}^{\mathcal{K}}(F)$*

*(11) $R_{\star\star}{}^{\mathcal{J}\vee\mathcal{K}}(F) = R_{\star\star}{}^{\mathcal{J}}(F) \cup R_{\star\star}{}^{\mathcal{K}}(F)$*

*Proof.* Similar to Proposition 11. □

*Remark 21.*

(i) In Proposition 19 and Proposition 20, the converse of parts 7 and 9 is generally incorrect. To show that, consider Example 1 (i).

(a) For part 7, take

(1) $F=\{e_1,e_3,e_4\}$, then $R^{\star\star\mathcal{J}}(F)=F$, but $F\notin\mathcal{J}$

(2) $F=\{e_2\}$, then $R_{\star\star}{}^{\mathcal{J}}(F)=F$, but $F^c\notin\mathcal{J}$

(b) For part 9, take

(1) $F=\{e_1,e_3,e_4\}$, then $R^{\star\star\mathcal{J}}(F)=F$, but $\mathcal{J}\neq P(E)$

(2) $F=\{e_2\}$, then $R_{\star\star}{}^{\mathcal{J}}(F)=F$, but $\mathcal{J}\neq P(E)$

(ii) In Proposition 19 and Proposition 20, the converse of part 8 is generally incorrect. To elucidate that consider Example 1 (ii)

(1) If $F=\{e_1,e_2\}$, then, $R^{\star\star\mathcal{K}}(F)=\{e_1,e_2\}\subseteq\{e_1,e_2\}=R^{\star\star\mathcal{J}}(F)$, but $\mathcal{J}\nsubseteq\mathcal{K}$

(2) If $F=\{e_3,e_4\}$, then, $R_{\star\star}{}^{\mathcal{J}}(F)=\{e_3,e_4\}\subseteq\{e_3,e_4\}=R_{\star\star}{}^{\mathcal{K}}(F)$, but $\mathcal{J}\nsubseteq\mathcal{K}$

(iii) In Example 1 (i), take $\mathcal{J}=\{\phi,\{e_1\}\}$. So, the converse of part 2 in Proposition 19 and Proposition 20 is generally incorrect. Additionally, the inclusion relations of parts 3 and 4 in Proposition 19 and Proposition 20 generally proper

(a) For part 2, take $F=\{e_1,e_4\},H=\{e_2,e_3\}$, then

(1) $R^{\star\star\mathcal{J}}(F)=\{e_1,e_3,e_4\}\subseteq E=R^{\star\star\mathcal{J}}(H)=E$, but $F\nsubseteq H$

(2) $R_{\star\star}{}^{\mathcal{J}}(F)=\phi\subseteq\{e_2\}=R_{\star\star}{}^{\mathcal{J}}(H)$, but $F\nsubseteq H$

(b) For part 3, take

(1) $F=\{e_1,e_4\}$, then, $R^{\star\star\mathcal{J}}(F)=\{e_1,e_3,e_4\}$, $R^{\star\star\mathcal{J}}(R^{\star\star\mathcal{J}}(F))=E$. Therefore, $R^{\star\star\mathcal{J}}(F)=\{e_1,e_3,e_4\}\neq E=R^{\star\star\mathcal{J}}(R^{\star\star\mathcal{J}}(F))$

(2) $F=\{e_2,e_3\}$, then, $R_{\star\star}{}^{\mathcal{J}}(F)=\{e_2\}$, $R_{\star\star}{}^{\mathcal{J}}(R_{\star\star}{}^{\mathcal{J}}(F))=\phi$. Therefore, $R_{\star\star}{}^{\mathcal{J}}(F)=\{e_2\}\neq\phi=R_{\star\star}{}^{\mathcal{J}}(R_{\star\star}{}^{\mathcal{J}}(F))$

(c) For part 4, take $F=\{e_1,e_4\},H=\{e_2,e_3\}$ thus

(1) $F\cap H=\phi$, then, $R^{\star\star\mathcal{J}}(F)=\{e_1,e_3,e_4\}$, $R^{\star\star\mathcal{J}}(H)=E$, $R^{\star\star\mathcal{J}}(F\cap H)=\phi$. Therefore, $R^{\star\star\mathcal{J}}(F)\cap R^{\star\star\mathcal{J}}(H)=\{e_1,e_3,e_4\}\neq\phi=R^{\star\star\mathcal{J}}(F\cap H)$

(2) $F\cup H=E$, then, $R^{\mathcal{J}}_{\star\star}(F)=\phi$, $R^{\mathcal{J}}_{\star\star}(H)=\{e_2\}$, $R^{\star\star\mathcal{J}}(F\cup H)=E$. Therefore, $R^{\mathcal{J}}_{\star\star}(F)\cup R^{\mathcal{J}}_{\star\star}(H)=\{e_2\}\neq E=R^{\mathcal{J}}_{\star\star}(F\cup H)$

*Remark 22.* Some properties of the lower and upper approximations given in the first method are not kept by this method as we demonstrate in the following.

(i) In Example 1 (i) take

(1) $F=\{u_1\}\in\mathcal{J}$, then, $R^{\star\star\mathcal{J}}(F)=F$. Hence, if $F\in\mathcal{J}\nRightarrow R^{\star\star\mathcal{J}}(F)=\phi$

(2) $F^c=\{u_1\}\in\mathcal{J}$, then, $R_{\star\star}{}^{\mathcal{J}}(F)=F$. Hence, if $F^c\in\mathcal{J}\nRightarrow R_{\star\star}{}^{\mathcal{J}}(F)=U$

(ii) In Example 1 (ii) take

(1) $\mathcal{K}=P(E)$, $F=\{e_1,e_3\}$, then, $R^{\star\star\mathcal{K}}(F)=F$. Hence, if $\mathcal{K}=P(E)\nRightarrow R^{\star\star\mathcal{K}}(F)=\phi$

(2) $\mathcal{K}=P(E)$, $F=\{e_2,e_4\}$, then, $R_{\star\star}{}^{\mathcal{K}}(F)=F$. Hence, if $\mathcal{K}=P(E)\nRightarrow R_{\star\star}{}^{\mathcal{K}}(F)=E$

*Remark 23.* Some Pawlak's properties are not kept by this method as we demonstrate in the following. In Example 1 (i) take $\mathcal{J}=\{\phi,\{e_1\}\}$,

(1) $F=\{e_1,e_2\}$, then, $R^{\star\star\mathcal{J}}(F)=\{e_1,e_2,e_3\}$ and $R^{\mathcal{J}}_{\star\star}(R^{\star\star\mathcal{J}}(F))=\{e_2\}$. Hence, $R^{\star\star\mathcal{J}}(F)\nsubseteq R^{\mathcal{J}}_{\star\star}(R^{\star\star\mathcal{J}}(F))$

(2) $F=\{e_3,e_4\}$, then, $R^{\mathcal{J}}_{\star\star}(F)=\{e_4\}$ and $R^{\star\star\mathcal{J}}(R^{\mathcal{J}}_{\star\star}(F))=\{e_1,e_3,e_4\}$. Hence, $R^{\star\star\mathcal{J}}(R^{\mathcal{J}}_{\star\star}(F))\nsubseteq R^{\mathcal{J}}_{\star\star}(F)$

**Proposition 24.** *Consider $R$ and $\mathcal{J}$ are, respectively, binary relation and ideal on $E\neq\phi$ and let $\phi\neq F\subseteq E$. Then,*

*(1) $0\leq ACC_R^{\star\star\mathcal{J}}(F)\leq 1$*

*(2) $ACC_R^{\star\star\mathcal{J}}(E)=1$*

*Proof.* As given in Proposition 14. □

**Theorem 25.** *Consider $R$ as a binary relation on $E$ and let $\mathcal{J},\mathcal{K}$ be ideals on $E$ such that $\mathcal{J}\subseteq\mathcal{K}$. For each $F\subseteq E$, the next properties hold.*

*(1) $BND_R^{\star\star\mathcal{K}}(F)\subseteq BND_R^{\star\star\mathcal{J}}(F)$*

*(2) $ACC_R^{\star\star\mathcal{J}}(F)\leq ACC_R^{\star\star\mathcal{K}}(F)$*

*(3) $Rough_R^{\star\star\mathcal{K}}(F)\leq Rough_R^{\star\star\mathcal{J}}(F)$*

*Proof.* Following similar arguments given in Theorem 15. □

*Remark 26.* In Theorem 25, the converses of parts 1, 2, and 3 are generally incorrect. To validate this notice, consider Example 1 (ii) and take $F=\{e_2,e_4\}$. Then,

(1) $\mathrm{BND}_R^{\star\star\mathcal{K}}(F)=\phi\subseteq\phi=\mathrm{BND}_R^{\star\star\mathcal{J}}(F)$, but $\mathcal{J}\nsubseteq\mathcal{K}$

(2) $\mathrm{ACC}_R^{\star\star\mathcal{K}}(F)=1\leq 1=\mathrm{ACC}_R^{\star\star\mathcal{J}}(F)$, but $\mathcal{J}\nsubseteq\mathcal{K}$

(3) $\mathrm{Rough}_R^{\star\star\mathcal{K}}(F)=0\leq 0=\mathrm{Rough}_R^{\star\star\mathcal{J}}(F)$, but $\mathcal{J}\nsubseteq\mathcal{K}$

**Theorem 27.** *Let $\phi \neq F \subseteq E$, $\mathcal{J}$ be an ideal on E and $R_1, R_2$ be two binary relations on E. If $R_1 \subseteq R_2$, then*

(1) $R_1{}^{**\mathcal{J}}(F) \subseteq R_2{}^{**\mathcal{J}}(F)$

(2) $R_{2**}{}^{\mathcal{J}}(F) \subseteq R_{1**}{}^{\mathcal{J}}(F)$

(3) $BND_{R_1}^{**\mathcal{J}}(F) \subseteq BND_{R_2}^{**\mathcal{J}}(F)$

(4) $ACC_{R_2}^{**\mathcal{J}}(F) \leq ACC_{R_1}^{**\mathcal{J}}(F)$

(5) $Rough_{R_1}^{**\mathcal{J}}(F) \leq Rough_{R_2}^{**\mathcal{J}}(F)$

*Proof.* Similar to Theorem 17. □

*Remark 28.* In Theorem 27, the inclusion and less than relations are generally proper. To illustrate that, consider Example 2 and take $F = \{e_1, e_2\}$. Then,

(1) $R_1{}^{**\mathcal{J}}(F) = \{e_1, e_4\} \neq E = R_2{}^{**\mathcal{J}}(F)$

(2) $\mathrm{BND}_{R_1}^{**\mathcal{J}}(F) = \{e_4\} \neq \{e_1, e_3, e_4\} = \mathrm{BND}_{R_2}^{**\mathcal{J}}(F)$

(3) $\mathrm{ACC}_{R_1}^{**\mathcal{J}}(F) = 2/3 \neq 1/4 = \mathrm{ACC}_{R_2}^{**\mathcal{J}}(F)$

(4) $\mathrm{Rough}_{R_1}^{**\mathcal{J}}(F) = 1/3 \neq 3/4 = \mathrm{Rough}_{R_2}^{**\mathcal{J}}(F)$

### *3.3. The Third Method of the Improvement of the Approximations and Accuracy Measure of Subsets*

*Definition 29.* Let $R$ and $\mathcal{J}$ be, respectively, binary relation and ideal on a nonempty set $E$. The third kind of the improvement of lower and upper approximations, boundary region, accuracy, and roughness of a nonempty subset $F$ of $E$ induced from $R$ and $\mathcal{J}$ is defined, respectively, by

$$R_{\star\star\star}{}^{\mathcal{J}}(F) = \bigcup_{e\in E}\{\mu_u(e)\colon \mu_u(e) \cap F^c \in \mathcal{J}\}, \tag{20}$$

$$R^{\star\star\star\mathcal{J}}(F) = \left(R_{\star\star\star}{}^{\mathcal{J}}(F^c)\right)^c, \tag{21}$$

$$\mathrm{BND}_R^{\star\star\star\mathcal{J}}(F) = R^{\star\star\star\mathcal{J}}(F) - R_{\star\star\star}{}^{\mathcal{J}}(F), \tag{22}$$

$$\mathrm{ACC}_R^{\star\star\star\mathcal{J}}(F) = \frac{|R_{\star\star\star}{}^{\mathcal{J}}(F) \cap F|}{|R^{\star\star\star}\mathcal{J}(F) \cup F|}, \tag{23}$$

$$\mathrm{Rough}_R^{\star\star\star\mathcal{J}}(F) = 1 - \mathrm{ACC}_R^{\star\star\star\mathcal{J}}(F). \tag{24}$$

**Proposition 30.** *Consider $F, H \subseteq E$ and let $\mathcal{J}, \mathcal{K}$ be ideals and R be a binary relation on E. Then,*

(1) $F \subseteq H \Rightarrow R_{\star\star\star}{}^{\mathcal{J}}(F) \subseteq R_{\star\star\star}{}^{\mathcal{J}}(H)$

(2) $R_{\star\star\star}{}^{\mathcal{J}}(F) \cup R_{\star\star\star}{}^{\mathcal{J}}(H) \subseteq R_{\star\star\star}{}^{\mathcal{J}}(F \cup H)$

(3) $R_{\star\star\star}{}^{\mathcal{J}}(F \cap H) \subseteq R_{\star\star\star}{}^{\mathcal{J}}(F) \cap R_{\star\star\star}{}^{\mathcal{J}}(H)$

(4) $R_{\star\star\star}{}^{\mathcal{J}}(F) = (R^{\star\star\star\mathcal{J}}(F^c))^c$

(5) *If $\mathcal{J} \subseteq \mathcal{K}$, then $R_{\star\star\star}{}^{\mathcal{J}}(F) \subseteq R_{\star\star\star}{}^{\mathcal{K}}(F)$*

(6) $R_{\star\star\star}{}^{\mathcal{J}\cap\mathcal{K}}(F) = R_{\star\star\star}{}^{\mathcal{J}}(F) \cap R_{\star\star\star}{}^{\mathcal{K}}(F)$

*Proof.*

(1) Let $F \subseteq H$ and $e \in R_{\star\star\star}{}^{\mathcal{J}}(F)$. Then, $\exists y \in E$ such that $e \in \mu_u(y) \cap F^c \in \mathcal{J}$. Hence, $e \in \mu_u(y) \cap H^c \in \mathcal{J}$ (by $H^c \subseteq F^c$, and the properties of ideal). Thus, $e \in R_{\star\star\star}{}^{\mathcal{J}}(H)$. Therefore, $R_{\star\star\star}{}^{\mathcal{J}}(F) \subseteq R_{\star\star\star}{}^{\mathcal{J}}(H)$

(2) Immediately by part (4)

(3) Immediately by part (4)

(4) Immediately obtains by Definition 29

(5) Let $\mathcal{J} \subseteq \mathcal{K}$ and $e \in R_{\star\star\star}{}^{\mathcal{J}}(F)$. Then, $\exists y \in E$ such that $e \in \mu_u(y) \cap F^c \in \mathcal{J} \subseteq \mathcal{K}$. So, $e \in R_{\star\star\star}{}^{\mathcal{K}}(F)$, and hence, $R_{\star\star\star}{}^{\mathcal{J}}(F) \subseteq R_{\star\star\star}{}^{\mathcal{K}}(F)$

(6) $R_{\star\star\star}{}^{\mathcal{J}\cap\mathcal{K}}(F) = \bigcup_{e\in E}\{\mu_u(e)\colon \mu_u(e) \cap F^c \in \mathcal{J} \cap \mathcal{K}\}. = (\bigcup_{e\in E}\{\mu_u(e)\colon \mu_u(e) \cap F^c \in \mathcal{J}\})$ and $(\bigcup_{e\in E}\{\mu_u(e)\colon \mu_u(e) \cap F^c \in \mathcal{K}\}). = (\bigcup_{e\in E}\{\mu_u(e)\colon \mu_u(e) \cap F^c \in \mathcal{J}\}) \cap (\bigcup_{e\in E}\{\mu_u(e)\colon \mu_u(e) \cap F^c \in \mathcal{K}\}). = R_{\star\star\star}{}^{\mathcal{J}}(F) \cap R_{\star\star\star}{}^{\mathcal{K}}(F)$

□

**Proposition 31.** *Consider $F, H \subseteq E$ and let $\mathcal{J}, \mathcal{K}$ be ideals and R be a binary relation on E. Then,*

(1) $F \subseteq H \Rightarrow R^{\star\star\star\mathcal{J}}(F) \subseteq R^{\star\star\star\mathcal{J}}(H)$

(2) $R^{\star\star\star\mathcal{J}}(F \cap H) \subseteq R^{\star\star\star\mathcal{J}}(F) \cap R^{\star\star\star\mathcal{J}}(H)$

(3) $R^{\star\star\star\mathcal{J}}(F) \cup R^{\star\star\star\mathcal{J}}(H) \subseteq R^{\star\star\star\mathcal{J}}(F \cup H)$

(4) $R^{\star\star\star\mathcal{J}}(F) = (R_{\star\star\star}{}^{\mathcal{J}}(F^c))^c$

(5) *If $\mathcal{J} \subseteq \mathcal{K}$, then $R^{\star\star\star\mathcal{K}}(F) \subseteq R^{\star\star\star\mathcal{J}}(F)$*

(6) $R^{\star\star\star\mathcal{J}\cap\mathcal{K}}(F) = R^{\star\star\star\mathcal{J}}(F) \cup R^{\star\star\star\mathcal{K}}(F)$

*Proof.*

(1) Let $F \subseteq H$. Thus, $H^c \subseteq F^c$, and hence, $R_{\star\star\star}{}^{\mathcal{J}}(H^c) \subseteq R_{\star\star\star}{}^{\mathcal{J}}(F^c)$ (by no. (4) in Proposition 30). So, $(R_{\star\star\star}{}^{\mathcal{J}}(F^c))^c \subseteq (R_{\star\star\star}{}^{\mathcal{J}}(H^c))^c$. Consequently, $R^{\star\star\star\mathcal{J}}(F) \subseteq R^{\star\star\star\mathcal{J}}(H)$

(2) Immediately by part (4)

(3) Immediately by part (4)

(4) Immediately obtains by Definition 29

(5) Let $\mathcal{J} \subseteq \mathcal{K}$ and $e \in R^{\star\star\star\mathcal{K}}(F)$. Then, $e \in (R_{\star\star\star}{}^{\mathcal{K}}(F^c))^c \subseteq (R_{\star\star\star}{}^{\mathcal{J}}(F^c))^c$, (by no. (5) in Proposition 30). Thus, $e \in (R_{\star\star\star}{}^{\mathcal{J}}(F^c))^c = R^{\star\star\star\mathcal{J}}(F)$. Therefore, $R^{\star\star\star\mathcal{K}}(F) \subseteq R^{\star\star\star\mathcal{J}}(F)$

(6) $R^{\star\star\star\mathcal{J}\cap\mathcal{K}}(F) = (R_{\star\star\star}{}^{\mathcal{J}\cap\mathcal{K}}(F^c))^c. = (R_{\star\star\star}{}^{\mathcal{J}}(F^c) \cap R_{\star\star\star}{}^{\mathcal{K}}(F^c))^c$ (by no. (6) in Proposition 30). $= (R_{\star\star\star}{}^{\mathcal{J}}(F^c))^c \cup (R_{\star\star\star}{}^{\mathcal{K}}(F^c))^c. = R^{\star\star\star\mathcal{J}}(F) \cup R^{\star\star\star\mathcal{K}}(F)$

□

*Remark 32.*

(1) The converse of part 1 in Proposition 30 and Proposition 31 is generally incorrect. Example 1 (i) points out this fact

(a) If $F = \{e_1\}, H = \{e_4\}$, then, $R^{\star\star\star\mathcal{J}}(F) = \phi, R^{\star\star\star\mathcal{J}}(H) = \{e_4\}$. Therefore, $R^{\star\star\star\mathcal{J}}(F) \subseteq R^{\star\star\star\mathcal{J}}(H)$, but $F \nsubseteq H$

(b) If $F = \{e_2\}, H = \{e_1, e_3, e_4\}$, then, $R_{\star\star\star}{}^{\mathcal{J}}(F) = \{e_1, e_2, e_3\}, R_{\star\star\star}{}^{\mathcal{J}}(H) = E$. Therefore, $R_{\star\star\star}{}^{\mathcal{J}}(F) \subseteq R_{\star\star\star}{}^{\mathcal{J}}(H)$, but $F \nsubseteq H$

(2) The inclusion relations of part 2 in Proposition 30 and Proposition 31 are generally proper. To show that consider Example 1 (iii) and take $F = \{e_1, e_4\}$, $H = \{e_2, e_3\}$. Then

(a) $R^{\star\star\star\mathcal{J}}(F) = R^{\star\star\star\mathcal{J}}(H) = E, R^{\star\star\star\mathcal{J}}(F \cap H) = \phi$. Therefore, $R^{\star\star\star\mathcal{J}}(F) \cap R^{\star\star\star\mathcal{J}}(H) = E \neq \phi = R^{\star\star\star\mathcal{J}}(F \cap H)$

(b) $R_{\star\star\star}{}^{\mathcal{J}}(F) = R_{\star\star\star}{}^{\mathcal{J}}(H) = \phi, R_{\star\star\star}{}^{\mathcal{J}}(F \cup H) = E$. Therefore, $R_{\star\star\star}{}^{\mathcal{J}}(F) \cup R_{\star\star\star}{}^{\mathcal{J}}(H) = \phi \neq E = R_{\star\star\star}{}^{\mathcal{J}}(F \cup H)$

(3) In Example 1 (i), if $\mathcal{J} = \{\phi, \{e_1\}\}$. So, the inclusion relations of part 3 in Proposition 30 and Proposition 31 are generally proper as we illustrate in the following

(a) $F = \{e_1, e_2, e_4\}, H = \{e_1, e_3, e_4\}, F \cap H = \{e_1, e_4\}$, then, $R_{\star\star\star}{}^{\mathcal{J}}(F) = F, R_{\star\star\star}{}^{\mathcal{J}}(H) = H, R_{\star\star\star}{}^{\mathcal{J}}(F \cap H) = \phi$. Therefore, $R_{\star\star\star}{}^{\mathcal{J}}(F) \cap R_{\star\star\star}{}^{\mathcal{J}}(H) = \{e_1, e_4\} \neq \phi = R_{\star\star\star}{}^{\mathcal{J}}(F \cap H)$

(b) $F = \{e_3\}, H = \{e_2\}, F \cup H = \{e_2, e_3\}$, then, $R^{\star\star\star\mathcal{J}}(F) = F, R^{\star\star\star\mathcal{J}}(H) = H, R^{\star\star\star\mathcal{J}}(F \cup H) = E$. Therefore, $R^{\star\star\star\mathcal{J}}(F) \cup R^{\star\star\star\mathcal{J}}(H) = \{e_2, e_3\} \neq E = R^{\star\star\star\mathcal{J}}(F \cup H)$

(4) In Proposition 30 and Proposition 31, the converse of part 5 is generally incorrect. To show that consider Example 1 (ii) and take

(a) $F = \{e_1, e_2\}$, then, $R^{\star\star\star\mathcal{J}}(F) = F, R^{\star\star\star\mathcal{K}}(F) = \{e_1\}$. Therefore, $R^{\star\star\star\mathcal{K}}(F) \subseteq R^{\star\star\star\mathcal{J}}(F)$, but $\mathcal{J} \nsubseteq \mathcal{K}$

(b) $F = \{e_1, e_2, e_3\}$, then, $R_{\star\star\star}{}^{\mathcal{J}}(F) = R_{\star\star\star}{}^{\mathcal{K}}(F) = \{e_2, e_3\}$. Therefore, $R_{\star\star\star}{}^{\mathcal{J}}(F) \subseteq R_{\star\star\star}{}^{\mathcal{K}}(F)$, but $\mathcal{J} \nsubseteq \mathcal{K}$

*Remark 33.* Some properties of the lower and upper approximations given in the second method are not kept by this method as we demonstrate in the following.

(i) In Example 1 (i) take

(1) $F = \{e_1\}$, then $R^{\star\star\star\mathcal{J}}(F) = \phi$. Hence, $F \nsubseteq R^{\star\star\star\mathcal{J}}(F)$

(2) $F = \{e_2\}$, then $R_{\star\star\star}{}^{\mathcal{J}}(F) = \{e_1, e_2, e_3\}$. Hence, $R_{\star\star\star}{}^{\mathcal{J}}(F) \nsubseteq F$

(3) $F = E$, then $R^{\star\star\star\mathcal{J}}(E) = \{e_4\}$. Hence, $R^{\star\star\star\mathcal{J}}(E) \neq E$

(4) $F = \phi$, then $R_{\star\star\star}{}^{\mathcal{J}}(\phi) = \{e_1, e_2, e_3\}$. Hence, $R_{\star\star\star}{}^{\mathcal{J}}(\phi) \neq \phi$

*Example 3.* Let $E = \{e_1, e_2, e_3, e_4\}$, $\mathcal{J} = \{\phi, \{e_2\}\}$ and $R = \{(e_2, e_2)\}$ be a binary relation defined on $E$ thus $\mu_u(e_2) = \{e_2\}, \mu_u(e_1) = \mu_u(e_3) = \mu_u(e_4) = \phi$. Take

(1) $F = E$, then $R_{\star\star\star}{}^{\mathcal{J}}(E) = \{e_2\}$. Hence, $R_{\star\star\star}{}^{\mathcal{J}}(E) \neq E$

(2) $F = \phi$, then $R^{\star\star\star\mathcal{J}}(\phi) = \{e_1, e_3, e_4\}$. Hence, $R^{\star\star\star\mathcal{J}}(\phi) \neq \phi$

*Remark 34.* Some properties of the lower and upper approximations given in the first/second method are not kept by this method as we demonstrate in the following. In Example 1, take

(1) $F = \{e_1, e_3, e_4\}$, then, $F^c \in \mathcal{J}$, then $R_{\star\star\star}{}^{\mathcal{J}}(F) = \{e_2\}$. Hence, if $F^c \in \mathcal{J} \nRightarrow R_{\star\star\star}{}^{\mathcal{J}}(F) = E$ or $F$

(2) $F = \{e_2\} \in \mathcal{J}$, then, $R^{\star\star\star\mathcal{J}}(F) = \{e_1, e_3, e_4\}$. Hence, if $F \in \mathcal{J} \nRightarrow R^{\star\star\star\mathcal{J}}(F) = \phi$ or $F$

(3) $F = \{e_1, e_3, e_4\}, \mathcal{J} = P(E)$, then, $R_{\star\star\star}{}^{\mathcal{J}}(F) = \{e_2\}$. Hence, if $\mathcal{J} = P(E) \nRightarrow R_{\star\star\star}{}^{\mathcal{J}}(F) = E$, or $F$

(4) $F = \{e_2\}, \mathcal{J} = P(E)$, then, $R^{\star\star\star\mathcal{J}}(F) = \{e_1, e_3, e_4\}$. Hence, if $\mathcal{J} = P(E) \nRightarrow R^{\star\star\star\mathcal{J}}(F) = \phi$, or $F$

*Remark 35.* Some Pawlak's properties are not kept by this method as we demonstrate in the following. In Example 1 (i), if $\mathcal{J} = \{\phi, \{e_1\}\}$,

(1) $F = \{e_1, e_2\}$, then $R^{\star\star\star\mathcal{J}}(F) = \{e_2\}, R_{\star\star\star}{}^{\mathcal{J}}(R^{\star\star\star\mathcal{J}}(F)) = \phi$. Hence, $R^{\star\star\star\mathcal{J}}(F) \nsubseteq R_{\star\star\star}{}^{\mathcal{J}}(R^{\star\star\star\mathcal{J}}(F))$

(2) $F = \{e_3, e_4\}$, then $R_{\star\star\star}{}^{\mathcal{J}}(F) = \{e_1, e_3, e_4\}, R^{\star\star\star\mathcal{J}}(R_{\star\star\star}{}^{\mathcal{J}}(F)) = E$. Hence, $R^{\star\star\star\mathcal{J}}(R_{\star\star\star}{}^{\mathcal{J}}(F)) \nsubseteq R_{\star\star\star}{}^{\mathcal{J}}(F)$

**Proposition 36.** *Consider $R$ and $\mathcal{J}$ are, respectively, binary relation and ideal on $E \neq \phi$ and let $\phi \neq F \subseteq E$. Then,*

*(1) $0 \leq ACC_R^{\star\star\star\mathcal{J}}(F) \leq 1$*

*(2) $ACC_R^{\star\star\mathcal{J}}(E) = 1$*

*Proof.* It is similar to Proposition 14. □

**Theorem 37.** *Consider $R$ as a binary relation on $E$ and let $\mathcal{J}, \mathcal{K}$ be ideals on $E$ such that $\mathcal{J} \subseteq \mathcal{K}$. For each $F \subseteq E$, the next properties hold.*

*(1) $BND_R^{\star\star\star\mathcal{K}}(F) \subseteq BND_R^{\star\star\star\mathcal{J}}(F)$*

*(2) $ACC_R^{\star\star\star\mathcal{J}}(F) \leq ACC_R^{\star\star\star\mathcal{K}}(F)$*

*(3) $Rough_R^{\star\star\star\mathcal{K}}(F) \leq Rough_R^{\star\star\star\mathcal{J}}(F)$*

*Proof.* Similar to Theorem 15. □

*Remark 38.* In Theorem 37, the converse of parts 1, 2, and 3 is generally false. To validate that take Example 1 (ii) and let $F = \{e_3, e_4\}$. Then,

(1) $\mathrm{BND}_R^{***\mathscr{K}}(F) = \{e_1\} \subseteq \{e_1, e_2\} = \mathrm{BND}_R^{***\mathscr{J}}(F)$, but $\mathscr{J} \nsubseteq \mathscr{K}$

(2) $\mathrm{ACC}_R^{***\mathscr{J}}(F) = 1/2 \leq 2/3 = \mathrm{ACC}_R^{***\mathscr{K}}(F)$, but $\mathscr{J} \nsubseteq \mathscr{K}$

(3) $\mathrm{Rough}_R^{***\mathscr{K}}(F) = 1/3 \leq 1/2 = \mathrm{Rough}_R^{***\mathscr{J}}(F)$, but $\mathscr{J} \nsubseteq \mathscr{K}$

The lower and upper approximations, boundary region, accuracy, and roughness generated by the third type does not have the monotonicity. The next example confirm this fact.

*Example 4.* Let $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, $\mathscr{J} = \{\phi, \{e_2\}\}$, $R_1, R_2$ be two binary relations on $E$ where $R_1 = \Delta \cup \{(e_1, e_3), (e_3, e_1), (e_3, e_7), (e_4, e_6), (e_5, e_7), (e_6, e_4), (e_7, e_3), (e_7, e_5)\}$, $R_2 = R_1 \cup \{(e_1, e_4), (e_1, e_5), (e_2, e_6), (e_4, e_1), (e_5, e_1), (e_6, e_2)\}$. Thus, $\mu_{1u}(e_1) = \{e_1, e_3, e_7\}$, $\mu_{1u}(e_2) = \{e_2\}$, $\mu_{1u}(e_3) = \mu_{1u}(e_7) = \{e_1, e_3, e_5, e_7\}$, $\mu_{1u}(e_4) = \mu_{1u}(e_6) = \{e_4, e_6\}$, $\mu_{1u}(e_5) = \{e_3, e_5, e_7\}$, $\mu_{2u}(e_1) = \{e_1, e_3, e_4, e_5, e_6, e_7\}$, $\mu_{2u}(e_2) = \{e_2, e_4, e_6\}$, $\mu_{2u}(e_3) = \mu_{2u}(e_5) = \{e_1, e_3, e_4, e_5, e_7\}$, $\mu_{2u}(e_4) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, $\mu_{2u}(e_6) = \{e_1, e_2, e_4, e_6\}$, $\mu_{2u}(e_7) = \{e_1, e_3, e_5, e_7\}$. Take

(1) $F = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, then, $R_{1***}{}^{\mathscr{J}}(F) = \{e_2, e_4, e_6\}$, $R_{2***}{}^{\mathscr{J}}(F) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. Therefore, $R_{1***}{}^{\mathscr{J}}(F) \hat{\cup} R_{2***}{}^{\mathscr{J}}(F)$

(2) $F = \{e_7\}$, then, $R_1{}^{***\mathscr{J}}(F) = \{e_1, e_3, e_5, e_7\}$, $R_2{}^{***\mathscr{J}}(F) = \{e_7\}$. Therefore, $R_1{}^{***\mathscr{J}}(F) \nsubseteq R_2{}^{***\mathscr{J}}(F)$

$F = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, then $R_{1**}{}^{\mathscr{J}}(F) = \{e_2, e_4, e_6\}$, $R_1{}^{***\mathscr{J}}(F) = E$, $R_{2**}{}^{\mathscr{J}}(F) = F$, $R_2{}^{***\mathscr{J}}(F) = E$. Therefore,

(a) $\mathrm{BND}_{R_1}^{***\mathscr{J}}(F) = \{e_1, e_3, e_5, e_7\} \nsubseteq \{e_7\} = \mathrm{BND}_{R_2}^{***\mathscr{J}}(F)$

(b) $\mathrm{ACC}_{R_1}^{***\mathscr{J}}(F) = 3/7 < 6/7 = \mathrm{ACC}_{R_2}^{***\mathscr{J}}(F)$

(c) $\mathrm{Rough}_{R_1}^{***\mathscr{J}}(F) = 4/7 > 1/7 = \mathrm{Rough}_{R_2}^{***\mathscr{J}}(F)$

Although, $R_1 \subseteq R_2$.

### *3.4. The Fourth Method of the Improvement of the Approximations and Accuracy Measure of Subsets*

*Definition 39.* Let $R$ and $\mathscr{J}$ be, respectively, binary relation and ideal on a nonempty set $E$. The fourth kind of the improvement of lower and upper approximations, boundary region, accuracy, and roughness of a nonempty subset $F$ of $E$ induced from $R$ and $\mathscr{J}$ is defined, respectively, by

$$R^{****\mathscr{J}}(F) = \underset{e\in E}{\cup}\{\mu_u(e): \mu_u(e) \cap F \notin \mathscr{J}\}, \tag{25}$$

$$R_{****}{}^{\mathscr{J}}(F) = \left(R^{****\mathscr{J}}(F^c)\right)^c, \tag{26}$$

$$\mathrm{BND}_R^{****\mathscr{J}}(F) = R^{****\mathscr{J}}(F) - R_{****}{}^{\mathscr{J}}(F), \tag{27}$$

$$\mathrm{ACC}_R^{****\mathscr{J}}(F) = \frac{|R_{****}{}^{\mathscr{J}}(F) \cap F|}{|R^{****\mathscr{J}}(F) \cup F|}, \tag{28}$$

$$\mathrm{Rough}_R^{****\mathscr{J}}(F) = 1 - \mathrm{ACC}_R^{****\mathscr{J}}(F). \tag{29}$$

**Proposition 40.** *Consider $F, H \subseteq E$ and let $\mathscr{J}, \mathscr{K}$ be ideals and $R$ be a binary relation on $E$. Then,*

*(1)* $R^{****\mathscr{J}}(\phi) = \phi$

*(2)* $F \subseteq H \Rightarrow R^{****\mathscr{J}}(F) \subseteq R^{****\mathscr{J}}(H)$

*(3)* $R^{****\mathscr{J}}(F \cap H) \subseteq R^{****\mathscr{J}}(F) \cap R^{****\mathscr{J}}(H)$

*(4)* $R^{****\mathscr{J}}(F \cup H) = R^{****\mathscr{J}}(F) \cup R^{****\mathscr{J}}(H)$

*(5)* $R^{****\mathscr{J}}(F) = (R_{****}{}^{\mathscr{J}}(F^c))^c$

*(6) If* $F \in \mathscr{J}$, *then* $R^{****\mathscr{J}}(F) = \phi$

*(7) If* $\mathscr{J} \subseteq \mathscr{K}$, *then* $R^{****\mathscr{K}}(F) \subseteq R^{****\mathscr{J}}(F)$

*(8) If* $\mathscr{J} = P(E)$, *then* $R^{****\mathscr{J}}(F) = \phi$

*(9)* $R^{****\mathscr{J}\cap\mathscr{K}}(F) = R^{****\mathscr{J}}(F) \cup R^{****\mathscr{K}}(F)$

*(10)* $R^{****\mathscr{J}\vee\mathscr{K}}(F) = R^{****\mathscr{J}}(F) \cap R^{****\mathscr{K}}(F)$

*Proof.*

(1) $R^{****\mathscr{J}}(\phi) = \underset{e\in E}{\cup}\{\mu_u(e): \mu_u(e) \cap \phi \notin \mathscr{J}\} = \phi$

(2) Let $F \subseteq H$ and $e \in R^{****\mathscr{J}}(F)$. Then, $\exists y \in E$ such that $e \in \mu_u(y)$ and $\mu_u(y) \cap F \notin \mathscr{J}$. Thus, $\mu_u(y) \cap H \notin \mathscr{J}$. So, $e \in R^{****\mathscr{J}}(H)$. Consequently, $R^{****\mathscr{J}}(F) \subseteq R^{****\mathscr{J}}(H)$

(3) Immediately by part (2)

(4) $R^{****\mathscr{J}}(F \cup H) = \underset{e\in E}{\cup}\{\mu_u(e): \mu_u(e) \cap (F \cup H) \notin \mathscr{J}\}$. $= (\underset{e\in E}{\cup}\{\mu_u(e): \mu_u(e) \cap F \notin \mathscr{J}\}) \cup (\underset{e\in E}{\cup}\{\mu_u(e): \mu_u(e) \cap H \notin \mathscr{J}\})$. $= (\underset{e\in E}{\cup}\{\mu_u(e): \mu_u(e) \cap F \notin \mathscr{J}\})$ or $(\underset{e\in E}{\cup}\{\mu_u(e): \mu_u(e) \cap H \notin \mathscr{J}\})$. $= R^{****\mathscr{J}}(F) \cup R^{****\mathscr{J}}(H)$

(5) $(R_{****}{}^{\mathscr{J}}(F^c))^c = ((R^{****\mathscr{J}}(F))^c)^c. = R^{****\mathscr{J}}(F)$

(6) Immediately obtains by Definition 39

(7) Let $\mathscr{J} \subseteq \mathscr{K}$, $e \in R^{****\mathscr{K}}(F)$. Then, $\exists y \in E$ such that $e \in \mu_u(y)$ and $\mu_u(y) \cap F \notin \mathscr{K}$. Thus, $\mu_u(y) \cap F \notin \mathscr{J}$ as $\mathscr{J} \subseteq \mathscr{K}$. So, $e \in R^{****\mathscr{J}}(F)$. Hence, $R^{****\mathscr{K}}(F) \subseteq R^{****\mathscr{J}}(F)$

(8) Immediately obtains by Definition 39

(9) $R^{****\mathscr{J}\cap\mathscr{K}}(F) = \underset{e\in E}{\cup}\{\mu_u(e): \mu_u(e) \cap F \notin \mathscr{J} \cap \mathscr{K}\}. = (\underset{e\in E}{\cup}\{\mu_u(e): \mu_u(e) \cap F \notin \mathscr{J}\})$ or $(\underset{e\in E}{\cup}\{\mu_u(e): \mu_u(e) \cap F \notin$

$\mathscr{K}\}).=(\cup_{e\in E}\{\mu_u(e)\colon \mu_u(e)\cap F\in\mathscr{J}\})\cup(\cup_{e\in E}\{\mu_u(e)\colon \mu_u(e)\cap F\in\mathscr{K}\}).=R^{\star\star\star\star\mathscr{J}}(F)\cup R^{\star\star\star\star\mathscr{K}}(F)$

(10) $R^{\star\star\star\star\mathscr{J}\vee\mathscr{K}}(F)=\cup_{e\in E}\{\mu_u(e)\colon \mu_u(e)\cap F\in\mathscr{J}\vee\mathscr{K}\}.=\cup_{e\in E}\{\mu_u(e)\colon \mu_u(e)\cap F\in\mathscr{J}\cup\mathscr{K}\}.=(\cup_{e\in E}\{\mu_u(e)\colon \mu_u(e)\cap F\in\mathscr{J}\})$and$(\cup_{e\in E}\{\mu_u(e)\colon \mu_u(e)\cap F\in\mathscr{K}\}).=(\cup_{e\in E}\{\mu_u(e)\colon \mu_u(e)\cap F\in\mathscr{J}\})\cap(\cup_{e\in E}\{\mu_u(e)\colon \mu_u(e)\cap F\in\mathscr{K}\}).=R^{\star\star\star\star\mathscr{J}}(F)\cap R^{\star\star\star\star\mathscr{K}}(F)$

□

**Proposition 41.** *Consider $F,H\subseteq E$ and let $\mathscr{J},\mathscr{K}$ be ideals and R be a binary relation on E. Then,*

*(1) $R_{\star\star\star\star}{}^{\mathscr{J}}(E)=E$*

*(2) $F\subseteq H\Rightarrow R_{\star\star\star\star}{}^{\mathscr{J}}(F)\subseteq R_{\star\star\star\star}{}^{\mathscr{J}}(H)$*

*(3) $R_{\star\star\star\star}{}^{\mathscr{J}}(F)\cup R_{\star\star\star\star}{}^{\mathscr{J}}(H)\subseteq R_{\star\star\star\star}{}^{\mathscr{J}}(F\cup H)$*

*(4) $R_{\star\star\star\star}{}^{\mathscr{J}}(F\cap H)=R_{\star\star\star\star}{}^{\mathscr{J}}(F)\cap R_{\star\star\star\star}{}^{\mathscr{J}}(H)$*

*(5) $R_{\star\star\star\star}{}^{\mathscr{J}}(F)=(R^{\star\star\star\star\mathscr{J}}(F^c))^c$*

*(6) If $F^c\in\mathscr{J}$, then, $R_{\star\star\star\star}{}^{\mathscr{J}}(F)=E$*

*(7) If $\mathscr{J}\subseteq\mathscr{K}$, then, $R_{\star\star\star\star}{}^{\mathscr{J}}(F)\subseteq R_{\star\star\star\star}{}^{\mathscr{K}}(F)$*

*(8) If $\mathscr{J}=P(E)$, then, $R_{\star\star\star\star}{}^{\mathscr{J}}(F)=E$*

*(9) $R_{\star\star\star\star}{}^{\mathscr{J}\cap\mathscr{K}}(F)=R_{\star\star\star\star}{}^{\mathscr{J}}(F)\cap R_{\star\star\star\star}{}^{\mathscr{K}}(F)$*

*(10) $R_{\star\star\star\star}{}^{\mathscr{J}\vee\mathscr{K}}(F)=R_{\star\star\star\star}{}^{\mathscr{J}}(F)\cup R_{\star\star\star\star}{}^{\mathscr{K}}(F)$*

*Proof.*

(1) $R_{\star\star\star\star}{}^{\mathscr{J}}(E)=(R^{\star\star\star\star\mathscr{J}}(\phi))^c=\phi^c=E$ by Proposition 40 part (1).

(2) Let $F\subseteq H$. Thus, $H^c\subseteq F^c$, and hence, $R^{\star\star\star\star\mathscr{J}}(H^c)\subseteq R^{\star\star\star\star\mathscr{J}}(F^c)$ (by Proposition 40 part (2)). Then, $(R^{\star\star\star\star\mathscr{J}}(F^c))^c\subseteq(R^{\star\star\star\star\mathscr{J}}(H^c))^c$. So, $R_{\star\star\star\star}{}^{\mathscr{J}}(F)\subseteq R_{\star\star\star\star}{}^{\mathscr{J}}(H)$

(3) Immediately by part (2)

(4) $R_{\star\star\star\star}{}^{\mathscr{J}}(F\cap H)=(R^{\star\star\star\star\mathscr{J}}(F\cap H)^c)^c.=(R^{\star\star\star\star\mathscr{J}}(F^c\cup H^c))^c.=(R^{\star\star\star\star\mathscr{J}}(F^c)\cup R^{\star\star\star\star\mathscr{J}}(H^c))^c$(by no. (4) in Proposition 40). $=(R^{\star\star\star\star\mathscr{J}}(F^c))^c\cap(R^{\star\star\star\star\mathscr{J}}(H^c))^c.=R_{\star\star\star\star}{}^{\mathscr{J}}(F)\cap R_{\star\star\star\star}{}^{\mathscr{J}}(H)$

(5) Immediately obtains by Definition 39

(6) Let $F^c\in\mathscr{J}$, then $R_{\star\star\star\star}{}^{\mathscr{J}}(F)=(R^{\star\star\star\star\mathscr{J}}(F^c))^c=(\phi)^c=E$ by Proposition 40 part (6)

(7) Let $\mathscr{J}\subseteq\mathscr{K}$. Then, $R^{\star\star\star\star\mathscr{K}}(F^c)\subseteq R^{\star\star\star\star\mathscr{J}}(F^c)$ by Proposition 40 part (7). Thus, $(R^{\star\star\star\star\mathscr{J}}(F^c))^c\subseteq(R^{\star\star\star\star\mathscr{K}}(F^c))^c$. Hence, $R_{\star\star\star\star}{}^{\mathscr{J}}(F)\subseteq R_{\star\star\star\star}{}^{\mathscr{K}}(F)$

(8) Let $\mathscr{J}=P(E)$, then, $R_{\star\star\star\star}{}^{\mathscr{J}}(F)=(R^{\star\star\star\star\mathscr{J}}(F^c))^c=(\phi)^c=E$ by Proposition 40 part (8)

(9) $R_{\star\star\star\star}{}^{\mathscr{J}\cap\mathscr{K}}(F)=(R^{\star\star\star\star\mathscr{J}\cap\mathscr{K}}(F^c))^c.=(R^{\star\star\star\star\mathscr{J}}(F^c)\cup R^{\star\star\star\star\mathscr{K}}(F^c))^c$ (by Proposition 40 part (9)). $=(R^{\star\star\star\star\mathscr{J}}(F^c))^c\cap(R^{\star\star\star\star\mathscr{K}}(F^c))^c.=R_{\star\star\star\star}{}^{\mathscr{J}}(F)\cap R_{\star\star\star\star}{}^{\mathscr{K}}(F)$

(10) $R_{\star\star\star\star}{}^{\mathscr{J}\vee\mathscr{K}}(F)=(R^{\star\star\star\star\mathscr{J}\vee\mathscr{K}}(F^c))^c.=(R^{\star\star\star\star\mathscr{J}}(F^c)\cap R^{\star\star\star\star\mathscr{K}}(F^c))^c$ (by Proposition 40 part (10)). $=(R^{\star\star\star\star\mathscr{J}}(F^c))^c\cup(R^{\star\star\star\star\mathscr{K}}(F^c))^c.=R_{\star\star\star\star}{}^{\mathscr{J}}(F)\cup R_{\star\star\star\star}{}^{\mathscr{K}}(F)$

□

*Remark 42.*

(1) In Proposition 40 and Proposition 41, the converse of part 2 is generally false. To validate that consider Example 1 (i)

(a) If $F=\{e_1\},H=\{e_4\}$, then, $R^{\star\star\star\star\mathscr{J}}(F)=\phi,R^{\star\star\star\star\mathscr{J}}(H)=E$. Therefore, $R^{\star\star\star\star\mathscr{J}}(F)\subseteq R^{\star\star\star\star\mathscr{J}}(H)$, but $F\nsubseteq H$

(b) If $F=\{e_1,e_2,e_3\}$ and $H=\{e_2,e_3,e_4\}$, then $R_{\star\star\star\star}{}^{\mathscr{J}}(F)=\phi,R_{\star\star\star\star}{}^{\mathscr{J}}(H)=E$. Therefore, $R_{\star\star\star\star}{}^{\mathscr{J}}(F)\subseteq R_{\star\star\star\star}{}^{\mathscr{J}}(H)$, but $F\nsubseteq H$

(2) In Proposition 40 and Proposition 41, the converse of parts 6, 7, and 8 is generally false. To show that consider Example 1 (ii).

(a) For part 6 take

(i) $F=\{e_1,e_2\}$, then, $R^{\star\star\star\star\mathscr{K}}(F)=\phi$. Therefore, $R^{\star\star\star\star\mathscr{K}}(F)=\phi$, but $F\in\mathscr{K}$

(ii) $F=\{e_3,e_4\}$, then, $R_{\star\star\star\star}{}^{\mathscr{K}}(F)=E$. Therefore, $R_{\star\star\star\star}{}^{\mathscr{K}}(F)=E$, but $F^c\in\mathscr{K}$.

(b) For part 7 take

(i) $F=\{e_1,e_2\}$, then, $R^{\star\star\star\star\mathscr{J}}(F)=\{e_2\},R^{\star\star\star\star\mathscr{K}}(F)=\phi$. Therefore, $R^{\star\star\star\star\mathscr{K}}(F)\subseteq R^{\star\star\star\star\mathscr{J}}(F)$, but $\mathscr{J}\nsubseteq\mathscr{K}$

(ii) $F=\{e_3,e_4\}$, then, $R_{\star\star\star\star}{}^{\mathscr{J}}(F)=\{e_1,e_3,e_4\},R_{\star\star\star\star}{}^{\mathscr{K}}(F)=E$. Therefore, $R_{\star\star\star\star}{}^{\mathscr{J}}(F)\subseteq R_{\star\star\star\star}{}^{\mathscr{K}}(F)$, but $\mathscr{J}\nsubseteq\mathscr{K}$

(c) For part 8 take

(i) $F=\{e_1,e_2\}$, then, $R^{\star\star\star\star\mathscr{K}}(F)=\phi$, but $\mathscr{K}\neq P(E)$

(ii) $F=\{e_3,e_4\}$, then, $R_{\star\star\star\star}{}^{\mathscr{K}}(F)=E$, but $\mathscr{K}\neq P(E)$

(3) In Proposition 40 and Proposition 41, the inclusion relations of part 3 are generally proper. To demonstrate

that consider Example 1 (iii), and let $F=\{e_1,e_4\}$ and $H=\{e_2,e_3\}$. Then

(a) $R^{\star\star\star\star\mathcal{J}}(F)=R^{\star\star\star\star\mathcal{J}}(H)=E, R^{\star\star\star\star\mathcal{J}}(F\cap H)=\phi$. Therefore, $R^{\star\star\star\star\mathcal{J}}(F)\cap R^{\star\star\star\star\mathcal{J}}(H)=E\neq\phi=R^{\star\star\star\mathcal{J}}(F\cap H)$

(b) $R_{\star\star\star\star}{}^{\mathcal{J}}(F)=R_{\star\star\star\star}{}^{\mathcal{J}}(H)=\phi, R_{\star\star\star\star}{}^{\mathcal{J}}(F\cup H)=E$. Therefore, $R_{\star\star\star\star}{}^{\mathcal{J}}(F)\cup R_{\star\star\star\star}{}^{\mathcal{J}}(H)=\phi\neq E=R_{\star\star\star\star}{}^{\mathcal{J}}(F\cup H)$

*Remark 43.* Some properties of the lower and upper approximations given in the second method are not kept by this method as we demonstrate in the following.

(i) In Example 1 (i) take

(a) $F=\{e_1\}$, then, $R^{\star\star\star\star\mathcal{J}}(F)=\phi$. Hence, $F\nsubseteq R^{\star\star\star\star\mathcal{J}}(F)$

(b) $F=\{e_2,e_3,e_4\}$, then, $R_{\star\star\star\star}{}^{\mathcal{J}}(F)=E$. Hence, $R_{\star\star\star\star}{}^{\mathcal{J}}(F)\nsubseteq F$

(ii) In Example 1 (ii) take

(a) $F=E$, then, $R^{\star\star\star\star\mathcal{J}}(E)=\{e_2,e_3,e_4\}$. Hence, $R^{\star\star\star\star\mathcal{J}}(E)\neq E$

(b) $F=\phi$, then, $R_{\star\star\star\star}{}^{\mathcal{J}}(\phi)=\{e_1\}$. Hence, $R_{\star\star\star\star}{}^{\mathcal{J}}(\phi)\neq\phi$

**Proposition 44.** *Consider R and $\mathcal{J}$ are, respectively, binary relation and ideal on $E\neq\phi$ and let $\phi\neq F\subseteq E$. Then,*

*(1)* $0\leq ACC_R^{\star\star\star\star\mathcal{J}}(F)\leq 1$

*(2)* $ACC_R^{\star\star\star\star\mathcal{J}}(E)=1$

*Proof.* Similar to Proposition 14. □

**Theorem 45.** *Consider R as a binary relation on E and let $\mathcal{J},\mathcal{K}$ be ideals on E such that $\mathcal{J}\subseteq\mathcal{K}$. For each $F\subseteq E$, the next properties hold.*

*(1)* $BND_R^{\star\star\star\star\mathcal{K}}(F)\subseteq BND_R^{\star\star\star\star\mathcal{J}}(F)$

*(2)* $ACC_R^{\star\star\star\star\mathcal{J}}(F)\leq ACC_R^{\star\star\star\star\mathcal{K}}(F)$

*(3)* $Rough_R^{\star\star\star\star\mathcal{K}}(F)\leq Rough_R^{\star\star\star\star\mathcal{J}}(F)$

*Proof.* Similar to Theorem 15. □

*Remark 46.* Example 1 (ii) shows that the converse of parts 1, 2, and 3 in Theorem 45 is not necessary to be true in general. Take, $F=\{e_3,e_4\}$, then

(1) $\mathrm{BND}_R^{\star\star\star\star\mathcal{K}}(F)=\phi\subseteq\phi=\mathrm{BND}_R^{\star\star\star\star\mathcal{J}}(F)$, but $\mathcal{J}\nsubseteq\mathcal{K}$

(2) $\mathrm{ACC}_R^{\star\star\star\star\mathcal{J}}(F)=1\leq 1=\mathrm{ACC}_R^{\star\star\star\star\mathcal{K}}(F)$, but $\mathcal{J}\nsubseteq\mathcal{K}$

(3) $\mathrm{Rough}_R^{\star\star\star\star\mathcal{K}}(F)=0\leq 0=\mathrm{Rough}_R^{\star\star\star\star\mathcal{J}}(F)$, but $\mathcal{J}\nsubseteq\mathcal{K}$

**Theorem 47.** *Let $\phi\neq F\subseteq E$, $\mathcal{J}$ be an ideal on E and $R_1,R_2$ be two binary relations on E. If $R_1\subseteq R_2$, then*

*(1)* $R_1^{\star\star\star\star\mathcal{J}}(F)\subseteq R_2^{\star\star\star\star\mathcal{J}}(F)$

*(2)* $R_{2\star\star\star\star}{}^{\mathcal{J}}(F)\subseteq R_{1\star\star\star\star}{}^{\mathcal{J}}(F)$

*(3)* $BND_{R_1}^{\star\star\star\star\mathcal{J}}(F)\subseteq BND_{R_2}^{\star\star\star\star\mathcal{J}}(F)$

*(4)* $ACC_{R_2}^{\star\star\star\star\mathcal{J}}(F)\leq ACC_{R_1}^{\star\star\star\star\mathcal{J}}(F)$

*(5)* $Rough_{R_1}^{\star\star\star\star\mathcal{J}}(F)\leq Rough_{R_2}^{\star\star\star\star\mathcal{J}}(F)$

*Proof.*

(1) Let $e\in R_1^{\star\star\star\star\mathcal{J}}(F)$. Then, $\exists y\in E$ such that $e\in\mu_{1u}(y)\cap F\in\mathcal{J}$. Since, $\mu_{1u}(y)\subseteq\mu_{2u}(y)$ (by Theorem 4 [20]). It follows that $e\in\mu_{2u}(y)\cap F\in\mathcal{J}$. Thus, $e\in R_2^{\star\star\star\star\mathcal{J}}(F)$. Hence, $R_1^{\star\star\star\star\mathcal{J}}(F)\subseteq R_2^{\star\star\star\star\mathcal{J}}(F)$

(2) $e\in R_{2\star\star\star\star}{}^{\mathcal{J}}(F)=(R_2^{\star\star\star\star\mathcal{J}}(F^c))^c.\subseteq(R_1^{\star\star\star\star\mathcal{J}}(F^c))^c$ (by part (1)).$=R_{1\star\star\star\star}{}^{\mathcal{J}}(F)$

(3) Let $e\in\mathrm{BND}_{R_1}^{\star\star\star\star\mathcal{J}}(F)$. Then, $e\in R_1^{\star\star\star\star\mathcal{J}}(F)-R_{1\star\star\star\star}{}^{\mathcal{J}}(F)$. So, $e\in R_1^{\star\star\star\star\mathcal{J}}(F)$ and $e\in(R_{1\star\star\star\star}{}^{\mathcal{J}}(F))^c$. Thus, $e\in R_2^{\star\star\star\star\mathcal{J}}(F)$ and $e\in(R_{2\star\star\star\star}{}^{\mathcal{J}}(F))^c$ by parts 1 and 2. Hence, $e\in\mathrm{BND}_{R_2}^{\star\star\star\star\mathcal{J}}(F)$. Therefore, $\mathrm{BND}_{R_1}^{\star\star\star\star\mathcal{J}}(F)\subseteq\mathrm{BND}_{R_2}^{\star\star\star\star\mathcal{J}}(F)$

(4) $\mathrm{ACC}_{R_2}^{\star\star\star\star\mathcal{J}}(F)=|R_{2\star\star\star\star}{}^{\mathcal{J}}(F)\cap F/R_2^{\star\star\star\star\mathcal{J}}(F)\cup F|\leq|R_{1\star\star\star\star}{}^{\mathcal{J}}(F)\cap F/R_1^{\star\star\star\star\mathcal{J}}(F)\cup F|=\mathrm{ACC}_{R_1}^{\star\star\star\star\mathcal{J}}(F)$

(5) Straightforward by (4)

□

*Remark 48.* In Theorem 47, the inclusion and less than relations are generally proper. To show this matter consider Example 2 and take

(i) $F=\{e_1,e_2\}$, then

(1) $R_1^{\star\star\star\star\mathcal{J}}(F)=\{e_1,e_4\}\neq\{e_1,e_3,e_4\}=R_2^{\star\star\star\star\mathcal{J}}(F)$

(2) $\mathrm{ACC}_{R_1}^{\star\star\star\star\mathcal{J}}(F)=2/3\neq 1/2=\mathrm{ACC}_{R_2}^{\star\star\star\star\mathcal{J}}(F)$

(3) $\mathrm{Rough}_{R_1}^{\star\star\star\star\mathcal{J}}(F)=1/3\neq 1/2=\mathrm{Rough}_{R_2}^{\star\star\star\star\mathcal{J}}(F)$

(ii) $F=\{e_2,e_3\}$, then, $R_{1\star\star\star\star}{}^{\mathcal{J}}(F)=F\neq\{e_2\}=R_{2\star\star\star\star}{}^{\mathcal{J}}(F)$

## 4. Comparison the Proposed Methods and Their Advantages Compared to the Previous Ones

Herein, we first compare between the current purposed methods and demonstrate that the method given in Subsection 3.3 is the best in terms of develop the approximation operators and values of accuracy. Then, we clarify that the first approach produces accuracy measures of subsets higher than their counterparts displayed in [20].

**Theorem 49.** *Let $F \subseteq E$, $\mathcal{J}$ be an ideal on E and R be a binary relation on E. Then,*

(1) $R^{\star\mathcal{J}}(F) \subseteq R^{\star\star\mathcal{J}}(F)$

(2) $R_{\star\star}{}^{\mathcal{J}}(F) \subseteq R_{\star}^{\mathcal{J}}(F)$

(3) $BND_R^{\star\star\mathcal{J}}(F) \subseteq BND_R^{\star\mathcal{J}}(F)$

(4) $ACC_R^{\star\star\mathcal{J}}(F) = ACC_R^{\star\mathcal{J}}(F)$

(5) $Rough_R^{\star\star\mathcal{J}}(F) = Rough_R^{\star\mathcal{J}}(F)$

*Proof.* Immediately by using the Definitions 9 and 18. □

*Remark 50.* Example 2 shows that the inclusion and less than in Theorem 49 cannot be replaced by equality relation in general. Take $F = \{e_1, e_2, e_3\}$, then

(1) $R_1^{\star\mathcal{J}}(F) = \{e_1, e_4\} \neq E = R_1^{\star\star\mathcal{J}}(F)$

(2) $R_{\star\star 1}{}^{\mathcal{J}}(F) = \{e_1, e_2, e_3\} \neq E = R_{\star 1}{}^{\mathcal{J}}(F)$

(3) $\mathrm{BND}_{R_1}^{\star}{}^{\mathcal{J}}(F) = \phi \neq \{e_4\} = \mathrm{BND}_{R_1}^{\star\star\mathcal{J}}(F)$

**Theorem 51.** *Let $F \subseteq E$, $\mathcal{J}$ be an ideal on E and R be a reflexive relation on E. Then,*

(1) $R_{\star\star}{}^{\mathcal{J}}(F) \subseteq R_{\star}^{\mathcal{J}}(F) \subseteq R_{\star\star\star}{}^{\mathcal{J}}(F)$

(2) $R^{\star\star\star\mathcal{J}}(F) \subseteq R^{\star\mathcal{J}}(F) \subseteq R^{\star\star\mathcal{J}}(F)$

(3) $BND_R^{\star\star\star\mathcal{J}}(F) \subseteq BND_R^{\star\mathcal{J}}(F) \subseteq BND_R^{\star\star\mathcal{J}}(F)$

(4) $ACC_R^{\star\star\mathcal{J}}(F) \leq ACC_R^{\star\mathcal{J}}(F) \leq ACC_R^{\star\star\star\mathcal{J}}(F)$

(5) $Rough_R^{\star\star\star\mathcal{J}}(F) \leq Rough_R^{\star\mathcal{J}}(F) \leq Rough_R^{\star\star\mathcal{J}}(F)$

*Proof.*

(1) By Theorem 49, we have $R_{\star\star}{}^{\mathcal{J}}(F) \subseteq R_{\star}^{\mathcal{J}}(F)$. To prove, $R_{\star}^{\mathcal{J}}(F) \subseteq R_{\star\star\star}{}^{\mathcal{J}}(F)$. Let $e \in R_{\star}^{\mathcal{J}}(F)$, then, $\mu_u(e) \cap F^c \in \mathcal{J}$. Hence, $\mu_u(e) \subseteq R_{\star\star\star}{}^{\mathcal{J}}(F)$. Since, $R$ is a reflexive relation, thus, $e \in \mu_u(e) \subseteq R_{\star\star\star}{}^{\mathcal{J}}(F)$. Therefore, $e \in R_{\star\star\star}{}^{\mathcal{J}}(F)$

(2) To prove, $R^{\star\star\star\mathcal{J}}(F) \subseteq R^{\star\mathcal{J}}$. Let $e \in R^{\star\star\star\mathcal{J}}(F) = (R_{\star\star\star}(F^c))^c$, then, $x \notin R_{\star\star\star}(F^c)$. Hence, by Definition 29, we get $\mu_u(e) \cap F \notin \mathcal{J}$. It follows that $e \in R^{\star\mathcal{J}}(F)$. By Theorem 49, we have $R^{\star\mathcal{J}}(F) \subseteq R^{\star\star\mathcal{J}}(F)$

(3) Straightforward from (4) and (2)

In Theorem 51, the inclusion and less than relation in Theorem 51 cannot be replaced by equality relation in general as the next example demonstrates. □

*Example 52.* Let $E = \{e_1, e_2, e_3, e_4\}$, $\mathcal{J} = \{\phi, \{e_4\}\}$, and $R = \Delta \cup \{(e_1, e_4), (e_3, e_4), (e_4, e_2)\}$ be a binary relation defined on $E$. By calculations, we obtain $\mu_u(e_1) = \mu_u(e_3) = \{e_1, e_3, e_4\}$, $\mu_u(e_2) = \{e_2, e_4\}$, $\mu_u(e_4) = E$. For part 3, take $F = \{e_1, e_3, e_4\}$,then

(1) $R^{\star\star\star\mathcal{J}}(F) = \{e_1, e_3\} \subsetneq \{e_1, e_3, e_4\} = R^{\star\mathcal{J}}(F)$

(2) $R_{\star}^{\mathcal{J}}(F) = \{e_1, e_3\} \subsetneq \{e_1, e_3, e_4\} = R_{\star\star\star}{}^{\mathcal{J}}(F)$

(3) $\mathrm{BND}_R^{\star\star\star\mathcal{J}}(F) = \phi \subsetneq \{e_4\} = \mathrm{BND}_R^{\star\mathcal{J}}(F)$

(4) $\mathrm{ACC}_R^{\star\mathcal{J}}(F) = 2/3 \lneq 1 = \mathrm{ACC}_R^{\star\star\star\mathcal{J}}(F)$

(5) $\mathrm{Rough}_R^{\star\star\star\mathcal{J}}(F) = 0 \lneq 1/3 = \mathrm{Rough}_R^{\star\mathcal{J}}(F)$

**Theorem 53.** *Let $F \subseteq E$, $\mathcal{J}$ be an ideal on E and R be a reflexive relation on E. Then,*

(1) $R_{\star\star\star\star}{}^{\mathcal{J}}(F) \subseteq R_{\star}^{\mathcal{J}}(F) \subseteq R_{\star\star\star}{}^{\mathcal{J}}(F)$

(2) $R^{\star\star\star\mathcal{J}}(F) \subseteq R^{\star\mathcal{J}} \subseteq R^{\star\star\star\star\mathcal{J}}(F)$

(3) $BND_R^{\star\star\star\mathcal{J}}(F) \subseteq BND_R^{\star\mathcal{J}}(F) \subseteq BND_R^{\star\star\star\star\mathcal{J}}(F)$

(4) $ACC_R^{\star\star\star\star\mathcal{J}}(F) \leq ACC_R^{\star\mathcal{J}}(F) \leq ACC_R^{\star\star\star\mathcal{J}}(F)$

(5) $Rough_R^{\star\star\star\mathcal{J}}(F) \leq Rough_R^{\star\mathcal{J}}(F) \leq Rough_R^{\star\star\star\star\mathcal{J}}(F)$

*Proof.*

(1) By Theorem 51, we have $R_{\star}^{\mathcal{J}}(F) \subseteq R_{\star\star\star}{}^{\mathcal{J}}(F)$. To prove, $R_{\star\star\star\star}{}^{\mathcal{J}}(F) \subseteq R_{\star}^{\mathcal{J}}(F)$, let $e \in R_{\star\star\star\star}{}^{\mathcal{J}}(F) = R^{\star\star\star\star\mathcal{J}}(F^c)^c$. Then, $x \notin R^{\star\star\star\star\mathcal{J}}(F^c)$. Thus, by Definition 39, $\mu_u(e) \cap F^c \in \mathcal{J}$. It follows that $\mu_u(e) \subseteq R_{\star}^{\mathcal{J}}(F)$. Since, $R$ is a reflexive relation, then $e \in \mu_u(e) \subseteq R_{\star}^{\mathcal{J}}(F)$. Therefore, $e \in R_{\star}^{\mathcal{J}}(F)$

(2) By Theorem 51, we have $R^{\star\star\star\mathcal{J}}(F) \subseteq R^{\star\mathcal{J}}(F)$. To prove $R^{\star\mathcal{J}}(F) \subseteq R^{\star\star\star\star\mathcal{J}}(F)$, let $e \in R^{\star\mathcal{J}}(F)$, then $\mu_u(e) \cap F \notin \mathcal{J}$. It follows that $\mu_u(e) \subseteq R^{\star\star\star\star\mathcal{J}}(F)$. Since, $R$ is a reflexive relation, then $e \in \mu_u(e) \subseteq R^{\star\star\star\star\mathcal{J}}(F)$. Therefore, $e \in R^{\star\star\star\star\mathcal{J}}(F)$

(3) Straightforward from (4) and (2)

□

*Remark 54.* In Theorem 53, the inclusion and less than relations are generally proper. To show this matter, consider Example 52 and take $F = \{e_1, e_3, e_4\}$. Then,

(1) $R^{\star\mathcal{J}} = \{e_1, e_3, e_4\} \subsetneq E = R^{\star\star\star\star\mathcal{J}}(F)$

(2) $R_{\star\star\star\star}{}^{\mathcal{J}}(F) = \phi \subsetneq \{e_1, e_3\} = R_{\star}^{\mathcal{J}}(F)$

(3) $\mathrm{BND}_R^{\star\mathcal{J}}(F) = \{e_4\} \subsetneq E = \mathrm{BND}_R^{\star\star\star\star\mathcal{J}}(F)$

(4) $\mathrm{ACC}_R^{\star\star\star\star\mathcal{J}}(F) = 0 \lneq 2/3 = \mathrm{ACC}_R^{\star\mathcal{J}}(F)$

(5) $\mathrm{Rough}_R^{\star\mathcal{J}}(F) = 1/3 \lneq 1 = \mathrm{Rough}_R^{\star\star\star\star\mathcal{J}}(F)$

*Remark 55.* It follows from Theorem 51 and Theorem 53 that the best method to improve the approximations and increase the accuracy values is that given in the third type in Subsection 3.3, since the boundary regions in this case are decreased (or canceled) by increasing the lower approximations and decreasing the upper approximations with the comparison of the other types in the other sections. Moreover, the accuracy is more accurate than the other types.

The following result elucidates that the first type of our approaches is better than the approximation spaces given in [20] (see Definition 5).

**Theorem 56.** *Let $F \subseteq E$, $\mathcal{J}$ be an ideal on E and R be a binary relation on a nonempty set E. Then,*

(1) $R^{\star \mathcal{J}}(F) \subseteq R^{\star}(F)$

(2) $R_{\star}(F) \subseteq R_{\star}^{\mathcal{J}}(F)$

(3) $BND_R^{\star \mathcal{J}}(F) \subseteq B_R^{\star}(F)$

(4) $Acc_R^{\star}(F) \leq ACC_R^{\star \mathcal{J}}(F)$

(5) $Rough_R^{\star \mathcal{J}}(F) \leq Rough_R^{\star}(F)$

*Proof.*

(1) Let $e \in R^{\star \mathcal{J}}(F)$. Then, $\mu_u(e) \cap F \in \mathcal{J}$. Therefore, $\mu_u(e) \cap F \neq \phi$. Hence, $e \in R^{\star\star}(F)$, which means that $R^{\star \mathcal{J}}(F) \subseteq R^{\star\star}(F)$

(2) Let $e \in R_{\star\star}(F)$. Then, $\mu_u(e) \subseteq F$. Therefore, $\mu_u(e) \cap F^c \in \mathcal{J}$. Hence, $e \in R_{\star}^{\mathcal{J}}(F)$, which means that $R_{\star\star}(F) \subseteq R_{\star}^{\mathcal{J}}(F)$

(3) Immediately by parts 1 and 2

□

*Remark 57.* In Theorem 56, the inclusion and less than relations are generally proper. To show this matter consider Example 2 and take $F = \{e_1, e_2\}$. Then,

(1) $R_1^{\star \mathcal{J}}(F) = \{e_1, e_4\} \neq \{e_1, e_2, e_4\} = R_1^{\star}(F)$

(2) $R_{\star 1}^{\ \mathcal{J}}(F) = E \neq \{e_2\} = R_{1\star}(F)$

(3) $\mathrm{BND}_{R_1}^{\star \mathcal{J}}(F) = \phi \neq \{e_1, e_4\} = B_{R_1}^{\star}(F)$

(4) $\mathrm{Acc}_{R_1}^{\star \mathcal{J}}(F) = 2/3 \neq 1/3 = \mathrm{Acc}_{R_1}^{\star}(F)$

(5) $\mathrm{Rough}_{R_1}^{\star \mathcal{J}}(F) = 1/3 \neq 2/3 = \mathrm{Rough}_{R_1}^{\star}(F)$

According to Theorem 56, it can be seen that the present methods reduce the boundary region by increasing the lower approximations and decreasing the upper approximations with the comparison of Al-shami's methods [20]. This means that the current approximation spaces are proper generalizations of Al-shami's approximations [20].

One can easily prove the next result which show that Al-shami's approximations [20] are special cases of the current approximations.

**Proposition 58.** *If the ideal $\mathcal{J}$ is the empty set, then, the approximation spaces given herein and the approximation spaces given in [20] are identical.*

## 5. Medical Application to Dengue Fever Disease

One of the global diseases that disturb humanity is dengue fever. According to the data from World Health Organization, it is common in many regions around the world, mainly in South America and Asia, and it causes about 60 million symptomatic infections and 13600 status deaths. Medically, this illness is transmitted to humans via virus-carrying dengue mosquitoes. Its symptoms begin from 3 to 4 days of infection. The average period of recovery lies from 2 days to a week.

In this section, we analyze this illness via the structures of maximal union neighborhoods and ideals with respect to the first type of the proposed approach. We also compare the current approach with one of the recent approaches introduced in [20] and show that the present approach is more accurate.

To start this discussion, we deal with the data of eight patients of dengue fever $E = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ as displayed in Table 1. The set of symptoms of this illness (called "attributes") is the set $\Sigma = \{F, H, J, S\}$, where $F$, $H$, $J$, and $S$, respectively, denote fever, headache, characteristic skin rash, and muscle and joint pains. The attributes have two values: 1 refers patient has symptoms, and 0 refers the patient has no symptoms. The made decision also has the same two values with the meaning of possessing dengue fever disease or not.

We associate each patient with his/her symptoms by a map $f : E \longrightarrow 2^{\Sigma}$ such that $f(v_i)$ equals symptoms of patient $v_i$. According to Table 1, we find the following.

$$f(v_1) = \{F, J, S\}, \tag{30}$$

$$f(v_2) = f(v_3) = \{J\}, \tag{31}$$

$$f(v_4) = \{H\}, \tag{32}$$

$$f(v_5) = \{F, S\}, \tag{33}$$

$$f(v_6) = f(v_8) = \{F, H, J\}, \tag{34}$$

$$f(v_7) = \{F, H\}. \tag{35}$$

The binary relation between these patients is specified by the system's experts; assume this relation is given as follows.

$$v_i R v_j \Leftrightarrow \left| f(v_i) \cap f\left(v_j\right) \right| \geq 2. \tag{36}$$

That is, the two patients are related if they have two similar symptoms at least. Of course, this relation is changed according to the standpoint of system's experts. It can be noted the relation in 1 is a symmetry relation. This means

Table 1: Information system of dengue fever.

| $E$ | $F$ | $H$ | $J$ | $S$ | Dengue fever |
|---|---|---|---|---|---|
| $v_1$ | 1 | 0 | 1 | 1 | 1 |
| $v_2$ | 0 | 0 | 1 | 0 | 0 |
| $v_3$ | 0 | 0 | 1 | 0 | 1 |
| $v_4$ | 0 | 1 | 0 | 0 | 0 |
| $v_5$ | 1 | 0 | 0 | 1 | 0 |
| $v_6$ | 1 | 1 | 1 | 0 | 1 |
| $v_7$ | 1 | 1 | 0 | 0 | 0 |
| $v_8$ | 1 | 1 | 1 | 0 | 1 |

that the right, left, and union neighborhoods are equal, i.e., $N_r(v) = N_l(v) = N_u(v)$ for each $v \in E$. On the other hand, $R$ is not a reflexive because $(v_2, v_2) \in R$, also, it is not a transitive relation because $(v_1, v_7) \in R$ in spite of $(v_1, v_6), (v_6, v_7) \in R$. This means we cannot apply Pawlak's approach to model this information system and because Pawlak's approach applied only when the relation is equivalence.

To build the ideal approximation space of dengue fever information system, we first list the relation as follows $R = \{(v_1, v_1), (v_1, v_5), (v_1, v_6), (v_1, v_8), (v_5, v_1), (v_5, v_5), (v_6, v_1), (v_6, v_6), (v_6, v_7), (v_6, v_8), (v_7, v_6), (v_7, v_7), (v_7, v_8), (v_8, v_1), (v_8, v_6), (v_8, v_7), (v_8, v_8)\}$.

Then, we calculate the union neighborhood $N_u$ of each element in $E$

$$N_u(v_1) = \{v_1, v_5, v_6, v_8\}, \tag{37}$$

$$N_u(v_2) = N_u(v_3) = N_u(v_4) = \phi, \tag{38}$$

$$N_u(v_5) = \{v_1, v_5\}, \tag{39}$$

$$N_u(v_6) = N_u(v_8) = \{v_1, v_6, v_7, v_8\}, \tag{40}$$

$$N_u(v_7) = \{v_6, v_7, v_8\}. \tag{41}$$

After that, we calculate the maximal union neighborhood $\mu_u$ of each element in $E$.

$$\mu_u(v_1) = \mu_u(v_6) = \mu_u(v_8) = \{v_1, v_5, v_6, v_7, v_8\}, \tag{42}$$

$$\mu_u(v_2) = \mu_u(v_3) = \mu_u(v_4) = \phi, \tag{43}$$

$$\mu_u(v_5) = \{v_1, v_5, v_6, v_8\}, \tag{44}$$

$$\mu_u(v_7) = \{v_1, v_6, v_7, v_8\}, \tag{45}$$

Finally, we consider the ideal is $\mathcal{J} = \{\phi, \{v_5\}, \{v_7\}, \{v_5, v_7\}\}$.

Now, we consider two sets: a set of patients infected with dengue fever $A_1 = \{v_1, v_3, v_6, v_8\}$, and a set of patients without infection with dengue fever $A_2 = \{v_2, v_4, v_5, v_7\}$. In what follows, we calculate their lower and upper approximations, boundary regions, and the accuracy measures utilizing a method of maximal union neighborhoods given in [20] and the first method given herein.

(i) Al-shami's approach [20] (see Definition 5)

*Case 1.* The set of patients infected with dengue fever $A_1 = \{v_1, v_3, v_6, v_8\}$.

By computing, we find the lower and upper approximations are $R_\star(A_1) = \{v_2, v_3, v_4\}$ and $R^\star(A_1) = \{v_1, v_5, v_6, v_7, v_8\}$, respectively. This means that the boundary region is $B_R^\star(A_1) = \{v_1, v_5, v_6, v_7, v_8\}$. Hence, the accuracy and roughness measures are $\mathrm{Acc}_R^\star(A_1) = 1/6$ and $\mathrm{Rough}_R^\star(A_1) = 5/6$, respectively.

(ii) Our approach given in Definition 9

By computing, we find the lower and upper approximations are $R_\star^{\mathcal{J}}(A_1) = \{v_1, v_2, v_3, v_4, v_6, v_7, v_8\}$ and $R^{\star\mathcal{J}}(A_1) = \phi$, respectively. This means that the boundary region is $\mathrm{BND}_R^{\star\mathcal{J}}(A_1) = \phi$. Hence, the accuracy and roughness measures are $\mathrm{ACC}_R^{\star\mathcal{J}}(A_1) = 3/4$ and $\mathrm{Rough}_R^{\star\mathcal{J}}(A_1) = 1/4$, respectively.

(i) Al-shami's approach [20] (see Definition 5)

*Case 2.* The set without infection with dengue fever $A_2 = \{v_2, v_4, v_5, v_7\}$.

By computing, we find the lower and upper approximations are $R_\star(A_2) = \{v_2, v_3, v_4\}$ and $R^\star(A_2) = \{v_1, v_5, v_6, v_7, v_8\}$, respectively. This means that the boundary region is $B_R^\star(A_2) = \{v_1, v_5, v_6, v_7, v_8\}$. Hence, the accuracy and roughness measures are $\mathrm{Acc}_R^\star(A_2) = 2/7$ and $\mathrm{Rough}_R^\star(A_2) = 5/7$, respectively.

(ii) Our approach given in Definition 9

By computing, we find the lower and upper approximations are $R_\star^{\mathcal{J}}(A_2) = \{v_2, v_3, v_4\}$ and $R^{\star\mathcal{J}}(A_2) = \phi$, respectively. This means that the boundary region is $\mathrm{BND}_R^{\star\mathcal{J}}(A_2) = \phi$. Hence, the accuracy and roughness measures are $\mathrm{ACC}_R^{\star\mathcal{J}}(A_2) = \mathrm{Rough}_R^{\star\mathcal{J}}(A_2) = 1/2$.

It follows from Case 1 and Case 2 that the boundary region of the infected patients with dengue fever using approach given in [20] is $\{v_1, v_5, v_6, v_7, v_8\}$, so we cannot decide whether these patients are infected with dengue fever or not. This matter expands the area of vagueness/uncertainty and affects the precision of made decision. On the other hand, by using the first method introduced in this work, we see the boundary region is the empty set, which means we decrease the vagueness in the data and then enhance the value of accuracy.

According to the above discussion and computations, we see that there are different techniques applied to approximate the sets. Our approach "maximal union neighborhoods and ideal" is one of the preferable techniques since it minimizes (or cancels) the boundary region by enlarging the lower approximation and dwindling the upper approximation, which leads to increase the value of the accuracy compared to the other types such those given in [20]. Hence, our approach eliminates the ambiguity of the data in the practical problems; particularly, in the medical diagnosis which needs accurate decisions.

## 6. Conclusion

In recent decades, rough set theory forms an important instrument for many authors interested to decision-making issues. The studies on this theory are mostly subjected to approximation spaces induced from new types of neighborhood systems or their interaction with other structures like ideals.

In this work, we have focused on the idea of approximation spaces with the aim to decrease the regions of boundary and increase the values of accuracy. Therefore, we have provided novel approximation spaces using the concepts of "maximal union neighborhoods" and "ideals." We have scrutinized the basic properties of Pawlak rough set model for the given approximation spaces. Also, we have discussed these properties in the current approximation spaces with respect to different ideals. One of the interesting obtained characterizations of the current models is preserving the monotonic property, which enables us to evaluate the vagueness in the data and enhance the confidence for the outcomes. With the help of examples, we have compared between the given approximation spaces with respect to the accuracy values. To demonstrate the advantages of the followed approach, we have proved that the first kind of our approximation spaces produces higher accuracy than the approach studied in [20] as well as we applied it to the analysis of the information system of dengue fever illness.

Our future roadmaps are to study these ideal approximation paradigms in the frame of soft rough set and fuzzy rough set. Also, we probe these ideal approximation paradigms from a topological view and elucidate the relationships between the two approaches. We can improve these ideal approximation paradigms via topological structures by using some generalizations of open sets such as $\alpha$-open and $b$-open sets. Moreover, we will research these approximation paradigms using other kinds of maximal neighborhoods and apply to handle real-life problems.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no competing interests.

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## References

[1] Z. Pawlak, "Rough sets," *International journal of computer & information sciences*, vol. 11, no. 5, pp. 341–356, 1982.

[2] Z. Pawlak, "Rough concept analysis," *Bulletin of the Polish Academy of Sciences Mathematics*, vol. 33, pp. 9-10, 1985.

[3] S. Akama, T. Murai, and Y. Kudo, *Reasoning with Rough Sets*, vol. 142, Springer, 2018.

[4] R. Bello and R. Falcon, "Rough sets in machine learning: a review," *Thriving Rough Sets*, pp. 87–118, 2017.

[5] M. Kryszkiewicz, "Rough set approach to incomplete information systems," *Information Sciences*, vol. 112, no. 1-4, pp. 39–49, 1998.

[6] Z. Pawlak, "Rough sets and decision analysis," *INFOR: Information Systems and Operational Research*, vol. 38, no. 3, pp. 132–144, 2000.

[7] J. Dai and Q. Xu, "Approximations and uncertainty measures in incomplete information systems," *Information Sciences*, vol. 198, pp. 62–80, 2012.

[8] J. A. Pomykala, "About tolerance and similarity relations in information systems," in *Lecture Notes in Artificial Intelligence 2475 (RSCTC 2002)*, J. Alpigini, J. Peters, A. Skowron, and N. Zhong, Eds., pp. 175–182, Springer, Berlin, 2002.

[9] Y. Y. Yao, "On generalizing Pawlak approximation operators," *Rough Sets and Current Trends in Computing*, vol. 1424, pp. 298–307, 1998.

[10] X. Zhang, J. Dai, and Y. Yu, "On the union and intersection operations of rough sets based on various approximation spaces," *Information Sciences*, vol. 292, pp. 214–229, 2015.

[11] Y. Y. Yao, "Two views of the theory of rough sets in finite universes," *International Journal of Approximate Reasoning*, vol. 15, no. 4, pp. 291–317, 1996.

[12] Y. Y. Yao, "Relational interpretations of neighborhood operators and rough set approximation operators," *Information Sciences*, vol. 111, no. 1-4, pp. 239–259, 1998.

[13] A. A. Allam, M. Y. Bakeir, and E. A. Abo-Tabl, "New approach for basic rough set concepts," in *International Workshop on Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing. Lecture Notes in Artificial Intelligence*, vol. 3641, pp. 64–73, Springer, Regina, 2005.

[14] A. A. Allam, M. Y. Bakeir, and E. A. Abo-Tabl, "New approach for closure spaces by relations," *Acta Mathematica Academiae Paedagogicae Nyiregyháziensis*, vol. 22, pp. 285–304, 2006.

[15] R. Mareay, "Generalized rough sets based on neighborhood systems and topological spaces," *Journal of the Egyptian Mathematical Society*, vol. 24, no. 4, pp. 603–608, 2016.

[16] H. M. Abu-Donia, "Comparison between different kinds of approximations by using a family of binary relations," *Knowledge-Based Systems*, vol. 21, no. 8, pp. 911–919, 2008.

[17] H. M. Abu-Donia, "Multi knowledge based rough approximations and applications," *Knowledge-Based Systems*, vol. 26, pp. 20–29, 2012.

[18] D. Ciucci, "Orthopairs and granular computing," *Granular computing*, vol. 1, no. 3, pp. 159–170, 2016.

[19] J. Dai, S. Gao, and G. Zheng, "Generalized rough set models determined by multiple neighborhoods generated from a similarity relation," *Soft Computing*, vol. 22, no. 7, pp. 2081–2094, 2018.

[20] T. M. Al-shami, "Maximal rough neighborhoods with a medical application," *Journal of Ambient Intelligence and Humanized Computing*, 2022.

[21] T. M. Al-shami, "An improvement of rough sets' accuracy measure using containment neighborhoods with a medical application," *Information Sciences*, vol. 569, pp. 110–124, 2021.

[22] T. M. Al-shami and D. Ciucci, "Subset neighborhood rough sets," *Knowledge-Based Systems*, vol. 237, 2022.

[23] A. Kandil, M. M. Yakout, and A. Zakaria, "Generalized rough sets via ideals," *Annals of Fuzzy Mathematics and Informatics*, vol. 5, no. 3, pp. 525–532, 2013.

[24] A. Ç. Güler, E. D. Yildirim, and O. Özbakir, "Rough approximations based on different topologies via ideals," *Turkish Journal of Mathematics*, vol. 46, no. 4, pp. 1177–1192, 2022.

[25] M. Hosny, "Idealization of *j*-approximation spaces," *Univerzitet u Nišu*, vol. 34, pp. 287–301, 2020.

[26] A. Kandil, S. A. El-Sheikh, M. Hosny, and M. Raafat, "Bi-ideal approximation spaces and their applications," *Soft Computing*, vol. 24, no. 17, pp. 12989–13001, 2020.

[27] M. Hosny, "*Topologies generated by two ideals and the corresponding j-approximations spaces with applications*," *Journal of Mathematics*, vol. 2021, Article ID 6391266, 13 pages, 2021.

[28] T. M. Al-shami and M. Hosny, "Improvement of approximation spaces using maximal left neighborhoods and ideals," *IEEE Access*, vol. 10, pp. 79379–79393, 2022.

[29] M. Hosny and T. M. Al-shami, "Rough set models in a more general manner with applications," *AIMS mathematics*, vol. 7, no. 10, pp. 18971–19017, 2022.

[30] D. Jankovic and T. R. Hamlet, "New topologies from old via ideals," *The american mathematical monthly*, vol. 97, no. 4, pp. 295–310, 1990.