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Research Article

Notes on Relative Extension Functors

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The objective of this paper is to study the relative homological properties of contravariantly and covariantly finite subcategories. Some sufficient conditions for $\underline{\mathscr{E}xt}^{\,n}_{\mathscr{Y}}(A,B) \simeq \bar{\mathscr{E}xt}^{n}_{\mathscr{X}}(A,B)$ are obtained. We also give the conditions under which the stable categories $(\mathscr{C}/\mathscr{W})/(\mathscr{X}/\mathscr{W})$ are one-side triangulated categories.

1. Introduction

The notions of contravariantly and covariantly finite subcategories were introduced in [1] by Auslander and Smalø in connection with studying the problem of which subcategories of an Artin algebra have almost split sequences. Since then, contravariantly and covariantly finite subcategories are widely used in representation theory and relative homological algebra. In [2], Beligiannis studied the relative homological algebra induced by a pair $(\mathscr{C},\mathscr{X})$ consisting of an additive category $\mathscr{C}$ and a contravariantly finite subcategory $\mathscr{X}$ of $\mathscr{C}$. Suppose that $\mathscr{X}$ is contravariantly finite and any $\mathscr{X}$-epic has a kernel in $\mathscr{C}$. Then, $\forall B \in \mathscr{C}$; the contravariantly $\mathscr{X}-$ extension functors $\underline{\mathscr{E}xt}^{\,n}_{\mathscr{X}}(-,B)$: $\mathscr{C}^{op} \longrightarrow \mathscr{A}b, \forall n \geq 0$ are defined as the right $\mathscr{X}$-derived functor of $\mathscr{C}(-,C)$. The covariantly $\mathscr{Y}$-extension functors $\bar{\mathscr{E}xt}^{n}_{\mathscr{Y}}(A,-)$ are defined dually. Under some conditions, if $\bar{\mathscr{E}xt}^{n}_{\mathscr{Y}}(\mathscr{Y},B) = \underline{\mathscr{E}xt}^{\,n}_{\mathscr{X}}(A,\mathscr{X}) = 0$, then $\bar{\mathscr{E}xt}^{n}_{\mathscr{Y}}(A,B) \simeq \underline{\mathscr{E}xt}^{\,n}_{\mathscr{X}}(A,B)$. In the present paper, some results in [2] will be generalized. In [3], Beligiannis and Marmaridis constructed the left and right triangulated structures on the stable categories of additive categories induced from some homological finite subcategories. Recently, Li extended their results to more general settings [4]. Let $\mathscr{C}$ be an additive category and $\mathscr{X}$ be a full additive subcategory of $\mathscr{C}$. If $\mathscr{X}$ is contravariantly finite in $\mathscr{C}$ and any special $\mathscr{X}$-epic has a kernel in $\mathscr{C}$, then the stable category $\mathscr{C}/\mathscr{X}$ has a left triangulated structure induced by $\mathscr{X}$. If $\mathscr{Y}$ is covariantly finite in $\mathscr{C}$ and any special $\mathscr{Y}$-monic has a cokernel in $\mathscr{C}$, then the stable category $\mathscr{C}/\mathscr{Y}$ has a right triangulated structure induced by $\mathscr{Y}$. In Section 3, let $\mathscr{C}$ be an abelian category, $\mathscr{W}$ be a contravariantly finite subcategory of $\mathscr{C}$ and $\mathscr{W} \subseteq \mathscr{X} \subseteq \mathscr{C}$. We prove that the stable category $(\mathscr{C}/\mathscr{W})/(\mathscr{X}/\mathscr{W})$ also has a left triangulated structure.

In this paper, unless otherwise stated, we assume that all considered categories are skeletally small and additive, and their subcategories are full, additive, closed under direct summands and isomorphisms. Functors between categories are supposed to be additive. The following undefined symbols can be referred in [2, 5]. The latest related profound research conclusions on this subject can be found in [6–10].

2. Relative Homology

Homology provides an algebraic picture of topological spaces, and complexes provide a mean of calculating homology. Let $\mathscr{C}$ be an abelian category, $\mathscr{X}$ is a full subcategory of $\mathscr{C}$, which is closed under direct summands and isomorphisms. Consider a complex

$$A^{\cdot}: \cdots \longrightarrow A_{i+1} \longrightarrow A_i \longrightarrow \cdots, \tag{1}$$

in $\mathscr{C}$. The complex $A^{\cdot}$ is called *covariantly $\mathscr{X}$ -exact*, if the induced complex

$$\mathscr{C}(\mathscr{X},A^{\cdot}):\ \cdots \longrightarrow \mathscr{C}(\mathscr{X},A_{i+1}) \longrightarrow \mathscr{C}(\mathscr{X},A_i) \longrightarrow \cdots \quad (2)$$

is exact in an abelian category. For example, when $\mathscr{X}$ is a contravariantly finite subcategory of $\mathscr{C}$, for any right $\mathscr{X}$ -approximation of an object A in $\mathscr{C}$, $0 \longrightarrow \Omega(A) \longrightarrow X \longrightarrow A$ is covariantly $\mathscr{X}$-exact, where $\Omega(A)$ is called the first sysygy of A. If $A \in \mathrm{Gen}(\mathscr{X}) = \{B \in \mathscr{C}|$ there exists an epimorphism $X \longrightarrow B$, for a $X \in \mathscr{X}\}$, then the right $\mathscr{X}$ -approximation of A, $0 \longrightarrow \Omega(A) \longrightarrow X \longrightarrow A \longrightarrow 0$ is an exact sequence. Dually, the complex $A^{\cdot}$ is *contravariantly $\mathscr{X}$ -exact*, if the induced complex

$$\mathscr{C}(A^{\cdot}\mathscr{X}):\ \cdots \longrightarrow \mathscr{C}(A_{i-1},\mathscr{X}) \longrightarrow \mathscr{C}(A_i,\mathscr{X}) \longrightarrow \cdots \quad (3)$$

is exact in an abelian category. If $\mathscr{C}(A^{\cdot},\mathscr{X})$ and $\mathscr{C}(A^{\cdot},\mathscr{X})$ are both exact, the complex $A^{\cdot}$ is called functorially $\mathscr{X}$-exact.

Let $\mathscr{X}$ be a contravariantly finite subcategory of $\mathscr{C}$ and $A \in \mathscr{C}$. The $\mathscr{X}$ *-resolution* of A is the following complex:

$$X_A^{\cdot}:\ \cdots \longrightarrow X_A^{i+1} \longrightarrow X_A^{i} \longrightarrow \cdots \longrightarrow X_A^{1} \longrightarrow X_A^{0} \longrightarrow A \longrightarrow 0, \quad (4)$$

where $X_A^{i+1} \longrightarrow X_A^{i}$ is the composition $X_A^{i+1} \longrightarrow \Omega^{i+1}(A) \longrightarrow X_A^{i}$. The morphism $X_A^{i+1} \longrightarrow \Omega^{i+1}(A)$ is a right $\mathscr{X}$ -approximation of the $(i+1)^{\text{th}}$ sysygy $\Omega^{i+1}(A)$ of A. Dually, if $\mathscr{Y}$ is a covariantly finite subcategory of $\mathscr{C}$ and $B \in \mathscr{C}$, the $\mathscr{Y}$ -coresolution of B is the complex

$$Y^B_{\cdot}:\ 0 \longrightarrow B \longrightarrow Y_0^B \longrightarrow Y_1^B \longrightarrow \cdots \longrightarrow Y_i^B \longrightarrow Y_{i+1}^B \longrightarrow \cdots, \quad (5)$$

where $Y_i^B \longrightarrow Y_{i+1}^B$ is the composition $Y_i^B \longrightarrow \Omega^{-1}(B) \longrightarrow Y_{i+1}^B$. The morphism $\Omega^{-1}(B) \longrightarrow Y_{i+1}^B$ is a left $\mathscr{Y}$-approximation of the i^{th} cosysygy $\Omega^{-1}(B)$ of B. So $\mathscr{X}$-resolution of A is covariant $\mathscr{X}$-exact complex, $\mathscr{Y}$-coresolution of B is contrvariant $\mathscr{Y}$-exact complex. If the category $\mathscr{C}$ has enough projective objects, and these projective objects are contained in $\mathscr{X}$, then any $\mathscr{X}$-resolution is an exact sequence. Similarly, if the category $\mathscr{C}$ has enough injective objects, which are contained in $\mathscr{Y}$, then any $\mathscr{Y}$-coresolution is an exact sequence.

For any object $A \in \mathscr{C}$, we denote $\widehat{X_A^{\cdot}}$ by the deleted complex of the $\mathscr{X}$-resolutions $X_A^{\cdot}$

$$\widehat{X_A^{\cdot}}:\ \cdots \longrightarrow X_A^{i+1} \xrightarrow{f^{i+1}} X_A^{i} \longrightarrow \cdots \longrightarrow X_A^{0} \longrightarrow 0. \quad (6)$$

For $B \in \mathscr{C}$, the right derived $\mathscr{X}$-functors $\underline{\mathscr{E}xt}^{\,n}_{\mathscr{X}}(-,B)$: $\mathscr{C}^{op} \longrightarrow \mathscr{A}b, \forall n \geq 0$ of $\mathscr{C}(-,B)$ are defined to be

$$\underline{\mathscr{E}xt}^{\,n}_{\mathscr{X}}(A,B) = \frac{\mathrm{Ker}\,\mathscr{C}(f^{n+1},B)}{\mathrm{Im}\,\mathscr{C}(f^{n},B)},\ n = 0,1,2,\cdots, \quad (7)$$

where the $\mathscr{C}^{op}$ denoted the opposite category of $\mathscr{C}$. Dually, $\widehat{Y^B_{\cdot}}$ denotes the deleted complex of $\mathscr{Y}$-coresolutions $Y^B_{\cdot}$

$$\widehat{Y^B_{\cdot}}:\ 0 \longrightarrow Y_0^B \longrightarrow Y_1^B \longrightarrow \cdots \longrightarrow Y_i^B \xrightarrow{g_i} Y_{i+1}^B \longrightarrow \cdots. \quad (8)$$

For $A \in \mathscr{C}$, the right derived $\mathscr{Y}$-functors $\bar{\mathscr{E}xt}^{n}_{\mathscr{Y}}(A,-)$: $\mathscr{C} \longrightarrow \mathscr{A}b, \forall n \geq 0$ of $\mathscr{C}(A,-)$ are defined to be

$$\bar{\mathscr{E}xt}^{n}_{\mathscr{Y}}(A,B) = \frac{\mathrm{Ker}\,\mathscr{C}(A,g_n)}{\mathrm{Im}\,\mathscr{C}(A,g_{n-1})},\ n = 0,1,2,\cdots. \quad (9)$$

It is similar to the extension functor in homological algebra, by the comparison theorem [11], $\underline{\mathscr{E}xt}^{\,n}_{\mathscr{X}}(A,B)$ does not depend on the choice of $\mathscr{X}$-resolutions of A. $\bar{\mathscr{E}xt}^{n}_{\mathscr{Y}}(A,B)$ has nothing to do with the choice of $\mathscr{Y}$-coresolutions of B. If $A \in \mathrm{Gen}(\mathscr{X})$ or $\mathscr{X}$ contains all projective objects, then $\underline{\mathscr{E}xt}^{\,0}_{\mathscr{X}}(A,B) \simeq \mathscr{C}(A,B)$. If $B \in \mathrm{Cogen}(\mathscr{Y}) = \{B \in \mathscr{C}|$ there exists $Y \in \mathscr{Y}$, such that $B \longrightarrow Y$ is a monomorphism$\}$ or $\mathscr{Y}$ contains all injective objects, then $\bar{\mathscr{E}xt}^{0}_{\mathscr{Y}}(A,B) \simeq \mathscr{C}(A,B)$.

Example Let R be an algebra over a field K determined the following quiver $_1\!\overset{\alpha}{\longrightarrow}{}_2 \cdot \overset{\beta}{\longrightarrow}{}_{\underline{3}}$ with relation $\beta\alpha = 0$. Then, its AR-quiver is Scheme 1

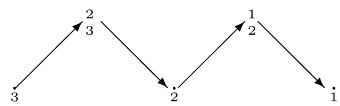
Let $S(i)$ and $P(i)$ be the indecomposable simple module and projective module at vertex i, respectively, $i = 1,2,3$. Put $T = S(1)$, $\mathscr{X} = \mathrm{add}(T)$ are direct sums of direct summand-s of T, then $\mathscr{X} = \mathrm{add}(T)$ is a contravariantly finite subcategory of R–mod categories and $\underline{\mathscr{E}xt}^{\,1}_{\mathscr{X}}(T,-) = 0$, but $\underline{\mathscr{E}xt}^{\,1}_{\mathscr{X}}(T,-) \neq 0$, since $0 \longrightarrow S(2) \longrightarrow P(1) \longrightarrow S(1) \longrightarrow 0$ is a nonsplit exact sequence. If $\mathscr{X} = \mathrm{add}\{S(3);P(2)\}$, then by Prop.1.2 in [12], $\mathscr{X}$ is a covariantly finite subcategory, $\underline{\mathscr{E}xt}^{\,1}_{\mathscr{X}}(P(1),P(3)) \simeq \mathrm{End}_R P(3) \neq 0$, $\bar{\mathscr{E}xt}^{1}_{\mathscr{X}}(P(1),P(3)) = 0$.

Proposition 1. *Let X, Y be full subcategories of C. Suppose to be closed under direct sums and direct summands. If X is contravariantly finite subcategory and Y is a covariantly finite subcategory, then*

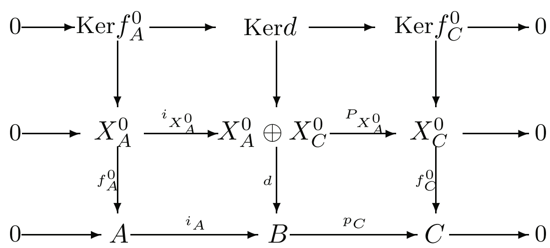
(1) for any covariantly X-exact complex $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, there is a corresponding X- resolution exact sequence, $0 \longrightarrow \widehat{X}_A^{\cdot} \longrightarrow \widehat{X}_B^{\cdot} \longrightarrow \widehat{X}_C^{\cdot} \longrightarrow 0$. In particular, for any n, $X_B^n \simeq X_A^n \bigoplus X_C^n$

(2) for any contravariantly Y-exact complex, $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, there is a corresponding Y-coresolution exact sequence, $0 \longrightarrow \widehat{Y}^A_{\cdot} \longrightarrow \widehat{Y}^B_{\cdot} \longrightarrow \widehat{Y}^C_{\cdot} \longrightarrow 0$, and for any n, $Y_n^B \simeq Y_n^A \oplus Y_n^C$

Proof. It is only necessary to prove (1) because (2) is its dual. Let $f_A^0 : X_A^0 \longrightarrow A, f_C^0 : X_C^0 \longrightarrow C$ be the right $\mathscr{X}$-approximations. The morphism $i_A : A \longrightarrow B$ and $pc : B \longrightarrow C$ are corresponding morphisms in X-exact complex $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$. By $\mathscr{X}$-exact property, there exists $g : X_C^0 \longrightarrow B$ such that $f_C^0 = pcg$. Since $i_A f_A^0 : X_A^0 \longrightarrow B$ by definition of direct sum, there exists a unique morphism d



Scheme 1



Scheme 2

$: X_A^0 \oplus X_C^0 \longrightarrow B$, such that $i_A f_A^0 = di_{x_A^0}$, $g = di_{x_C^0}$. So, we have the following commutative diagram Scheme 2.

Next, it is proved that $d : X_A^0 \oplus X_C^0 \longrightarrow B$ is a right $\mathscr{X}$-approximation.

Put $s : X \longrightarrow B$ and $X \in X$, there is $l : X \longrightarrow X_C^0$, such that $pcs = f_C^0 l$. Since $pcdi_{x_C^0} l = f_C^0 l = pcs$, $u : X \longrightarrow A$, so $di_{x_C^0} l - s = i_A u$, thus, there is $t : X \longrightarrow X_A^0$ such that $u = f_A^0 t$, hence $d(i_{x_C^0} l - i_{x_A^0} t) = s$. □

Finally, it is proved that the complex $0 \longrightarrow \operatorname{Ker} f_A^0 \longrightarrow \operatorname{Ker} d \longrightarrow \operatorname{Ker} f_C^0 \longrightarrow 0$ is covariantly X-exact. Obviously, we just need to prove that for all $X \in \mathscr{X}$, $C(X, \operatorname{Ker} d) \longrightarrow C(X, \operatorname{Ker} f_C^0)$ is an epimorphism. Then, $X_B^0 \simeq X_A^0 \oplus X_C^0$ is obtained by the fact that $0 \longrightarrow C(X, X_A^0) \longrightarrow C(X, X_A^0 \oplus X_C^0) \longrightarrow C(X, X_C^0) \longrightarrow 0$ is a split exact sequence. Repeating the above procedure Scheme 3.

We have $X_B^n \simeq X_A^n \oplus X_C^n$.

By Proposition 1, if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a covariantly $\mathscr{X}$-exact complex, then for any $D \in C$, there is a long exact sequence

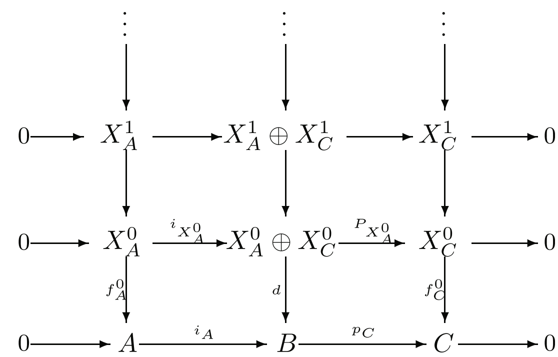
$$0 \longrightarrow \underline{\mathscr{E}xt}_{\mathscr{X}}^{0}(C, D) \longrightarrow \underline{\mathscr{E}xt}_{\mathscr{X}}^{0}(B, D) \longrightarrow \underline{\mathscr{E}xt}_{\mathscr{X}}^{0}(A, D) \longrightarrow \underline{\mathscr{E}xt}_{\mathscr{X}}^{1}(C, D) \longrightarrow \cdots \tag{10}$$

If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a contravariantly $\mathscr{Y}$-exact complex, then for any $D \in C$, there is a long exact sequence

$$0 \longrightarrow \bar{\mathscr{E}xt}_{\mathscr{Y}}^{0}(D, A) \longrightarrow \bar{\mathscr{E}xt}_{\mathscr{Y}}^{0}(D, B) \longrightarrow \bar{\mathscr{E}xt}_{\mathscr{Y}}^{0}(D, C) \longrightarrow \bar{\mathscr{E}xt}_{\mathscr{Y}}^{1}(D, A) \longrightarrow \cdots. \tag{11}$$

Example 1. has shown that, in general, $\bar{\mathscr{E}xt}_{\mathscr{Y}}^{n}(D, C) \neq \underline{\mathscr{E}xt}_{\mathscr{X}}^{n}(D, C)$, even though $\mathscr{X} = \mathscr{Y}$ is a functorially finite subcategory. But, we have the following conclusions.

Lemma 2. *Let $\mathscr{X}$ be a contravariantly finite subcategory, $\mathscr{Y}$ be a covariantly finite subcategory. If $A, \Omega(A) \in Gen\mathscr{X}$ and $B, \Omega^{-1}(B) \in Cogen\mathscr{Y}$, then*



Scheme 3

(1) If $\bar{\mathscr{E}xt}_{\mathscr{Y}}^{1}(\mathscr{X}, B) = 0$ and $\underline{\mathscr{E}xt}_{\mathscr{X}}^{1}(A, \mathscr{Y}) = 0$, then there are isomorphisms

$$\bar{\mathscr{E}xt}_{\mathscr{Y}}^{1}(A, B) \simeq \underline{\mathscr{E}xt}_{\mathscr{X}}^{1}(A, B), \underline{\mathscr{E}xt}_{\mathscr{X}}^{1}(A, \Omega^{-1}(B)) \simeq \bar{\mathscr{E}xt}_{\mathscr{Y}}^{1}(\Omega(A), B) \tag{12}$$

(2) If $\underline{\mathscr{E}xt}_{\mathscr{X}}^{1}(A, \mathscr{Y}) = 0$, then there are epimorphism

$$\bar{\mathscr{E}xt}_{\mathscr{Y}}^{1}(\Omega(A), B) \longrightarrow \underline{\mathscr{E}xt}_{\mathscr{X}}^{1}(A, \Omega^{-1}(B)), \tag{13}$$

and morphism

$$\underline{\mathscr{E}xt}_{\mathscr{X}}^{1}(A, B) \longrightarrow \bar{\mathscr{E}xt}_{\mathscr{Y}}^{1}(A, B) \tag{14}$$

(3) If $\bar{\mathscr{E}xt}_{\mathscr{Y}}^{1}(\mathscr{X}, B) = 0$, then there are epimorphism

$$\underline{\mathscr{E}xt}_{\mathscr{X}}^{1}(A, \Omega^{-1}(B)) \longrightarrow \bar{\mathscr{E}xt}_{\mathscr{Y}}^{1}(\Omega(A), B), \tag{15}$$

and morphism

$$\bar{\mathscr{E}xt}_{\mathscr{Y}}^{1}(A, B) \longrightarrow \underline{\mathscr{E}xt}_{\mathscr{X}}^{1}(A, B) \tag{16}$$

Proof. We only prove (1). Similarly, (2) and (3) can be proved. Let $K_0 = \Omega(A), L^1 = \Omega^{-1}(B)$, $0 \longrightarrow K_0 \longrightarrow X^0 \longrightarrow A \longrightarrow 0$ be a right $\mathscr{X}$-approximation of A, and $0 \longrightarrow B \longrightarrow Y_0 \longrightarrow L^1 \longrightarrow 0$ be a left $\mathscr{Y}$-approximation of B. Therefore, there is the following commutative diagram Scheme 4.

By the snake lemma, we have Scheme 5.

Since $C(A, B) \xrightarrow{\tau} C(K_0, B) \longrightarrow \bar{\mathscr{E}xt}_{\mathscr{X}}^{1}(A, B) \longrightarrow 0$ is exact, so

$$\operatorname{Coker}\alpha \simeq \bar{\mathscr{E}xt}_{\mathscr{Y}}^{1}(A, B) \simeq \operatorname{Coker}\tau \simeq \bar{\mathscr{E}xt}_{\mathscr{X}}^{1}(A, B). \tag{17}$$

By the above commutative diagram, $\gamma\sigma = \eta\beta$. Since $\bar{\mathscr{E}xt}_{\mathscr{Y}}^{1}(\mathscr{X}, B) = 0$ and $\underline{\mathscr{E}xt}_{\mathscr{X}}^{1}(A, \mathscr{Y}) = 0$, so σ, β are epimorphism. Then,

$$\mathrm{Coker}\gamma \simeq \bar{\mathscr{E}xt}^1_{\mathscr{Y}}(K_0, B) \simeq \mathrm{Coker}\eta \simeq \bar{\mathscr{E}xt}^1_{\mathscr{X}}(A, L^1). \tag{18}$$

□

Theorem 3. *Let $\mathscr{X}$ be a contravariantly finite subcategory, $\mathscr{Y}$ be a covariantly finite subcategory. If $A, \Omega^n(A) \in Gen\mathscr{X}$; B, $\Omega^{-n}(B) \in Cogen\mathscr{Y}$, $n = 1, 2, 3, \cdots$; $\bar{\mathscr{E}xt}^1_{\mathscr{Y}}(\mathscr{X}, Cogen(\mathscr{Y})) = 0$ and $\underline{\mathscr{E}xt}^1_{\mathscr{X}}(Gen(\mathscr{X}), \mathscr{Y}) = 0$, then for all natural number n, there is an isomorphism*

$$\underline{\mathscr{E}xt}^n_{\mathscr{Y}}(A, B) \simeq \bar{\mathscr{E}xt}^n_{\mathscr{X}}(A, B). \tag{19}$$

Proof. Let $0 \longrightarrow \Omega^j(A) \longrightarrow X^j \longrightarrow \Omega^{j-1}(A) \longrightarrow 0$ be a covariantly $\mathscr{X}$-exact sequence, where $\Omega^{-1}(A) = A$. $0 \longrightarrow \Omega^{-i}(B) \longrightarrow Y_i \longrightarrow \Omega^{-i-1}(B) \longrightarrow 0$ be a contravariantly $\mathscr{Y}$-exact sequence, where $\Omega^0(B) = B$. Repeating the procedure of Lemma 2, we have

$$\underline{\mathscr{E}xt}^1_{\mathscr{X}}(\Omega^j(A), \Omega^{-i}(B)) \simeq \bar{\mathscr{E}xt}^1_{\mathscr{Y}}(\Omega^j(A), \Omega^{-i}(B)) \simeq \underline{\mathscr{E}xt}^1_{\mathscr{X}}(\Omega^{j-1}(A), \Omega^{-i-1}(B)). \tag{20}$$

Hence,

$$\begin{aligned}\bar{\mathscr{E}xt}^{n+1}_{\mathscr{Y}}(A, B) &\simeq \bar{\mathscr{E}xt}^n_{\mathscr{Y}}(A, \Omega^{-1}(B)) \simeq \cdots \bar{\mathscr{E}xt}^1_{\mathscr{Y}}(A, \Omega^{-n}(B)) \\ &\simeq \underline{\mathscr{E}xt}^1_{\mathscr{X}}(A, \Omega^{-n}(B)) \simeq \underline{\mathscr{E}xt}^{n+1}_{\mathscr{X}}(A, B).\end{aligned} \tag{21}$$

□

The subcategory $\mathscr{A}$ of $\mathscr{C}$ is called *pre- $\mathscr{X}$ -resolving* if $\mathscr{X} \subseteq \mathscr{A}$, and $\mathscr{A}$ is closed under kernels of $\mathscr{X}$-epic. The subcategory $\mathscr{B}$ of $\mathscr{C}$ is called *pre- $\mathscr{Y}$ -coresolving* if $\mathscr{Y} \subseteq \mathscr{B}$, and $\mathscr{B}$ is closed under cokernels of $\mathscr{Y}$-monic.

Corollary 4. *Let $\mathscr{X}$ be a contravariantly finite subcategory, $\mathscr{Y}$ be a covariantly finite subcategory. If $Gen(\mathscr{X})$ is pre- $\mathscr{X}$ -resolving, $Cogen(\mathscr{Y})$ is pre- $\mathscr{Y}$ -coresolving, $A \in Gen(\mathscr{X})$, $B \in Cogen(\mathscr{Y})$, $\bar{\mathscr{E}xt}^1_{\mathscr{Y}}(\mathscr{X}, Cogen(\mathscr{Y})) = 0$, $\underline{\mathscr{E}xt}^1_{\mathscr{X}}(Gen(\mathscr{X}), \mathscr{Y}) = 0$, then for any natural number n, there is an isomorphism*

$$\underline{\mathscr{E}xt}^n_{\mathscr{Y}}(A, B) \simeq \bar{\mathscr{E}xt}^n_{\mathscr{X}}(A, B). \tag{22}$$

The concepts of dimension and codimension are given in [2]. Put $A \in \mathscr{C}$, if A has $\mathscr{X}$-resolution

$$0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow 0, \tag{23}$$

in this case, the least such integer n is called $\mathscr{X}$-dimension of A, denote by $\dim_{\mathscr{X}}(A) = n$. If this nonnegative integer n does not exist, we call $\dim_{\mathscr{X}}(A) = \infty$. Obviously, if $A \in \mathscr{X}$, then $\dim_{\mathscr{X}}(A) = 0$. But the verse is not true. The global $\mathscr{X}$-dimension of the category $\mathscr{C}$ is defined by $\mathrm{gl.dim}_{\mathscr{X}}(\mathscr{C}) = \sup\{\dim_{\mathscr{X}}(A) | A \in \mathscr{C}\}$. Dually, using the definition of $\mathscr{Y}$-coresolution, we can define the $\mathscr{Y}$-codimension of object $B \in \mathscr{C}$, denoted by $\dim^{\mathscr{Y}}(B)$. The global $\mathscr{Y}$-codimension of the category $\mathscr{C}$, denoted by $\mathrm{gl.dim}^{\mathscr{Y}}(\mathscr{C}) = \sup\{\dim^{\mathscr{Y}}(B) | B \in \mathscr{C}\}$. The following proposition can be found in [2].

Proposition 5. *(1)$gl.\dim_X(C) = 0$ if and only if the inclusion functor $X \circ C$ admits right adjoint functor $R : C \longrightarrow X$*

(2)$gl.\dim^Y(C) = 0$ if and only if the inclusion functor $Y \circ C$ admits left adjoint functor $L : C \longrightarrow Y$

Corollary 6. *If $(\mathscr{X}, \mathscr{Y})$ is a torsion pair in $\mathscr{C}$, then $gl.\dim_{\mathscr{X}}(\mathscr{C}) = gl.\dim^{\mathscr{Y}}(\mathscr{C}) = 0$.*

Proposition 7. *Let $\mathscr{X}$ be a contravariantly finite subcategory, if $\Omega^{n+1}(A) \in Gen(\mathscr{X})$, then $\dim_{\mathscr{X}}(A) \le n$ if and only if $\Omega^{n+1}(A) = 0$.*

Proof. Since $0 \longrightarrow \Omega^{n+1}(A) \longrightarrow X_n \longrightarrow \Omega^n(A)$ is a covariantly $\mathscr{X}$-exact sequence, where $X_n \longrightarrow \Omega^n(A)$ is a right $\mathscr{X}$-approximation, thus $\mathscr{C}(\mathscr{X}, \Omega^{n+1}(A)) = 0$ if and only if $\Omega^{n+1}(A) = 0$. □

Similarly, we have the following proposition.

Proposition 8. *Let $\mathscr{Y}$ be a covariantly finite subcategory. If $\Omega^{-n-1}(A) \in Cogen(\mathscr{Y})$, then $\dim^{\mathscr{Y}}(A) \le n$ if and only if $\Omega^{-n-1}(A) = 0$.*

There should be an equivalence relation on the set of short exact sequences. So Y ext should be the set of equivalence classes; If $\underline{Y\ \mathrm{ext}}^1_{\mathscr{X}}(A, B)$ represents all the covariantly $\mathscr{X}$-short exact sequence of the form of $0 \longrightarrow B \longrightarrow Z \longrightarrow A \longrightarrow 0$, $Y\,\bar{\mathrm{ext}}^1_{\mathscr{Y}}(A, B)$ represents all the contravariantly $\mathscr{Y}$-short exact sequence of the form of $0 \longrightarrow B \longrightarrow Z \longrightarrow A \longrightarrow 0$, then there is the corresponding Yoneda lemma.

Lemma 9 (N. Yoneda). *If $A, \Omega(A), \Omega^2(A) \in Gen(\mathscr{X})$, there is a one-to-one correspondence*

$$\underline{Y\ \mathrm{ext}}^1_{\mathscr{X}}(A, B) \Leftrightarrow \underline{\mathscr{E}xt}^1_{\mathscr{X}}(A, B). \tag{24}$$

If $B, \Omega^{-1}(B), \Omega^{-2}(B) \in Cogen(\mathscr{Y})$, there is a one-to-one correspondence

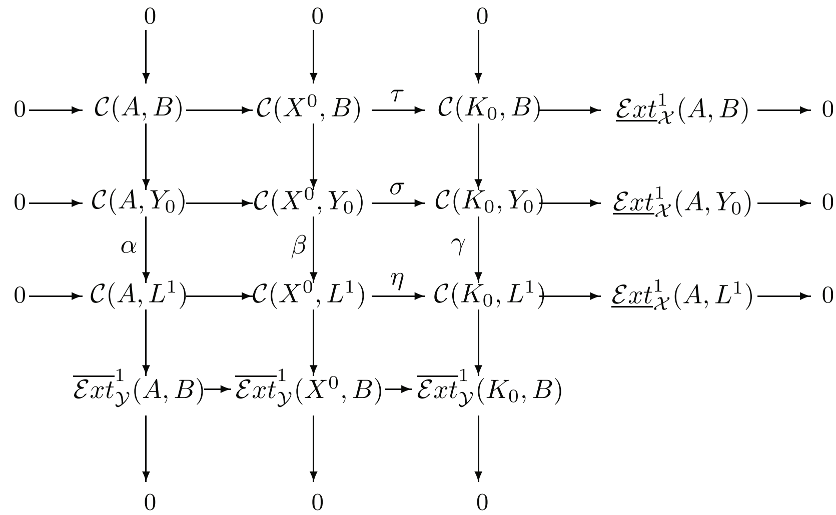
$$Y\,\bar{\mathrm{ext}}^1_{\mathscr{Y}}(A, B) \Leftrightarrow \bar{\mathscr{E}xt}^1_{\mathscr{Y}}(A, B). \tag{25}$$

Note: if $\mathscr{C}$ is a left R-module category, $\mathscr{X}$ is a subcategory of projective module category, $\mathscr{Y}$ is a subcategory of injective module category, then $\mathscr{C} = \mathrm{Gen}(\mathscr{X}) = \mathrm{Cogen}(\mathscr{Y})$.

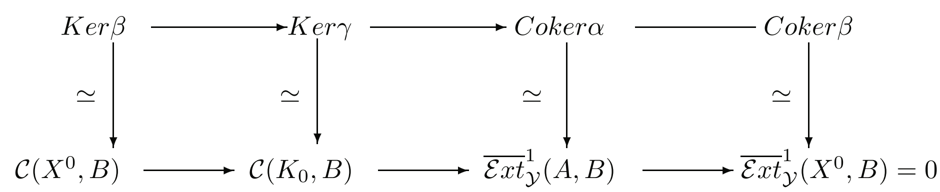
3. Stable Categories

Let us recall the stable category $\mathscr{C}/\mathscr{X}$. The objects of $\mathscr{C}/\mathscr{X}$ are the objects of $\mathscr{C}$. If A, B are objects of $\mathscr{C}/\mathscr{X}$, $(\mathscr{C}/\mathscr{X})(A, B) = \mathscr{C}(A, B)/\mathscr{X}(A, B)$.

Lemma 10. *Let $\mathscr{X}$ be a contravariantly finite subcategory, $\mathscr{Y}$ be a covariantly finite subcategory. $A \in Gen(\mathscr{X})$, $B \in Cogen($*



Scheme 4



Scheme 5

$\mathscr{Y}$), *for all* $C \in \mathscr{C}$,*there is an epimorphism* $\phi : \underline{\mathscr{E}xt}^{\,1}_{\mathscr{X}}(A, C) \longrightarrow (\mathscr{C}/\mathscr{X})(\Omega(A), C)$, *and there is an epimorphism* $\psi : \bar{\mathscr{E}xt}^{1}_{\mathscr{Y}}(C, B) \longrightarrow (\mathscr{C}/\mathscr{Y})(C, \Omega^{-1}(B))$.

Proof. (1) Let $0 \longrightarrow \Omega(A) \longrightarrow X_0 \longrightarrow A \longrightarrow 0$ be a covariantly $\mathscr{X}$-exact sequence, where$X_0 \in \mathscr{X}$, so there is a long exact sequence:

$$0 \longrightarrow \mathscr{C}(A, C) \longrightarrow \mathscr{C}(X_0, C) \longrightarrow \mathscr{C}(\Omega(A), C) \longrightarrow \underline{\mathscr{E}xt}^{\,1}_{\mathscr{X}}(A, C) \longrightarrow 0. \tag{26}$$

If let $g : \mathscr{C}(X_0, C) \longrightarrow \mathscr{C}(\Omega(A), C))$ be a corresponding morphism, then Im $(g) \subseteq \mathscr{X}(\Omega(A), C)$, thus

$$\phi : \underline{\mathscr{E}xt}^{\,1}_{\mathscr{X}}(A, C) = \mathscr{C}\frac{\Omega(A), C}{\text{Im }(g)} \longrightarrow \left(\frac{\mathscr{C}}{\mathscr{X}}\right)(A, C) = \mathscr{C}\frac{\Omega(A), C}{\mathscr{X}(\Omega(A), C)}. \tag{27}$$

(2) is obtained dually

Let $\mathscr{W} \subseteq \mathscr{X} \subseteq \mathscr{C}$ be the contravariantly finite subcategories. $\alpha : \mathscr{C} \longrightarrow \mathscr{C}/\mathscr{W}$, $\beta : \mathscr{C} \longrightarrow \mathscr{C}/\mathscr{X}$, $\gamma : \mathscr{X}/\mathscr{W} \longrightarrow (\mathscr{C}/\mathscr{W})/(\mathscr{X}/\mathscr{W})$ are the canonical functors, for any $A, B \in \mathscr{C}$, let $\alpha(A) = A_\alpha$ and $\beta(B) = B_\beta$, $f : A \longrightarrow B$ is denoted by $\alpha(f) = f_\alpha : A_\alpha \longrightarrow B_\alpha$, we have the following proposition. □

Proposition 11. *There is an isomorphism* $F : \mathscr{X}/\mathscr{W}(A_\alpha, B_\beta) \longrightarrow \mathscr{X}(A, B)/\mathscr{W}(A, B)$.

Proof. Put $f' \in \mathscr{X}/\mathscr{W}(A_\alpha, B_\beta)$, there is $f \in \mathscr{C}(A, B)$ such that $f_\alpha = f'$. Then $X \in \mathscr{X}$, such that f' factor through X_α, i.e. $h \in \mathscr{C}(A, X)$ and $t \in \mathscr{C}(X, B)$ such that $f_\alpha = t_\alpha h_\alpha = (th)_\alpha$. Thus, $f-$th $\in \mathscr{W}(A, B)$. Put

$$F : \begin{gathered}\frac{\mathscr{X}}{\mathscr{W}}(A_\alpha, B_\beta) \longrightarrow \frac{\mathscr{X}(A, B)}{\mathscr{W}(A, B)},\\ f' \mapsto f + \mathscr{W}(A, B).\end{gathered} \tag{28}$$

□

Conversely, put $f + \mathscr{W} \in \mathscr{X}(A, B)/\mathscr{W}(A, B)$, where $f \in \mathscr{X}(A, B)$. Thus, there exists $X \in \mathscr{X}$ such that $f = th$, where $h \in \mathscr{C}(A, X)$, $t \in \mathscr{C}(X, B)$. Therefore, $f_\alpha = t_\alpha h_\alpha \in \mathscr{X}/\mathscr{W}(A_\alpha, B_\beta)$. Put $G : \mathscr{X}(A, B)/\mathscr{W}(A, B) \longrightarrow \mathscr{X}/\mathscr{W}(A_\alpha, B_\beta)$ such that $G(f + \mathscr{W}(A, B)) = f_\alpha$. Hence, $GF = 1$ and $FG = 1$.

Theorem 12. $F : \mathscr{C}/\mathscr{X} \longrightarrow (\mathscr{C}/\mathscr{W})/(\mathscr{X}/\mathscr{W})$ *is an equivalence of an additional category.*

Proof. Put $F : A_\beta \mapsto A_{\gamma\alpha}$, for all $A, B \in \mathscr{C}$,

$$\begin{aligned}\frac{(\mathscr{C}/\mathscr{W})}{(\mathscr{X}/\mathscr{W})\left(A_{\gamma\alpha}, B_{\gamma\alpha}\right)} &\simeq \frac{(\mathscr{C}/\mathscr{W})\left(A_\alpha, B_\beta\right)}{(\mathscr{X}/\mathscr{W})\left(A_\alpha, B_\alpha\right)}\\ &\simeq \frac{(\mathscr{C}(A,B)/\mathscr{W}(A,B))}{(\mathscr{X}(A,B)/\mathscr{W}(A,B))}\\ &\simeq \frac{\mathscr{C}(A,B)}{\mathscr{X}(A,B)}\\ &\simeq (\mathscr{C}/\mathscr{X})\left(A_\beta, B_\beta\right).\end{aligned} \tag{29}$$

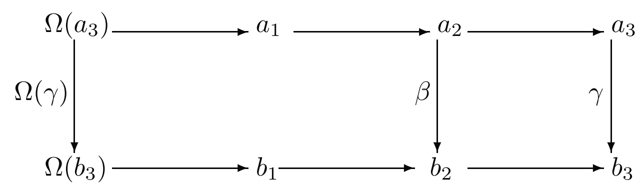
□

Let $\mathscr{C}$ be an additive category and Ω an additive covariant endofunctor on $\mathscr{C}$. Let Δ be a class of left triangles of the form $\Omega(w) \longrightarrow^f u \longrightarrow^g v \longrightarrow^h w$. The pair (Ω, Δ) is called a left triangulated structure on $\mathscr{C}$ if Δ is closed under isomorphisms and satisfies the following four axioms:

(Lt1) For any morphism $f : v \longrightarrow w$ there is a left triangle in Δ of the form $\Omega(w) \longrightarrow u \longrightarrow v \longrightarrow^f w$. For any object $u \in \mathscr{C}$, the left triangle $0 \longrightarrow u \longrightarrow^{1_u} u \longrightarrow 0$ is in Δ

(Lt2) Rotation axiom: for any left triangle $\Omega(w) \longrightarrow^f u \longrightarrow^g v \longrightarrow^h w$ in Δ, the left triangle $\Omega(v) \longrightarrow^{-\Omega(h)} \Omega(w) \longrightarrow^f u \longrightarrow^g v$ is also in Δ

(Lt3)If the figure below is the commutative diagram of left triangle inΔScheme 6



Scheme 6

Then, there is morphism $\alpha : a_1 \longrightarrow b_1$, which makes the figure above continuous to be a commutative diagram.

(Lt4) Octahedral axiom: for any two left triangles $\Omega(w) \longrightarrow^f u \longrightarrow^g v \longrightarrow^h w$ and $\Omega(z) \longrightarrow^i x \longrightarrow^l w \longrightarrow^k z$ in Δ, there is a left triangle $\Omega(z) \longrightarrow^j p \longrightarrow^m v \longrightarrow^{kh} z$ and two morphisms $\alpha : u \longrightarrow p, \beta : p \longrightarrow x$, such that the graph below is a commutative diagram Scheme 7, where the second column from the left is a left triangle in Δ

Dually, we can define the right triangulated structure (Σ, Δ') on $\mathscr{C}$, where Σ is a covariant additive endofunctor of $\mathscr{C}$ and Δ' is a class of right triangles of the form $u \longrightarrow v \longrightarrow w \longrightarrow \Sigma(u)$. which satisfies the dual right triangulated axioms.

An additive category $\mathscr{C}$ is called a left triangulated category if there is a left triangulated structure on $\mathscr{C}$. Dually, an additive category $\mathscr{C}$ is called a right triangulated category [13] if there is a right triangulated structure on it. The data of a right triangulated category was first introduced by Bernhard Keller in [14]. If the endofunctor Ω (respectively, Σ) is an autoequivalence, the left (respectively, right) triangulated category $(\mathscr{C}, \Omega, \Delta)$ (respectively, $(\mathscr{C}, \Sigma, \Delta')$) is a triangulated category. Left and right triangulated categories are natural generalization of triangulated categories.

Let $\mathscr{X}$ be a contravariantly finite subcategory of an additive category $\mathscr{C}$. A morphism $f : B \longrightarrow A$ in $\mathscr{C}$ is called an $\mathscr{X}$-epic if for any $X \in \mathscr{X}$ the induced map $\mathrm{Hom}_{\mathscr{C}}(X, g)$: $\mathrm{Hom}_{\mathscr{C}}(X, B) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X, A)$ is surjective. A morphism $f : B \longrightarrow A$ in $\mathscr{C}$ is called a special $\mathscr{X}$-epic if it is of the following form:

$$B \oplus X_A \overset{(g,p_A)}{\longrightarrow} A, \tag{30}$$

where g is a morphism of $\mathscr{C}$ and p_A is a right $\mathscr{X}$-approximation of A. By definition, a right $\mathscr{X}$-apprixomation is a special $\mathscr{X}$-epic and a special $\mathscr{X}$-epic is an $\mathscr{X}$-epic. Dually, if $\mathscr{Y}$ is covariantly finite in $\mathscr{C}$, we have the notions of an $\mathscr{Y}$-monic and a special $\mathscr{Y}$-monic. Let $\mathscr{C}$ be an additive category and $\mathscr{X}$ an additive subcategory of $\mathscr{C}$. Assume that $\mathscr{X}$ is contravariantly finite in $\mathscr{C}$ and any special $\mathscr{X}$-epic has a kernel. Then $\mathscr{C}/\mathscr{X}$ is a left triangulated category. If $\mathscr{Y}$ is covariantly finite in $\mathscr{C}$ and any special $\mathscr{Y}$-monic has a cokernel. Then, $\mathscr{C}/\mathscr{Y}$ is a right triangulated category.

Let $\mathscr{X}$ be a contravariantly finite subcategory of additional category $\mathscr{C}$, by [4], the stable category $\mathscr{C}/\mathscr{X}$ has a natural left triangulated structure. $\Omega_{\mathscr{X}} : \mathscr{C}/\mathscr{X} \longrightarrow \mathscr{C}/\mathscr{X}$ is the loop functor, which is defined as follows: $\Omega_{\mathscr{X}}(A_\beta) = \Omega(A)_\beta$, for any morphism $f : A \longrightarrow B$ in $\mathscr{C}$, $\Omega_{\mathscr{X}}(f_\beta)$: $\Omega_{\mathscr{X}}(A_\beta) \longrightarrow \Omega_{\mathscr{X}}(B_\beta)$ such that $\Omega_{\mathscr{X}}(f_\beta) = \Omega(f)_\beta$. For any covariantly $\mathscr{X}$ − exact complex in $\mathscr{C}/\mathscr{X}$,

$$0 \longrightarrow C \overset{h}{\longrightarrow} A \overset{g}{\longrightarrow} B, \tag{31}$$

if

$$0 \longrightarrow \Omega(B) \overset{i}{\longrightarrow} X_B^0 \overset{d^0}{\longrightarrow} B, \tag{32}$$

is a right $\mathscr{X}$-approximation of B, then we have the following commutative diagram Scheme 8.

So,

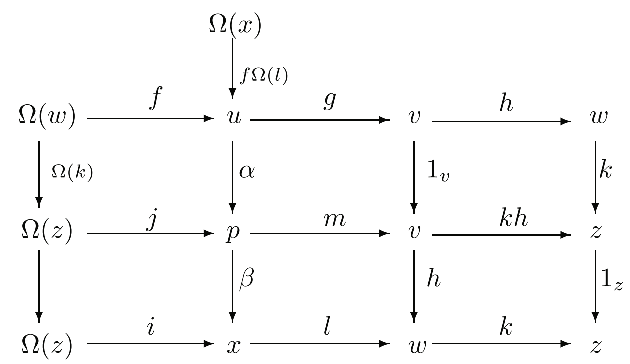
$$\Omega_{\mathscr{X}}(B_\beta) \overset{\gamma\beta}{\longrightarrow} C_\beta \overset{i\beta}{\longrightarrow} A_\beta \overset{d_\beta^0}{\longrightarrow} B_\beta, \tag{33}$$

is a standard triangle in $\mathscr{C}/\mathscr{X}$.

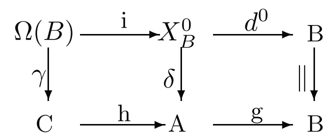
Theorem 13. *Let $\mathscr{C}$ be an abelian category, $\mathscr{X}$ and $\mathscr{Y}$ are subcategories of $\mathscr{C}$.*

(1) If $\mathscr{W} \subseteq \mathscr{X}$ is a contravariantly finite subcategory of $\mathscr{C}$ and any special X-epic has a kernel in $\mathscr{C}$, then $F : \mathscr{C}/\mathscr{X} \longrightarrow (\mathscr{C}/\mathscr{W})/(\mathscr{X}/\mathscr{W})$ is the equivalence of a left triangulated category

(2) If $\mathscr{V} \subseteq \mathscr{Y}$ is a covariantly finite subcategory of $\mathscr{C}$ and any special Y-monic has a cokernel in $\mathscr{C}$, then $\bar{F} : \mathscr{C}/\mathscr{Y} \longrightarrow (\mathscr{C}/\mathscr{V})/(\mathscr{Y}/\mathscr{V})$ is the equivalence of a right triangulated category



Scheme 7



Scheme 8

Proof. We only prove (1), and (2) can be obtained similarly. For any $A \in \mathscr{C}$,

$$F\Omega_{\mathscr{X}}\left(A_\beta\right) = F(\Omega A)_\beta = (\Omega A)_{\gamma\alpha} = \Omega_{\mathscr{X}/\mathscr{W}}\left(A_{\gamma\alpha}\right) = \Omega_{\mathscr{X}/\mathscr{W}}F\left(A_\beta\right), \tag{34}$$

if $f : A \longrightarrow B$, then

$$F\Omega_{\mathscr{X}}\left(f_\beta\right) = F(\Omega f)_\beta = (\Omega f)_{\gamma\alpha} = \Omega_{\mathscr{X}/\mathscr{W}}\left(f_{\gamma\alpha}\right) = \Omega_{\mathscr{X}/\mathscr{W}}F\left(f_\beta\right), \tag{35}$$

if

$$\Omega_{\mathscr{X}}B_\beta \xrightarrow{ch(g)} C_\beta \xrightarrow{h_\beta} A_\beta \xrightarrow{g_\beta} B_\beta, \tag{36}$$

is a triangle induced by special $\mathscr{X}$-epimorphism $g : A \longrightarrow B$ in $\mathscr{C}/\mathscr{X}$, then

$$\Omega_{\mathscr{X}/w}B_{\gamma\alpha} \longrightarrow C_{\gamma\alpha} \xrightarrow{h_{\gamma\alpha}} A_{\gamma\alpha} \xrightarrow{g_{\gamma\alpha}} B_{\gamma\alpha}, \tag{37}$$

is the left triangle in $(\mathscr{C}/\mathscr{W})/(\mathscr{X}/\mathscr{W})$, and that is

$$F\Omega_{\mathscr{X}}B_\beta \longrightarrow FC_\beta \xrightarrow{Fh_\beta} FA_\beta \xrightarrow{Fg_\beta} FB_\beta. \tag{38}$$

□

4. Conclusions

As a further generalization of the Proposition 2.8 in [2], we introduced the notion of Gen$(\mathscr{X})$ and Cogen$(\mathscr{Y})$, some sufficient conditions for $\underline{\mathscr{E}xt}^n_{\mathscr{Y}}(A,B) \simeq \bar{\mathscr{E}xt}^n_{\mathscr{X}}(A,B)$ are given. The left and right triangulated structures on the stable categories induced from some homological finite subcategories are discussed. Let $\mathscr{C}$ be an abelian category, $\mathscr{W}(\mathscr{V})$ be a contravariantly (covariantly) finite subcategory of $\mathscr{C}$ and $\mathscr{W} \subseteq \mathscr{X} \subseteq \mathscr{C}(\mathscr{V} \subseteq \mathscr{Y} \subseteq \mathscr{C})$, we have that the stable category $(\mathscr{C}/\mathscr{W})/(\mathscr{X}/\mathscr{W})$ $((\mathscr{C}/\mathscr{V})/(\mathscr{X}/\mathscr{V}))$ also has a left triangulated structure (right triangulated structure).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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