Hindawi
Journal of Mathematics
Volume 2022, Article ID 5649969, 19 pages
https://doi.org/10.1155/2022/5649969

Research Article

Upper Bound for Lebesgue Constant of Bivariate Lagrange Interpolation Polynomial on the Second Kind Chebyshev Points

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Received 31 July 2021; Accepted 10 November 2021; Published 4 January 2022

Academic Editor: Efthymios G. Tsionas

In the paper, we study the upper bound estimation of the Lebesgue constant of the bivariate Lagrange interpolation polynomial based on the common zeros of product Chebyshev polynomials of the second kind on the square $[-1,1]^2$. And, we prove that the growth order of the Lebesgue constant is $O((n+2)^2)$. This result is different from the Lebesgue constant of Lagrange interpolation polynomial on the unit disk, the growth order of which is $O(\sqrt{n})$. And, it is different from the Lebesgue constant of the Lagrange interpolation polynomial based on the common zeros of product Chebyshev polynomials of the first kind on the square $[-1,1]^2$, the growth order of which is $O((\ln n)^2)$.

1. Introduction

Chebyshev polynomials play an important role in modern developments, including orthogonal polynomials, polynomial approximation, numerical integration, and spectral methods for partial differential equations (cf. [1]). Especially, the zeros of Chebyshev polynomials are often used in the studies of one-variable Lagrange interpolation polynomials. Many good approximation properties have been obtained over the past decades (cf. [2]). Since multivariate Lagrange interpolation polynomials are difficult to express concretely, many scholars are interested to study them (cf. [3–15]).

Let $K \subset R^d$ be a nonempty compact set and V be a subspace of Π_n^d, where Π_n^d denotes the space of polynomials with d variables whose degrees do not exceed n and the dimension $\dim V = N$. Then, based on the nodes $X := \{\mathbf{x}_k\}_{k=1}^N \subset K$, the Lagrange interpolation problem related to V and X can be described as follows: for any function $f \in C(K)$, where $C(K)$ represents the continuous function space on K, we can find a unique polynomial $p \in V$ to satisfy the equation

$$p(\mathbf{x}_k) = f(\mathbf{x}_k), \quad k = 1, \ldots, N. \tag{1}$$

This polynomial is the so-called Lagrange interpolation polynomial and can be expressed as

$$L_n(f, \mathbf{x}) = \sum_{k=1}^{N} f(\mathbf{x}_k) l_k(\mathbf{x}), \tag{2}$$

where $l_k(\mathbf{x})$ are the Lagrange interpolation basis functions that satisfy the following formula:

$$l_k(\mathbf{x_j}) = \delta_{kj}. \tag{3}$$

The mapping $f \longrightarrow L_n f$ can be regarded as an operator from $C(K)$ to itself, and the norm of the operator is defined as

$$\begin{aligned}\lambda_n &= \|L_n\| \\ &= \max_{\mathbf{x}\in K} \sum_{k=1}^{N} |l_k(\mathbf{x})|,\end{aligned} \tag{4}$$

which is called the Lebesgue constant. We know that the uniform convergence of $L_n(f, \mathbf{x})$ for $f \in C(K)$ is closely related to the Lebesgue constant.

The univariate Lagrange interpolation polynomial and its Lebesgue constant have been extensively studied (cf. [2, 16]). Specially, for $K = [-1,1]$ and $V = \Pi_n^1$, the Lebesgue constant $\|L_n\| \geq C\log n$ and the order of the Lebesgue constant is $O(\log n)$ when the Chebyshev points are taken as the nodes (cf. [16]).

There are relatively few research results on multivariate Lagrange interpolation polynomials. In [3], from Berman's Theorem, it is shown that for $K = B^d$, the unit ball in R^d, $d \geq 2$, and $V = \prod_n^d$, the order of the Lebesgue constant is $O(n^{(d-1)/2})$.

It is well known that the Lagrange interpolation polynomial is closely related to cubature formula. Möller (cf. [4]) stated that for centrally symmetric weight functions, the node number of cubature formula satisfies

$$N \geq \dim \Pi_{n-1}^2 + \left[\frac{n}{2}\right] = \binom{n+1}{2} + \left[\frac{n}{2}\right], \tag{5}$$

and it is the so-called minimal cubature formula if the number of nodes reaches the lower bound. In [5], Xu studied the relationship between the compact cubature formula and the Lagrange interpolation polynomial. By using this relationship, Xu in [6] established the quadrature formula and the Lagrange interpolation polynomial on $K = [-1,1]^2$, based on the common zeros of the product Chebyshev polynomial of the first kind, which are called minimal cubature formula and Xu-type Lagrange interpolation polynomial on the first kind Chebyshev polynomial. Moreover, for $0 < p \leq \infty$, the mean convergence of the interpolation polynomial is also obtained.

Bos et al. [7] gave the numerical study of the upper bound of Lebesgue constant of the Xu-type Lagrange interpolation polynomial on the first kind Chebyshev polynomial, the order of which lies in $(\ln n)^2$, and they gave detailed proof of the order in [8]. And, Vecchia et al. [9] gave that the order of the lower bound estimate is $(\ln n)^2$.

In [10], for $K = [-1,1]^2$, we gave the compact formulae of the cubature formula and the Lagrange interpolation polynomial based on the common zeros of product Chebyshev polynomials of the second kind, which are called minimal cubature formula and Xu-type Lagrange interpolation polynomial on the second kind Chebyshev polynomial. Furthermore, for $0 < p \leq 2$, we studied the mean convergence of the Lagrange interpolation polynomials.

In this paper, we study the growth order of the Lebesgue constant and provide a direct elementary proof.

Theorem 1. *For $K = [-1,1]^2$, the upper bound estimate of the Lebesgue constant of Xu-type Lagrange interpolation polynomial on the second kind Chebyshev polynomial in [10] is*

$$\lambda_n \leq 160\sqrt{2}(n+2)^2. \tag{6}$$

Our result gives that the growth order of the Lebesgue constant of Xu-type Lagrange interpolation polynomial on the second kind Chebyshev polynomial on the square $[-1,1]^2$ is $O((n+2)^2)$. Obviously, it is different from the Lebesgue constant on the disk B^2, the growth order of which is $O(\sqrt{n})$, and is different from the Lebesgue constant of Xu-type Lagrange interpolation polynomial on the first kind Chebyshev polynomial on $[-1,1]^2$, the growth order of which is $O((\ln n)^2)$.

2. The Lebesgue Constant of Xu-Type Lagrange Interpolation Polynomial on the Second Kind Chebyshev Polynomial

In order to prove Theorem 1, by using reproducing kernel, we give the expression of the Lebesgue constant λ_n in this section.

First, we briefly introduce the Xu-type Lagrange interpolation polynomial on the second kind Chebyshev polynomial in [10].

Let $\mathbb{N}_0$ denote the set of nonnegative integers. For $n \in \mathbb{N}_0$, Chebyshev polynomial of the second kind $U_n(x)$ (cf. [17]) is defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta,\ \theta \in [0,\pi], \tag{7}$$

and they are orthogonal polynomials with respect to the second kind Chebyshev weight $w_1(x) = (2/\pi)\sqrt{1-x^2}$.

The product Chebyshev polynomial of the second kind of degree n on $[-1,1]^2$ (cf. [5]) is defined by

$$P_k^n(x,y) = U_{n-k}(x)U_k(y), \quad (x,y) \in [-1,1]^2, \quad k = 0,1,\ldots,n,\ n \in \mathbb{N}_0, \tag{8}$$

and correspondingly, the product Chebyshev weight function of the second kind is

$$\begin{aligned} W_1(x,y) &= w_1(x)w_1(y) \\ &= \frac{4}{\pi^2}\sqrt{1-x^2}\sqrt{1-y^2}, \quad (x,y) \in [-1,1]^2. \end{aligned} \tag{9}$$

For $\mathbf{x}, \mathbf{y} \in [-1,1]^2$, the reproducing kernel of the product Chebyshev polynomials is defined by

$$K_n(\mathbf{x},\mathbf{y}) = \sum_{k=0}^{n-1}\sum_{j=0}^{k} P_j^k(\mathbf{x})P_j^k(\mathbf{y}), \quad n = 1,2,\ldots. \tag{10}$$

Let $\mathbb{X} = \{\mathbf{x}_{k,l} \mid (k,l) = (2i, 2j+1), i = 1,2,\ldots,[(n+1)/2], j = 0,1,\ldots,[n/2]\} \cup \{\mathbf{x}_{k,l} \mid (k,l) = (2i+1, 2j), i = 0,1,\ldots,[n/2], j = 1,2,\ldots,[(n+1)/2]\}$, where $\mathbf{x}_{k,l} = (z_k, z_l)$, $z_k = \cos\theta_k$, $\theta_k = k\pi/(n+2)$, $k = 0,1,\ldots,n+2$, be nodes; the Xu-type Lagrange interpolation polynomial on the second kind Chebyshev polynomial is

$$
\begin{aligned}
L_n(f,\mathbf{x}) &= \sum_{k=1}^{N} f(\mathbf{x}_k) l_k(\mathbf{x}) \\
&= \sum_{k=1}^{N} f(\mathbf{x}_k) \frac{K_n^*(\mathbf{x},\mathbf{x}_k)}{K_n^*(\mathbf{x}_k,\mathbf{x}_k)} \\
&= \sum_{i=1}^{[(n+1)/2]} \sum_{j=0}^{[n/2]} \lambda_{2i,2j+1} f(\mathbf{x}_{2i,2j+1}) K_n^*(\mathbf{x},\mathbf{x}_{2i,2j+1}) + \sum_{i=0}^{[n/2]} \sum_{j=1}^{[(n+1)/2]} \lambda_{2i+1,2j} f(\mathbf{x}_{2i+1,2j}) K_n^*(\mathbf{x},\mathbf{x}_{2i+1,2j}),
\end{aligned}
\tag{11}
$$

where $\mathbf{x} = (x, y) \in [-1,1]^2$, $K_n^*(\mathbf{x},\mathbf{y}) = (1/2)[K_{n+1}(\mathbf{x},\mathbf{y}) + K_n(\mathbf{x},\mathbf{y})]$,

$$
\begin{aligned}
\lambda_{2i,2j+1} &= \left[K_n^*(\mathbf{x}_{2i,2j+1},\mathbf{x}_{2i,2j+1})\right]^{-1} \\
&= \frac{8(\sin\theta_{2i}\sin\theta_{2j+1})^2}{(n+2)^2}, \\
\lambda_{2i+1,2j} &= \left[K_n^*(\mathbf{x}_{2i+1,2j},\mathbf{x}_{2i+1,2j})\right]^{-1} \\
&= \frac{8(\sin\theta_{2i+1}\sin\theta_{2j})^2}{(n+2)^2}, \quad i,j = 0,1,\ldots,\left[\frac{n+1}{2}\right].
\end{aligned}
\tag{12}
$$

Obviously, the node number N of formula (11) is $2[(n+1)/2]([n/2]+1)$. When $n = 2m$, $N = n(n+2)/2$, which reaches the lower bound $\dim\Pi_{n-1}^2 + [n/2]$. When $n = 2m-1$, $N = (n+1)^2/2$, which is one more than the lower bound (cf. [4]).

$$
\lambda_n = \max_{\mathbf{x}\in[-1,1]^2} \sum_{k=1}^{N} \left| \frac{K_n^*(\mathbf{x},\mathbf{x}_k)}{K_n^*(\mathbf{x}_k,\mathbf{x}_k)} \right|
\tag{13}
$$

is called Lebesgue constant of Xu-type Lagrange interpolation polynomial on the second kind Chebyshev polynomial. Writing

$$
\lambda_n = \max_{\mathbf{x}\in[-1,1]^2} \left[\sum_{i=1}^{[(n+1)/2]} \sum_{j=0}^{[n/2]} \left| \frac{K_n^*(\mathbf{x},\mathbf{x}_{2i,2j+1})}{K_n^*(\mathbf{x}_{2i,2j+1},\mathbf{x}_{2i,2j+1})} \right| + \sum_{i=0}^{[n/2]} \sum_{j=1}^{[(n+1)/2]} \left| \frac{K_n^*(\mathbf{x},\mathbf{x}_{2i+1,2j})}{K_n^*(\mathbf{x}_{2i+1,2j},\mathbf{x}_{2i+1,2j})} \right| \right],
\tag{14}
$$

$$
\begin{aligned}
\Lambda_n^{(1)}(\mathbf{x}) &:= \sum_{i=1}^{[(n+1)/2]} \sum_{j=0}^{[n/2]} \left| \frac{K_n^*(\mathbf{x},\mathbf{x}_{2i,2j+1})}{K_n^*(\mathbf{x}_{2i,2j+1},\mathbf{x}_{2i,2j+1})} \right|, \\
\Lambda_n^{(2)}(\mathbf{x}) &:= \sum_{i=0}^{[n/2]} \sum_{j=1}^{[(n+1)/2]} \left| \frac{K_n^*(\mathbf{x},\mathbf{x}_{2i+1,2j})}{K_n^*(\mathbf{x}_{2i+1,2j},\mathbf{x}_{2i+1,2j})} \right|, \\
\Lambda_n(\mathbf{x}) &:= \Lambda_n^{(1)}(\mathbf{x}) + \Lambda_n^{(2)}(\mathbf{x}),
\end{aligned}
\tag{15}
$$

then

$$
\begin{aligned}
\Lambda_n &= \max_{\mathbf{x}\in[-1,1]^2} \Lambda_n(\mathbf{x}) \\
&= \max_{\mathbf{x}\in[-1,1]^2} \left[\Lambda_n^{(1)}(\mathbf{x}) + \Lambda_n^{(2)}(\mathbf{x})\right].
\end{aligned}
\tag{16}
$$

The expression of $K_n^*(\mathbf{x},\mathbf{y})$ given in [10] is

$$
K_n^*(\mathbf{x},\mathbf{y}) = \frac{1}{8\sin\varphi_1\sin\varphi_2\sin\phi_1\sin\phi_2} \sum_{k=1}^{2}\sum_{l=1}^{2} \varepsilon_i\varepsilon_j \left[F_n(\varphi_1+\varepsilon_k\phi_1, \varphi_2+\varepsilon_l\phi_2) + F_{n+1}(\varphi_1+\varepsilon_k\phi_1, \varphi_2+\varepsilon_l\phi_2)\right],
\tag{17}
$$

where $\varepsilon_k = (-1)^k$, $k = 1, 2$, $\mathbf{x} = (\cos\varphi_1, \cos\varphi_2)$, and $\mathbf{y} = (\cos\phi_1, \cos\phi_2)$. Then,

$$F_n(\tau_1, \tau_2) = \frac{[\cos(n+2)\tau_1 - \cos(n+2)\tau_2] + [\cos(n+1)\tau_1 - \cos(n+1)\tau_2]}{4(\cos\tau_1 - \cos\tau_2)}. \tag{18}$$

Lemma 1. *If $\phi_1 = \theta_{2i}$ and $\phi_2 = \theta_{2j+1}$, $i = 1, 2, \ldots, [(n+1)/2]$ and $j = 0, 1, \ldots, [n/2]$, then*

$$\sin(n+2)\frac{\varphi_1 + \varepsilon_k\phi_1 + \varphi_2 + \varepsilon_l\phi_2}{2} = (-1)^{i\varepsilon_k + j\varepsilon_l}\varepsilon_l\cos(n+2)\frac{\varphi_1 + \varphi_2}{2}, \tag{19}$$

$$\sin(n+2)\frac{\varphi_1 + \varepsilon_k\phi_1 - \varphi_2 - \varepsilon_l\phi_2}{2} = (-1)^{i\varepsilon_k - j\varepsilon_l}(-\varepsilon_l)\cos(n+2)\frac{\varphi_1 - \varphi_2}{2}. \tag{20}$$

If $\phi_1 = \theta_{2i+1}$ and $\phi_2 = \theta_{2j}$, $i = 0, 1, 2, \ldots, [n/2]$ and $j = 1, 2, \ldots, [(n+1)/2]$, then

$$\begin{aligned}\sin(n+2)\frac{\varphi_1 + \varepsilon_k\phi_1 + \varphi_2 + \varepsilon_l\phi_2}{2} &= (-1)^{i\varepsilon_k + j\varepsilon_l}\varepsilon_k\cos(n+2)\frac{\varphi_1 + \varphi_2}{2},\\ \sin(n+2)\frac{\varphi_1 + \varepsilon_k\phi_1 - \varphi_2 - \varepsilon_l\phi_2}{2} &= (-1)^{i\varepsilon_k - j\varepsilon_l}\varepsilon_k\cos(n+2)\frac{\varphi_1 - \varphi_2}{2},\end{aligned} \tag{21}$$

where $\varepsilon_k = (-1)^k$, $k = 1, 2$, and $\varepsilon_l = (-1)^l$, $l = 1, 2$.

We only prove formula (19); other formulae can be proved similarly. For $\phi_1 = \theta_{2i}$, $\phi_2 = \theta_{2j+1}$, $\varepsilon_k = (-1)^k$, and $\varepsilon_l = (-1)^l$, we have

$$\cos(n+2)\frac{\varepsilon_k\phi_1 + \varepsilon_l\phi_2}{2} = 0. \tag{22}$$

Hence,

$$\begin{aligned}\sin(n+2)\frac{\varphi_1 + \varepsilon_k\phi_1 + \varphi_2 + \varepsilon_l\phi_2}{2} &= \cos(n+2)\frac{\varphi_1 + \varphi_2}{2}\sin\left[(i\varepsilon_k + j\varepsilon_l)\pi + \frac{\pi}{2}\varepsilon_l\right]\\ &= (-1)^{i\varepsilon_k + j\varepsilon_l}\varepsilon_l\cos(n+2)\frac{\varphi_1 + \varphi_2}{2}.\end{aligned} \tag{23}$$

□

Lemma 2. *If $\phi_1 = \theta_{2i}$ and $\phi_2 = \theta_{2j+1}$, $i = 1, 2, \ldots, [(n+1)/2]$ and $j = 0, 1, \ldots, [n/2]$, then*

$$\begin{aligned}&\sum_{k=1}^{2}\sum_{l=1}^{2}\varepsilon_k\varepsilon_l\left[F_n(\varphi_1 + \varepsilon_k\phi_1, \varphi_2 + \varepsilon_l\phi_2) + F_{n+1}(\varphi_1 + \varepsilon_k\phi_1, \varphi_2 + \varepsilon_l\phi_2)\right]\\ &\quad = \sum_{k=1}^{2}\sum_{l=1}^{2}\varepsilon_k\varepsilon_l\cos^2\frac{\varphi_1 + \varepsilon_k\phi_1}{2}U_{n+1}\left(\frac{\varphi_1 + \varepsilon_k\phi_1 + \varphi_2 + \varepsilon_l\phi_2}{2}\right)\times U_{n+1}\left(\frac{\varphi_1 + \varepsilon_k\phi_1 - \varphi_2 - \varepsilon_l\phi_2}{2}\right)\\ &\quad = \sum_{k=1}^{2}\sum_{l=1}^{2}\varepsilon_k\varepsilon_l\frac{2\cos^2((\varphi_1 + \varepsilon_k\phi_1)/2)\cos((n+2)(\varphi_1 + \varphi_2)/2)\cos((n+2)(\varphi_1 - \varphi_2)/2)}{\cos(\varphi_1 + \varepsilon_k\phi_1) - \cos(\varphi_2 + \varepsilon_l\phi_2)}.\end{aligned} \tag{24}$$

If $\phi_1 = \theta_{2i+1}$ *and* $\phi_2 = \theta_{2j}, i = 0, 1, 2, \ldots, [n/2]$ *and* $j = 1, 2, \ldots, [(n+1)/2]$, *then*

$$\begin{aligned}
&\sum_{k=1}^{2}\sum_{l=1}^{2}\varepsilon_k\varepsilon_l\left[F_n\left(\varphi_1+\varepsilon_k\phi_1,\varphi_2+\varepsilon_l\phi_2\right)+F_{n+1}\left(\varphi_1+\varepsilon_k\phi_1,\varphi_2+\varepsilon_l\phi_2\right)\right] \\
&\quad=\sum_{k=1}^{2}\sum_{l=1}^{2}\varepsilon_k\varepsilon_l\cos^2\frac{\varphi_1+\varepsilon_k\phi_1}{2}U_{n+1}\left(\frac{\varphi_1+\varepsilon_k\phi_1+\varphi_2+\varepsilon_l\phi_2}{2}\right)\times U_{n+1}\left(\frac{\varphi_1+\varepsilon_k\phi_1-\varphi_2-\varepsilon_l\phi_2}{2}\right) \\
&\quad=-\sum_{k=1}^{2}\sum_{l=1}^{2}\varepsilon_k\varepsilon_l\frac{2\cos^2\left(\left(\varphi_1+\varepsilon_k\phi_1\right)/2\right)\cos\left(n+2\right)\left(\left(\varphi_1+\varphi_2\right)/2\right)\cos\left(n+2\right)\left(\left(\varphi_1-\varphi_2\right)/2\right)}{\cos\left(\varphi_1+\varepsilon_k\phi_1\right)-\cos\left(\varphi_2+\varepsilon_l\phi_2\right)}.
\end{aligned} \tag{25}$$

where $\varepsilon_k = (-1)^k, k = 1, 2,$ *and* $\varepsilon_l = (-1)^l, l = 1, 2.$

By (18), we have

$$\begin{aligned}
&F_n\left(\tau_1,\tau_2\right)+F_{n+1}\left(\tau_1,\tau_2\right) \\
&=-\frac{\cos\left(\tau_1/2\right)\left(\cos\left(\left(2n+3\right)/2\right)\tau_1+\cos\left(\left(2n+5\right)/2\right)\tau_1\right)-\cos\left(\tau_2/2\right)\left(\cos\left(\left(2n+3\right)/2\right)\tau_2+\cos\left(\left(2n+5\right)/2\right)\tau_2\right)}{4\sin\left(\left(\tau_1+\tau_2\right)/2\right)\sin\left(\left(\tau_1-\tau_2\right)/2\right)} \\
&=-\frac{\cos\left(n+2\right)\tau_1\cos^2\left(\tau_1/2\right)-\cos\left(n+2\right)\tau_2\cos^2\left(\tau_2/2\right)}{2\sin\left(\left(\tau_1+\tau_2\right)/2\right)\sin\left(\left(\tau_1-\tau_2\right)/2\right)} \\
&=-\frac{\left[\cos\left(n+2\right)\tau_1-\cos\left(n+2\right)\tau_2\right]\cos^2\left(\tau_1/2\right)+\cos\left(n+2\right)\tau_2\left(\cos^2\left(\tau_1/2\right)-\cos^2\left(\tau_2/2\right)\right)}{2\sin\left(\left(\tau_1+\tau_2\right)/2\right)\sin\left(\left(\tau_1-\tau_2\right)/2\right)} \\
&=\cos^2\frac{\tau_1}{2}U_{n+1}\left(\frac{\tau_1+\tau_2}{2}\right)U_{n+1}\left(\frac{\tau_1-\tau_2}{2}\right)+\frac{1}{2}\cos\left(n+2\right)\tau_2.
\end{aligned} \tag{26}$$

From (19) and (20), we can obtain (24). And, we can similarly prove (25). □

Lemma 3. *The following relation holds:*

$$\begin{aligned}
I&=\sum_{k=1}^{2}\sum_{l=1}^{2}\varepsilon_k\varepsilon_l\frac{\cos^2\left(\left(\varphi_1+\varepsilon_k\phi_1\right)/2\right)}{\cos\left(\varphi_1+\varepsilon_k\phi_1\right)-\cos\left(\varphi_2+\varepsilon_l\phi_2\right)} \\
&=4\sin\varphi_1\sin\phi_1\sin\varphi_2\sin\phi_2, \\
&\quad\frac{\left\{-\cos^2\left(\left(\varphi_1-\phi_1\right)/2\right)\left[\cos\left(\varphi_1+\phi_1\right)-\cos\left(\varphi_2-\phi_2\right)\right]-\cos^2\left(\left(\varphi_2-\phi_2\right)/2\right)\left[\cos\left(\varphi_1-\phi_1\right)-\cos\left(\varphi_2+\phi_2\right)\right]\right\}}{\prod_{k=1}^{2}\prod_{l=1}^{2}\left[\cos\left(\varphi_1+\varepsilon_k\phi_1\right)-\cos\left(\varphi_2+\varepsilon_l\phi_2\right)\right]},
\end{aligned} \tag{27}$$

where $\varepsilon_k = (-1)^k$, $k = 1, 2$, *and* $\varepsilon_l = (-1)^l, l = 1, 2$. *We have*

$$\begin{aligned}
I &= \sum_{k=1}^{2}\sum_{l=1}^{2}\varepsilon_k\varepsilon_l \frac{\cos^2((\varphi_1+\varepsilon_k\phi_1)/2)}{\cos(\varphi_1+\varepsilon_k\phi_1)-\cos(\varphi_2+\varepsilon_l\phi_2)} \\
&= \cos^2\frac{\varphi_1+\phi_1}{2}\cdot\frac{-\cos(\varphi_2-\phi_2)+\cos(\varphi_2+\phi_2)}{[\cos(\varphi_1+\phi_1)-\cos(\varphi_2+\phi_2)][\cos(\varphi_1+\phi_1)-\cos(\varphi_2-\phi_2)]} \\
&\quad -\cos^2\frac{\varphi_1-\phi_1}{2}\cdot\frac{-\cos(\varphi_2-\phi_2)+\cos(\varphi_2+\phi_2)}{[\cos(\varphi_1-\phi_1)-\cos(\varphi_2+\phi_2)][\cos(\varphi_1-\phi_1)-\cos(\varphi_2-\phi_2)]} \\
&= -2\sin\varphi_2\sin\phi_2\left\{\frac{\cos^2((\varphi_1+\phi_1)/2)}{[\cos(\varphi_1+\phi_1)-\cos(\varphi_2+\phi_2)][\cos(\varphi_1+\phi_1)-\cos(\varphi_2-\phi_2)]}\right. \\
&\quad \left.-\frac{\cos^2((\varphi_1-\phi_1)/2)}{[\cos(\varphi_1-\phi_1)-\cos(\varphi_2+\phi_2)][\cos(\varphi_1-\phi_1)-\cos(\varphi_2-\phi_2)]}\right\} \\
&= 2\sin\varphi_1\sin\phi_1\sin\varphi_2\sin\phi_2\left\{\frac{1}{[\cos(\varphi_1+\phi_1)-\cos(\varphi_2+\phi_2)][\cos(\varphi_1+\phi_1)-\cos(\varphi_2-\phi_2)]}\right. \\
&\quad \left.-\frac{4\cos^2((\varphi_1-\phi_1)/2)(\cos\varphi_1\cos\phi_1-\cos\varphi_2\cos\phi_2)}{\prod_{k=1}^{2}\prod_{l=1}^{2}[\cos(\varphi_1+\varepsilon_k\phi_1)-\cos(\varphi_2+\varepsilon_l\phi_2)]}\right\} \\
&= 4\sin\varphi_1\sin\phi_1\sin\varphi_2\sin\phi_2, \\
&\quad \frac{\{-\cos^2((\varphi_1-\phi_1)/2)[\cos(\varphi_1+\phi_1)-\cos(\varphi_2-\phi_2)]-\cos^2((\varphi_2-\phi_2)/2)[\cos(\varphi_1-\phi_1)-\cos(\varphi_2+\phi_2)]\}}{\prod_{k=1}^{2}\prod_{l=1}^{2}[\cos(\varphi_1+\varepsilon_k\phi_1)-\cos(\varphi_2+\varepsilon_l\phi_2)]}.
\end{aligned} \tag{28}$$

By Lemmas 2 and 3, the following result can be obtained. □

Lemma 4. *If* $\mathbf{x} = (\cos\varphi_1, \cos\varphi_2)$, $\mathbf{y} = (\cos\phi_1, \cos\phi_2)$, $\phi_1 = \theta_{2i}$, *and* $\phi_2 = \theta_{2j+1}$, $i = 1, 2, \ldots, [(n+1)/2]$ *and* $j = 0, 1, \ldots, [n/2]$, *then*

$$\begin{aligned}
K_n^*(\mathbf{x},\mathbf{y}) &= \frac{\cos(n+2)((\varphi_1+\varphi_2)/2)\cos(n+2)((\varphi_1-\varphi_2)/2)}{4\sin\varphi_1\sin\varphi_2\sin\phi_1\sin\phi_2} I \\
&= \cos(n+2)\frac{\varphi_1+\varphi_2}{2}\cos(n+2)\frac{\varphi_1-\varphi_2}{2}, \\
&\quad \frac{\{-\cos^2((\varphi_1-\phi_1)/2)[\cos(\varphi_1+\phi_1)-\cos(\varphi_2-\phi_2)]-\cos^2((\varphi_2-\phi_2)/2)[\cos(\varphi_1-\phi_1)-\cos(\varphi_2+\phi_2)]\}}{\prod_{k=1}^{2}\prod_{l=1}^{2}[\cos(\varphi_1+\varepsilon_k\phi_1)-\cos(\varphi_2+\varepsilon_l\phi_2)]}.
\end{aligned} \tag{29}$$

If $\phi_1 = \theta_{2i+1}$ *and* $\phi_2 = \theta_{2j}$, $i = 0, 1, 2, \ldots,$ $[n/2]$ *and* $j = 1, 2, \ldots, [(n+1)/2]$, *then*

$$K_n^*(\mathbf{x},\mathbf{y}) = \frac{-\cos(n+2)((\varphi_1+\varphi_2)/2)\cos(n+2)((\varphi_1-\varphi_2)/2)}{4\sin\varphi_1\sin\varphi_2\sin\phi_1\sin\phi_2} I$$
$$= \cos(n+2)\frac{\varphi_1+\varphi_2}{2}\cos(n+2)\frac{\varphi_1-\varphi_2}{2}\cdot \frac{\left\{\cos^2((\varphi_1-\phi_1)/2)[\cos(\varphi_1+\phi_1)-\cos(\varphi_2-\phi_2)] + \cos^2((\varphi_2-\phi_2)/2)[\cos(\varphi_1-\phi_1)-\cos(\varphi_2+\phi_2)]\right\}}{\prod_{k=1}^{2}\prod_{l=1}^{2}[\cos(\varphi_1+\varepsilon_k\phi_1)-\cos(\varphi_2+\varepsilon_l\phi_2)]}. \tag{30}$$

Furthermore, we can obtain the following lemma.

Lemma 5. *Let* $\mathbf{x} = (\cos\varphi_1, \cos\varphi_2)$ *and* $\mathbf{y} = (\cos\phi_1, \cos\phi_2)$. *If* $\phi_1 = \theta_{2i}$ *and* $\phi_2 = \theta_{2j+1}$, $i = 1, 2, \ldots, [(n+1)/2]$ *and* $j = 0, 1, \ldots, [n/2]$, *then*

$$\Lambda_n^1(\mathbf{x}) = \sum_{i=1}^{[(n+1)/2]}\sum_{j=0}^{[n/2]}\left|\frac{(\sin\phi_1\sin\phi_2)^2}{(n+2)^2}\cos(n+2)\frac{\varphi_1+\varphi_2}{2}\cos(n+2)\frac{\varphi_1-\varphi_2}{2}\left(\frac{\cos^2((\varphi_1-\phi_1)/2)}{\sin((\varphi_1+\phi_1+\varphi_2+\phi_2)/2)\sin((\varphi_1+\phi_1-\varphi_2-\phi_2)/2)\sin((\varphi_1-\phi_1+\varphi_2+\phi_2)/2)\sin((\varphi_1-\phi_1-\varphi_2-\phi_2)/2)}\right.\right.$$
$$\times\frac{1}{\sin((\varphi_1-\phi_1+\varphi_2-\phi_2)/2)\sin((\varphi_1-\phi_1-\varphi_2+\phi_2)/2)} + \frac{\cos^2(\varphi_2-\phi_2/2)}{\sin((\varphi_1+\phi_1+\varphi_2+\phi_2)/2)\sin((\varphi_1+\phi_1-\varphi_2-\phi_2)/2)}$$
$$\left.\left.\times\frac{1}{\sin((\varphi_1+\phi_1+\varphi_2-\phi_2)/2)\sin((\varphi_1+\phi_1-\varphi_2+\phi_2)/2)\sin((\varphi_1-\phi_1+\varphi_2-\phi_2)/2)\sin((\varphi_1-\phi_1-\varphi_2+\phi_2)/2)}\right)\right|. \tag{31}$$

If $\phi_1 = \theta_{2i+1}$ *and* $\phi_2 = \theta_{2j}$, $i = 0, 1, \ldots, (n/2)$ *and* $j = 1, 2\ldots, [(n+1)/2]$, *then*

$$\Lambda_n^2(\mathbf{x}) = \sum_{i=0}^{[n/2]}\sum_{j=1}^{[(n+1)/2]}\left|\frac{(\sin\phi_1\sin\phi_2)^2}{(n+2)^2}\cos(n+2)\frac{\varphi_1+\varphi_2}{2}\cos(n+2)\frac{\varphi_1-\varphi_2}{2}\left(\frac{\cos^2((\varphi_1-\phi_1)/2)}{\sin((\varphi_1+\phi_1+\varphi_2+\phi_2)/2)\sin((\varphi_1+\phi_1-\varphi_2-\phi_2)/2)\sin((\varphi_1-\phi_1+\varphi_2+\phi_2)/2)\sin((\varphi_1-\phi_1-\varphi_2-\phi_2)/2)}\right.\right.$$
$$\times\frac{1}{\sin((\varphi_1-\phi_1+\varphi_2-\phi_2)/2)\sin((\varphi_1-\phi_1-\varphi_2+\phi_2)/2)} + \frac{\cos^2((\varphi_2-\phi_2)/2)}{\sin((\varphi_1+\phi_1+\varphi_2+\phi_2)/2)\sin((\varphi_1+\phi_1-\varphi_2-\phi_2)/2)}$$
$$\left.\left.\times\frac{1}{\sin((\varphi_1+\phi_1+\varphi_2-\phi_2)/2)\sin((\varphi_1+\phi_1-\varphi_2+\phi_2)/2)\sin((\varphi_1-\phi_1+\varphi_2-\phi_2)/2)\sin((\varphi_1-\phi_1-\varphi_2+\phi_2)/2)}\right)\right|. \tag{32}$$

3. Proof of Theorem 1

The proof of Theorem 1 is given in this section. And, since the estimates of $\Lambda_n^1(\mathbf{x})$ and $\Lambda_n^2(\mathbf{x})$ are similar, we need to only estimate $\Lambda_n^1(\mathbf{x})$.

Setting $\tau_1 = (\varphi_1 + \varphi_2)/2 \in [0, \pi]$, $\tau_2 = (\varphi_1 - \varphi_2)/2 \in [-(\pi/2), (\pi/2)]$, we have

$$\frac{K_n^*(\mathbf{x}, \mathbf{x}_{2i,2j+1})}{K_n^*(\mathbf{x}_{2i,2j+1}, \mathbf{x}_{2i,2j+1})} := A_{i,j}^{(1)} + B_{i,j}^{(1)} := A_{i,j}^{(11)} A_{i,j}^{(12)} + B_{i,j}^{(11)} B_{i,j}^{(12)}, \tag{33}$$

where

$$\begin{aligned}
A_{i,j}^{(11)} &= \frac{\sin\theta_{2i}\sin\theta_{2j+1}}{n+2}\cdot\frac{\cos(n+2)\tau_2\cos((\varphi_1-\theta_{2i})/2)}{\sin(\tau_2+(\theta_{2i}-\theta_{2j+1})/2)\sin(\tau_2-(\theta_{2i}+\theta_{2j+1})/2)\sin(\tau_2-(\theta_{2i}-\theta_{2j+1})/2)},\\
A_{i,j}^{(12)} &= \frac{\sin\theta_{2i}\sin\theta_{2j+1}}{n+2}\cdot\frac{\cos(n+2)\tau_1\cos((\varphi_1-\theta_{2i})/2)}{\sin(\tau_1+(\theta_{2i}+\theta_{2j+1})/2)\sin(\tau_1-(\theta_{2i}-\theta_{2j+1})/2)\sin(\tau_1-(\theta_{2i}+\theta_{2j+1})/2)},\\
B_{i,j}^{(11)} &= \frac{\sin\theta_{2i}\sin\theta_{2j+1}}{n+2}\cdot\frac{\cos(n+2)\tau_2\cos((\varphi_2-\theta_{2j+1})/2)}{\sin(\tau_2+(\theta_{2i}-\theta_{2j+1})/2)\sin(\tau_2+(\theta_{2i}+\theta_{2j+1})/2)\sin(\tau_2-(\theta_{2i}-\theta_{2j+1})/2)},\\
B_{i,j}^{(12)} &= \frac{\sin\theta_{2i}\sin\theta_{2j+1}}{n+2}\cdot\frac{\cos(n+2)\tau_1\cos((\varphi_2-\theta_{2j+1})/2)}{\sin(\tau_1+(\theta_{2i}+\theta_{2j+1})/2)\sin(\tau_1+(\theta_{2i}-\theta_{2j+1})/2)\sin(\tau_1-(\theta_{2i}+\theta_{2j+1})/2)}.
\end{aligned} \tag{34}$$

To prove Theorem 1, we first prove some lemmas.

Lemma 6. *If $|\tau_2| \in [0, (\theta_1/2)]$ and $\tau_1 \in [0, (\theta_1/2)] \cup [\pi - (\theta_1/2), \pi]$, we have the following:*

(1) For $0 \le j \le i-2$ or $i+1 \le j \le [n/2]$,

$$\left|A_{i,j}^{(11)}\right| \le \frac{4\sqrt{\sin\theta_{2i}\sin\theta_{2j+1}}\left|\cos(n+2)\tau_2\right|}{(n+2)\sin^2((\theta_{2i}-\theta_{2j+1})/2)}. \tag{35}$$

For $j = i$ or $j = i-1$,

$$\left|A_{i,j}^{(11)}\right| \le 2(n+2). \tag{36}$$

(2) For $0 \le j \le i-2$,

$$\left|A_{i,j}^{(12)}\right| \le \frac{8\left|\cos(n+2)\tau_1\right|}{(n+2)\left|\sin((\theta_{2i}-\theta_{2j+1})/2)\right|}. \tag{37}$$

For $i+1 \le j \le [n/2]$,

$$\left|A_{i,j}^{(12)}\right| \le \frac{4\left|\cos(n+2)\tau_1\right|}{(n+2)\left|\sin((\theta_{2i}-\theta_{2j+1})/2)\right|}. \tag{38}$$

For $j = i$ or $j = i-1$,

$$\left|A_{i,j}^{(12)}\right| \le 4. \tag{39}$$

(1) We first consider the case of $\tau_2 \in [0, (\theta_1/2)]$.

For $0 \le j \le i-2$, since $((\theta_{2i}-\theta_{2j+1})/2) \le ((\theta_{2i}-\theta_{2j+1})/2) + \tau_2 \le (\pi/2)$, we obtain

$$\sin\left(\tau_2 + \frac{\theta_{2i}-\theta_{2j+1}}{2}\right) \ge \sin\left(\frac{\theta_{2i}-\theta_{2j+1}}{2}\right). \tag{40}$$

Noticing that $(\theta_{2i}+\theta_{2j+1})/2 - \theta_1/2 \le (\theta_{2i}+\theta_{2j+1})/2 - \tau_2 \le (\theta_{2i}+\theta_{2j+1})/2 \le \pi$ and $\sin x$ is a convex function on $[0, \pi]$, we have

$$\sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2} - \tau_2\right) \ge \min\left\{\sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2}\right), \sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2} - \frac{\theta_1}{2}\right)\right\} \ge \frac{1}{2}\sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2}\right) \ge \frac{1}{2}\sqrt{\sin\theta_{2i}\sin\theta_{2j+1}}. \tag{41}$$

For $(\theta_{2i}-\theta_{2j+1})/2 - \theta_1/2 \le (\theta_{2i}-\theta_{2j+1})/2 - \tau_2 \le (\theta_{2i}-\theta_{2j+1})/2 \le \pi/2$, then

$$\sin\left(\frac{\theta_{2i}-\theta_{2j+1}}{2} - \tau_2\right) \ge \sin\left(\frac{\theta_{2i}-\theta_{2j+1}}{2} - \frac{\theta_1}{2}\right) \ge \frac{1}{2}\sin\left(\frac{\theta_{2i}-\theta_{2j+1}}{2}\right). \tag{42}$$

By (40)–(42), we can obtain (35).

For $i+1 \le j \le [n/2]$, *since* $(\theta_3/2) \le (\theta_{2j+1}-\theta_{2i})/2 - \theta_1/2 \le (\theta_{2j+1}-\theta_{2i})/2 - \tau_2 < (\pi/2)$, *then*

$$\sin\left(\frac{\theta_{2j+1}-\theta_{2i}}{2}-\tau_2\right) \ge \sin\left(\frac{\theta_{2j+1}-\theta_{2i}}{2}-\frac{\theta_1}{2}\right) \ge \frac{1}{2}\sin\left(\frac{\theta_{2j+1}-\theta_{2i}}{2}\right). \quad (43)$$

For $(\theta_{2j+1}-\theta_{2i})/2 \le (\theta_{2j+1}-\theta_{2i})/2+\tau_2 \le (\theta_{2j+1}-\theta_{2i})/2+\theta_1/2 \le \theta_j \le (\pi/2)$, *we have*

$$\sin\left(\frac{\theta_{2j+1}-\theta_{2i}}{2}+\tau_2\right) \ge \sin\left(\frac{\theta_{2j+1}-\theta_{2i}}{2}\right), \quad (44)$$

and combining (41), we can obtain (35).

For $j=i$ *or* $j=i-1$, *it is easy to prove that* $|A_{i,j}^{(11)}| \le 2(n+2)$.

When $\tau_2 \in [-(\theta_1/2), 0]$, *by setting* $\tau_2' = -\tau_2 \in [0, (\theta_1/2)]$, *we can similarly prove (35) and (36).*

(2) If $\tau_1 \in [0, (\theta_1/2)]$, *for* $0 \le j \le i-2$, *since* $(\theta_{2i}+\theta_{2j+1})/2 \le (\theta_{2i}+\theta_{2j+1})/2+\tau_1 \le (\theta_{2i}+\theta_{2j+1})/2+\theta_1/2 \le \pi-\theta_2$, *then*

$$\sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2}+\tau_1\right) \ge \min\left\{\sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2}\right), \sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2}+\frac{\theta_1}{2}\right)\right\} \ge \frac{1}{2}\sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2}\right) \ge \frac{1}{2}\sqrt{\sin\theta_{2i}\sin\theta_{2j+1}}. \quad (45)$$

And, we have

$$\begin{aligned}\sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2}-\tau_1\right) &\ge \frac{1}{2}\sqrt{\sin\theta_{2i}\sin\theta_{2j+1}},\\ \sin\left(\frac{\theta_{2i}-\theta_{2j+1}}{2}-\tau_1\right) &\ge \frac{1}{2}\sin\left(\frac{\theta_{2i}-\theta_{2j+1}}{2}\right),\end{aligned} \quad (46)$$

so we obtain (37).

For $i+1 \le j \le [n/2]$, *considering*

$$\begin{aligned}\sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2}-\tau_1\right) &\ge \frac{1}{2}\sqrt{\sin\theta_{2i}\sin\theta_{2j+1}},\\ \sin\left(\frac{\theta_{2j+1}-\theta_{2i}}{2}+\tau_1\right) &\ge \sin\left(\frac{\theta_{2j+1}-\theta_{2i}}{2}\right),\end{aligned} \quad (47)$$

we have (38).

For $j=i$ *or* $j=i-1$, *it is easy to prove that* $|A_{i,j}^{(12)}| \le 4$.

When $\tau_1 \in [\pi-(\theta_1/2), \pi]$, *setting* $\tau_1' = \pi-\tau_1 \in [0, (\theta_1/2)]$, *similar to the case of* $\tau_1 \in [0, (\theta_1/2)]$, *the estimation of* $|A_{i,j}^{(12)}|$ *can be obtained.*

In the same way, we can obtain the following estimates of $|B_{i,j}^{(11)}|$ *and* $|B_{i,j}^{(12)}|$.

Lemma 7. *If* $|\tau_2| \in [0, (\theta_1/2)]$ *and* $\tau_1 \in [0, (\theta_1/2)] \cup [\pi-(\theta_1/2), \pi]$, *then we have the following:*

(1) For $0 \le j \le i-2$ *or* $i+1 \le j \le [n/2]$,

$$\left|B_{i,j}^{(11)}\right| \le \frac{4\sqrt{\sin\theta_{2i}\sin\theta_{2j+1}}\left|\cos(n+2)\tau_2\right|}{(n+2)\sin^2\left(\left(\theta_{2i}-\theta_{2j+1}\right)/2\right)}. \quad (48)$$

For $j=i$ *or* $j=i-1$, $|B_{i,j}^{(11)}| \le 2(n+2)$.

(2) For $0 \le j \le i-2$,

$$\left|B_{i,j}^{(12)}\right| \le \frac{4\left|\cos(n+2)\tau_1\right|}{(n+2)\left|\sin\left(\left(\theta_{2i}-\theta_{2j+1}\right)/2\right)\right|}. \quad (49)$$

For $i+1 \le j \le [n/2]$,

$$\left|B_{i,j}^{(12)}\right| \le \frac{8\left|\cos(n+2)\tau_1\right|}{(n+2)\left|\sin\left(\left(\theta_{2i}-\theta_{2j+1}\right)/2\right)\right|}. \quad (50)$$

For $j=i$ *or* $j=i-1$, $|B_{i,j}^{(12)}| \le 4$.

Lemma 8. *If* $\tau_1 \in [(\theta_1/2), (\pi/4)] \cup [(3\pi/4), \pi-(\theta_1/2)]$, *then* $|A_{i,j}^{(12)}| \le 4\sqrt{2}$, $|B_{i,j}^{(12)}| \le 4\sqrt{2}$.

(1) We first prove the case of $\tau_1 \in [(\theta_1/2), (\pi/4)]$. *If* $0 < (\theta_{2i}+\theta_{2j+1})/2 < \pi/2$, *we have*

$$\sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2}+\tau_1\right) \ge \frac{\sqrt{2}}{2}\sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2}\right) \ge \frac{\sqrt{2}}{4}\sin\theta_{2i}, \quad (51)$$

$$\begin{aligned}&\left|\frac{\cos(n+2)\tau_1\sin\theta_{2j+1}}{\sin\left(\tau_1-\left(\theta_{2i}-\theta_{2j+1}\right)/2\right)\sin\left(\tau_1-\left(\theta_{2i}+\theta_{2j+1}\right)/2\right)}\right|\\ &\le \left|\frac{\cos(n+2)\tau_1\cos\left(\tau_1-\left(\theta_{2i}+\theta_{2j+1}\right)/2\right)}{\sin\left(\tau_1-\left(\theta_{2i}+\theta_{2j+1}\right)/2\right)}\right|\\ &\quad+\left|\frac{\cos(n+2)\tau_1\cos\left(\tau_1-\left(\theta_{2i}-\theta_{2j+1}\right)/2\right)}{\sin\left(\tau_1-\left(\theta_{2i}-\theta_{2j+1}\right)/2\right)}\right| \le 2(n+2).\end{aligned} \quad (52)$$

If $(\pi/2) \le (\theta_{2i}+\theta_{2j+1})/2 \le \pi$, *we obtain*

$$\sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2}-\tau_1\right)\geq\frac{\sqrt{2}}{2}\sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2}\right)\geq\frac{\sqrt{2}}{4}\sin\,\theta_{2j+1}, \tag{53}$$

$$\left|\frac{\cos(n+2)\tau_1\sin\,\theta_{2i}}{\sin\left(\tau_1+\left(\theta_{2i}+\theta_{2j+1}\right)/2\right)\sin\left(\tau_1-\left(\theta_{2i}-\theta_{2j+1}\right)/2\right)}\right|\leq 2(n+2). \tag{54}$$

In summary, we can obtain $|A_{i,j}^{(12)}|\leq 4\sqrt{2}$.

(2) *If* $\tau_1\in[(3\pi/4),\pi-(\theta_1/2)]$, *by setting* $\tau_1'=\pi-\tau_1\in[\theta_1/2,\pi/2]$ *and the same as the case of* $\tau_1\in[(\theta_1/2),(\pi/4)]$, *the conclusion of the lemma can be proved.*

Similar to Lemma 8, we can obtain the following conclusion. □

Lemma 9. *If* $\tau_1\in[(\theta_1/2),(\pi/3)]\cup[(2\pi/3),\pi-(\theta_1/2)]$, *then* $|A_{i,j}^{(12)}|\leq 8, |B_{i,j}^{(12)}|\leq 8$.

Lemma 10. *If* $|\tau_2|\in[(\theta_1/2),(\pi/4)]$, *then* $|A_{i,j}^{(11)}|\leq 4, |B_{i,j}^{(11)}|\leq 4$.

The estimates of $|A_{i,j}^{(11)}|$ *and* $|B_{i,j}^{(11)}|$ *are similar, so we only take* $|A_{i,j}^{(11)}|$ *as an example.*

We first prove the case of $\tau_2\in[(\theta_1/2),(\pi/4)]$. *For every i, there is* j_0 *so that* $|\tau_2-(\theta_{2i}+\theta_{2j_0+1})/2|\leq(\theta_1/2)$ *holds, that is,* $(\theta_{2i}+\theta_{2j_0+1})/2-(\theta_1/2)\leq\tau_2\leq(\theta_{2i}+\theta_{2j_0+1})/2+(\theta_1/2)$.

(i) *For* $j=j_0$, *we have* $\theta_{2i}-(\theta_1/2)\leq\tau_2+(\theta_{2i}-\theta_{2j_0+1})/2\leq\theta_{2i}+(\theta_1/2)$; *then,*

$$\sin\left(\tau_2+\frac{\theta_{2i}-\theta_{2j_0+1}}{2}\right)\geq\min\left\{\sin\left(\theta_{2i}+\frac{\theta_1}{2}\right),\ \sin\left(\theta_{2i}-\frac{\theta_1}{2}\right)\right\}\geq\frac{1}{2}\sin\,\theta_{2i}. \tag{55}$$

Because of $\theta_{2j_0+1}-(\theta_1/2)\leq\tau_2-(\theta_{2i}-\theta_{2j_0+1})/2\leq\theta_{2j_0+1}+(\theta_1/2)$, *we have*

$$\sin\left(\tau_2-\frac{\theta_{2i}-\theta_{2j_0+1}}{2}\right)\geq\min\left\{\sin\left(\theta_{2j_0+1}+\frac{\theta_1}{2}\right),\ \sin\left(\theta_{2j_0+1}-\frac{\theta_1}{2}\right)\right\}\geq\frac{1}{2}\sin\,\theta_{2j_0+1}, \tag{56}$$

and on account of $|(\cos(n+2)\tau_2/\sin(\tau_2-(\theta_{2i}+\theta_{2j+1})/2)|\leq n+2$, *we obtain* $|A_{i,j}^{(11)}|\leq 4$.

(ii) *The remaining part will be discussed in two situations:* $j_0\leq i-1$ *and* $j_0\geq i$.

(a) *The case* $j_0\leq i-1$.

For $j\leq j_0-1\leq i-2$, *since* $(\pi/2)\geq\tau_2-(\theta_{2i}-\theta_{2j+1})/2\geq\theta_{2j+1}$, *we have* $\sin(\tau_2-(\theta_{2i}-\theta_{2j+1})/2)\geq\sin\,\theta_{2j+1}$. *And, because of*

$$\left|\frac{\sin\,\theta_{2i}\cos(n+2)\tau_2}{\sin\left(\tau_2+\left(\theta_{2i}-\theta_{2j+1}\right)/2\right)\sin\left(\tau_2-\left(\theta_{2i}+\theta_{2j+1}\right)/2\right)}\right|\leq 2(n+2), \tag{57}$$

we obtain $|A_{i,j}^{(11)}|\leq 2$.

For $j_0+1\leq j\leq i-1\leq[(n+1)/2]-1$, *considering that* $\theta_i\leq\tau_2+(\theta_{2i}-\theta_{2j+1})/2<\theta_{2i}$, *we have* $\sin(\tau_2+(\theta_{2i}-\theta_{2j+1})/2)\geq\min\{\sin\,\theta_{2i},\ \sin\,\theta_i\}$, *so*

$$\left|\frac{\sin\,\theta_{2j+1}\cos(n+2)\tau_2}{\sin\left(\tau_2-\left(\theta_{2j+1}-\theta_{2i}\right)/2\right)\sin\left(\tau_2-\left(\theta_{2i}-\theta_{2j+1}\right)/2\right)}\right|\leq 2(n+2). \tag{58}$$

And, using the following (61), similar to (57), we obtain $|A_{i,j}^{(11)}| \le 4$.

For $j \ge i \ge j_0 + 1$, *since* $(\theta_{2j+1}/2) \le \tau_2 + (\theta_{2j+1} - \theta_{2i})/2 \le \theta_{2j+1} - (\theta_1/2)$, *then*

$$\sin\left(\tau_2 + \frac{\theta_{2j+1} - \theta_{2i}}{2}\right) \ge \min\left\{\sin\frac{\theta_{2j+1}}{2},\ \sin\theta_{2j+1}\right\}. \tag{59}$$

Therefore,

$$\left|\frac{\sin\theta_{2j+1}}{\sin\left(\tau_2 + \left(\theta_{2j+1} - \theta_{2i}\right)/2\right)}\right| \le 2. \tag{60}$$

And, on account of (57), we have $|A_{i,j}^{(11)}| \le 4$.

(b) The case $j_0 \ge i$.

For $0 \le j \le i - 1 \le j_0 - 1$, *on account of* $\theta_{2i} \le \tau_2 + (\theta_{2i} - \theta_{2j+1})/2 \le (\pi/2)$, *we obtain* $\sin(\tau_2 + (\theta_{2i} - \theta_{2j+1})/2) \ge \sin\theta_{2i}$. *And, considering that*

$$\left|\frac{\sin\theta_{2j+1}\cos(n+2)\tau_2}{\sin\left(\tau_2 - \left(\theta_{2j+1} + \theta_{2i}\right)/2\right)\sin\left(\tau_2 - \left(\theta_{2i} - \theta_{2j+1}\right)/2\right)}\right| \le 2(n+2), \tag{61}$$

we obtain $|A_{i,j}^{(11)}| \le 2$.

For $i \le j < j_0 - 1$, *on account of* $(\pi/4) \ge \tau_2 - (\theta_{2j+1} - \theta_{2i})/2 > \theta_{2i}$, *we have* $\sin(\tau_2 - (\theta_{2j+1} - \theta_{2i})/2) \ge \sin\theta_{2i}$. *Since* $\theta_{2j+1} \le \tau_2 + (\theta_{2j+1} - \theta_{2i})/2 < (\pi/2)$, *we obtain* $\sin(\tau_2 + (\theta_{2j+1} - \theta_{2i})/2) \ge \sin\theta_{2j+1}$. *And, considering that* $|\cos(n+2)\tau_2/\sin(\tau_2 - (\theta_{2i} + \theta_{2j+1})/2)| \le n+2$, *we obtain* $|A_{i,j}^{(11)}| \le 1$.

For $i + 1 \le j_0 + 1 \le j \le [n/2]$, *since* $(\theta_{2j+1}/2) \le \tau_2 + (\theta_{2j+1} - \theta_{2i})/2 < \theta_{2j+1}$, *we obtain*

$$\sin\left(\tau_2 + \frac{\theta_{2j+1} - \theta_{2i}}{2}\right) \ge \min\left\{\sin\theta_{2j+1},\ \sin\frac{\theta_{2j+1}}{2}\right\}. \tag{62}$$

Thus,

$$\left|\frac{\sin\theta_{2j+1}}{\sin\left(\tau_2 + \left(\theta_{2i} - \theta_{2j+1}\right)/2\right)}\right| \le \max\left\{\left|\frac{\sin\theta_{2j+1}}{\sin\theta_{2j+1}}\right|, \left|\frac{\sin\theta_{2j+1}}{\sin\left(\theta_{2j+1}/2\right)}\right|\right\} \le 2. \tag{63}$$

And on account of (57), we have $|A_{i,j}^{(11)}| \le 4$.

To sum up, if $\tau_2 \in [\theta_1/2, \pi/4]$, *we obtain* $|A_{i,j}^{(11)}| \le 4$.

When $\tau_2 \in [-(\pi/4), -(\theta_1/2)]$, *by setting* $\tau_2' = -\tau_2 \in [(\theta_1/2), (\pi/2)]$, *we can similarly prove the conclusion.* □

Lemma 11. *If* $|\tau_2| \in [(\pi/4), (\pi/2)]$, *then* $|A_{i,j}^{(11)}| \le 4\sqrt{2}$, $|B_{i,j}^{(11)}| \le 4\sqrt{2}$.

The estimates of $|A_{i,j}^{(11)}|$ *and* $|B_{i,j}^{(11)}|$ *are similar, so we only take* $|A_{i,j}^{(11)}|$ *as an example.*

For $j \le i - 1$, *we*

$$\sin\left(\tau_2 + \frac{\theta_{2i} - \theta_{2j+1}}{2}\right) \ge \frac{\sqrt{2}}{2}\cos\frac{\theta_{2i} - \theta_{2j+1}}{2} \ge \frac{\sqrt{2}}{4}\sin\theta_{2i}. \tag{64}$$

And, using (61), we obtain $|A_{i,j}^{(11)}| \le 4\sqrt{2}$.

For $i \le j$, *we have* $\sin(\tau_2 + (\theta_{2j+1} - \theta_{2i})/2) \ge (\sqrt{2}/4)\sin\theta_{2j+1}$ *and (57). So, we obtain* $|A_{i,j}^{(11)}| \le 4\sqrt{2}$.

If $\tau_2 \in [-(\pi/2), -(\pi/4)]$, *by setting* $\tau_2' = -\tau_2 \in [(\pi/4), (\pi/2)]$, *we can similarly prove the conclusion.*

To sum up, when $|\tau_2| \in [(\pi/4), (\pi/2)]$, *we have* $|A_{i,j}^{(11)}| \le 4\sqrt{2}$. □

Proof of Theorem 1. Since $\tau_1 \in [0, \pi]$, $\tau_2 \in [-(\pi/2), (\pi/2)]$ and $|\tau_2| \le \tau_1$, it will be convenient to divide the argument into several cases as follows:

(1) If $|\tau_2| \in [0, (\theta_1/2)]$, then τ_1 may belong to the following intervals $[0, (\theta_1/2)] \cup [\pi - (\theta_1/2), \pi]$, $[(\theta_1/2), (\pi/4)] \cup [(3\pi/4), \pi - (\theta_1/2)]$, and $[(\pi/4), (3\pi/4)]$

(2) If $|\tau_2| \in [(\theta_1/2), (\pi/4)]$, then τ_1 may belong to the following intervals $[(\theta_1/2), (\pi/3)] \cup [(2\pi/3), \pi - (\theta_1/2)]$, $[(\pi/3), (2/3)\pi]$, and $[\pi - (\theta_1/2), \pi]$

(3) If $|\tau_2| \in [(\pi/4), (\pi/2)]$, then τ_1 may belong to the following intervals $[(\pi/4), (\pi/2) - (\theta_1/2)] \cup [(\pi/2) + (\theta_1/2), (3/4)\pi]$, $[(\pi/2) - (\theta_1/2), (\pi/2)]$, $[(\pi/2), (\pi/2) + (\theta_1/2)]$, $[(3/4)\pi, \pi - (\theta_1/2)]$, and $[\pi - (\theta_1/2), \pi]$

Next, we will discuss each case separately.

Case 1. If $|\tau_2| \in [0, (\theta_1/2)], \tau_1 \in [0, (\theta_1/2)] \cup [\pi - (\theta_1/2), \pi]$, then

$$\Lambda_n^1(\mathbf{x}) \le 28(n+2)^2. \tag{65}$$

By Lemma 6, we can obtain

$$\begin{aligned}
\sum_{i=1}^{[(n+1)/2]}\sum_{j=0}^{i-2}\left|A_{i,j}^{(1)}\right| &\le \sum_{i=1}^{[(n+1)/2]}\sum_{j=0}^{i-2}\left|\frac{32\sqrt{\sin\theta_{2i}\sin\theta_{2j+1}}\cos(n+2)\tau_1\cos(n+2)\tau_2}{(n+2)^2\sin^3\left(\left(\theta_{2i}-\theta_{2j+1}\right)/2\right)}\right| \\
&\le \sum_{i=2}^{[(n+1)/2]}\sum_{j=0}^{i-2}\frac{32}{(n+2)^2\sin^3\left(\left(\theta_{2i}-\theta_{2j+1}\right)/2\right)} \le \sum_{i=2}^{[(n+1)/2]}\sum_{j=0}^{i-2}\frac{32(n+2)}{(2i-2j-1)^3} \\
&\le \sum_{i=2}^{[(n+1)/2]} 32(n+2)\int_1^{i-2}\frac{1}{(2x-1)^3}\mathrm{d}x = \sum_{i=2}^{[(n+1)/2]} 8(n+2)\left[1-\frac{1}{(2i-3)^2}\right] \\
&\le 4(n+2)^2,
\end{aligned} \tag{66}$$

$$\begin{aligned}
\sum_{i=1}^{[(n+1/2)]}\sum_{j=i+1}^{[(n/2)]}\left|A_{i,j}^{(1)}\right| &\le \sum_{i=1}^{[(n/2)]-1}\sum_{j=i+1}^{[n/2]}\left|\frac{16\sqrt{\sin\theta_{2i}\sin\theta_{2j+1}}\cos(n+2)\tau_1\cos(n+2)\tau_2}{(n+2)^2\sin^3\left(\left(\theta_{2j+1}-\theta_{2i}\right)/2\right)}\right| \\
&\le \sum_{i=1}^{[n/2]-1}\sum_{j=i+1}^{[n/2]}\frac{16}{(n+2)^2\sin^3\left(\left(\theta_{2j+1}-\theta_{2i}\right)/2\right)} \le \sum_{i=1}^{[n/2]-1}\sum_{j=i+1}^{[n/2]}\frac{16(n+2)}{(2j-2i+1)^3} \le 2(n+2)^2.
\end{aligned} \tag{67}$$

For $j = i$ or $j = i-1$, by Lemma 6, we obtain

$$\sum_{i=1}^{[n/2]}\left|A_{i,j}^{(1)}\right| \le \sum_{i=1}^{[n/2]} 8(n+2) \le 4(n+2)^2. \tag{68}$$

From (66)–(68), we can obtain

$$\sum_{i=1}^{[(n+1)/2]}\sum_{j=0}^{[n/2]}\left|A_{i,j}^{(1)}\right| \le 14(n+2)^2, \quad |\tau_2| \in \left[0, \frac{\theta_1}{2}\right], \tau_1 \in \left[0, \frac{\theta_1}{2}\right] \cup \left[\pi - \frac{\theta_1}{2}, \pi\right]. \tag{69}$$

Similarly, by Lemma 7, we have

$$\sum_{i=1}^{[(n+1)/2]}\sum_{j=0}^{[n/2]}\left|B_{i,j}^{(1)}\right| \le 14(n+2)^2, \quad |\tau_2| \in \left[0, \frac{\theta_1}{2}\right], \tau_1 \in \left[0, \frac{\theta_1}{2}\right] \cup \left[\pi - \frac{\theta_1}{2}, \pi\right]. \tag{70}$$

Therefore,

$$\Lambda_n^1(\mathbf{x}) \le 28(n+2)^2, \quad |\tau_2| \in \left[0, \frac{\theta_1}{2}\right], \tau_1 \in \left[0, \frac{\theta_1}{2}\right] \cup \left[\pi - \frac{\theta_1}{2}, \pi\right]. \tag{71}$$

Case 2. If $|\tau_2| \in [0, (\theta_1/2)]$ and $\tau_1 \in [(\theta_1/2), (\pi/4)] \cup [(3\pi/4), \pi - (\theta_1/2)]$, then

$$\Lambda_n^1(\mathbf{x}) \le 32\sqrt{2}(n+2)^2. \tag{72}$$

By Lemmas 6 and 8, it can be proved that

$$\sum_{i=1}^{[(n+1)/2]}\sum_{j=0}^{i-2}\left|A_{i,j}^{(1)}\right| \leq \sum_{i=1}^{[(n+1)/2]}\sum_{j=0}^{i-2}\frac{16\sqrt{2}\sqrt{\sin\,\theta_{2i}\sin\,\theta_{2j+1}}\left|\cos(n+2)\tau_2\right|}{(n+2)\sin^2\left(\left(\theta_{2i}-\theta_{2j+1}\right)/2\right)}$$
$$\leq \sum_{i=2}^{[(n+1)/2]}\sum_{j=0}^{i-2}\frac{16\sqrt{2}}{(n+2)\sin^2\left(\left(\theta_{2i}-\theta_{2j+1}\right)/2\right)} \leq \sum_{i=2}^{[(n+1)/2]}\sum_{j=0}^{i-2}\frac{16\sqrt{2}(n+2)}{(2i-2j-1)^2} \tag{73}$$
$$\leq 4\sqrt{2}(n+2)^2,$$

$$\sum_{i=1}^{[(n+1)/2]}\sum_{j=i+1}^{[n/2]}\left|A_{i,j}^{(1)}\right| \leq \sum_{i=1}^{[n/2]-1}\sum_{j=i+1}^{[n/2]}\frac{16\sqrt{2}\sqrt{\sin\,\theta_{2i}\sin\,\theta_{2j+1}}\left|\cos(n+2)\tau_2\right|}{(n+2)\sin^2\left(\left(\theta_{2j+1}-\theta_{2i}\right)/2\right)}$$
$$\leq \sum_{i=1}^{[n/2]-1}\sum_{j=i+1}^{[n/2]}\frac{16\sqrt{2}}{(n+2)\sin^2\left(\left(\theta_{2j+1}-\theta_{2i}\right)/2\right)} \leq \sum_{i=1}^{[n/2]-1}\sum_{j=i+1}^{[n/2]}\frac{16\sqrt{2}(n+2)}{(2j-2i+1)^2} \tag{74}$$
$$\leq 4\sqrt{2}(n+2)^2.$$

For $j=i$ or $j=i-1$, we have

$$\sum_{i=1}^{[n/2]}\left|A_{i,j}^{(1)}\right| \leq \sum_{i=1}^{[n/2]} 8\sqrt{2}(n+2) \leq 4\sqrt{2}(n+2)^2. \tag{75}$$

Combining (73)–(75), we obtain

$$\sum_{i=1}^{[(n+1)/2]}\sum_{j=0}^{[n/2]}\left|A_{i,j}^{(1)}\right| \leq 16\sqrt{2}(n+2)^2,$$
$$\left|\tau_2\right| \in \left[0,\frac{\theta_1}{2}\right], \tau_1 \in \left[\frac{\theta_1}{2},\frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4},\pi-\frac{\theta_1}{2}\right]. \tag{76}$$

In the same way, we have

$$\sum_{i=1}^{[(n+1)/2]}\sum_{j=0}^{[n/2]}\left|B_{i,j}^{(1)}\right| \leq 16\sqrt{2}(n+2)^2,$$
$$\left|\tau_2\right| \in \left[0,\frac{\theta_1}{2}\right], \tau_1 \in \left[\frac{\theta_1}{2},\frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4},\pi-\frac{\theta_1}{2}\right]. \tag{77}$$

In conclusion, it can be seen that

$$\Lambda_n^1(\mathbf{x}) \leq 32\sqrt{2}(n+2)^2,$$
$$\left|\tau_2\right| \in \left[0,\frac{\theta_1}{2}\right], \tau_1 \in \left[\frac{\theta_1}{2},\frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4},\pi-\frac{\theta_1}{2}\right]. \tag{78}$$

Case 3. If $|\tau_2| \in [0,\theta_1/2], \tau_1 \in [\pi/4, 3\pi/4]$, then

$$\Lambda_n^{(1)}(\mathbf{x}) \leq 4(n+2)^2. \tag{79}$$

Since $|\tau_2| \leq \theta_1/2 \leq \pi/6$, we have

$$\sin\,\varphi_1 \sin\,\varphi_2 = \sin^2\tau_1 - \sin^2\tau_2 \geq \frac{1}{4}. \tag{80}$$

Furthermore, we can obtain

$$\Lambda_n^{(1)}(\mathbf{x}) \leq \sum_{i=1}^{[(n+1)/2]}\sum_{j=0}^{[n/2]}\frac{\sin\,\theta_{2i}\sin\,\theta_{2j+1}}{(n+2)^2\sin\,\varphi_1\sin\,\varphi_2} \times \sum_{k=1}^{2}\sum_{l=1}^{2}\left|U_{n+1}\left(\tau_1+\frac{\varepsilon_k\theta_{2i}+\varepsilon_l\theta_{2j+1}}{2}\right)U_{n+1}\left(\tau_2+\frac{\varepsilon_k\theta_{2i}+\varepsilon_l\theta_{2j+1}}{2}\right)\right| \tag{81}$$
$$\leq 4(n+2)^2.$$

Case 4. If $|\tau_2| \in [(\theta_1/2), (\pi/4)]$ and $\tau_1 \in [(\theta_1/2), (\pi/3)] \cup [(2\pi/3), \pi-(\theta_1/2)]$, then

$$\Lambda_n^{(1)}(\mathbf{x}) \leq 16(n+2)^2. \tag{82}$$

By Lemmas 9 and 10, it is easy to prove this conclusion.

Case 5. If $|\tau_2| \in [(\theta_1/2), (\pi/4)]$ and $\tau_1 \in [(\pi/3), (2/3)\pi]$, then

$$\Lambda_n^{(1)}(\mathbf{x}) \leq 4(n+2)^2. \tag{83}$$

If $|\tau_2| \in [(\theta_1/2), (\pi/4)]$ and $\tau_1 \in [\pi/3, 2\pi/3]$, since $\sin\varphi_1 \sin\varphi_2 = \sin^2\tau_1 - \sin^2\tau_2 \geq (1/4)$, then similar to (81), we obtain $\Lambda_n^{(1)}(\mathbf{x}) \leq 4(n+2)^2$.

Case 6. If $|\tau_2| \in [(\theta_1/2), (\pi/4)]$ and $\tau_1 \in [\pi - (\theta_1/2), \pi]$, then

$$\Lambda_n^{(1)}(\mathbf{x}) \leq 8(n+2)^2. \tag{84}$$

If $\tau_1 \in [\pi - (\theta_1/2), \pi]$, it is easy to prove that $|A_{i,j}^{(12)}| \leq 4$. Combining the results of Lemma 10, we can obtain

$$\sum_{i=1}^{[(n+1)/2]} \sum_{j=0}^{[n/2]} \left|A_{i,j}^{(1)}\right| \leq \sum_{i=1}^{[(n+1)/2]} \sum_{j=0}^{[n/2]} 16 \leq 4(n+2)^2. \tag{85}$$

Similarly, we can obtain

$$\sum_{i=1}^{[(n+1)/2]} \sum_{j=0}^{[n/2]} \left|B_{i,j}^{(1)}\right| \leq \sum_{i=1}^{[(n+1)/2]} \sum_{j=0}^{[n/2]} 16 \leq 4(n+2)^2. \tag{86}$$

Thus, $\Lambda_n^1(\mathbf{x}) \leq 8(n+2)^2, |\tau_2| \in [(\theta_1/2), (\pi/4)]$, and $\tau_1 \in [\pi - (\theta_1/2), \pi]$.

Case 7. If $|\tau_2| \in [(\pi/4), (\pi/2)]$ for $\tau_1 \in [(\pi/4), (\pi/2) - (\theta_1/2)] \cup [(\pi/2) + (\theta_1/2), (3/4)\pi]$, then

$$\Lambda_n^{(1)}(\mathbf{x}) \leq 80\sqrt{2}(n+2)^2. \tag{87}$$

By Lemma 11, we know $|A_{i,j}^{(11)}| \leq 4\sqrt{2}$ and $|\tau_2| \in [(\pi/4), (\pi/2)]$. Next, let us estimate $|A_{i,j}^{(12)}|$, for $\tau_1 \in [(\pi/4), (\pi/2) - (\theta_1/2)]$. For every j, there is i_0 such that $|\tau_1 - (\theta_{2i_0} + \theta_{2j+1})/2| \leq (\theta_1/2)$ holds; that is, $(\theta_{2i_0} + \theta_{2j+1})/2 - \theta_1/2 \leq \tau_1 \leq (\theta_{2i_0} + \theta_{2j+1})/2 + (\theta_1/2)$.

(i) For $i = i_0$, since $(\pi/2) - (\theta_1/2) \leq \tau_1 + (\theta_{2i_0} + \theta_{2j+1})/2 \leq \pi - (\theta_1/2)$, then

$$\sin\left(\tau_1 + \frac{\theta_{2i_0} + \theta_{2j+1}}{2}\right) \geq \sin\frac{\theta_1}{2} \geq \frac{1}{n+2}. \tag{88}$$

And, because of $\theta_{2j+1} - (\theta_1/2) \leq \tau_1 - (\theta_{2i_0} - \theta_{2j+1})/2 \leq \theta_{2j+1} + (\theta_1/2)$, we can obtain

$$\sin\left(\tau_1 - \frac{\theta_{2i_0} - \theta_{2j+1}}{2}\right) \geq \min\left\{\sin\left(\theta_{2j+1} + \frac{\theta_1}{2}\right), \sin\left(\theta_{2j+1} - \frac{\theta_1}{2}\right)\right\} \geq \frac{1}{2}\sin\theta_{2j+1}. \tag{89}$$

Furthermore, considering that $|\cos(n+2)\tau_1/\sin(\tau_1 - (\theta_{2i_0} + \theta_{2j+1})/2)| \leq n+2$, we have $|A_{i_0,j}^{(12)}| \leq 2(n+2)$. Thus,

$$\sum_{j=0}^{[n/2]} \left|A_{i_0,j}^{(1)}\right| \leq \sum_{j=0}^{[n/2]} 8\sqrt{2}(n+2) \leq 4\sqrt{2}(n+2)^2. \tag{90}$$

(ii) For $0 \leq i \leq i_0 - 1$, the two cases $j \leq i-1$ and $j \geq i$ are discussed separately.

(a) If $j \leq i-1$, for $\theta_{2j+1} \leq \tau_1 - (\theta_{2i} - \theta_{2j+1})/2 \leq (\theta_{2i} + \theta_{2i_0+1})/2$, we have

$$\sin\left(\tau_1 - \frac{\theta_{2i} - \theta_{2j+1}}{2}\right) \geq \min\left\{\sin\frac{\theta_{2i} + \theta_{2i_0+1}}{2}, \sin\theta_{2j+1}\right\} \geq \min\left\{\frac{1}{2}\sin\theta_{2i}, \sin\theta_{2j+1}\right\}, \tag{91}$$

so $|(\sin\theta_{2i}\sin\theta_{2j+1}/\sin(\tau_1 - (\theta_{2i} - \theta_{2j+1})/2)| \leq 2$. And, considering that

$$\begin{aligned} \sin\left(\tau_1 + \frac{\theta_{2i} + \theta_{2j+1}}{2}\right)\sin\left(\tau_1 - \frac{\theta_{2i} + \theta_{2j+1}}{2}\right) &= \sin^2\tau_1 - \sin^2\left(\frac{\theta_{2i} + \theta_{2j+1}}{2}\right) \\ &= \left(\sin\tau_1 + \sin\frac{\theta_{2i} + \theta_{2j+1}}{2}\right)\left(\sin\tau_1 - \sin\frac{\theta_{2i} + \theta_{2j+1}}{2}\right) \geq \sin^2\left(\frac{\tau_1}{2} - \frac{\theta_{2i} + \theta_{2j+1}}{4}\right), \end{aligned} \tag{92}$$

we obtain

$$\left|A_{i,j}^{(12)}\right| \leq \frac{2}{(n+2)\sin^2\left(\tau_1/2 - \left(\theta_{2i} + \theta_{2j+1}\right)/4\right)}. \quad (93)$$

(b) If $j \geq i$, we have

$$\sin\left(\tau_1 + \frac{\theta_{2j+1} - \theta_{2i}}{2}\right) \geq \sin\frac{\pi}{4}\cos\frac{\theta_{2j+1} - \theta_{2i}}{2} \geq \frac{\sqrt{2}}{2}\sqrt{\sin\theta_{2j+1}\sin\theta_{2i}}. \quad (94)$$

And, considering that

$$\begin{aligned}\sin\left(\tau_1 + \frac{\theta_{2i} + \theta_{2j+1}}{2}\right)\sin\left(\tau_1 - \frac{\theta_{2i} + \theta_{2j+1}}{2}\right) &= \left(\sin\tau_1 + \sin\frac{\theta_{2i} + \theta_{2j+1}}{2}\right)\left(\sin\tau_1 - \sin\frac{\theta_{2i} + \theta_{2j+1}}{2}\right) \\ &\geq \sqrt{2}\sqrt{\sin\theta_{2i}\sin\theta_{2j+1}}\sin^2\left(\frac{\tau_1}{2} - \frac{\theta_{2i} + \theta_{2j+1}}{4}\right),\end{aligned} \quad (95)$$

we obtain

$$\left|A_{i,j}^{(12)}\right| \leq \frac{1}{(n+2)\sin^2\left(\tau_1/2 - \left(\theta_{2i} + \theta_{2j+1}\right)/4\right)}. \quad (96)$$

In summary, we obtain

$$\begin{aligned}\sum_{j=0}^{[n/2]}\sum_{i=1}^{i_0-1}\left|A_{i,j}^{(1)}\right| &\leq \sum_{j=0}^{[n/2]}\sum_{i=1}^{i_0-1}\frac{4\sqrt{2}}{n+2}\cdot\frac{2}{\sin^2\left(\tau_1/2 - \left(\theta_{2i} + \theta_{2j+1}\right)/2\right)} \\ &\leq \sum_{j=0}^{[n/2]}\sum_{i=1}^{i_0-1}\frac{32\sqrt{2}(n+2)}{\left[2\left(i_0 - i\right) - 1\right]^2} \leq 32\sqrt{2}(n+2)\sum_{j=0}^{[n/2]}\left(1 + \int_1^{i_0-1}\frac{1}{(2x-1)^2}\mathrm{d}x\right) \\ &\leq 24\sqrt{2}(n+2)^2.\end{aligned} \quad (97)$$

(iii) For $i_0 + 1 \leq i \leq [(n+1)/2]$, the following two cases are discussed separately.

(a) If $j \geq i$, from (96), we have

$$\sum_{j=0}^{[n/2]}\sum_{i=i_0+1}^{[n/2]}\left|A_{i,j}^{(1)}\right| \leq \sum_{j=0}^{[n/2]}\sum_{i=i_0+1}^{[n/2]}\frac{4\sqrt{2}}{n+2}\cdot\frac{1}{\sin^2\left(\left(\theta_{2i} + \theta_{2j+1}\right)/4 - \tau_1/2\right)} \leq \sum_{j=0}^{[n/2]}\sum_{i=i_0+1}^{[n/2]}\frac{16\sqrt{2}(n+2)^2}{\left[2\left(i_0 - i\right) - 1\right]^2} \leq 12\sqrt{2}(n+2)^2. \quad (98)$$

(b) If $j \le i-1$, for every j, there is i_1 such that $|\tau_1 - (\theta_{2i_1} - \theta_{2j+1})/2| \le (\theta_1/2)$ holds, that is,

$$\frac{\theta_{2i_1} - \theta_{2j+1}}{2} - \frac{\theta_1}{2} \le \tau_1 \le \frac{\theta_{2i_1} - \theta_{2j+1}}{2} + \frac{\theta_1}{2}. \quad (99)$$

For $i = i_1$, since $\theta_{2i_1} - (\theta_1/2) \le \tau_1 + (\theta_{2i_1} + \theta_{2j+1})/2 \le \theta_{2i_1} + (\theta_1/2)$, then

$$\sin\left(\tau_1 + \frac{\theta_{2i} + \theta_{2j+1}}{2}\right) \ge \min\left\{\sin\left(\theta_{2i_1} + \frac{\theta_1}{2}\right), \ \sin\left(\theta_{2i_1} - \frac{\theta_1}{2}\right)\right\} \ge \frac{1}{2}\sin\,\theta_{2i_1}. \quad (100)$$

On account of $-\theta_{2j+1} - (\theta_1/2) \le \tau_1 - (\theta_{2i_1} + \theta_{2j+1})/2 \le -\theta_{2j+1} + (\theta_1/2)$, we obtain

$$\left|\sin\left(\tau_1 - \frac{\theta_{2i} + \theta_{2j+1}}{2}\right)\right| \ge \min\left\{\sin\left(\theta_{2j+1} - \frac{\theta_1}{2}\right), \ \sin\left(\theta_{2j+1} + \frac{\theta_1}{2}\right)\right\} \ge \frac{1}{2}\sin\,\theta_{2j+1}. \quad (101)$$

And because of $|\cos(n+2)\tau_1/\sin(\tau_1 - (\theta_{2i} - \theta_{2j+1})/2)| \le n+2$, we obtain $|A_{i,j}^{(12)}| \le 4$.

For $i \ne i_1$, since $(\theta_{2i_1-1} + \theta_{2i})/2 \le \tau_1 + (\theta_{2i} + \theta_{2j+1})/2 \le (\theta_{2i_1+1} + \theta_{2i})/2$, then

$$\left|\sin\left(\tau_1 + \frac{\theta_{2i} + \theta_{2j+1}}{2}\right)\right| \ge \min\left\{\sin\left(\frac{\theta_{2i_1+1} + \theta_{2i}}{2}\right), \ \sin\left(\frac{\theta_{2i_1-1} + \theta_{2i}}{2}\right)\right\} \ge \frac{1}{2}\sin\,\theta_{2i}. \quad (102)$$

And, because of

$$\left|\frac{\cos(n+2)\tau_1 \sin\,\theta_{2j+1}}{\sin\left(\tau_1 - \left(\theta_{2i} - \theta_{2j+1}\right)/2\right)\sin\left(\tau_1 - \left(\theta_{2i} + \theta_{2j+1}\right)/2\right)}\right| \le 2(n+2), \quad (103)$$

we obtain $|A_{i,j}^{(12)}| \le 4$.

Thus, for $j \le i-1$, we have

$$\sum_{j=0}^{[n/2]} \sum_{i=i_0+1}^{[(n+1)/2]} \left|A_{i,j}^{(1)}\right| \le 4\sqrt{2}(n+2)^2. \quad (104)$$

Based on the above results, if $|\tau_2| \in [\pi/4, \pi/2]$ and $\tau_1 \in [\pi/4, \pi/2 - \theta_1/2]$, we can obtain

$$\sum_{j=0}^{[n/2]} \sum_{i=1}^{[(n+1)/2]} \left|A_{i,j}^{(1)}\right| \le 40\sqrt{2}(n+2)^2. \quad (105)$$

Similarly, if $|\tau_2| \in [\pi/4, \pi/2]$ and $\tau_1 \in [\pi/4, \pi/2 - \theta_1/2]$, we can obtain

$$\sum_{j=0}^{[n/2]} \sum_{i=1}^{[(n+1)/2]} \left|B_{i,j}^{(1)}\right| \le 40\sqrt{2}(n+2)^2. \quad (106)$$

Therefore, we obtain $\Lambda_n^1(\mathbf{x}) \le 80\sqrt{2}(n+2)^2$, $|\tau_2| \in [(\pi/4), (\pi/2)]$, and $\tau_1 \in [(\pi/4), (\pi/2) - (\theta_1/2)]$.

And, we can similarly prove the case of $|\tau_2| \in [(\pi/4), (\pi/2)]$ and $\tau_1 \in [(\pi/2) + (\theta_1/2), (3/4)\pi]$. Then, the conclusion of Case 7 is obtained.

Case 8. If $|\tau_2| \in [(\pi/4), (\pi/2)]$ and $\tau_1 \in [(\pi/2) - (\theta_1/2), (\pi/2)]$, then

$$\Lambda_n^{(1)}(\mathbf{x}) \le 66\sqrt{2}(n+2)^2. \quad (107)$$

If $\tau_1 \in [(\pi/2) - (\theta_1/2), (\pi/2)]$, for every j, there is $i_0 = [(n+1)/2] - j$ such that $|\tau_1 - (\theta_{2i_0} + \theta_{2j+1})/2| \le (\theta_1/2)$ holds.

(i) For $1 \le i \le i_0 - 1$, the following two cases are discussed separately:

(a) If $j \le i-1$, since $\theta_{2j+1} \le \tau_1 - (\theta_{2i} - \theta_{2j+1})/2 < (\pi/2)$, then $\sin(\tau_1 - (\theta_{2i} - \theta_{2j+1})/2) \ge \sin\,\theta_{2j+1}$.

And, because of $(\pi/2)+\theta_1 \le \tau_1 + (\theta_{2i} + \theta_{2j+1})/2 \le \pi - (\theta_1/2)$, we have

$$\sin\left(\tau_1 + \frac{\theta_{2i}+\theta_{2j+1}}{2}\right) \ge \sin\left(\frac{\pi}{2} + \frac{\theta_{2i}+\theta_{2j+1}}{2}\right) \ge 1 - \frac{2i+2j+1}{n+2}. \tag{108}$$

And, considering that $(\theta_1/2) \le \tau_1 - (\theta_{2i} + \theta_{2j+1})/2 < (\pi/2)$, we obtain

$$\sin\left(\tau_1 - \frac{\theta_{2i}+\theta_{2j+1}}{2}\right) \ge \sin\left(\frac{\pi}{2} - \frac{\theta_1}{2} - \frac{\theta_{2i}+\theta_{2j+1}}{2}\right) \ge 1 - \frac{2i+2j+2}{n+2}. \tag{109}$$

Hence,

$$\left|A_{i,j}^{(12)}\right| \le \frac{n+2}{(n-2i-2j)^2}. \tag{110}$$

(b) If $j \ge i$, we have

$$\sin\left(\tau_1 + \frac{\theta_{2j+1}-\theta_{2i}}{2}\right) \ge \sin\frac{\pi}{6}\cos\frac{\theta_{2j+1}-\theta_{2i}}{2} \ge \frac{1}{2}\sqrt{\sin\theta_{2j+1}\sin\theta_{2i}}. \tag{111}$$

And. by (108) and (109), we obtain

$$\left|A_{i,j}^{(12)}\right| \le \frac{2(n+2)}{(n-2i-2j)^2}. \tag{112}$$

Therefore,

$$\begin{aligned}\sum_{j=0}^{[n/2]}\sum_{i=1}^{i_0-1}\left|A_{i,j}^{(1)}\right| &\le 8\sqrt{2}(n+2)\sum_{j=0}^{[(n+1)/2]-2}\sum_{i=1}^{[(n+1)/2]-j-1}\frac{1}{(n-2i-2j)^2}\\ &\le 8\sqrt{2}(n+2)\sum_{j=0}^{[(n+1)/2]-2}\left[\int_1^{[(n+1)/2]-j-1}\frac{1}{(n-2j-2x)^2}\mathrm{d}x+1\right]\\ &\le 6\sqrt{2}(n+2)^2.\end{aligned} \tag{113}$$

(ii) For $i_0+2 \le i \le [(n+1)/2]$, the following two cases are discussed separately:

(a) If $j \le i-1$, since $(\pi/2)-\theta_1 \le \tau_1 - (\theta_{2i}-\theta_{2j+1})/2 \le (\pi/2) - (\theta_1/2)$, then

$$\left|\frac{\sin\theta_{2i}\sin\theta_{2j+1}}{\sin\left(\tau_1 - \left(\theta_{2i}-\theta_{2j+1}\right)/2\right)}\right| \le \frac{\sin\theta_{2i}\sin\theta_{2j+1}}{\sin\left(\theta_i + (\theta_1/2)\right)\sin\left(\theta_j + (\theta_1/2)\right)} \le 4. \tag{114}$$

Because of $0 \le \theta_{i-i_0} - (\theta_1/2) \le (\theta_{2i}+\theta_{2j+1})/2 - \tau_1 \le \theta_{i-i_0} + (\theta_1/2) < (\pi/2)$, we can obtain

$$\sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2} - \tau_1\right) \ge \sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2} - \frac{\pi}{2}\right) \ge \frac{2i+2j+1}{n+2} - 1. \tag{115}$$

Owing to $\theta_1/2 \le \tau_1 + (\theta_{2i}+\theta_{2j+1})/2 - \pi \le \pi/2 - 3\theta_1/2$, we have

$$\sin\left(\tau_1 + \frac{\theta_{2i}+\theta_{2j+1}}{2} - \pi\right) \ge \sin\left(\frac{\theta_{2i}+\theta_{2j+1}}{2} - \frac{\theta_1}{2} - \frac{\pi}{2}\right) \ge \frac{2i+2j}{n+2} - 1. \quad (116)$$

Then,

$$\left|A_{i,j}^{(11)}\right| \le \frac{4(n+2)}{(2i+2j-n-1)[2(i+j)-(n+2)]}. \quad (117)$$

(b) If $j \ge i$, from (111), (115), (116), we know that

$$\left|A_{i,j}^{(11)}\right| \le \frac{2(n+2)}{(2i+2j-n-1)[2(i+j)-(n+2)]}. \quad (118)$$

Hence,

$$\begin{aligned}\sum_{j=0}^{[n/2]}\sum_{i=i_0+2}^{[(n+1)/2]}\left|A_{i,j}^{(1)}\right| &\le \sum_{j=0}^{[n/2]}\sum_{i=i_0+2}^{[(n+1)/2]}\frac{4\sqrt{2}}{n+2}\cdot\frac{4(n+2)}{(2i+2j-n-1)[2(i+j)-(n+2)]}\\ &\le \sum_{j=0}^{[(n+1)/2]-1}\sum_{i=i_0+2}^{[(n+1)/2]}\frac{16\sqrt{2}(n+2)}{[2(i+j)-(n+2)]^2}\\ &\le \sum_{j=0}^{[(n+1)/2]-1}16\sqrt{2}(n+2)\left[\int_{i_0+2}^{[(n+1)/2]}\frac{1}{(2x+2j-n-2)^2}dx+1\right]\\ &\le 12\sqrt{2}(n+2)^2.\end{aligned} \quad (119)$$

(iii) For $i = i_0$ or $i = i_0+1$, it is easy to prove that $|A_{i,j}^{(12)}| \le 2(n+2)$. Then, we have

$$\sum_{j=0}^{[n/2]}\left|A_{i,j}\right| \le \sum_{j=0}^{[n/2]}8\sqrt{2}(n+2) \le 4\sqrt{2}(n+2)^2. \quad (120)$$

Based on the above conclusions, we can obtain

$$\sum_{i=1}^{[(n+1)/2]}\sum_{j=0}^{[n/2]}\left|A_{i,j}\right| \le 26\sqrt{2}(n+2)^2. \quad (121)$$

Similarly, we can prove that

$$\sum_{i=1}^{[(n+1)]}\sum_{j=0}^{[n/2]}\left|B_{i,j}\right| \le 40\sqrt{2}(n+2)^2. \quad (122)$$

Then, $\Lambda_n^1(\mathbf{x}) \le 66\sqrt{2}(n+2)^2, |\tau_2| \in [(\pi/4), (\pi/2)]$, and $\tau_1 \in [(\pi/2)-(\theta_1/2), (\pi/2)]$.

Case 9. If $|\tau_2| \in [\pi/4, \pi/2]$ and $\tau_1 \in [\pi/2, \pi/2 + (\theta_1/2)]$, then

$$\Lambda_n^{(1)}(\mathbf{x}) \le 38\sqrt{2}(n+2)^2. \quad (123)$$

If $|\tau_2| \in [(\pi/4), (\pi/2)]$ and $\tau_1 \in [(\pi/2), (\pi/2)+(\theta_1/2)]$, similar to the estimates of (121) and (122), we can obtain

$$\begin{aligned}\sum_{i=1}^{[(n+1)/2]}\sum_{j=0}^{[n/2]}\left|A_{i,j}\right| &\le 14\sqrt{2}(2+n)^2,\\ \sum_{i=1}^{[(n+1)/2]}\sum_{j=0}^{[n/2]}\left|B_{i,j}\right| &\le 24\sqrt{2}(n+2)^2.\end{aligned} \quad (124)$$

Therefore, $\Lambda_n^1(\mathbf{x}) \le 38\sqrt{2}(n+2)^2, |\tau_2| \in [(\pi/4), (\pi/2)]$, and $\tau_1 \in [(\pi/2), (\pi/2)+(\theta_1/2)]$.

Case 10. If $|\tau_2| \in [\pi/4, \pi/2]$ and $\tau_1 \in [3\pi/4, \pi - \theta_1/2]$, then

$$\Lambda_n^{(1)}(\mathbf{x}) \le 16(n+2)^2. \quad (125)$$

By Lemma 8 and 11, we can easily obtain the conclusion.

Case 11. If $|\tau_2| \in [\pi/4, \pi/2]$ and $\tau_1 \in [\pi - \theta_1/2, \pi]$, then

$$\Lambda_n^1(\mathbf{x}) \le 8\sqrt{2}(n+2)^2. \quad (126)$$

Proof. If $\tau_1 \in [\pi - \theta_1/2, \pi]$, it is easy to prove that $|A_{i,j}^{(12)}| \le 4$, $|B_{i,j}^{(12)}| \le 4$. And, by Lemma 11, we can obtain the conclusion.

Based on the conclusion of Case 1–Case 11, we obtain $\Lambda_n^1(\mathbf{x}) \le 80\sqrt{2}(n+2)^2, |\tau_2| \in [0, \pi/2], \tau_1 \in [0, \pi]$.

And similarly, we have $\Lambda_n^2(\mathbf{x}) \le 80\sqrt{2}(n+2)^2$, $|\tau_2| \in [0, \pi/2]$, and $\tau_1 \in [0, \pi]$.

Then, the proof of Theorem 1 is completed. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (grant number 11571362); Natural Science Foundation of Hebei Province (grant number A2019109076); the Research and Development Program for Science and Technology of Handan (grant number 1523103064-6); the Nursery Engineering of Handan University (grant number 2018303); and Project of Spectral Science and Technology Innovation Team of Handan University (grant number XKYTD202002).

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