

Research Article

Continued Fraction Interpolation of Preserving Horizontal Asymptote

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Received 26 March 2022; Accepted 7 May 2022; Published 1 June 2022

Academic Editor: Hassan Raza

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The classical Thiele-type continued fraction interpolation is an important method of rational interpolation. However, the rational interpolation based on the classical Thiele-type continued fractions cannot maintain the horizontal asymptote when the interpolated function is of a horizontal asymptote. By means of the relationship between the leading coefficients of the numerator and the denominator and the reciprocal differences of the continued fraction interpolation, a novel algorithm for the continued fraction interpolation is constructed in an effort to preserve the horizontal asymptote while approximating the given function with a horizontal asymptote. The uniqueness of the interpolation problem is proved, an error estimation is given, and numerical examples are provided to verify the effectiveness of the presented algorithm.

1. Introduction

The classical Thiele-type continued fraction interpolation [1–9] is an important method of rational interpolation. Suppose $y = f(x)$ is the interpolated function and $y_j = f(x_j)$, $j = 0, 1, \dots, n$ where x_0, x_1, \dots, x_n are $n + 1$ different interpolating nodes. The classical Thiele-type continued fraction has the form as follows:

$$R_n(x) = b_0 + \frac{x - x_0}{b_1} + \frac{x - x_1}{b_2} + \dots + \frac{x - x_{n-1}}{b_n}, \quad (1)$$

where

$$b_j = \varphi[x_0, x_1, x_2, \dots, x_j], \quad j = 0, 1, \dots, n, \quad (2)$$

is the j th inverse difference of the function $f(x)$ with respect to x_0, x_1, \dots, x_j , which can be calculated recursively as follows:

$$\varphi[x_j] = f(x_j), \quad j = 0, 1, \dots, n,$$

$$\varphi[x_p, x_q] = \frac{x_q - x_p}{\varphi[x_q] - \varphi[x_p]},$$

$$\varphi[x_u, \dots, x_v, x_r, x_s] = \frac{x_s - x_r}{\varphi[x_u, \dots, x_v, x_s] - \varphi[x_u, \dots, x_v, x_r]}. \quad (3)$$

It is not difficult to show that $R_n(x)$ is a rational function whose numerator and denominator are polynomials of degrees not exceeding $[(n + 1)/2]$ and $[n/2]$, respectively, where $[u]$ denotes the largest integer not exceeding u , and $R_n(x)$ satisfies

$$R_n(x_j) = y_j, \quad j = 0, 1, 2, \dots, n. \quad (4)$$

The n th reciprocal difference $\rho[x_0, x_1, \dots, x_n]$ of the function $f(x)$ with respect to x_0, x_1, \dots, x_n is defined recursively as follows:

$$\rho[x_j] = f(x_j), \quad j = 0, 1, \dots, n,$$

$$\rho[x_p, x_q] = \frac{x_q - x_p}{\rho[x_q] - \rho[x_p]}, \tag{5}$$

$$\rho[x_u, \dots, x_v, x_r, x_s] = \frac{x_s - x_r}{\rho[x_u, \dots, x_v, x_s] - \rho[x_u, \dots, x_v, x_r]} + \rho[x_u, \dots, x_v].$$

The inverse differences can be calculated via the reciprocal differences as follows [10, 11]:

$$\varphi[x_u, \dots, x_v, x_r, x_s] = \rho[x_u, \dots, x_v, x_r, x_s] - \rho[x_u, \dots, x_v]. \tag{6}$$

Let

$$R_n(x) = b_0 + \frac{x - x_0}{b_1} + \frac{x - x_1}{b_2} + \dots + \frac{x - x_{n-1}}{b_n} = \frac{P_n(x)}{Q_n(x)}. \tag{7}$$

Denote by $L(P_n(x))$ the leading coefficient of the polynomial $P_n(x)$, then when n is odd, the reciprocal differences and the leading coefficients of the numerator polynomial and denominator polynomial of continued fraction interpolation have the following identity relationship [12]:

$$\begin{aligned} L(P_n(x)) &= 1, \\ L(Q_n(x)) &= \rho[x_0, x_1, \dots, x_n], \end{aligned} \tag{8}$$

when n is even,

$$L(P_n(x)) = \rho[x_0, x_1, \dots, x_n], \tag{9}$$

$$L(Q_n(x)) = 1. \tag{10}$$

The classical Thiele-type continued fraction interpolation may not necessarily maintain the original horizontal asymptote of the interpolated function when the interpolated function has a horizontal asymptote. This paper presents an algorithm to construct the continued fraction interpolation preserving the horizontal asymptote that the interpolated function possesses. The uniqueness of solution of the numerical problem is proved, an error estimation is worked out, and numerical examples are provided to show the effectiveness of the new algorithm.

2. The Algorithm for Continued Fraction Interpolation of Preserving Horizontal Asymptote

The problem for continued fraction interpolation of preserving horizontal asymptote: Let $y = f(x)$ be defined in I and $\{x_0, x_1, \dots, x_{2m-1}\} \subset I$ be $2m$ distinct interpolation nodes such that $y_j = f(x_j)$, $j = 0, 1, \dots, 2m - 1$. Suppose $y = f(x)$ has a horizontal asymptote $y = A$, i.e., $\lim_{x \rightarrow \infty} f(x) = A$, where A is a constant. Our purpose is to seek for a rational function of the following form:

$$R_{2m}(x) = b_0 + \frac{x - x_0}{b_1} + \frac{x - x_1}{b_2} + \dots + \frac{x - x_{2m-1}}{b_{2m}} = \frac{P_{2m}(x)}{Q_{2m}(x)}. \tag{11}$$

such that

$$\begin{aligned} R_{2m}(x_j) &= y_j, \quad i = 0, 1, \dots, 2m - 1, \\ \lim_{x \rightarrow \infty} R_{2m}(x) &= A, \end{aligned} \tag{12}$$

where

$$b_l = \varphi[x_0, x_1, \dots, x_l], \quad l = 0, 1, \dots, 2m. \tag{13}$$

Since x_{2m} is unknown, the formula of inverse differences cannot be used to calculate $b_{2m} = \varphi[x_0, x_1, \dots, x_{2m}]$ directly. It is not difficult to show

$$\deg P_{2m}(x) \leq \left\lfloor \frac{2m + 1}{2} \right\rfloor = m, \tag{14}$$

$$\deg Q_{2m}(x) \leq \left\lfloor \frac{2m}{2} \right\rfloor = m.$$

$R_{2m}(x)$ can be written as in the following form:

$$\begin{aligned} R_{2m}(x) &= b_0 + \frac{x - x_0}{b_1} + \frac{x - x_1}{b_2} + \dots + \frac{x - x_{2m-1}}{b_{2m}} = \frac{P_{2m}(x)}{Q_{2m}(x)} \\ &= \frac{c_mx^m + c_{m-1}x^{m-1} + \dots + c_1x + c_0}{d_mx^m + d_{m-1}x^{m-1} + \dots + d_1x + d_0}. \end{aligned} \tag{15}$$

It follows from (9) and (10),

$$\begin{aligned} \frac{L(P_{2m}(x))}{L(Q_{2m}(x))} &= \frac{\rho[x_0, x_1, \dots, x_{2m}]}{1} = \rho[x_0, x_1, \dots, x_{2m}] \\ &= \frac{c_m}{d_m} = \lim_{x \rightarrow \infty} R_{2m}(x). \end{aligned} \tag{16}$$

Using the relationship between the inverse differences and the reciprocal differences gives

$$\rho[x_0, x_1, \dots, x_{2m}] = b_0 + b_2 + \dots + b_{2m-2} + b_{2m}. \tag{17}$$

With (12), (16), and (17) in mind, we have

$$\begin{aligned} \lim_{x \rightarrow \infty} R_{2m}(x) &= A = \rho[x_0, x_1, \dots, x_{2m}] \\ &= b_0 + b_2 + \dots + b_{2m-2} + b_{2m}, \end{aligned} \tag{18}$$

i.e.,

$$b_{2m} = A - b_0 - b_2 - \dots - b_{2m-4} - b_{2m-2}. \tag{19}$$

As a result, the continued fraction interpolation with the preserved horizontal asymptote is given by

$$R_{2m}(x) = b_0 + \frac{x - x_0}{b_1} + \frac{x - x_1}{b_2} + \dots + \frac{x - x_{2m-2}}{b_{2m-1}} + \frac{x - x_{2m-1}}{A - b_0 - b_2 - \dots - b_{2m-4} - b_{2m-2}}. \tag{20}$$

3. The Uniqueness of Interpolant

Theorem 1. *Let*

$$R_{2m}(x) = b_0 + \frac{x - x_0}{b_1} + \frac{x - x_1}{b_2} + \dots + \frac{x - x_{2m-2}}{b_{2m-1}} + \frac{x - x_{2m-1}}{A - b_0 - b_2 - \dots - b_{2m-4} - b_{2m-2}}. \tag{21}$$

If all the i th inverse differences $b_j = \varphi[x_0, x_1, x_2, \dots, x_j]$, $j = 0, 1, 2, \dots, 2m - 1$ exist and $A - b_0 - b_2 - \dots - b_{2m-4} - b_{2m-2} \neq 0$, then set (see [13, 14])

$$t_{2m}(x) = A - b_0 - b_2 - \dots - b_{2m-4} - b_{2m-2}, \tag{22}$$

$$t_k(x) = b_k + \frac{x - x_k}{t_{k+1}(x)}, \quad k = 2m - 1, \dots, 1, 0. \tag{23}$$

If

$$t_{k+1}(x_k) \neq 0, \quad k = 2m - 1, \dots, 1, 0, \tag{24}$$

then, we have

$$R_{2m}(x_j) = y_j, \quad j = 0, 1, \dots, 2m - 1, \tag{25}$$

$$\lim_{x \rightarrow \infty} R_{2m}(x) = A.$$

Proof. $\forall i \in \{0, 1, 2, \dots, 2m - 1\}$, using (20), (22)–(24) gives

$$\begin{aligned} R_{2m}(x_j) &= b_0 + \frac{x_j - x_0}{b_1} + \frac{x_j - x_1}{b_2} + \dots + \frac{x_j - x_{j-1}}{b_j} \\ &= b_0 + \frac{x_j - x_0}{b_1} + \frac{x_j - x_1}{b_2} + \dots + \frac{x_j - x_{j-2}}{b_{j-1}} \\ &= b_0 + \frac{x_j - x_0}{b_1} + \frac{x_j - x_1}{b_2} + \dots + \frac{x_j - x_{j-2}}{\varphi[x_0, x_1, \dots, x_{j-2}, x_j]} \\ &= \dots = b_0 + \frac{x_j - x_0}{\varphi[x_0, x_j]} \\ &= \varphi[x_j] = f(x_j). \end{aligned} \tag{26}$$

The following can be obtained with the formulas (16), (17), and (19):

$$\begin{aligned} \lim_{x \rightarrow \infty} R_{2m}(x) &= \rho[x_0, x_1, \dots, x_{2m}] \\ &= b_0 + b_2 + \dots + b_{2m-2} + b_{2m} = A. \end{aligned} \tag{27}$$

The proof is completed. \square

Theorem 2. *If a rational interpolation function of type $[m, m]$ with the preserved horizontal asymptote exists, it must be the unique one.*

Proof. Suppose the two rational functions of type $[m, m]$.

$$R_{2m}(x) = \frac{P_{2m}(x)}{Q_{2m}(x)} = \frac{c_m x^m + c_{m-1} x^{m-1} + \dots + c_1 x + c_0}{d_m x^m + d_{m-1} x^{m-1} + \dots + d_1 x + d_0},$$

$$R_{2m}^*(x) = \frac{P_{2m}^*(x)}{Q_{2m}^*(x)} = \frac{c_m^* x^m + c_{m-1}^* x^{m-1} + \dots + c_1^* x + c_0^*}{d_m^* x^m + d_{m-1}^* x^{m-1} + \dots + d_1^* x + d_0^*}. \tag{28}$$

Both meet the interpolation conditions in formula (12), we have

$$\begin{cases} \frac{P_{2m}(x_j)}{Q_{2m}(x_j)} = \frac{P_{2m}^*(x_j)}{Q_{2m}^*(x_j)}, & j = 0, 1, \dots, 2m - 1, \\ \lim_{x \rightarrow \infty} \frac{P_{2m}(x)}{Q_{2m}(x)} = \frac{c_m}{d_m} = \lim_{x \rightarrow \infty} \frac{P_{2m}^*(x)}{Q_{2m}^*(x)} = \frac{c_m^*}{d_m^*} = A. \end{cases} \tag{29}$$

That is,

$$\begin{cases} P_{2m}(x_j)Q_{2m}^*(x_j) - P_{2m}^*(x_j)Q_{2m}(x_j) = 0, & j = 0, 1, \dots, 2m - 1, \\ c_m d_m^* - c_m^* d_m = 0. \end{cases} \tag{30}$$

Since the leading term of $P_{2m}(x)Q_{2m}^*(x) - P_{2m}^*(x)Q_{2m}(x)$ is $(c_m d_m^* - c_m^* d_m)x^{2m} = 0$, $P_{2m}(x)Q_{2m}^*(x) - P_{2m}^*(x)Q_{2m}(x)$ turns out to be a polynomial of degree not exceeding $2m - 1$, which has $2m$ distinct zeros. Therefore,

$$P_{2m}(x)Q_{2m}^*(x) \equiv P_{2m}^*(x)Q_{2m}(x), \tag{31}$$

namely,

$$R_{2m}(x) \equiv R_{2m}^*(x). \tag{32}$$

\square

4. The Error Estimation

Theorem 3. Suppose $[c, d]$ is the smallest interval containing $X_{2m-1} = \{x_0, x_1, \dots, x_{2m-1}\}$ and $f(x)$ is $2m$ times differentiable in $[c, d]$. Let

$$R_{2m}(x) = b_0 + \frac{x - x_0}{b_1} + \frac{x - x_1}{b_2} + \dots + \frac{x - x_{2m-2}}{b_{2m-1}} + \frac{x - x_{2m-1}}{A - b_0 - b_2 - \dots - b_{2m-4} - b_{2m-2}} = \frac{P_{2m}(x)}{Q_{2m}(x)}, \tag{33}$$

satisfy $R_{2m}(x_j) = y_j, j = 0, 1, \dots, 2m - 1$, and $\lim_{x \rightarrow \infty} R_{2m}(x) = A$. Then, for each $x \in [c, d]$, there exists a point $\xi \in (c, d)$ such that

$$f(x) - R_{2m}(x) = \frac{\omega_{2m}(x)}{Q_{2m}(x)} \cdot \frac{[f(x)Q_{2m}(x)]_{x=\xi}^{(2m)}}{(2m)!}, \tag{34}$$

and

$$\lim_{x \rightarrow \infty} [f(x) - R_{2m}(x)] = 0, \tag{35}$$

where $\omega_{2m}(x) = (x - x_0)(x - x_1) \dots (x - x_{2m-1})$.

Proof. Let $E(x) = f(x)Q_{2m}(x) - P_{2m}(x)$. Then, from $R_{2m}(x_j) = y_j (j = 0, 1, \dots, 2m - 1)$, it follows,

$$E(x_j) = 0, \quad j = 0, 1, \dots, 2m - 1. \tag{36}$$

Using the Lagrange interpolation formula with remainder term yields (see [15]),

$$\begin{aligned} E(x) &= \omega_{2m}(x) \cdot \frac{[f(x)Q_{2m}(x) - P_{2m}(x)]_{x=\xi}^{(2m)}}{(2m)!} \\ &= \omega_{2m}(x) \cdot \frac{[f(x)Q_{2m}(x)]_{x=\xi}^{(2m)}}{(2m)!}. \end{aligned} \tag{37}$$

Therefore,

$$f(x) - R_{2m}(x) = \frac{\omega_{2m}(x)}{Q_{2m}(x)} \cdot \frac{[f(x)Q_{2m}(x)]_{x=\xi}^{(2m)}}{(2m)!}, \tag{38}$$

and

$$\lim_{x \rightarrow \infty} [f(x) - R_{2m}(x)] = \lim_{x \rightarrow \infty} f(x) - \lim_{x \rightarrow \infty} R_{2m}(x) = A - A = 0. \tag{39}$$

5. Numerical Examples

Example 1. Given six interpolation nodes $x_0 = 1, x_1 = 4, x_2 = 8, x_3 = 12, x_4 = 16, x_5 = 20$. Suppose $f(x) = |\arctan x|$, then, $f(x_0) = 0.78539816, f(x_1) = 1.32581766, f(x_2) = 1.44644133, f(x_3) = 1.48765509, f(x_4) = 1.50837752, f(x_5) = 1.52083793$ and $\lim_{x \rightarrow \infty} |\arctan x| = 1.57079633$. We want to construct the continued fraction interpolant $R_6(x)$ such that it meets the interpolation conditions and $\lim_{x \rightarrow \infty} R_6(x) = 1.57079633$ (keep eight decimal places).

According to what is known, the involved inverse differences can be calculated as shown in Table 1.

By equation (19), we have

$$\begin{aligned} b_6 &= A - b_0 - b_2 - b_4 \\ &= 1.57079633 - 0.78539816 - 0.79395317 - (-0.00856673) \\ &= 0.00001173. \end{aligned} \tag{40}$$

Substituting b_6 into $R_6(x)$ gives

$$\begin{aligned} R_6(x) &= 0.78539816 + \frac{x - 1}{5.55124306} + \frac{x - 4}{0.79395317} + \frac{x - 8}{-1400.48475216} + \frac{x - 12}{-0.00856673} \\ &\quad + \frac{x - 16}{1707699.07647493} + \frac{x - 20}{0.00001173}, \end{aligned} \tag{41}$$

which can be simplified as

$$\begin{aligned} P_6(x) &= 1.57079633x^3 - 0.415046576808782x^2 + 0.641166706097807x - 0.112286748066322, \\ Q_6(x) &= x^3 + 0.372393353974841x^2 + 0.645238679864088x + 0.127305141976055, \end{aligned} \tag{42}$$

TABLE 1: Table of inverse differences.

j	x_j	$f(x_j)$	$\varphi[x_0, x_j]$	$\varphi[x_0, x_1, x_j]$	$\varphi[x_0, x_1, x_j]$	$\varphi[x_0, x_1, x_3, x_j]$	$\varphi[x_0, x_1, x_3, x_4, x_5]$
0	1	0.78539816					
1	4	1.32581766	5.55124306				
2	8	1.44644133	10.58932356	0.79395317			
3	12	1.48765509	15.66378274	0.79109702	-1400.48475216		
4	16	1.50837752	20.74748045	0.78966916	-1867.40726226	-0.00856673	
5	20	1.52083793	25.83488252	0.78881307	-2334.58517597	-0.00856439	1707699.07647493

so,

$$R_6(x) = \frac{1.57079633x^3 - 0.415046576808782x^2 + 0.641166706097807x - 0.112286748066322}{x^3 + 0.372393353974841x^2 + 0.645238679864088x + 0.127305141976055} \tag{43}$$

It is obvious that

$$\lim_{x \rightarrow \infty} R_6(x) = 1.57079633. \tag{44}$$

Using the classical Thiele-type continued fraction interpolation, one can get

$$R_5(x) = 0.78539816 + \frac{x-1}{5.55124306} + \frac{x-4}{0.79395317} + \frac{x-8}{-1400.48475216} + \frac{x-12}{-0.00856673} + \frac{x-16}{1707699.07647493}, \tag{45}$$

which can be simplified as

$$R_5(x) = \frac{0.5855832647x^3 + 1569477.287x^2 - 416900.8348x + 225451.515}{999183.1503x^2 + 370155.227x + 385222.0773} \tag{46}$$

A comparison is made between the curves $y = f(x)$ and $y = R_6(x)$ as shown in Figure 1. The values of $|f(x) - R_6(x)|$ and $|f(x) - R_5(x)|$ at certain points are calculated as shown in Table 2. The errors $|f(x) - R_6(x)|$ and $|f(x) - R_5(x)|$ are illustrated in Figure 2.

Example 2. Given six interpolation nodes $x_j = j + 2$, $j = 0, 1, \dots, 5$. Suppose $f(x) = (\sqrt[3]{1+x^3})/x$, then, $f(x_0) = 1.04004191$, $f(x_1) = 1.01219632$, $f(x_2) = 1.00518144$, $f(x_3) = 1.00265959$, $f(x_4) = 1.00154083$, $f(x_5) = 1.00097087$, and $\lim_{x \rightarrow \infty} ((\sqrt[3]{1+x^3})/x) = 1$. We want to construct the continued fraction interpolant $R_6(x)$

such that it meets the interpolation conditions and $\lim_{x \rightarrow \infty} R_6(x) = 1$.

According to what is known, the involved inverse differences can be calculated as shown in Table 3.

From equation (19), it follows:

$$\begin{aligned} b_6 &= A - b_0 - b_2 - b_4 \\ &= 1 - 1.04004191 - (-0.04659998) - 0.00713454 \\ &= -0.00057647. \end{aligned} \tag{47}$$

Substituting b_6 into $R_6(x)$ gives

$$\begin{aligned} R_6(x) &= 1.04004191 + \frac{x-2}{-35.91233252} + \frac{x-3}{-0.04659998} + \frac{x-4}{669.58185365} \\ &\quad + \frac{x-5}{0.00713454} + \frac{x-6}{-15861.70879642} + \frac{x-7}{-0.00057647}, \end{aligned} \tag{48}$$

which can be simplified as

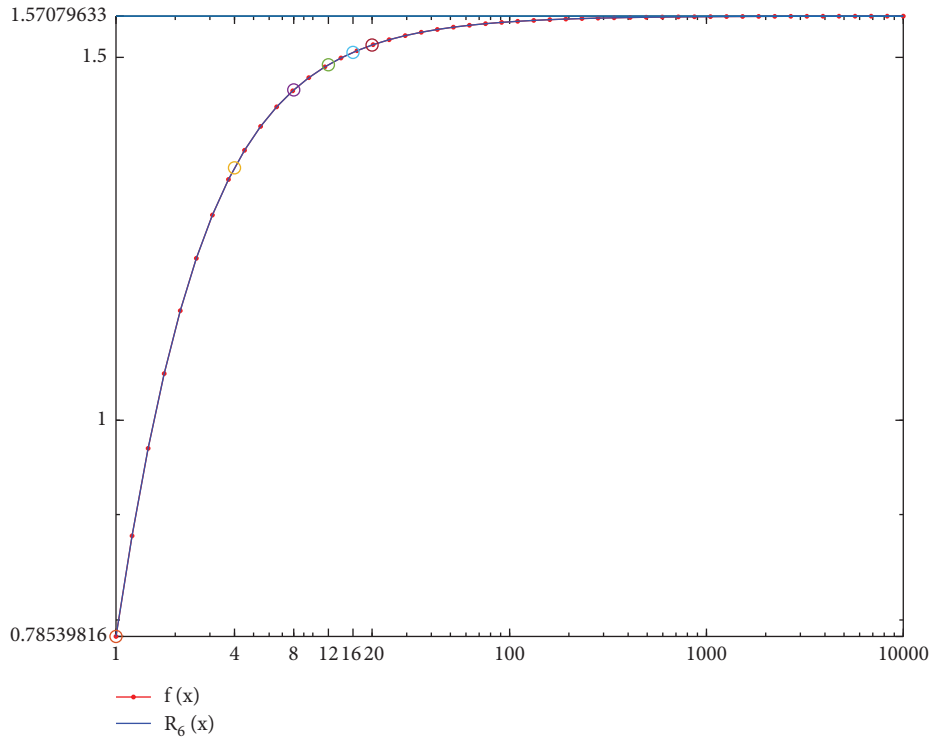


FIGURE 1: Function plots of $f(x)$ and $R_6(x)$.

TABLE 2: Numerical experiments about the error.

x	$ f(x) - R_6(x) $	$ f(x) - R_5(x) $
10	0.0000000052	0.0000000750
50	0.0000000043	0.0000067734
100	0.0000000019	0.0000300746
1000	0.0000000025	0.0005507282
10000	0.0000000031	0.0058245238

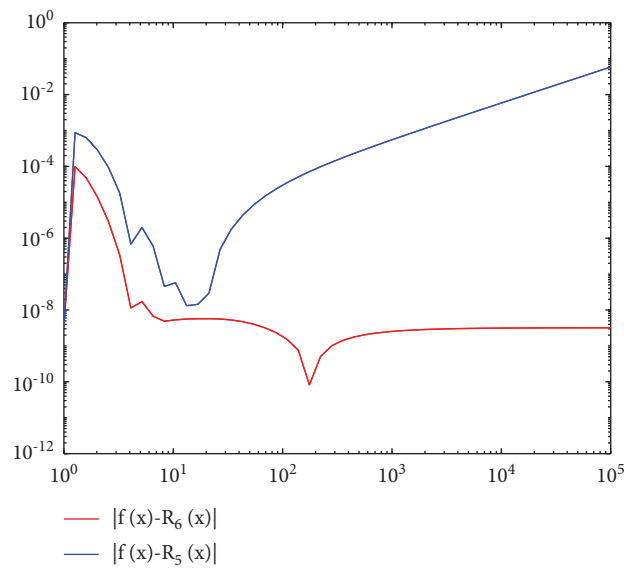


FIGURE 2: Function plots of $|f(x) - R_6(x)|$ and $|f(x) - R_5(x)|$.

TABLE 3: Table of inverse differences.

j	x_j	$f(x_j)$	$\varphi[x_0, x_j]$	$\varphi[x_0, x_1, x_j]$	$\varphi[x_0, x_1, x_j]$	$\varphi[x_0, x_1, x_3, x_j]$	$\varphi[x_0, x_1, x_3, x_4, x_5]$
0	2	1.04004191					
1	3	1.01219632	-35.91233252				
2	4	1.00518144	-57.37156992	-0.04659998			
3	5	1.00265959	-80.25183125	-0.04510651	669.58185365		
4	6	1.00154083	-103.89319748	-0.04413007	809.74503269	0.00713454	
5	7	1.00097087	-127.97203165	-0.04345007	952.40742106	0.00707150	-15861.70879642

$$\begin{aligned}
 P_6(x) &= x^3 - 0.0270442257372862x^2 + 0.0708647913780895x + 0.587171073596883, \\
 Q_6(x) &= x^3 - 0.0270376783697333x^2 + 0.0706956613059999x + 0.255582822560929.
 \end{aligned}
 \tag{49}$$

So,

$$R_6(x) = \frac{x^3 - 0.0270442257372862x^2 + 0.0708647913780895x + 0.587171073596883}{x^3 - 0.0270376783697333x^2 + 0.0706956613059999x + 0.255582822560929}.
 \tag{50}$$

Clearly,

$$\lim_{x \rightarrow \infty} R_6(x) = 1.
 \tag{51}$$

Using the classical Thiele-type continued fraction interpolation, one can get

$$\begin{aligned}
 R_5(x) &= 1.04004191 + \frac{x-2}{-35.91233252} + \frac{x-3}{-0.04659998} + \frac{x-4}{669.58185365} \\
 &\quad + \frac{x-5}{0.00713454} + \frac{x-6}{-15861.70879642},
 \end{aligned}
 \tag{52}$$

which can be simplified as

$$R_5(x) = \frac{-0.00006304490976x^3 + 0.9617508703x^2 - 1.53746219x + 1.143366968}{0.96005036232x^2 - 1.518908091x + 1.03883064}.
 \tag{53}$$

A comparison is made between the curves $y = f(x)$ and $y = R_6(x)$, as shown in Figure 3. The values of $|f(x) - R_6(x)|$ and $|f(x) - R_5(x)|$ at certain points are calculated, as shown in Table 4. The errors $|f(x) - R_6(x)|$ and $|f(x) - R_5(x)|$ are shown in Figure 4.

Example 3. Given six interpolation nodes $x_j = (j/2) + 2$, $j = 0, 1, \dots, 5$. Suppose $f(x) = (1/\sqrt{2\pi}) e^{-(x^2/2)} + 1$, then, $f(x_0) = 1.05399097$, $f(x_1) = 1.01752830$, $f(x_2) = 1.00443185$, $f(x_3) = 1.00087268$, $f(x_4) = 1.00013383$, $f(x_5) = 1.00001598$ and $\lim_{x \rightarrow \infty} (1/\sqrt{2\pi})e^{-(x^2/2)} + 1 = 1$. We try to construct a continued fraction interpolant $R_6(x)$

such that it meets the interpolation conditions and $\lim_{x \rightarrow \infty} R_6(x) = 1$.

According to what is known, the involved inverse differences can be calculated and listed in the following Table 5. From (19), it follows:

$$\begin{aligned}
 b_6 &= A - b_0 - b_2 - b_4 \\
 &= 1 - 1.05399097 - (-0.07733634) - 0.03057007 \\
 &= -0.00722470.
 \end{aligned}
 \tag{54}$$

Substituting b_6 into $R_6(x)$ gives

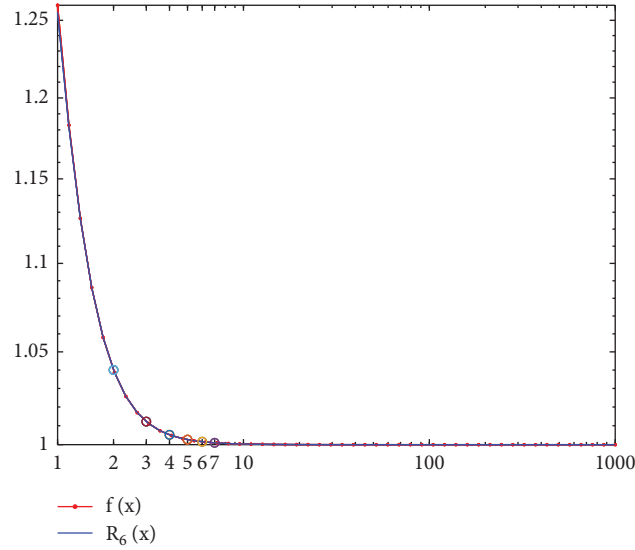


FIGURE 3: Function plots of $f(x)$ and $R_6(x)$.

TABLE 4: Numerical experiments about the error.

x	$ f(x) - R_6(x) $	$ f(x) - R_5(x) $
10	0.00000000173	0.0000155716
50	0.0000000759	0.0019175451
100	0.0000000502	0.0050577370
1000	0.0000000064	0.0640174977
10000	0.0000000001	0.6550176460

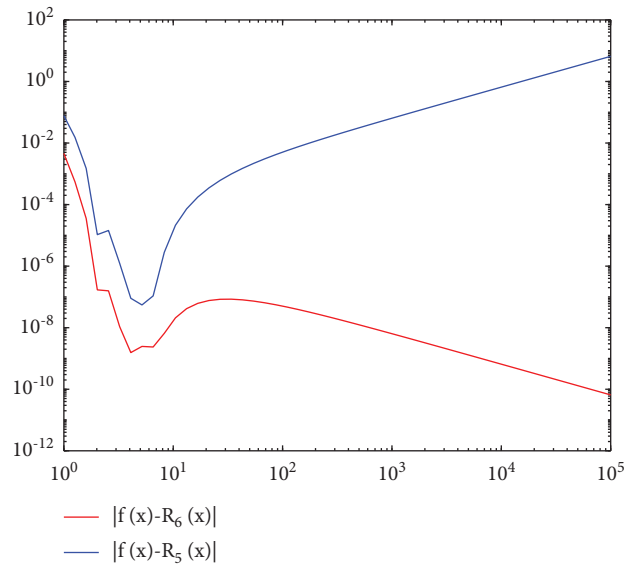


FIGURE 4: Function plots of $|f(x) - R_6(x)|$ and $|f(x) - R_5(x)|$.

$$\begin{aligned}
 R_6(x) = & 1.05399097 + \frac{x - 2}{-13.71265611} + \frac{x - 2.5}{-0.07733634} + \frac{x - 3}{58.85639112} \\
 & + \frac{x - 3.5}{0.03057007} + \frac{x - 4}{-259.12153694} + \frac{x - 4.5}{-0.00722470}
 \end{aligned}
 \tag{55}$$

TABLE 5: Table of inverse differences.

j	x_j	$f(x_j)$	$\varphi[x_0, x_j]$	$\varphi[x_0, x_1, x_j]$	$\varphi[x_0, x_1, x_j]$	$\varphi[x_0, x_1, x_3, x_j]$	$\varphi[x_0, x_1, x_3, x_4, x_5]$
0	2	1.05399097					
1	2.5	1.01752830	-13.71265611				
2	3	1.00443185	-20.17792161	-0.07733634			
3	3.5	1.00087268	-28.23886414	-0.06884109	58.85639112		
4	4	1.00013383	-37.13528304	-0.06404064	75.21225636	0.03057007	
5	4.5	1.00001598	-46.31775448	-0.06134010	93.77201086	0.02864048	-259.12153694

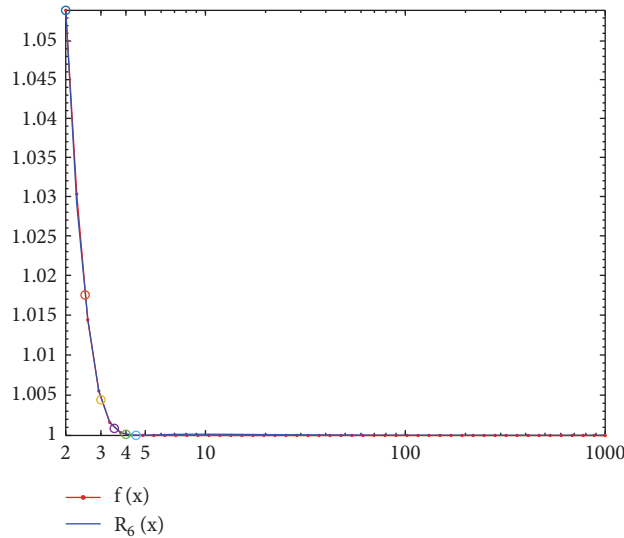


FIGURE 5: Function plots of $f(x)$ and $R_6(x)$.

which can be simplified as

$$\begin{aligned}
 P_6(x) &= x^3 - 6.51105513479376x^2 + 14.6566808471970x - 10.9463472624099, \\
 Q_6(x) &= x^3 - 6.51354079716349x^2 + 14.6793603725717x - 10.9982988663582.
 \end{aligned}
 \tag{56}$$

So,

$$R_6(x) = \frac{x^3 - 6.51105513479376x^2 + 14.6566808471970x - 10.9463472624099}{x^3 - 6.51354079716349x^2 + 14.6793603725717x - 10.9982988663582}.
 \tag{57}$$

Obviously,

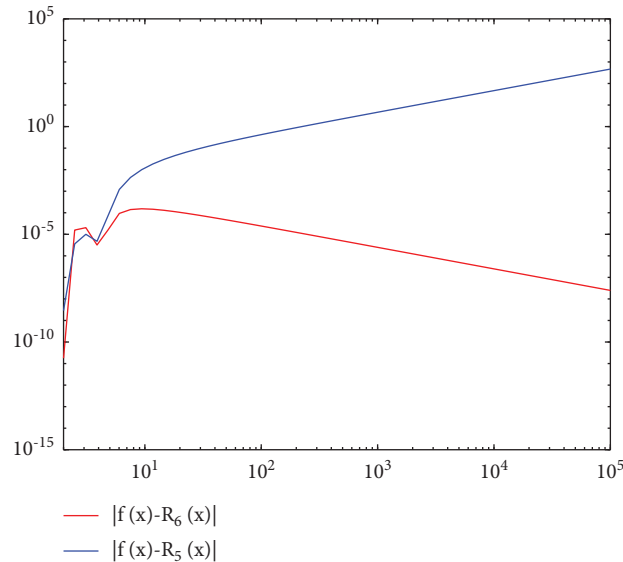
$$\lim_{x \rightarrow \infty} R_6(x) = 1.
 \tag{58}$$

Using the classical Thiele-type continued fraction interpolation, we have

$$\begin{aligned}
 R_5(x) &= 1.05399097 + \frac{x-2}{-13.71265611} + \frac{x-2.5}{-0.07733634} + \frac{x-3}{58.85639112} \\
 &\quad + \frac{x-3.5}{0.03057007} + \frac{x-4}{-259.12153694},
 \end{aligned}
 \tag{59}$$

TABLE 6: Numerical experiments about the error.

x	$ f(x) - R_6(x) $	$ f(x) - R_5(x) $
10	0.0001521807	0.0120808928
50	0.0000468949	0.1902357123
100	0.0000241804	0.4228901150
1000	0.0000024791	4.6280382082
10000	0.0000002485	46.688387239

FIGURE 6: Function plots of $|f(x) - R_6(x)|$ and $|f(x) - R_5(x)|$.

which can be simplified as

$$R_5(x) = \frac{-0.003859192918x^3 + 0.8758988689x^2 - 2.912238657x + 3.350279554}{0.8257816176x^2 - 2.694383623x + 3.03301528}. \quad (60)$$

A comparison is conducted between the curves $y = f(x)$ and $y = R_6(x)$ as shown in Figure 5. The values of $|f(x) - R_6(x)|$ and $|f(x) - R_5(x)|$ at certain points are calculated and listed in Table 6. The errors $|f(x) - R_6(x)|$ and $|f(x) - R_5(x)|$ are shown in Figure 6.

6. Conclusion

As classical approximation tool, continued fractions have been playing an important role in numerical rational approximation. However, continued fractions are rarely involved in shape-preserving design which is an interesting research topic in geometric modeling. In this paper, we construct an interpolating rational function based on the continued fractions, which serves to approximate the functions with the horizontal asymptotes. An algorithm is presented for the interpolating rational function to preserve the horizontal asymptote, the uniqueness of the interpolating rational function is proved and the error is analyzed. Numerical examples are given to verify the effectiveness of the new method.

Data Availability

The data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work is supported by the Anhui University Postgraduate Scientific Research Project under Grant No. YJS20210368 and the National Natural Science Foundation of China under Grant No. 62172135.

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