Hindawi
Journal of Mathematics
Volume 2022, Article ID 5537768, 16 pages
https://doi.org/10.1155/2022/5537768

Research Article

Thakur's Iterative Scheme for Approximating Common Fixed Points to a Pair of Relatively Nonexpansive Mappings

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Received 13 January 2021; Revised 6 February 2021; Accepted 9 February 2022; Published 22 April 2022

Academic Editor: A. Ghareeb

In this work, we propose the three-step Thakur iterative process associated with two mappings in the setting of Banach space. Using this Thakur iteration, we approximate a common fixed point for a pair of noncyclic, relatively nonexpansive mappings. And we support our main result with a numerical example. Also, we give a stronger version of our main result by using von Neumann sequences. Finally, we provide some corollaries on the convergence of common best proximity points in uniformly convex Banach space.

1. Introduction and Preliminaries

In recent years, the convergence of iterative processes for fixed points and common fixed points has become an attractive problem in the theory of nonlinear analysis. In the literature, there are many articles that provide different kinds of iterative processes and their convergence results. At this point, Picard and Mann's iterative processes are well-known iterative procedures that often help to find fixed points of a mapping of the form $F\colon X \longrightarrow X$, where X is Banach space. Here, we recall the following:

(i) Picard iteration: let $w_0 \in X$. Then iteration is defined by

$$w_{n+1} = Fw_n. \tag{1}$$

(ii) Mann iteration: let $w_0 \in X$. Then iteration is defined by

$$w_{n+1} = (1-\eta_n)w_n + \eta_n F w_n, \quad \eta_n \in [0,1]. \tag{2}$$

The Picard iteration is a basic tool to find fixed points, and it was an important starting point for the improvement of other iterative processes. At the same time, the Picard iteration fails to converge a fixed point for the class of nonexpansive mappings (see [1]).

Later on, Ishikawa [2] iteration, a two-step iteration process helps to approximate fixed points of nonexpansive mappings. For a starting point $w_0 \in X$, this iterative scheme is defined by

$$\begin{cases} w_{n+1} = (1-\eta_n)\, w_n + \eta_n F u_n, \\ u_n = (1-\gamma_n) w_n + \gamma_n F w_n, \end{cases} \tag{3}$$

where $\{\eta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$.

Agarwal et al. [3] introduced two-step iteration process in 2007 for an arbitrary $w_0 \in X$, it is defined as

$$\begin{cases} w_{n+1} = (1-\eta_n)\, F w_n + \eta_n F u_n, \\ u_n = (1-\gamma_n) w_n + \gamma_n F w_n, \end{cases} \tag{4}$$

where $\{\eta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$.

In 2000, Noor [4] introduced the following iteration scheme: starting with $w_0 \in X$, we define $\{w_n\}$ iteratively by

$$\begin{cases} w_{n+1} = (1-\eta_n)w_n + \eta_n F v_n, \\ v_n = (1-\delta_n)w_n + \delta_n F u_n, \\ u_n = (1-\gamma_n)w_n + \gamma_n F w_n, \end{cases} \tag{5}$$

where $\{\eta_n\}, \{\delta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$.

In the sequel, the following iterative process is defined by Thakur et al. in [5]: for an arbitrarily chosen element $w_0 \in X$, the sequence $\{w_n\}$ is generated by

$$\begin{cases} w_{n+1} = (1-\eta_n)F u_n + \eta_n F v_n, \\ v_n = (1-\delta_n)u_n + \delta_n F u_n, \\ u_n = (1-\gamma_n)w_n + \gamma_n F w_n, \end{cases} \tag{6}$$

where $\{\eta_n\}, \{\delta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$.

Using the Mann iteration process, Eldred et al. [6] proved the convergence result of the fixed point for noncyclic, relatively nonexpansive mappings in the uniformly convex Banach space. One can note that the relatively nonexpansive mappings need not be continuous. Also, Gabeleh et al. [7] proved strong and weak convergence of the Ishikawa iterative scheme for noncyclic relatively nonexpansive mappings in uniformly convex Banach spaces. Gabriela et al. [8] proved the convergence of Thakur iteration for Suzuki-type nonexpansive mappings. This class of mappings properly contains the class of nonexpansive mappings. Recently, Abdeljawad et al. [9] approximated fixed points and best proximity points for relatively nonexpansive mappings through the Agarwal iterative process.

On the other hand, while solving systems of equations of the form $Fx = x, Gx = x$ (F and G are selfmappings), one needs to study common fixed points and their convergence theorems. So, the researchers are showing an interest in finding common fixed points for different kinds of mappings through the well-known iterative processes. In the literature, Rashwan [10] proved the convergence of Mann iteration to a common fixed point for a pair of mappings defined on Banach space. Later on, Ćirić et al. [11] proved the convergence of Ishikawa iteration to common fixed points of two self-mappings in complete convex metric space.

In the sequel, the researchers showed interest in approximating common fixed points for a pair of nonexpansive mappings in the setting of Banach space. For example, Maingé [12] approximated common fixed points for nonexpansive mappings in Hilbert space. Also, Song et al. [13] provided a strong convergence result of common fixed points for a family of nonexpansive mappings in the setting of reflexive Banach spaces. Later on, Gu et al. [14] proved the convergence of Ishikawa iterations associated with two mappings to the common fixed point in uniformly convex Banach space. Also, Gopi et al. [15] found a common fixed point for a pair of relatively nonexpansive mappings and found a common best proximity point for a pair of non-self relatively nonexpansive mappings via the Ishikawa iterative process. And, Pragadeeswarar et al. [16] proved the convergence of a common best proximity point for a pair of mean nonexpansive mappings.

In the light of the above literature survey, one can think of how the three-step Thakur iterative process will approach the common fixed point for a pair of noncyclic, relatively nonexpansive mappings. So, we want to approximate such a common fixed point using Thakur's iterative process.

The purpose of this paper is to present convergence results of the Thakur iterative process for common fixed points of a pair of noncyclic, relatively nonexpansive mappings in uniformly convex Banach space. Using the von Neumann sequence, we prove the strong convergence result of the Thakur iterative process. To support our main result, we provide a numerical example and we compare the Thakur iteration is how faster than other known iterative processes. Finally, we use projective operators to find the best common proximity point.

The following notations are used subsequently: let M and N be nonempty subsets of a Banach space X.

$$\begin{aligned} d(w,N) &= \inf\{\|w-z\|: z \in N\}, \\ d(M,N) &= \inf\{\|w-z\|: w \in M, z \in N\}, \\ P_M(w) &= \{z \in M: \|w-z\| = d(w,M)\}, \\ M_0 &= \left\{w \in M: \left\|w-z'\right\| = d(M,N) \text{for some } z' \in N\right\}, \\ N_0 &= \left\{z \in N: \left\|w'-z\right\| = d(M,N) \text{for some } w' \in M\right\}. \end{aligned} \tag{7}$$

If M is convex, a closed subset of a reflexive and strictly convex space, then $P_M(w)$ contains one element and if M and N are convex, closed subsets of a reflexive space, with either M or N is bounded, then $M_0 \neq \varnothing$.

First, we reconstruct the Thakur iteration associated with two noncyclic mappings $F, G: M \cup N \longrightarrow M \cup N$, with M is convex, as follows: for an arbitrary chosen element $w_0 \in M$, the sequence $\{w_n\}$ is generated by

$$\begin{cases} w_{n+1} = (1-\eta_n)G u_n + \eta_n F v_n, \\ v_n = (1-\delta_n)u_n + \delta_n G u_n, \\ u_n = (1-\gamma_n)w_n + \gamma_n F w_n, \end{cases} \tag{8}$$

where $\{\eta_n\}, \{\delta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following condition: (R) $0 < \varepsilon \leq \gamma_n(1-\gamma_n)$.

Also, we provide the Abbas, Noor, Agarwal, and Ishikawa iterations associated with two noncyclic mappings $F, G: M \cup N \longrightarrow M \cup N$, with M is convex, as follows:

(i) Abbas: let $w_0 \in M$. Then the iteration is defined by

$$\begin{cases} w_{n+1} = (1-\eta_n)F u_n + \eta_n G v_n, \\ v_n = (1-\delta_n)F w_n + \delta_n G u_n, \\ u_n = (1-\gamma_n)w_n + \gamma_n F w_n, \quad \eta_n, \delta_n, \gamma_n \in [0,1]. \end{cases} \tag{9}$$

(ii) Noor: let $w_0 \in M$. Then the iteration is defined by

$$\begin{cases} w_{n+1} = (1-\eta_n)w_n + \eta_n F v_n, \\ v_n = (1-\delta_n)w_n + \delta_n G u_n, \\ u_n = (1-\gamma_n)w_n + \gamma_n F w_n, \quad \eta_n, \delta_n, \gamma_n \in [0,1]. \end{cases} \tag{10}$$

(iii) Agarwal: let $w_0 \in M$. Then the iteration is defined by

$$\begin{cases} w_{n+1} = (1-\eta_n)Fw_n + \eta_n Gu_n, \\ u_n = (1-\gamma_n)w_n + \gamma_n Fw_n, \quad \eta_n, \gamma_n \in [0,1]. \end{cases} \tag{11}$$

(iv) Ishikawa: let $w_0 \in M$. Then the iteration is defined by

$$\begin{cases} w_{n+1} = (1-\eta_n)w_n + \eta_n Gu_n, \\ u_n = (1-\gamma_n)w_n + \gamma_n Fw_n, \quad \eta_n, \gamma_n \in [0,1]. \end{cases} \tag{12}$$

The following definitions and theorems are very useful to our results:

Definition 1. Let M and N be nonempty subsets of a metric space (X, d). An element $w \in M$ is said to be the best proximity points of the nonself mapping $F\colon M \longrightarrow N$ if it satisfies the condition that

$$d(w, Fw) = d(M, N). \tag{13}$$

Definition 2 (see [6]). Let M and N be nonempty subsets of a Banach space X. A mapping $F\colon M \cup N \longrightarrow M \cup N$ is relatively nonexpansive, if

$$\|Fw - Fz\| \le \|w - z\|, \text{for all } w \in M, z \in N. \tag{14}$$

Definition 3 (see [17]). Let $(X, \|\cdot\|)$ be a Banach space. For every $\varepsilon \in [0,2]$, we define the modulus of convexity of $\|\cdot\|$ by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X, \|x - y\| \ge \varepsilon\right\}, \tag{15}$$

where B_X is the unit ball of Banach space X.

The norm is called uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\epsilon \in [0,2]$. The space $(X, \|\cdot\|)$ is then called uniformly convex space.

Definition 4 (see [18]). Let X be a Banach space. The pair of mappings $F, G\colon X \longrightarrow X$ is said to be mean nonexpansive if

$$\begin{aligned} \|Fu - Gv\| \le{} & a\|u - v\| + b\{\|u - Fu\| + \|v - Gv\|\} \\ & + c\{\|u - Gv\| + \|v - Fu\|\}, \end{aligned} \tag{16}$$

for all $u, v \in X, a, b, c \in [0,1]$ and $a + 2b + 2c \le 1$.

Remark 1. In Definition 4, for $a = 1, b = c = 0$, then the pair of mappings $F, G\colon X \longrightarrow X$ is said to be nonexpansive.

Lemma 1 (see [19]). *Suppose X be a uniformly convex Banach space. Suppose $0 < a < b < 1$, and $\{t_n\}$ is a sequence in $[a, b]$. Suppose $\{w_n\}, \{v_n\}$ are sequences in X such that $\|w_n\| \le 1, \|v_n\| \le 1$ for all n. We define $\{z_n\}$ in X by $z_n = (1-t_n)w_n + t_n v_n$. If $\lim_{n\longrightarrow\infty}\|z_n\| = 1$, then $\lim_{n\longrightarrow\infty}\|w_n - v_n\| = 0$.*

Lemma 2 (see [5]). *Suppose E is a uniformly convex Banach space and $0 < p \le t_n \le q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\limsup_{n\longrightarrow\infty}\|x_n\| \le r$, $\limsup_{n\longrightarrow\infty}\|y_n\| \le r$, and $\limsup_{n\longrightarrow\infty}\|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \ge 0$. Then $\lim_{n\longrightarrow\infty}\|x_n - y_n\| = 0$.*

Proposition 1 (see [20]). *If X is a uniformly convex space and $\eta \in (0,1)$ and $\varepsilon > 0$, then for any $d > 0$, if $w, z \in X$ are such that $\|w\| \le d, \|z\| \le d, \|w - z\| \ge \varepsilon$, then there exists $\delta = \delta(\varepsilon/d) > 0$, such that $\|\eta w + (1-\eta)z\| \le (1 - 2\delta(\varepsilon/d) \min(\eta, 1-\eta))d$.*

Here, we prove a result that shows Thakur iteration converges to the common fixed point of a pair of nonexpansive self-mappings. This result helps to prove our main theorem.

Theorem 1. *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space X and suppose $G, F\colon K \longrightarrow K$ is a pair of nonexpansive mappings with a nonempty common fixed point set. For an arbitrary chosen $w_0 \in K$, let the sequence $\{w_n\}$ be generated by (7) where $\eta_n, \delta_n, \gamma_n \in (\varepsilon, 1-\varepsilon)$, and $\varepsilon \in (0, (1/2))$. Then $\lim_{n\longrightarrow\infty}\|w_n - Fw_n\| = 0$ and $\lim_{n\longrightarrow\infty}\|Gu_n - Fv_n\| = 0$. Moreover, if $F(K)$ lies in a compact set, then $\{w_n\}$ converges to a common fixed point of G and F.*

Proof. By assumption, there exists $z \in K$ such that $Gz = Fz = z$. Now, from (7), we have

$$\begin{aligned} \|u_n - z\| &= \|(1-\gamma_n)w_n + \gamma_n Fw_n - z\| \\ &= \|(1-\gamma_n)(w_n - z) + \gamma_n(Fw_n - z)\| \\ &\le (1-\gamma_n)\|w_n - z\| + \gamma_n\|Fw_n - Gz\| \\ &\le (1-\gamma_n)\|w_n - z\| + \gamma_n\|w_n - z\| \\ &= \|w_n - z\|. \end{aligned} \tag{17}$$

In the same way, we can obtain

$$\begin{aligned} \|v_n - z\| &= \|(1-\delta_n)u_n + \delta_n Gu_n - z\| \\ &= \|(1-\delta_n)(u_n - z) + \delta_n(Gu_n - z)\| \\ &\le (1-\delta_n)\|u_n - z\| + \delta_n\|Gu_n - Fz\| \\ &\le (1-\delta_n)\|u_n - z\| + \delta_n\|u_n - z\| \\ &= \|u_n - z\|. \end{aligned} \tag{18}$$

Now, using inequality equation (17), one gets

$$\|v_n - z\| \le \|w_n - z\|. \tag{19}$$

Therefore, by equations (17) and (19), we obtain

$$\begin{aligned}\|w_{n+1}-z\| &= \|(1-\eta_n)Gu_n+\eta_n Fv_n - z\|\\ &= \|(1-\eta_n)(Gu_n-z)+\eta_n(Fv_n-z)\|\\ &\le (1-\eta_n)\|Gu_n-Fz\|+\eta_n\|Fv_n-Gz\|\\ &\le (1-\eta_n)\|u_n-z\|+\eta_n\|v_n-z\|\\ &\le (1-\eta_n)\|w_n-z\|+\eta_n\|w_n-z\|\\ &= \|w_n-z\|.\end{aligned} \tag{20}$$

This implies that the sequence $\{\|w_n-z\|\}$ is nonincreasing and bounded below by 0. Hence there exists $d\ge 0$, such that $\|w_n-z\|\longrightarrow d$. □

Case 1. Suppose $d=0$. First

$$\begin{aligned}\|w_n-Fw_n\| &\le \|w_n-z\|+\|z-Fw_n\|\\ &\le \|w_n-z\|+\|Gz-Fw_n\|\\ &\le \|w_n-z\|+\|z-w_n\|.\end{aligned} \tag{21}$$

As $n\longrightarrow\infty$, we get $\|w_n-Fw_n\|\longrightarrow 0$.From the Thakur iteration, we obtain

$$\begin{aligned}\|w_{n+1}-w_n\| &= \|(1-\eta_n)Gu_n+\eta_n Fv_n-w_n\|\\ &= \|Gu_n-w_n+\eta_n(Fv_n-Gu_n)\|\\ &\le \|Gu_n-w_n\|+\eta_n\|Fv_n-Gu_n\|.\end{aligned} \tag{22}$$

Now, by equation (17), we have.

$$\begin{aligned}\|Gu_n-w_n\| &\le \|Gu_n-z\|+\|z-w_n\|\\ &= \|Gu_n-Fz\|+\|z-w_n\|\\ &\le \|u_n-z\|+\|z-w_n\|\\ &\le 2\|w_n-z\|.\end{aligned} \tag{23}$$

As $n\longrightarrow\infty$, we obtain $\|Gu_n-w_n\|\longrightarrow 0$.Also, by equations (17) and (19), we obtain

$$\begin{aligned}\|Fv_n-Gu_n\| &\le \|v_n-u_n\|\\ &= \|v_n-z\|+\|z-u_n\|\\ &\le 2\|w_n-z\|.\end{aligned} \tag{24}$$

As $n\longrightarrow\infty$, we obtain $\|Fv_n-Gu_n\|\longrightarrow 0$. So, we get $\|w_{n+1}-w_n\|\longrightarrow 0$.

Since $F(K)$ is contained in a compact set, $\{Fw_n\}$ has a subsequence $\{Fw_{n_k}\}$ that converges to a point $z\in K$. Also $\{w_{n_k}\}$ and $\{w_{n_k+1}\}$ converge to z. This implies that $\{w_n\}$ converges to z. From the Thakur iteration, we can deduce that $\|u_n-w_n\|=\gamma_n\|Fw_n-w_n\|$, implies $\|u_n-w_n\|\longrightarrow 0$. Then $u_n\longrightarrow z$. And also $Gu_n\longrightarrow z, Fw_n\longrightarrow z$. Since F and G are continuous, it implies that $Gu_n\longrightarrow Gz, Fw_n\longrightarrow Fz$. Therefore $Fz=Gz=z$, which completes the proof.

Case 2. If $\|w_n-z\|\longrightarrow d>0$. Suppose there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and an $\epsilon>0$ such that $\|w_{n_k}-Fw_{n_k}\|\ge\varepsilon>0$ for all k.

Since the modulus of convexity of δ of X is a continuous and increasing function, we choose $\xi>0$ as small that $(1-c\delta(\varepsilon/(d+\xi)))(d+\xi)<d$, where $c>0$.

Now we choose k, such that $\|w_{n_k}-z\|\le d+\xi$. Now we have

$$\begin{aligned}\|z-w_{n_k+1}\| &= \|z-((1-\eta_{n_k})Gu_{n_k}+\eta_{n_k}Fv_{n_k})\|\\ &= \|(1-\eta_{n_k})z+\eta_{n_k}z-((1-\eta_{n_k})G((1-\gamma_{n_k})w_{n_k}+\gamma_{n_k}Fw_{n_k})+\eta_{n_k}Fv_{n_k})\|\\ &\le (1-\eta_{n_k})\|z-G((1-\gamma_{n_k})w_{n_k}+\gamma_{n_k}Fw_{n_k})\|+\eta_{n_k}\|z-Fv_{n_k}\|\\ &= (1-\eta_{n_k})\|Fz-G((1-\gamma_{n_k})w_{n_k}+\gamma_{n_k}Fw_{n_k})\|+\eta_{n_k}\|Gz-Fv_{n_k}\|\\ &\le (1-\eta_{n_k})\|z-((1-\gamma_{n_k})w_{n_k}+\gamma_{n_k}Fw_{n_k})\|+\eta_{n_k}\|z-v_{n_k}\|.\end{aligned} \tag{25}$$

Now, by Proposition 1, we can obtain

$$\begin{aligned}&\left\|z-\left(\left(1-\gamma_{n_k}\right)w_{n_k}+\gamma_{n_k}Fw_{n_k}\right)\right\|\\&=\left\|\left(1-\gamma_{n_k}\right)\left(z-w_{n_k}\right)+\gamma_{n_k}\left(z-Fw_{n_k}\right)\right\|\\&\leq\left(1-2\delta\left(\frac{\varepsilon}{d+\xi}\right)\min\left\{\gamma_{n_k},1-\gamma_{n_k}\right\}\right)(d+\xi).\end{aligned} \tag{26}$$

Also, using equation (26), we get

$$\begin{aligned}\left\|z-v_{n_k}\right\|&=\left\|z-\left(\left(1-\delta_{n_k}\right)u_{n_k}+\delta_{n_k}Gu_{n_k}\right)\right\|\\&=\left\|\left(1-\delta_{n_k}\right)\left(z-u_{n_k}\right)+\delta_{n_k}\left(z-Gu_{n_k}\right)\right\|\\&\leq\left(1-\delta_{n_k}\right)\left\|z-u_{n_k}\right\|+\delta_{n_k}\left\|z-Gu_{n_k}\right\|\\&=\left(1-\delta_{n_k}\right)\left\|z-u_{n_k}\right\|+\delta_{n_k}\left\|Fz-Gu_{n_k}\right\|\\&\leq\left(1-\delta_{n_k}\right)\left\|z-u_{n_k}\right\|+\delta_{n_k}\left\|z-u_{n_k}\right\|\\&=\left\|z-u_{n_k}\right\|\\&\leq\left(1-2\delta\left(\frac{\varepsilon}{d+\xi}\right)\min\left\{\gamma_{n_k},1-\gamma_{n_k}\right\}\right)(d+\xi).\end{aligned} \tag{27}$$

Therefore, the equation (25) becomes

$$\left\|z-w_{n_k+1}\right\|\leq\left(1-2\delta\left(\frac{\varepsilon}{d+\xi}\right)\min\left\{\gamma_{n_k},1-\gamma_{n_k}\right\}\right)(d+\xi). \tag{28}$$

Since there exists $l>0$ such that $2\min\left\{\gamma_{n_k},1-\gamma_{n_k}\right\}\geq l$,

$$\begin{aligned}&\left(1-2\delta\left(\frac{\varepsilon}{d+\xi}\right)\min\left\{\gamma_{n_k},1-\gamma_{n_k}\right\}\right)(d+\xi)\\&\leq\left(1-l\delta\left(\frac{\varepsilon}{d+\xi}\right)\right)(d+\xi).\end{aligned} \tag{29}$$

Suppose that we choose very small $\xi>0$, we have $(1-l\delta(\varepsilon/d+\xi))(d+\xi)<d$, which is a contradiction. This implies that $\lim_{n\longrightarrow\infty}\|w_n-Fw_n\|=0$.

Now we prove that $\|Gu_n-Fv_n\|\longrightarrow 0$. For, we define $a_n=(w_{n+1}-z)/\|w_n-z\|$, $b_n=(Gu_n-z)/\|w_n-z\|$ and $c_n=(Fv_n-z)/\|w_n-z\|$. Now, using equation (17), we get.

$$\begin{aligned}\left\|Gu_n-z\right\|&=\left\|Gu_n-Fz\right\|\\&\leq\left\|u_n-z\right\|\\&\leq\left\|w_n-z\right\|,\end{aligned} \tag{30}$$

also, by (19), we obtain

$$\begin{aligned}\left\|Fv_n-z\right\|&=\left\|Fv_n-Gz\right\|\\&\leq\left\|v_n-z\right\|\\&\leq\left\|w_n-z\right\|.\end{aligned} \tag{31}$$

Therefore $\|b_n\|=(\|Gu_n-z\|/\|w_n-z\|)\leq(\|w_n-z\|/\|w_n-z\|)=1$ and also $\|c_n\|=(\|Fv_n-z\|/\|w_n-z\|)\leq(\|w_n-z\|/\|w_n-z\|)=1$. From Thakur's iteration, we obtain $w_{n+1}-z=(1-\eta_n)(Gu_n-z)+\eta_n(Fv_n-z)$. Dividing by $\|w_n-z\|$, we get

$$\frac{w_{n+1}-z}{\left\|w_n-z\right\|}=(1-\eta_n)\frac{(Gu_n-z)}{\left\|w_n-z\right\|}+\eta_n\frac{(Fv_n-z)}{\left\|w_n-z\right\|}. \tag{32}$$

Then $a_n=(1-\eta_n)b_n+\eta_nc_n$. Now we prove that $\|a_n\|\longrightarrow 1$. Now,

$$\lim_{n\longrightarrow\infty}\left\|a_n\right\|=\lim_{n\longrightarrow\infty}\frac{\left\|w_{n+1}-z\right\|}{\left\|w_n-z\right\|}=\frac{d}{d}=1. \tag{33}$$

By Lemma 1, $\|b_n-c_n\|\longrightarrow 0$. Therefore $\|Gu_n-Fv_n\|\longrightarrow 0$.

Since $F(K)$ is contained in a compact set, $\{Fw_n\}$ has a subsequence $\left\{Fw_{n_k}\right\}$ that converges to a point $a\in K$. Also $\left\{w_{n_k}\right\}$ converges to a. Now, $\|Fw_{n_k}-Ga\|\leq\|w_{n_k}-a\|$. As $k\longrightarrow\infty$, we obtain $a=Ga$. Since G is continuous, $Gw_{n_k}\longrightarrow Ga$. So, we have $\|Gw_{n_k}-Fa\|\leq\|w_{n_k}-a\|$. As $k\longrightarrow\infty$, we get $Fa=Ga$. Therefore, $Fa=Ga=a$. Since a is a common fixed point, implies $\lim_{n\longrightarrow\infty}\|w_n-a\|$ exists. Therefore, $\lim_{n\longrightarrow\infty}\|w_n-a\|=\lim_{k\longrightarrow\infty}\|w_{n_k}-a\|=0$. So, $w_n\longrightarrow a$, which completes the proof.

Let M be a convex closed subset of a Hilbert Space X. Then for $w\in X$, we know that $P_M(w)$ is the nearest point to w and unique point of M. And also P_M is nonexpansive and distinguished by Kolmogorov's criterion as $\langle w-P_Mw,P_Mw-a\rangle\geq 0$, for all $w\in X$ and $a\in M$.

Let M and N be two convex closed subsets of X. We define.

$$P(w)=P_M(P_N(w))\text{for each }w\in X. \tag{34}$$

Then $\{P^n(w)\}\subset M$ and $\{P_N(P^n(w))\}\subset N$. When M and N are closed, the convergence of these sequences in norm was proved by von Neumann [21]. The sequences $\{P^n(w)\}$ and $\{P_N(P^n(w))\}$ are called von Neumann sequences or alternating projection algorithms for two sets.

Definition 5 (see [22]). Let M and N be nonempty closed convex subsets of a Hilbert space X. We say that (M,N) is boundedly regular if for each bounded subset S of X and for each $\varepsilon>0$ there exists $\delta>0$ such that.

$$\max\{d(w,M),d(w,N-v)\}\leq\delta\quad d(w,N)\leq\varepsilon,\ \forall w\in X, \tag{35}$$

where $v=P_{N-M}(0)$ is the displacement vector from M to N. (v is the unique vector satisfying $\|v\|=d(M,N)$).

Theorem 2 (see [22]). *If (M, N) is boundedly regular, then the von Neumann sequences converge in the norm.*

Theorem 3 (see [22]). *If M or N is boundedly compact, then (M, N) is boundedly regular.*

Lemma 3 (see [19]). *Let M be a nonempty closed and convex subset and N be a nonempty closed subset of a uniformly convex Banach space. Let $\{w_n\}$ and $\{a_n\}$ be sequences in M and $\{z_n\}$ be a sequence in N satisfying the following:*

(1) $\|w_n - z_n\| \longrightarrow d(M, N)$,

(2) $\|a_n - z_n\| \longrightarrow d(M, N)$.

Then $\|w_n - a_n\|$ converges to zero.

Corollary 1 (see [19]). *Let M be a nonempty closed convex subset and N be a nonempty closed subset of uniformly convex Banach space. Let $\{w_n\}$ be a sequence in M and $z_0 \in N$ such that $\|w_n - z_0\| \longrightarrow d(M, N)$. Then $\{w_n\}$ converges to $P_M(z_0)$.*

Proposition 2 (see [23]). *Let M and N be two closed and convex subsets of a Hilbert space X. Then $P_N(M) \subseteq N, P_M(N) \subseteq M$, and $\|P_N w - P_M z\| \leq \|w - z\|$ for $w \in M$ and $z \in N$.*

Lemma 4. *Let M and N be two closed and convex subsets of a Hilbert space X. For each $w \in X$.*

$$\left\|P^{n+1}(w) - a\right\| \leq \left\|P^{n}(w) - a\right\|, \text{for each } a \in M_0 \cup N_0. \quad (36)$$

Lemma 5 (see [24]). *Let (M, N) be a nonempty, bounded, closed, and convex pair in a reflexive and strictly convex Banach space X. We define $\mathrm{P}\colon M_0 \cup N_0 \longrightarrow M_0 \cup N_0$ as*

$$\mathrm{P}(x) = \begin{cases} \mathrm{P}_{M_0}(x), & \text{if } x \in N_0, \\ \mathrm{P}_{N_0}(x), & \text{if } x \in M_0, \end{cases} \quad (37)$$

Then the following statements hold:

(1)$\|x - \mathrm{P}x\| = d(M, N)$ for any $x \in M_0 \cup N_0$ and $\mathrm{P}(M_0) \subseteq N_0, \mathrm{P}(N_0) \subseteq M_0$.

(2)P is an isometry, that is, $\|\mathrm{P}x - \mathrm{P}y\| = \|x - y\|$ for all $(x, y) \in M_0 \times N_0$.

(3)P is affine.

Definition 6 (see [25]). If $M_0 \neq \emptyset$ then the pair (M, N) is said to have P-property if for any $u_1, u_2 \in M_0$ and $v_1, v_2 \in N_0$

$$\begin{cases} d(u_1, v_1) = d(M, N) \\ d(u_2, v_2) = d(M, N) \end{cases} \quad d(u_1, u_2) = d(v_1, v_2). \quad (38)$$

Lemma 6 (see [26]). *Every, nonempty, bounded, closed and convex pair in a uniformly convex Banach space X has the P-property.*

Lemma 7 (see [27]). *Let (M, N) be a nonempty, closed, and convex pair in a uniformly convex Banach space X. Then for the projection mapping $\mathrm{P}\colon M_0 \cup N_0 \longrightarrow M_0 \cup N_0$ defined in equation (17) we have both $\mathrm{P}|_{M_0}$ and $\mathrm{P}|_{N_0}$ are continuous.*

2. Main Results

Theorem 4. *Let M and N be nonempty bounded closed convex subsets of a uniformly convex Banach space and suppose $F, G\colon M \cup N \longrightarrow M \cup N$ satisfy*

(1) $G(M) \subseteq M, G(N) \subseteq N, F(M) \subseteq M$ and $F(N) \subseteq N$;

(2) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in M, v \in N$; and

(3) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in N, v \in M$,

with a nonempty common fixed-point set. For an arbitrary chosen $w_0 \in M$, let the sequence $\{w_n\}$ be generated by (7) where $\eta_n, \delta_n, \gamma_n \in (\varepsilon, 1 - \varepsilon)$, where $\varepsilon \in (0, (1/2))$ and $n = 0, 1, 2, \ldots$ Suppose $d(w_n, M_0) \longrightarrow 0$, then $\lim_{n \longrightarrow \infty} \|Gu_n - Fv_n\| = 0, \lim_{n \longrightarrow \infty} \|u_n - Gu_n\| = 0$ and $\lim_{n \longrightarrow \infty} \|w_n - Fw_n\| = 0$. Moreover, if $F(M)$ lies in a compact set, then $\{w_n\}, \{v_n\}$ and $\{u_n\}$ converge to a common fixed point of G and F.

Proof. If $d(M, N) = 0$, then $M_0 = N_0 = M \cap N$ and by Theorem 1 we can prove the result from the truth that $F, G\colon M \cap N \longrightarrow M \cap N$ is nonexpansive. Therefore, let us take that $d(M, N) > 0$. For a common fixed point $z \in N$ of F and G, we get

$$\begin{aligned} \left\|u_n - z\right\| &= \left\|(1 - \gamma_n) w_n + \gamma_n F w_n - z\right\| \\ &= \left\|(1 - \gamma_n)(w_n - z) + \gamma_n (F w_n - z)\right\| \\ &\leq (1 - \gamma_n)\left\|w_n - z\right\| + \gamma_n \left\|F w_n - z\right\| \\ &= (1 - \gamma_n)\left\|w_n - z\right\| + \gamma_n \left\|F w_n - G z\right\| \\ &\leq (1 - \gamma_n)\left\|w_n - z\right\| + \gamma_n \left\|w_n - z\right\| \\ &= \left\|w_n - z\right\|. \end{aligned} \quad (39)$$

In the same way, we can obtain.

$$\begin{aligned} \left\|v_n - z\right\| &= \left\|(1 - \delta_n) u_n + \delta_n G u_n - z\right\| \\ &= \left\|(1 - \delta_n)(u_n - z) + \delta_n (G u_n - z)\right\| \\ &\leq (1 - \delta_n)\left\|u_n - z\right\| + \delta_n \left\|G u_n - z\right\| \\ &= (1 - \delta_n)\left\|u_n - z\right\| + \delta_n \left\|G u_n - F z\right\| \\ &\leq (1 - \delta_n)\left\|u_n - z\right\| + \delta_n \left\|u_n - z\right\| \\ &= \left\|u_n - z\right\|. \end{aligned} \quad (40)$$

Now, using inequality equation (39), one gets

$$\|v_n - z\| \le \|w_n - z\|. \tag{41}$$

Therefore, by equations (39) and (41), we obtain

$$\begin{aligned}\|w_{n+1} - z\| &= \|(1-\eta_n)Gu_n + \eta_n F v_n - z\| \\ &= \|(1-\eta_n)(Gu_n - z) + \eta_n (Fv_n - z)\| \\ &\le (1-\eta_n)\|Gu_n - Fz\| + \eta_n\|Fv_n - Gz\| \\ &\le (1-\eta_n)\|u_n - z\| + \eta_n\|v_n - z\| \\ &\le (1-\eta_n)\|w_n - z\| + \eta_n\|w_n - z\| \\ &= \|w_n - z\|.\end{aligned} \tag{42}$$

This implies that the sequence $\{\|w_n - z\|\}$ is nonincreasing. Then we can find $d > 0$ such that $\lim_{n\longrightarrow\infty}\|w_n - z\| = d$.

Suppose there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and an $\varepsilon > 0$ such that $\|w_{n_k} - Fw_{n_k}\| \ge \varepsilon > 0$ for all k. Since the modulus of convexity of δ of X is continuous and increasing function, we choose $\xi > 0$ as small that $(1 - c\delta(\varepsilon/(d+\xi)))(d+\xi) < d$, where $c > 0$. Now we choose k, such that $\|w_{n_k} - z\| \le d + \xi$. Now we have

$$\begin{aligned}\|z - w_{n_k+1}\| &= \|z - ((1-\eta_{n_k})Gu_{n_k} + \eta_{n_k}Fv_{n_k})\| \\ &= \|(1-\eta_{n_k})z + \eta_{n_k}z - ((1-\eta_{n_k})G((1-\gamma_{n_k})w_{n_k} + \gamma_{n_k}Fw_{n_k}) + \eta_{n_k}Fv_{n_k})\| \\ &\le (1-\eta_{n_k})\|z - G((1-\gamma_{n_k})w_{n_k} + \gamma_{n_k}Fw_{n_k})\| + \eta_{n_k}\|z - Fv_{n_k}\| \\ &= (1-\eta_{n_k})\|Fz - G((1-\gamma_{n_k})w_{n_k} + \gamma_{n_k}Fw_{n_k})\| + \eta_{n_k}\|Gz - Fv_{n_k}\| \\ &\le (1-\eta_{n_k})\|z - ((1-\gamma_{n_k})w_{n_k} + \gamma_{n_k}Fw_{n_k})\| + \eta_{n_k}\|z - v_{n_k}\|.\end{aligned} \tag{43}$$

Now, by Proposition 1, we can obtain

$$\begin{aligned}\|z - ((1-\gamma_{n_k})w_{n_k} + \gamma_{n_k}Fw_{n_k})\| &= \|(1-\gamma_{n_k})(z - w_{n_k}) + \gamma_{n_k}(z - Fw_{n_k})\| \\ &\le \left(1 - 2\delta\left(\frac{\varepsilon}{d+\xi}\right)\min\{\gamma_{n_k}, 1-\gamma_{n_k}\}\right)(d+\xi).\end{aligned} \tag{44}$$

Also, using equation (44), we get

$$\begin{aligned}\|z - v_{n_k}\| &= \|z - ((1-\delta_{n_k})u_{n_k} + \delta_{n_k}Gu_{n_k})\| \\ &= \|(1-\delta_{n_k})(z - u_{n_k}) + \delta_{n_k}(z - Gu_{n_k})\| \\ &\le (1-\delta_{n_k})\|z - u_{n_k}\| + \delta_{n_k}\|z - Gu_{n_k}\| \\ &= (1-\delta_{n_k})\|z - u_{n_k}\| + \delta_{n_k}\|Fz - Gu_{n_k}\| \\ &\le (1-\delta_{n_k})\|z - u_{n_k}\| + \delta_{n_k}\|z - u_{n_k}\| \\ &= \|z - u_{n_k}\| \\ &\le \left(1 - 2\delta\left(\frac{\varepsilon}{d+\xi}\right)\min\{\gamma_{n_k}, 1-\gamma_{n_k}\}\right)(d+\xi).\end{aligned} \tag{45}$$

Therefore, the equation (43) becomes

$$\|z - w_{n_k+1}\| \le \left(1 - 2\delta\left(\frac{\varepsilon}{d+\xi}\right)\min\{\gamma_{n_k}, 1-\gamma_{n_k}\}\right)(d+\xi). \tag{46}$$

Since there exists $l > 0$ such that $2\min\{\gamma_{n_k}, 1-\gamma_{n_k}\} \ge l$,

$$\begin{aligned}&\left(1 - 2\delta\left(\frac{\varepsilon}{d+\xi}\right)\min\{\gamma_{n_k}, 1-\gamma_{n_k}\}\right)(d+\xi) \\ &\quad\le \left(1 - l\delta\left(\frac{\varepsilon}{d+\xi}\right)\right)(d+\xi).\end{aligned} \tag{47}$$

Suppose that we choose very small $\xi > 0$, we have $(1 - l\delta(\varepsilon/(d+\xi)))(d+\xi) < d$, which is a contradiction. This implies that $\lim_{n\longrightarrow\infty}\|w_n - Fw_n\| = 0$. From the Thakur iteration, we have $\|u_n - w_n\| = \gamma_n\|w_n - Fw_n\|$, which implies $\|u_n - w_n\| \longrightarrow 0$.

Now we prove that $\|Gu_n - Fv_n\| \longrightarrow 0$. For, we define $a_n = (w_{n+1} - z)/\|w_n - z\|$, $b_n = (Gu_n - z)/\|w_n - z\|$ and $c_n = (Fv_n - z)/\|w_n - z\|$. Now, using equation (17), we get

$$\begin{aligned}\|Gu_n - z\| &= \|Gu_n - Fz\| \\ &\le \|u_n - z\| \\ &\le \|w_n - z\|,\end{aligned} \tag{48}$$

also, by equation (19), we obtain

$$\begin{aligned}\|Fv_n - z\| &= \|Fv_n - Gz\| \\ &\le \|v_n - z\| \\ &\le \|w_n - z\|.\end{aligned} \tag{49}$$

Therefore $\|b_n\| = (\|Gu_n - z\|/\|w_n - z\|) \le (\|w_n - z\|/\|w_n - z\|) = 1$ and also $\|c_n\| = (\|Fv_n - z\|/\|w_n - z\|) \le (\|w_n - z\|/\|w_n - z\|) = 1$. From Thakur's iteration, we obtain $w_{n+1} - z = (1-\eta_n)(Gu_n - z) + \eta_n(Fv_n - z)$. Dividing by $\|w_n - z\|$, we get

$$\frac{w_{n+1} - z}{\|w_n - z\|} = (1-\eta_n)\frac{(Gu_n - z)}{\|w_n - z\|} + \eta_n\frac{(Fv_n - z)}{\|w_n - z\|}. \tag{50}$$

Then $a_n = (1-\eta_n)b_n + \eta_n c_n$. Now we prove that $\|a_n\| \longrightarrow 1$. Now,

$$\lim_{n\longrightarrow\infty}\|a_n\| = \lim_{n\longrightarrow\infty}\frac{\|w_{n+1} - z\|}{\|w_n - z\|} = \frac{d}{d} = 1. \tag{51}$$

By Lemma 1, $\|b_n - c_n\| \longrightarrow 0$. Therefore $\|Gu_n - Fv_n\| \longrightarrow 0$.

Since $\lim_{n\longrightarrow\infty}\|w_n - z\| = d$, and from equations (39) and (41), we can obtain

$$\limsup_{n\longrightarrow\infty}\|u_n - z\| \le d, \tag{52}$$

and

$$\limsup_{n\longrightarrow\infty}\|v_n - z\| \le d. \tag{53}$$

Also, we have $\|Gu_n - z\| = \|Gu_n - Fz\| \le \|u_n - z\|$. Taking lim sup on both sides, we obtain

$$\limsup_{n\longrightarrow\infty}\|Gu_n - z\| \le d. \tag{54}$$

Now

$$\begin{aligned}\|w_{n+1} - z\| &= \|(1-\eta_n)Gu_n + \eta_n Fv_n - z\| \\ &= \|Gu_n - z\| + \eta_n\|Fv_n - Gu_n\|,\end{aligned} \tag{55}$$

as $n \longrightarrow \infty$, by $\|Fv_n - Gu_n\| \longrightarrow 0$, we get

$$d \le \lim\inf_{n\longrightarrow\infty}\|Gu_n - z\|. \tag{56}$$

So, from equations (54) and (56), we obtain $\lim_{n\longrightarrow\infty}\|Gu_n - z\| = d$. On the other hand, we have

$$\begin{aligned}\|Gu_n - z\| &\le \|Gu_n - Fv_n\| + \|Fv_n - z\| \\ &= \|Gu_n - Fv_n\| + \|Fv_n - Gz\| \\ &= \|Gu_n - Fv_n\| + \|v_n - z\|,\end{aligned} \tag{57}$$

and this yields that

$$d \le \lim\inf_{n\longrightarrow\infty}\|v_n - z\|. \tag{58}$$

So, by equations (53) and (58), we deduce

$$\lim_{n\longrightarrow\infty}\|v_n - z\| = d. \tag{59}$$

Using Lemma 2, we get $\|u_n - Gu_n\| \longrightarrow 0$, as $n \longrightarrow \infty$. From the Thakur iteration, we have $\|v_n - u_n\| = \delta_n\|u_n - Gu_n\|$, which implies $\|v_n - u_n\| \longrightarrow 0$.

Since $F(M)$ is contained in a compact set, $\{Fw_n\}$ has a subsequence $\{Fw_{n_k}\}$ that converges to a point $a \in M$. Also $\{w_{n_k}\}$ and $\{u_{n_k}\}$ converge to a. Since $d(w_n, M_0) \longrightarrow 0$, there exists $\{a_n\} \subseteq M_0$ such that $\|w_n - a_n\| \longrightarrow 0$. Therefore, $a_{n_k} \longrightarrow a$, which gives that $a \in M_0$. Let $D = d(M, N)$ and choose $b \in N_0$ such that $\|a - b\| = D$. So, we have $\|w_{n_k} - b\| \longrightarrow \|a - b\| = D$ and $\|w_{n_k} - b\| \ge \|Fw_{n_k} - Gb\| \longrightarrow \|a - Gb\|$. So $\|a - Gb\| = D$. By strict convexity of the norm, $Gb = b$. From $\|Fa - Gb\| \le \|a - b\|$, follows $\|Fa - Gb\| = D$. Then $Fa = a$.

On the other hand, $\|u_{n_k} - b\| \longrightarrow \|a - b\| = D$, and $\|u_{n_k} - b\| \ge \|Gu_{n_k} - Fb\| \longrightarrow \|a - Fb\|$. So $\|a - Fb\| = D$. By strict convexity of the norm, $Fb = b$. From $\|Ga - Fb\| \le \|a - b\|$, follows $\|Ga - Fb\| = D$. Then $Ga = a$. Therefore, $Fa = Ga = a$. Let $x \in M_0$. Then we have

$$\|Fx - GPx\| \le \|x - Px\| = d(M, N). \tag{60}$$

Therefore, $\|Fx - GPx\| = d(M, N) = \|Fx - PFx\|$. By Lemma 6, we get $GPx = PFx$. In particular, $GPa = PFa$. In the same way, we can prove that $FPa = PGa$. So $F(Pa) = Pa$ and $GPa = Pa$. Since $Pa \in N_0$, we can obtain $\lim_{n\longrightarrow\infty}\|w_n - Pa\|$ exists. Therefore,

$$\lim_{n\longrightarrow\infty}\|w_n - Pa\| = \lim_{k\longrightarrow\infty}\|w_{n_k} - Pa\| = \|a - Pa\| = d(M, N). \tag{61}$$

This implies $w_n \longrightarrow a$. Also, $v_n \longrightarrow a, u_n \longrightarrow a$. □

Corollary 2. *Let M and N be nonempty bounded closed convex subsets of a uniformly convex Banach space and suppose F, G: $M \cup N \longrightarrow M \cup N$ satisfy*

(1) $G(M) \subseteq M, G(N) \subseteq N, F(M) \subseteq M$ and $F(N) \subseteq N$;

(2) for $u \in M, v \in N$; and

(3) $\|Fu - Gv\| \le \|u - v\|$ for $u \in N, v \in M$,

with a nonempty common fixed-point set. For an arbitrary chosen $w_0 \in M_0$, let the sequence $\{w_n\}$ be generated by (7) where $\eta_n, \delta_n, \gamma_n \in (\varepsilon, 1-\varepsilon)$, where $\varepsilon \in (0, (1/2))$ and $n = 0, 1, 2, \ldots$, then $\lim_{n\longrightarrow\infty}\|Gu_n - Fv_n\| = 0, \lim_{n\longrightarrow\infty}\|u_n - Gu_n\| = 0$, and $\lim_{n\longrightarrow\infty}\|w_n - Fw_n\| = 0$. Moreover, if $F(M)$ lies in a compact set, then $\{w_n\}, \{v_n\}$, and $\{u_n\}$ converge to a common fixed point of G and F.

Corollary 3. *Let M and N be nonempty bounded closed convex subsets of a uniformly convex Banach space and suppose $F, G\colon M \cup N \longrightarrow M \cup N$ satisfy*

(1) $G(M)\subseteq M, G(N)\subseteq N, F(M)\subseteq M$ and $F(N)\subseteq N$;

(2) $\|Fu - Gv\| \le \|u - v\|$ for $u \in M, v \in N$; and

(3) $\|Fu - Gv\| \le \|u - v\|$ for $u \in N, v \in M$,

with a nonempty common fixed-point set. Let $w_0 \in M_0$, and define $w_{n+1} = P^n((1-\eta_n)Gu_n + \eta_n Fv_n)$ where $v_n = (1-\delta_n)u_n + \delta_n Gu_n$, $u_n = (1-\gamma_n)w_n + \gamma_n Fw_n$, $\eta_n, \delta_n, \gamma_n \in (\varepsilon, 1-\varepsilon)$, where $\varepsilon \in (0, (1/2))$ and $n = 0, 1, 2, \ldots$ then $\lim_{n\longrightarrow\infty}\|Gu_n - Fv_n\| = 0$, $\lim_{n\longrightarrow\infty}\|u_n - Gu_n\| = 0$, and $\lim_{n\longrightarrow\infty}\|w_n - Fw_n\| = 0$. Moreover, if $F(M)$ lies in a compact set, then $\{w_n\}, \{v_n\}$, and $\{u_n\}$ converge to a common fixed point of G and F.

Proof. One can note that $P^n((1-\eta_n)Gu_n + \eta_n Fv_n) = (1-\eta_n)Gu_n + \eta_n Fv_n$. By Theorem 4, the result follows.

We illustrate the above theorem through the following example. □

Example 1. Let $(\mathbb{R}^2, \|\cdot\|)$ with $\|(u_1, u_2) - (v_1, v_2)\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$. Let $M = \{(0, u) \in \mathbb{R}^2\colon u \in [0, 1]\}$ and $N = \{(1, u) \in \mathbb{R}^2\colon u \in [2, 3]\}$, then $d(M, N) = \sqrt{2}$. And we define a pair of mappings $F, G\colon M \cup N \longrightarrow M \cup N$ by $F(0, u) = (0, 1), F(1, u) = (1, u)$ and $G(0, u) = (0, u), G(1, u) = (1, 2)$. For $(0, u) \in M$, $(1, v) \in N$, we have

$$\|G(0, u) - F(1, v)\| = \|(0, u) - (1, v)\|. \tag{62}$$

For $(0, u) \in M$, $(1, v) \in N$, we have

$$\|F(0, u) - G(1, v)\| = \|(0, 1) - (1, 2)\| = \sqrt{2} \le \|(0, u) - (1, v)\|. \tag{63}$$

Clearly, the set $\{(0, 1), (1, 2)\}$ is common fixed points of F and G Fix $\eta_n = (3/4), \delta_n = (3/4), \gamma_n = (3/4)\forall n$. Let $(0, w_0) \in M$, then the Thakur iteration becomes

$$\begin{aligned}
(0, u_n) &= \left(1 - \frac{3}{4}\right)(0, w_n) + \frac{3}{4}F(0, w_n) \\
&= \frac{1}{4}(0, w_n) + \frac{3}{4}(0, 1) \\
&= \left(0, \frac{w_n}{4}\right) + \left(0, \frac{3}{4}\right) \\
&= \left(0, \frac{w_n + 3}{4}\right) \\
(0, v_n) &= \left(1 - \frac{3}{4}\right)(0, u_n) + \frac{3}{4}G(0, u_n) \\
&= \frac{1}{4}\left(0, \frac{w_n + 3}{4}\right) + \frac{3}{4}G\left(0, \frac{w_n + 3}{4}\right) \\
&= \frac{1}{4}\left(0, \frac{w_n + 3}{4}\right) + \frac{3}{4}\left(0, \frac{w_n + 3}{4}\right) \\
&= \left(0, \frac{w_n + 3}{4}\right) \\
(0, w_{n+1}) &= \left(1 - \frac{3}{4}\right)G(0, u_n) + \frac{3}{4}F(0, v_n) \\
&= \frac{1}{4}G\left(0, \frac{w_n + 3}{4}\right) + \frac{3}{4}(0, 1) \\
&= \frac{1}{4}\left(0, \frac{w_n + 3}{4}\right) + \left(0, \frac{3}{4}\right) \\
&= \left(0, \frac{w_n + 3}{16}\right) + \left(0, \frac{3}{4}\right) \\
&= \left(0, \frac{w_n + 15}{16}\right).
\end{aligned} \tag{64}$$

Using MATLAB coding, we give the following Table 1 to show that the iteration $\{(0, w_{n+1})\}, \{(0, u_n)\}$, and $\{(0, v_n)\}$, converge to a common fixed point of F, G for an initial point $(0, w_0) = (0, 0.1) \in M$.

In the same way, for the above example, the iterations (9), (10), (11), and (12) become

Table 1: Thakur iteration

n	$(0, v_n) = (0, v_n)$	$(0, w_{n+1})$
09	(0,0.999996566772461)	(0,0.999999999986903)
10	(0,0.999999141693115)	(0,0.999999999999181)
11	(0,0.999999785423279)	(0,0.999999999999949)
12	(0,0.999999946355820)	(0,0.999999999999997)
13	(0, 0.999999986588955)	(0,1.000000000000000)
⋮	⋮	⋮
24	(0,0.999999999999997)	
25	(0,0.999999999999999)	
26	(0,1.000000000000000)	

Table 2: Comparative results.

n	Ishikawa	Noor	Agarwal	Abbas	Thakur
10	(0,0.999768781231887)	(0,0.999999141693115)	(0,0.999999951665723)	(0,0.999999997282962)	(0,0.999999999999181)
11	(0,0.999898841788951)	(0,0.999999785423279)	(0,0.999999990937323)	(0,0.999999999617984)	(0,0.999999999999949)
12	(0, 0.999955743282666)	(0,0.999999946355820)	(0,0.999999998300748)	(0,0.999999999946289)	(0,0.999999999999997)
13	(0,0.999980637686166)	(0,0.999999986588955)	(0,0.999999999681390)	(0,0.999999999992448)	(0,1.000000000000000)
14	(0,0.999991528987698)	(0,0.999999996647239)	(0,0.999999999940261)	(0,0.999999999998938)	(0,1.000000000000000)
15	(0, 0.999996293932118)	(0,0.999999999161810)	(0,0.999999999988799)	(0,0.999999999999851)	(0,1.000000000000000)
16	(0,0.999998378595301)	(0,0.999999999790452)	(0,0.999999999997900)	(0,0.999999999999979)	(0,1.000000000000000)
17	(0, 0.999999290635444)	(0,0.999999999947613)	(0,0.999999999999606)	(0,0.999999999999997)	(0,1.000000000000000)
18	(0,0.999999689653007)	(0,0.999999999986903)	(0,0.999999999999926)	(0,1.000000000000000)	(0,1.000000000000000)
19	(0,0.999999864223191)	(0,0.999999999996726)	(0,0.999999999999986)	(0,1.000000000000000)	(0,1.000000000000000)
20	(0,0.999999940597646)	(0,0.999999999999181)	(0,0.999999999999997)	(0,1.000000000000000)	(0,1.000000000000000)
21	(0, 0.999999974011470)	(0,0.999999999999795)	(0,1.000000000000000)	(0,1.000000000000000)	(0,1.000000000000000)
22	(0,0.999999988630018)	(0,0.999999999999949)	(0,1.000000000000000)	(0,1.000000000000000)	(0,1.000000000000000)
23	(0,0.999999995025633)	(0,0.999999999999987)	(0,1.000000000000000)	(0,1.000000000000000)	(0,1.000000000000000)
24	(0,0.999999997823714)	(0,0.999999999999997)	(0,1.000000000000000)	(0,1.000000000000000)	(0,1.000000000000000)
25	(0, 0.999999999047875)	(0,0.999999999999999)	(0,1.000000000000000)	(0,1.000000000000000)	(0,1.000000000000000)
26	(0,0.999999999583445)	(0,1.000000000000000)	(0,1.000000000000000)	(0,1.000000000000000)	(0,1.000000000000000)
⋮	⋮	⋮	⋮	⋮	⋮
41	(0,0.999999999999998)				
42	(0,0.999999999999999)				
43	(0,1.000000000000000)				

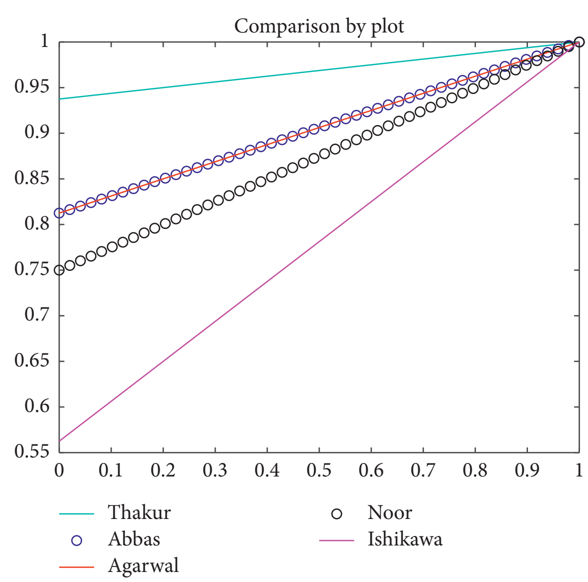

Figure 1: Convergence results.

(i) Abbas: for an initial point $(0, w_0) \in M$,

$$(0, w_{n+1}) = \left(0, \frac{9w_n + 55}{64}\right), \tag{65}$$

(ii) Agarwal: for an initial point $(0, w_0) \in M$,

$$(0, w_{n+1}) = \left(0, \frac{3w_n + 13}{16}\right), \tag{66}$$

(iii) Noor: for an initial point $(0, w_0) \in M$,

$$(0, w_{n+1}) = \left(0, \frac{w_n + 3}{4}\right), \tag{67}$$

(iv) Ishikawa: for an initial point $(0, w_0) \in M$,

$$(0, w_{n+1}) = \left(0, \frac{7w_n + 9}{16}\right). \tag{68}$$

Using MATLAB coding, we give the following Table 2, which compares Thakur iteration with Abbas, Agarwal, Noor, and Ishikawa iterations.

Using MATLAB coding, we give the following Figure 1, which compares convergence of Thakur iteration with Abbas, Agarwal, Noor, and Ishikawa iterations by the plot.

Now we omit the assumptions on constants $\{\eta_n\}, \{\delta_n\}, \{\gamma_n\}$, and $d(w_n, M_0) \longrightarrow 0$ in the above theorem and we provide the following theorem by using the condition (R) on constants $\{\eta_n\}, \{\delta_n\}$, and $\{\gamma_n\}$.

Lemma 8 (see [7]). *A Banach space X is uniformly convex if and only if for each fixed number $r > 0$, there exists a continuous strictly increasing function $\phi\colon [0,\infty) \longrightarrow [0,\infty), \phi(t) = 0$ if and only if $t = 0$, such that*

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\phi(\|x-y\|), \tag{69}$$

for all $\lambda \in [0,1]$ and all $x, y \in X$ such that $\|x\| \le r$ and $\|y\| \le r$.

Lemma 9 (see [7]). *We consider a strictly increasing function $\phi\colon [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$. If a sequence $\{r_n\}$ in $[0,\infty)$ satisfies $\lim_{n\longrightarrow\infty}\phi(r_n) = 0$, then $\lim_{n\longrightarrow\infty} r_n = 0$.*

Lemma 10 (see [7]). *Let (A, B) be a nonempty and closed pair in a uniformly convex Banach space X such that A is convex. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B such that $\lim_{n\longrightarrow\infty}\|x_n - y_n\| = d(A, B)$ and $\lim_{n\longrightarrow\infty}\|z_n - y_n\| = d(A, B)$, then we have $\lim_{n\longrightarrow\infty}\|x_n - z_n\| = 0$.*

Theorem 5. *Let M and N be nonempty bounded closed convex subsets of a uniformly convex Banach space and suppose $F, G\colon M \cup N \longrightarrow M \cup N$ satisfy*

(1) $G(M) \subseteq M, G(N) \subseteq N, F(M) \subseteq M$ and $F(N) \subseteq N$;

(2) $\|Fu - Gv\| \le \|u - v\|$ for $u \in M, v \in N$; and

(3) $\|Fu - Gv\| \le \|u - v\|$ for $u \in N, v \in M$,

with a nonempty common fixed-point set. For an arbitrary chosen $w_0 \in M_0$, let the sequence $\{w_n\}$ be generated by (7) where $\{\eta_n\}, \{\delta_n\}, \{\gamma_n\}$ satisfy (R) and $n = 0, 1, 2, \ldots$. Then $\lim_{n\longrightarrow\infty}\|Gu_n - Fv_n\| = 0, \lim_{n\longrightarrow\infty}\|u_n - Gu_n\| = 0$ and $\lim_{n\longrightarrow\infty}\|w_n - Fw_n\| = 0$. Moreover, if $F(M)$ lies in a compact set, then $\{w_n\}, \{v_n\}$, and $\{u_n\}$ converge to a common fixed point of G and F.

Proof. Let $z \in N_0$ be a common fixed point of F and G. Then from Lemma 8, there exists continuous strictly increasing function $\phi\colon [0,\infty) \longrightarrow [0,\infty)$ such that

$$\begin{aligned}
&\left\|w_{n+1}-z\right\|^2\\
&=\left\|(1-\eta_n)Gu_n+\eta_nFv_n-z\right\|^2\\
&=\left\|\eta_n(Fv_n-z)+(1-\eta_n)(Gu_n-z)\right\|^2\\
&\le\eta_n\left\|Fv_n-z\right\|^2+(1-\eta_n)\left\|Gu_n-z\right\|^2-\eta_n(1-\eta_n)\phi\left(\left\|Fv_n-Gu_n\right\|\right)\\
&=\eta_n\left\|Fv_n-Gz\right\|^2+(1-\eta_n)\left\|Gu_n-Fz\right\|^2-\eta_n(1-\eta_n)\phi\left(\left\|Fv_n-Gu_n\right\|\right)\\
&\le\eta_n\left\|v_n-z\right\|^2+(1-\eta_n)\left\|u_n-z\right\|^2\\
&=\eta_n\left\|(1-\delta_n)u_n+\delta_nGu_n-z\right\|^2+(1-\eta_n)\left\|(1-\gamma_n)w_n+\gamma_nFw_n-z\right\|^2\\
&=\eta_n\left\|\delta_n(Gu_n-z)+(1-\delta_n)(u_n-z)\right\|^2+(1-\eta_n)\left\|\gamma_n(Fw_n-z)+(1-\gamma_n)(w_n-z)\right\|^2\\
&\le\eta_n\delta_n\left\|Gu_n-z\right\|^2+\eta_n(1-\delta_n)\left\|u_n-z\right\|^2-\eta_n\delta_n(1-\delta_n)\phi\left(\left\|Gu_n-u_n\right\|\right)+(1-\eta_n)\gamma_n\left\|Fw_n-z\right\|^2\\
&\quad+(1-\eta_n)(1-\gamma_n)\left\|w_n-z\right\|^2-(1-\eta_n)\gamma_n(1-\gamma_n)\phi\left(\left\|Fw_n-w_n\right\|\right)\\
&\le\eta_n\delta_n\left\|u_n-z\right\|^2+\eta_n(1-\delta_n)\left\|u_n-z\right\|^2+(1-\eta_n)\gamma_n\left\|w_n-z\right\|^2+(1-\eta_n)(1-\gamma_n)\left\|w_n-z\right\|^2\\
&\quad-(1-\eta_n)\gamma_n(1-\gamma_n)\phi\left(\left\|Fw_n-w_n\right\|\right)\\
&\le\eta_n\left\|u_n-z\right\|^2+(1-\eta_n)\left\|w_n-z\right\|^2-(1-\eta_n)\gamma_n(1-\gamma_n)\phi\left(\left\|Fw_n-w_n\right\|\right)\\
&=\eta_n\left\|(1-\gamma_n)w_n+\gamma_nFw_n-z\right\|^2+(1-\eta_n)\left\|w_n-z\right\|^2-(1-\eta_n)\gamma_n(1-\gamma_n)\phi\left(\left\|Fw_n-w_n\right\|\right)\\
&=\eta_n\left\|\gamma_n(Fw_n-z)+(1-\gamma_n)(w_n-z)\right\|^2+(1-\eta_n)\left\|w_n-z\right\|^2-(1-\eta_n)\gamma_n(1-\gamma_n)\phi\left(\left\|Fw_n-w_n\right\|\right)\\
&\le\eta_n\gamma_n\left\|Fw_n-z\right\|^2+\eta_n(1-\gamma_n)\left\|w_n-z\right\|^2-\eta_n\gamma_n(1-\gamma_n)\phi\left(\left\|Fw_n-w_n\right\|\right)+(1-\eta_n)\left\|w_n-z\right\|^2\\
&\quad-(1-\eta_n)\gamma_n(1-\gamma_n)\phi\left(\left\|Fw_n-w_n\right\|\right)\\
&\le\left\|w_n-z\right\|^2-\gamma_n(1-\gamma_n)\phi\left(\left\|Fw_n-w_n\right\|\right).
\end{aligned}\tag{70}$$

Therefore, we can deduce the following inequality:

$$\gamma_n(1-\gamma_n)\,\phi\left(\left\|Fw_n-w_n\right\|\right)\le\left\|w_n-z\right\|^2-\left\|w_{n+1}-z\right\|^2. \tag{71}$$

Now, we proceed with the following:

Suppose $\{\eta_n\},\{\delta_n\}$, and $\{\gamma_n\}$ satisfy (R). From equation (73), we get

$$\sum_{n=1}^{m}\gamma_n(1-\gamma_n)\,\phi\left(\left\|Fw_n-w_n\right\|\right)\le\left\|w_1-z\right\|^2-\left\|w_{m+1}-z\right\|^2. \tag{72}$$

As $m\longrightarrow\infty$, we get $\sum_{\infty}^{\infty}\gamma_n(1-\gamma_n)\phi(\|Fw_n-w_n\|)<\infty$. In view of the fact that $\gamma_n(1-\gamma_n)\ge\varepsilon$, implies $\phi(\|Fw_n-w_n\|)\longrightarrow 0$, so $\|Fw_n-w_n\|\longrightarrow 0$.

As in Theorem 4, we can prove $\|u_n-w_n\|\longrightarrow 0, \|Gu_n-Fv_n\|\longrightarrow 0, \|Gu_n-u_n\|\longrightarrow 0$, and $\|v_n-u_n\|\longrightarrow 0$. Now, since $F(M)$ lies in a compact subset then $\{Fw_n\}$ has a convergent subsequence $\{Fw_{n_k}\}$, converging to some point $u\in M_0$. Also, we have $w_{n_k}\longrightarrow u, u_{n_k}\longrightarrow u, Gu_{n_k}\longrightarrow u$.

Now $\|w_{n_k}-Pu\|\ge\|Fw_{n_k}-GPu\|\longrightarrow\|u-GPu\|$. So $\|u-GPu\|=D$. From $\|Fu-GPu\|\le\|u-Pu\|$, follows $\|Fu-GPu\|=D$. Then, $Fu=u$.

On the other hand, $\|u_{n_k}-Pu\|\ge\|Gu_{n_k}-FPu\|\longrightarrow\|u-FPu\|$. So $\|u-FPu\|=D$. From $\|Gu-FPu\|\le\|u-Pu\|$, follows $\|Gu-FPu\|=D$. Then $Gu=u$. Therefore, $Fu=Gu=u$.

Let $x\in M_0$. Then we have

$$\|Fx-GPx\|\le\|x-Px\|=d(M,N). \tag{73}$$

Therefore $\|Fx-GPx\|=d(M,N)=\|Fx-PFx\|$. By Lemma 6, we get $GPx=PFx$. In particular $GPu=PFu$. In the same way, we can prove that $FPu=\mathrm{P}Gu$.

Since $F(Pu)=P(Gu)=Pu$ and $G(Pu)=P(Fu)=Pu$, we get that $\lim_{n\longrightarrow\infty}\|w_n-Pu\|$ exists. So

$$\lim_{n\longrightarrow\infty}\left\|w_n-Pu\right\|=\lim_{k\longrightarrow\infty}\left\|w_{n_k}-Pu\right\|=\|u-Pu\|=d(M,N), \tag{74}$$

which gives $w_n\longrightarrow u$.

In the next result, we provide a stronger version to approximate the common fixed point via von Neumann sequences. □

Theorem 6. *Let M and N be nonempty bounded closed convex subsets of a Hilbert space and suppose F,G: $M\cup N\longrightarrow M\cup N$ satisfy*

(1) $G(M)\subseteq M, G(N)\subseteq N, F(M)\subseteq M$ and $F(N)\subseteq N$;

(2) $\|Fu-Gv\|\le\|u-v\|$ for $u\in M, v\in N$; and

(3) $\|Fu-Gv\|\le\|u-v\|$ for $u\in N, v\in M$,

with a nonempty common fixed-point set. Let $w_0 \in M$, *and define* $w_{n+1} = P^n((1-\eta_n)Gu_n + \eta_n Fv_n)$ *where* $v_n = (1-\delta_n)u_n + \delta_n Gu_n, u_n = (1-\gamma_n)w_n + \gamma_n Fw_n, \eta_n, \delta_n, \gamma_n \in (\varepsilon, 1-\varepsilon)$, *where* $\varepsilon \in (0,(1/2))$ *and* $n = 0,1,2,\ldots$, *then* $\lim_{n\longrightarrow\infty}\|w_n - Fw_n\| = 0$. *Moreover, if* $F(M)$ *lies in a compact set and* $\|u_n - Gu_n\| \longrightarrow 0, \|v_n - Fv_n\| \longrightarrow 0$ *then* $\{w_n\}$ *converges to a common fixed point of* F, G.

Proof. If $d(M,N) = 0$, then $M_0 = N_0 = M\cap N$ and $F,G\colon M\cap N \longrightarrow M\cap N$ is a pair of nonexpansive with $w_{n+1} = P^n((1-\eta_n)Gu_n + \eta_n Fv_n) = (1-\eta_n)Gu_n + \eta_n Fv_n$, the usual Thakur iteration. So, let us take that $d(M,N) > 0$. Let $z \in N_0$ be a common fixed point of F and G. Now, by equations (39) and (41), we obtain

$$\begin{aligned}\|w_{n+1} - z\| &= \|P^n((1-\eta_n)Gu_n + \eta_n Fv_n) - z\| \\ &\le \|(1-\eta_n)Gu_n + \eta_n Fv_n - z\| \\ &= \|(1-\eta_n)(Gu_n - z) + \eta_n(Fv_n - z)\| \\ &\le (1-\eta_n)\|u_n - z\| + \eta_n\|v_n - z\| \\ &\le (1-\eta_n)\|w_n - z\| + \eta_n\|w_n - z\| \\ &= \|w_n - z\|.\end{aligned} \tag{75}$$

This implies that the sequence $\{\|w_n - z\|\}$ is nonincreasing. Then we can find $d > 0$ such that $\lim_{n\longrightarrow\infty}\|w_n - z\| = d$.

Suppose there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and an $\epsilon > 0$ such that $\|w_{n_k} - Fw_{n_k}\| \ge \varepsilon > 0$ for all k.

Since the modulus of convexity of δ of X is continuous and increasing function we choose $\xi > 0$ as small that $(1 - c\delta(\varepsilon/(d+\xi)))(d+\xi) < d$, where $c > 0$.

Now we choose k, such that $\|w_{n_k} - z\| \le d + \xi$. Now we have

$$\begin{aligned}\|z - w_{n_k+1}\| &= \left\|z - P^{n_k}\left(\left(1-\eta_{n_k}\right)Gu_{n_k} + \eta_{n_k}Fv_{n_k}\right)\right\| \\ &\le \left\|z - \left(\left(1-\eta_{n_k}\right)Gu_{n_k} + \eta_{n_k}Fv_{n_k}\right)\right\| \\ &= \left\|\left(1-\eta_{n_k}\right)z + \eta_{n_k}z - \left(\left(1-\eta_{n_k}\right)G\left(\left(1-\gamma_{n_k}\right)w_{n_k} + \gamma_{n_k}Fw_{n_k}\right) + \eta_{n_k}Fv_{n_k}\right)\right\| \\ &\le \left(1-\eta_{n_k}\right)\left\|z - G\left(\left(1-\gamma_{n_k}\right)w_{n_k} + \gamma_{n_k}Fw_{n_k}\right)\right\| + \eta_{n_k}\left\|z - Fv_{n_k}\right\| \\ &\le \left(1-\eta_{n_k}\right)\left\|z - \left(\left(1-\gamma_{n_k}\right)w_{n_k} + \gamma_{n_k}Fw_{n_k}\right)\right\| + \eta_{n_k}\left\|z - v_{n_k}\right\|.\end{aligned} \tag{76}$$

Now

$$\begin{aligned}&\left\|z - \left(\left(1-\gamma_{n_k}\right)w_{n_k} + \gamma_{n_k}Fw_{n_k}\right)\right\| \\ &\quad = \left\|\left(1-\gamma_{n_k}\right)\left(z - w_{n_k}\right) + \gamma_{n_k}\left(z - Fw_{n_k}\right)\right\| \\ &\quad \le \left(1 - 2\delta\left(\frac{\varepsilon}{d+\xi}\right)\min\left\{\gamma_{n_k}, 1-\gamma_{n_k}\right\}\right)(d+\xi).\end{aligned} \tag{77}$$

Also, using equation (79), we get

$$\begin{aligned}\left\|z - v_{n_k}\right\| &= \left\|z - \left(\left(1-\delta_{n_k}\right)u_{n_k} + \delta_{n_k}Gu_{n_k}\right)\right\| \\ &= \left\|\left(1-\delta_{n_k}\right)\left(z - u_{n_k}\right) + \delta_{n_k}\left(z - Gu_{n_k}\right)\right\| \\ &\le \left(1-\delta_{n_k}\right)\left\|z - u_{n_k}\right\| + \delta_{n_k}\left\|z - Gu_{n_k}\right\| \\ &\le \left(1-\delta_{n_k}\right)\left\|z - u_{n_k}\right\| + \delta_{n_k}\left\|z - u_{n_k}\right\| \\ &= \left\|z - u_{n_k}\right\| \\ &\le \left(1 - 2\delta\left(\frac{\varepsilon}{d+\xi}\right)\min\left\{\gamma_{n_k}, 1-\gamma_{n_k}\right\}\right)(d+\xi).\end{aligned} \tag{78}$$

Therefore, the equation (78) becomes

$$\left\|z - w_{n_k+1}\right\| \leq \left(1 - 2\delta\left(\frac{\varepsilon}{d+\xi}\right)\min\left\{\gamma_{n_k}, 1-\gamma_{n_k}\right\}\right)(d+\xi). \tag{79}$$

Since there exists $l > 0$ such that $2\min\left\{\gamma_{n_k}, 1-\gamma_{n_k}\right\} \geq l$,

$$\left(1 - 2\delta\left(\frac{\varepsilon}{d+\xi}\right)\min\left\{\gamma_{n_k}, 1-\gamma_{n_k}\right\}\right)(d+\xi) \leq \left(1 - l\delta\left(\frac{\varepsilon}{d+\xi}\right)\right)(d+\xi). \tag{80}$$

Suppose that we choose very small $\xi > 0$, we have $(1 - l\delta(\varepsilon/(d+\xi)))(d+\xi) < d$, which is a contradiction. This implies that $\lim_{n\longrightarrow\infty}\|w_n - Fw_n\| = 0$. Now we prove that $\|w_{n+1} - w_n\| \longrightarrow 0$. From the Thakur iteration, we get $\|u_n - w_n\| = \gamma_n\|Fw_n - w_n\|$. Since $\lim_{n\longrightarrow\infty}\|w_n - Fw_n\| = 0$ we obtain $\|u_n - w_n\| \longrightarrow 0$.

Since $F(M)$ is contained in a compact set, $\{Fw_n\}$ has a subsequence $\left\{Fw_{n_k}\right\}$ that converges to a point $v_0 \in M$. Also $\left\{w_{n_k}\right\}$ converges to v_0. From the given sequence, we obtain

$$\begin{aligned}
\left\|w_{n_k+1} - w_{n_k}\right\| &= \left\|P^{n_k}\left(\left(1-\eta_{n_k}\right)Gu_{n_k} + \eta_{n_k}Fv_{n_k}\right) - w_{n_k}\right\| \\
&\leq \left\|\left(1-\eta_{n_k}\right)Gu_{n_k} + \eta_{n_k}Fv_{n_k} - w_{n_k}\right\| \\
&= \left\|Gu_{n_k} - w_{n_k}\right\| + \eta_{n_k}\left\|Gu_{n_k} - Fv_{n_k}\right\| \\
&\leq \left\|Gu_{n_k} - u_{n_k}\right\| + \left\|u_{n_k} - w_{n_k}\right\| + \eta_{n_k}\left(\left\|Gu_{n_k} - u_{n_k}\right\| + \left\|u_{n_k} - v_{n_k}\right\| + \left\|v_{n_k} - Fv_{n_k}\right\|\right).
\end{aligned} \tag{81}$$

Since $\|Gu_{n_k} - u_{n_k}\| \longrightarrow 0$ implies $\|u_{n_k} - v_{n_k}\| \longrightarrow 0$. Then $\|w_{n_k+1} - w_{n_k}\| \longrightarrow 0$. Therefore, $w_{n_k+1} \longrightarrow v_0$, which implies that $w_n \longrightarrow v_0$. Also, we have $u_n \longrightarrow v_0, Gu_n \longrightarrow v_0, Fw_n \longrightarrow v_0$ as $n \longrightarrow \infty$.

Now, $\|Fw_n - G(P_N(v_0))\| \leq \|w_n - P_N(v_0)\|$, which implies that.

$\|v_0 - G(P_N(v_0))\| \leq \|v_0 - P_N(v_0)\|$. Hence, $G(P_N(v_0)) = P_N(v_0)$.

Similarly, $\|Gu_n - F(P_N(v_0))\| \leq \|u_n - P_N(v_0)\|$, which implies that.

$\|v_0 - F(P_N(v_0))\| \leq \|v_0 - P_N(v_0)\|$. Hence, $F(P_N(v_0)) = P_N(v_0)$.

Also, $\|G(P(v_0)) - P_N(v_0)\| = \|G(P(v_0)) - F(P_N(v_0))\| \leq \|P(v_0) - P_N(v_0)\|$.

So $G(P(v_0)) = P(v_0)$.

And also $\|F(P(v_0)) - P_N(v_0)\| = \|F(P(v_0)) - G(P_N(v_0))\| \leq \|P(v_0) - P_N(v_0)\|$.

So $F(P(v_0)) = P(v_0)$.

Now $\|GP_N(P(v_0)) - P(v_0)\| = \|GP_N(P(v_0)) - F(P(v_0))\| \leq \|P_N(P(v_0)) - P(v_0)\|$. Thus $GP_N(P(v_0)) = P_N(P(v_0))$.

For any n, we have $F(P^n(v_0)) = P^n(v_0)$ and $GP_N(P^n(v_0)) = P_N(P^n(v_0))$.

Similarly, $\|FP_N(P(v_0)) - P(v_0)\| = \|FP_N(P(v_0)) - G(P(v_0))\| \leq \|P_N(P(v_0)) - P(v_0)\|$. Thus $FP_N(P(v_0)) = P_N(P(v_0))$.

For any n, we have $G(P^n(v_0)) = P^n(v_0)$ and $FP_N(P^n(v_0)) = P_N(P^n(v_0))$. By Theorem 2, for each $u \in M$ the sequence $\{P^n(u)\}$ converges to some $r(u) \in M_0$. Now,

$$\begin{aligned}
\|G(r(v_0)) - P_N(r(v_0))\| &\leq \lim_{n\longrightarrow\infty}\|G(r(v_0)) - P_N(P^n(v_0))\| \\
&= \lim_{n\longrightarrow\infty}\|G(r(v_0)) - F(P_N(P^n(v_0)))\| \\
&\leq \lim_{n\longrightarrow\infty}\|r(v_0) - P_N(P^n(v_0))\| \\
&= \|r(v_0) - P_N(r(v_0))\|.
\end{aligned} \tag{82}$$

So $\|G(r(v_0)) - P_N(r(v_0))\| \leq \|r(v_0) - P_N(r(v_0))\|$.

Therefore $G(r(v_0)) = r(v_0)$ and similarly, we get $GP_N(r(v_0)) = P_N(r(v_0))$.

In the same way, we prove that $F(r(v_0)) = r(v_0)$ and $FP_N(r(v_0)) = P_N(r(v_0))$.

Now we define $g_n\colon M \longrightarrow \mathbb{R}$ by $g_n(u) = \|P^n(u) - r(u)\|$.

Since $\|r(u) - r(v)\| = \lim_{n\longrightarrow\infty}\|P^n(u) - P^n(v)\| \leq \|u - v\|$, then we conclude that r is continuous. Therefore $g_n(u)$ is continuous and converges pointwise to zero. Since $r(u) \in M_0$, by Lemma 4, we obtain $g_{n+1} \leq g_n$. Therefore g_n converges uniformly on the compact set.

$$F = \left\{\left(1-\eta_{n_k}\right)Gu_{n_k} + \eta_{n_k}Fv_{n_k}\right\} \cup \{v_0\}. \tag{83}$$

Therefore,

$$\begin{aligned}
\lim_{k\longrightarrow\infty}\Big\|P^{n_k}&\left(\left(1-\eta_{n_k}\right)Gu_{n_k} + \eta_{n_k}Fv_{n_k}\right) \\
&- r\left(\left(1-\eta_{n_k}\right)Gu_{n_k} + \eta_{n_k}Fv_{n_k}\right)\Big\| = 0.
\end{aligned} \tag{84}$$

Since $r((1-\eta_{n_k})Gu_{n_k}+\eta_{n_k}Fv_{n_k}) \longrightarrow r(v_0)$, we get $w_{n_k+1} \longrightarrow r(v_0)$, which gives that $r(v_0)=v_0$. Therefore $Gv_0 = G(r(v_0)) = r(v_0) = v_0$ and $Fv_0 = F(r(v_0)) = r(v_0) = v_0$, which completes the proof.

Suppose X is a Hilbert space and let M and N be nonempty bounded closed convex subsets of X and suppose $F, G\colon M\cup N \longrightarrow M\cup N$ satisfy

(1) $G(M)\subseteq N, G(N)\subseteq M, F(M)\subseteq N$ and $F(N)\subseteq M$;

(2) $\|Fu-Gv\|\leq\|u-v\|$ for $u\in M, v\in N$; and

(3) $\|Fu-Gv\|\leq\|u-v\|$ for $u\in N, v\in M$.

We consider $P_MG\colon M\longrightarrow M, P_NF\colon N\longrightarrow N$, $P_NG\colon N\longrightarrow N$ and $P_MF\colon M\longrightarrow M$. From Proposition 2, $\|P_MF(u)-P_NG(v)\|\leq\|u-v\|$ for $u\in M$ and $v\in N$ and $\|P_NF(u)-P_MG(v)\|\leq\|u-v\|$ for $u\in N$ and $v\in M$, by Theorem 4 and Theorem 6, we give the following results on the convergence of best proximity points. □

Corollary 4. *Let M and N be nonempty bounded closed convex subsets of a Hilbert space and suppose $F, G\colon M\cup N\longrightarrow M\cup N$ satisfy*

(1) $G(M)\subseteq N, G(N)\subseteq M, F(M)\subseteq N$ and $F(N)\subseteq M$;

(2) $\|Fu-Gv\|\leq\|u-v\|$ for $u\in M, v\in N$; and

(3) $\|Fu-Gv\|\leq\|u-v\|$ for $u\in N, v\in M$.

If M is mapped into a compact subset of N, then for any $w_0\in M_0$ the sequence is defined by $w_{n+1}=(1-\eta_n)P_MGu_n+\eta_nP_MFv_n$, where $v_n=(1-\delta_n)u_n+\delta_nP_MGu_n$, $u_n=(1-\gamma_n)w_n+\gamma_nP_MFw_n$, converges to w in M_0 such that $\|w-Fw\|=\|w-Gw\|=d(M,N)$.

Corollary 5. *Let M and N be nonempty bounded closed convex subsets of a Hilbert space and suppose $F, G\colon M\cup N\longrightarrow M\cup N$ satisfy*

(1) $G(M)\subseteq N, G(N)\subseteq M, F(M)\subseteq N$ and $F(N)\subseteq M$;

(2) $\|Fu-Gv\|\leq\|u-v\|$ for $u\in M, v\in N$; and

(3) $\|Fu-Gv\|\leq\|u-v\|$ for $u\in N, v\in M$.

If M is mapped into a compact subset of N, then for any $w_0\in M$ the sequence defined by $w_{n+1}=(1-\eta_n)P_MGu_n+\eta_nP_MFv_n$, where $v_n=(1-\delta_n)u_n+\delta_nP_MGu_n, u_n=(1-\gamma_n)w_n+\gamma_nP_MFw_n$ converges to w in M_0 such that $\|w-Fw\|=\|w-Gw\|=d(M,N)$, provided $d(w_n, M_0)\longrightarrow 0$.

Corollary 6. *Let M and N be nonempty bounded closed convex subsets of a Hilbert space and suppose $F, G\colon M\cup N\longrightarrow M\cup N$ satisfy*

(1) $G(M)\subseteq N, G(N)\subseteq M, F(M)\subseteq N$ and $F(N)\subseteq M$;

(2) $\|Fu-Gv\|\leq\|u-v\|$ for $u\in M, v\in N$; and

(3) for $u\in N, v\in M$.

If M is mapped into a compact subset of N, then for any $w_0\in M_0$ the sequence defined by $w_{n+1}=P^n((1-\eta_n)P_MGu_n+\eta_nP_MFv_n)$, where $v_n=(1-\delta_n)u_n+\delta_nP_MGu_n, u_n=(1-\gamma_n)w_n+\gamma_nP_MFw_n$ converges to w in M_0 such that $\|w-Fw\|=\|w-Gw\|=d(M,N)$.

Proof.. The result follows from Corollary 4. □

Corollary 7. *Let M and N be nonempty bounded closed convex subsets of a Hilbert space and suppose $F, G\colon M\cup N\longrightarrow M\cup N$ satisfy*

(1) $G(M)\subseteq N, G(N)\subseteq M, F(M)\subseteq N$ and $F(N)\subseteq M$;

(2) $\|Fu-Gv\|\leq\|u-v\|$ for $u\in M, v\in N$; and

(3) $\|Fu-Gv\|\leq\|u-v\|$ for $u\in N, v\in M$.

Let $w_0\in M$, and define $w_{n+1}=P^n((1-\eta_n)P_MGu_n+\eta_nP_MFv_n)$, where $v_n=(1-\delta_n)u_n+\delta_nP_MGu_n, u_n=(1-\delta_n)\ w_n+\delta_nP_MFw_n, \eta_n, \delta_n\in(\varepsilon, 1-\varepsilon)$, where $\varepsilon\in(0,(1/2))$ and $n=0,1,2,\ldots$. If M is mapped into a compact subset of N and $\|u_n-P_MGu_n\|\longrightarrow 0, \|v_n-P_MFv_n\|\longrightarrow 0$, then $\{w_n\}$ converges to w in M_0 such that $\|w-Fw\|=\|w-Gw\|=d(M,N)$.

Proof. The result follows from Theorem 6. □

3. Conclusions

The fixed-point theorems provide sufficient conditions to ensure the existence of fixed points in different domains. Briefly, the fixed-point theorem possesses the solution of equations of the form $Fx=x$, where F is self-mapping. On the other hand, researchers want to find numerically such a fixed point by using different types of iterative processes for selfcontractive type operators in metric spaces, Hilbert spaces, or several classes of Banach spaces. One of the most famous iterative schemes is Picard's iterative process. Many research papers were presented for approaching the fixed point through Picard's iterative process. Later, for fast convergence, many iterative processes were found to approximate fixed points numerically. In this article, we consider the Thakur iterative process for fast convergence of common fixed points for relatively nonexpansive mappings in uniformly convex Banach spaces. Also, we approximate the common fixed point via the von Neumann iterative process in Hilbert space settings. We provide an example to illustrate our main result. As a consequence of our main results, we find common best proximity points for cyclic relatively nonexpansive mappings in Hilbert space.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest with this study.

Acknowledgments

This work has been partially funded by the Basque Government through Grant IT1207-19.

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