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Research Article

A Semianalytical Approach for the Solution of Nonlinear Modified Camassa–Holm Equation with Fractional Order

Jiahua Fang[1] **Muhammad Nadeem**[2] **and Hanan A. Wahash**[3]

[1] *Yibin University, Yibin 644000, China*
[2] *Faculty of Science, Yibin University, Yibin 644000, China*
[3] *Department of Mathematics, Albaydaa University, Albayda, Yemen*

Correspondence should be addressed to Muhammad Nadeem; nadeem@yibinu.edu.cn and Hanan A. Wahash; hawahash@baydaauniv.net

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This paper presents the approximate solution of the nonlinear acoustic wave propagation model is known as the modified Camassa–Holm (mCH) equation with the Caputo fractional derivative. We examine this study utilizing the Laplace transform ($\mathscr{L}$ T) coupled with the homotopy perturbation method (HPM) to construct the strategy of the Laplace transform homotopy perturbation method ($\mathscr{L}$ T-HPM). Since the Laplace transform is suitable only for a linear differential equation, therefore $\mathscr{L}$ T-HPM is the suitable approach to decompose the nonlinear problems. This scheme produces an iterative formula for finding the approximate solution of illustrated problems that leads to a convergent series without any small perturbation and restriction. Graphical results demonstrate that $\mathscr{L}$ T-HPM is simple, straightforward, and suitable for other nonlinear problems of fractional order in science and engineering.

1. Introduction

In the recent century, fractional differential problems have caught much attention towards the researchers and scientists due to their precise representation of the physical appearance. Many physical phenomena have been reported across the nonlinear models such as engineering, geophysics, astronomy, medicine, hydrology, chemical engineering, and astrophysics [1–4]. Most of the nonlinear problems of fractional order are difficult to solve. Therefore, these models are very much important to examine the exact and numerical solutions. Currently, many authors have examined the direct correlation and interrelated work on nonlinear problems and symmetry [5]. Integral transform methods are extremely effective in reducing the complexity of these nonlinear fractional problems. There are a number of popular and effective schemes to tackle the nonlinear appearance of these models with fractional order such as the Laplace transform [6], Fourier series approach [7], F-Expansion scheme [8], Residual power series method [9], (G/G)-expansion approach [10], Trial equation approach [11], Sinc–Bernoulli collocation method [12], Variational iteration scheme [13], Subequation [14], Homotopy perturbation method [15], spline collocation approach [16], and so on.

In this paper, we consider a family of modified β-model of the aspect [17].

$$D^{\alpha}u_t - u_{xxt} + (\beta+1)u^2u_x - \beta u_x u_{xx} - uu_{xxx} = 0, \quad (1)$$

Setting $\beta = 2$ in equation (1), and we obtain the fractional modified Camassa–Holm (mCH) model of the shape.

$$D^{\alpha}u_t - u_{xxt} + 3u^2u_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad (2)$$

where u represents the horizontal component of the fluid velocity, x and t indicate the spatial and temporal elements. The mCH model appears in shallow water that was discovered to be entirely integrable with a Lax pair as an approximation to the incompressible Euler equation [18]. Islam et al. [19] obtained the solitary wave solution of the simplified modified Camassa–Holm equation. Zulfiqar and

Ahmad [20] used the Exp-function scheme to investigate the solitary wave solutions of the fractional simplified mCH model. Khatun et al. [21] studied the explicit solutions of the mCH equation with fractional order. Labidi and Omrani [22] studied the variational iteration method and the homotopy perturbation method for solving the mCH equation and found the results in good agreement.

Another powerful technique was introduced to solve the nonlinear problem by He [23, 24] with some recent developments. Kashkari and El-Tantawy [25] applied the homotopy perturbation method for the dissipative soliton collisions in a collisional complex unmagnetized plasma. Later many authors showed the validity and accuracy of this approach [26, 27]. Gupta et al. [28] obtained the approximate solution of the family of the mCH equation with fractional time derivative. Khuri and Sayfy [29] introduced a strategy for specific kinds of differential problems. Later, Anjum and He [30] adopted this scheme for the solution of the nonlinear oscillator problem. Nadeem and Li [31] present a hybrid approach for the solution of nonlinear vibration systems and then Zhang et al. [32] extended this approach for obtaining the solution of nonlinear time fractional differential problems but all these have some limitations and assumptions.

In the present study, we propose an approach, called $\mathscr{L}$ T-HPM which removes these disadvantages and elaborates our scheme to achieve the approximate solution of this nonlinear problem. The implementation of $\mathscr{L}$ T coupled with HPM makes them easier for the construction of this approach for the solution of mCH with fractional order. This approach can also be considered for fractals theory [33, 34]. This article is summarized as follows: in Section 2, we recall the definition of fractional calculus theory. In Section 3, we construct the idea of $\mathscr{L}$ T-HPM to solve the mCH equation. In Section 4, we test the validity and accuracy of $\mathscr{L}$ T-HPM illustrating a numerical problem with the help of graphs. At last, we represent the conclusion in Section 5.

2. Basic Concept of Fractional Theory

In this section, we present some fractional properties to understand the physical nature of calculus theory.

Definition 1. The fractional view of $u(t)$ is described as follows [32]:

$$D^{\alpha}u(x) = J^{h-\alpha}D^{h}u(x) = \frac{1}{\Gamma(h-\alpha)}\int_{0}^{t}(t-\tau)^{h-\alpha-1}f^{h}(t)\mathrm{d}t,$$
$$\text{for } h-1<\alpha\leq h,\ h\in N,\ t>0,\ u\in C_{-1}^{h}. \tag{3}$$

Definition 2. The fractional view of $\mathscr{L}[u(t)]$ is [1, 35]

$$\mathscr{L}[D_{x}^{n\alpha}u(x,t)] = s^{n\alpha}F(s) - \sum_{k=0}^{n-1}s^{n\alpha-k-1}u_{x}^{(k)}(0,t),$$
$$n-1<\alpha\leq n. \tag{4}$$

Definition 3. Let $u(t)=t^{\alpha}$, so $\mathscr{L}$ T is [32]

$$\mathscr{L}[t^{\alpha}] = \int_{0}^{\infty}e^{-st}t^{\alpha}\mathrm{d}t = \frac{\Gamma(\alpha+1)}{s^{(\alpha+1)}}. \tag{5}$$

Definition 4. The Caputo-sense becomes as for order $\alpha>0$,

$$D^{\gamma}u(x,t) = \begin{cases} \dfrac{1}{\Gamma(h-\alpha)}\displaystyle\int_{0}^{t}(t-\tau)^{h-\alpha-1}\frac{\partial^{h}u(x,t)}{\partial\tau^{h}}\mathrm{d}\tau, & h-1<\alpha<h,\\[2ex] \dfrac{\partial^{h}u(x,t)}{\partial t^{h}}, & \alpha=h\in N. \end{cases} \tag{6}$$

3. Fundamental Concept of $\mathscr{L}$ T-HPM

In this segment, we construct the fundamental concept of $\mathscr{L}$ T-HPM. Let us consider the following FPDEs:

$$D_{t}^{\alpha}u(x,t) = T_{1}[u(x,t)] + T_{2}[u(x,t)] + g(x,t),$$
$$x\in\mathbb{R},\ n-1<\alpha\leq n, \tag{7}$$

where $D_{t}^{\alpha}=(\partial^{\alpha}/\partial t^{\alpha})$ is taken in Caputo sense, T_1 and T_2 are linear and nonlinear operators whereas $g(x,t)$ represents as a source term.

By applying $\mathscr{L}$ T to equation (7), it follows,

$$\mathscr{L}[D_{t}^{\alpha}u(x,t)] = \mathscr{L}[T_{1}u(x,t) + T_{2}u(x,t) + g(x,t)]. \tag{8}$$

Applying $\mathscr{L}$ T, we obtain the following equation:

$$s^{\alpha}\mathscr{L}[u(x,t)] - s^{\alpha-1}[u(x,0)]$$
$$= \mathscr{L}[T_{1}u(x,t) + T_{2}u(x,t) + g(x,t)]. \tag{9}$$

On applying Inverse $\mathscr{L}$ T, we receive,

$$u(x,t) = W(x,t) + \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}\{T_{1}u(x,t) + T_{2}u(x,t)\}\right], \tag{10}$$

where $W(x,t)=\mathscr{L}^{-1}[(1/s)u(x,0)+(1/s^{\alpha})\mathscr{L}\{g(x,t)\}]$.

The approximate solution of equation (7) can be expressed in terms of the following power series:

$$u(x,t) = \sum_{n=0}^{\infty}p^{n}u_{n}(x,t), \tag{11}$$

where p is called the homotopy parameter. According to HPM [23], The nonlinear terms can be calculated as follows:

$$T_2 u(x,t) = \sum_{n=0}^{\infty} p^n H_n(u). \tag{12}$$

Then, He's polynomials $H_n(u)$ can be obtained using the following formula:

$$H_n(u_0 + u_1 + \cdots + u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(T_2\left(\sum_{i=0}^{\infty} p^i u_i\right)\right)_{p=0}, \quad n = 0, 1, 2, \ldots. \tag{13}$$

Now putting equations (11) and (12) in equation (10), we obtain the following equation:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = W(x,t) + p\left[\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left(T_1\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\right)\right\}\right]. \tag{14}$$

Equating the values of p, we obtain the following equation:

$$\begin{aligned}
p^0&: u_0(x,t) = W(x,t),\\
p^1&: u_1(x,t) = -\mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\{T_1 u_0(x,t) + H_0\}\right],\\
p^2&: u_2(x,t) = -\mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\{T_1 u_1(x,t) + H_1\}\right],\\
p^3&: u_3(x,t) = -\mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\{T_1 u_2(x,t) + H_2\}\right],\\
&\vdots .
\end{aligned} \tag{15}$$

by continuing this process, we are able to identify the exact solution of this problems such as

$$u(x,t) = \lim_{N\longrightarrow\infty}\sum_{n=0}^{N} u_n(x,t). \tag{16}$$

Generally, this series converges very rapidly.

4. Numerical Applications

In this section, we implement the idea of $\mathscr{L}$ T-HPM for obtaining the smooth solitary wave and singular wave solutions. We see that this scheme presents good results only after a few terms. We compute the values of iterations with the help of Mathematical Software 11.0.1. We present some 2D and 3D graphs for a better understanding of the behavior of the mCH model.

4.1. Example 1. Considering the mCH equation with fractional order α such as

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) + 3u^2\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - u\frac{\partial^3 u}{\partial x^3} = 0, \tag{17}$$

with initial condition

$$u(x,0) = \frac{1}{3}\left[1 - 4\operatorname{sech}^2\left(\frac{x}{\sqrt{6}}\right)\right]. \tag{18}$$

Employing the $\mathscr{L}$ T on equation (17), we get the following equation:

$$\begin{aligned}
\mathscr{L}\left[\frac{\partial^\alpha u}{\partial t^\alpha}\right] &= \mathscr{L}\left[\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) - 3u^2\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^3 u}{\partial x^3}\right],\\
s^\alpha\mathscr{L}[u(x,t)] - s^{\alpha-1}[u(x,0)] &= \mathscr{L}\left[\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) - 3u^2\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^3 u}{\partial x^3}\right],\\
\mathscr{L}[u] &= \frac{u(x,0)}{s} + \frac{1}{s^\alpha}\mathscr{L}\left[\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) - 3u^2\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^3 u}{\partial x^3}\right].
\end{aligned} \tag{19}$$

Using the inverse $\mathscr{L}$ T property,

$$u = u(x,0) + \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}\left\{\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) - 3u^2\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^3 u}{\partial x^3}\right\}\right]. \tag{20}$$

The description of $\mathscr{L}$ T-HPM presents as follows:

$$\sum_{n=0}^{\infty} p^n u_n = u(x,0) + \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}\left\{\sum_{n=0}^{\infty} p^n\frac{\partial}{\partial t}\left(\frac{\partial^2 u_n}{\partial x^2}\right) - 3\sum_{n=0}^{\infty} p^n u_n^2 \sum_{n=0}^{\infty} p^n\frac{\partial u_n}{\partial x} + 2\sum_{n=0}^{\infty} p^n\frac{\partial u_n}{\partial x}\sum_{n=0}^{\infty} p^n\frac{\partial^2 u_n}{\partial x^2} + \sum_{n=0}^{\infty} p^n u_n \sum_{n=0}^{\infty} p^n\frac{\partial^3 u_n}{\partial x^3}\right\}\right]. \tag{21}$$

Equating the values of p, we obtain the following equation:

$$\begin{aligned}
p^0\colon u_0 &= u(x,0) \\
&= \frac{1}{3}\left[1 - 4\text{sech}^2\left(\frac{x}{\sqrt{6}}\right)\right], \\
p^1\colon u_1 &= \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}\left\{\frac{\partial}{\partial t}\left(\frac{\partial^2 u_0}{\partial x^2}\right) - 3u_0^2\frac{\partial u_0}{\partial x} + 2\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2} + u_0\frac{\partial^3 u_0}{\partial x^3}\right\}\right] \\
&= -\frac{1}{27}\sqrt{\frac{2}{3}}\left[\sin h\left(\sqrt{\frac{3}{2}}x\right) + 25\sin h\left(\frac{x}{\sqrt{6}}\right)\right]\text{sech}^5\left(\frac{x}{\sqrt{6}}\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)}, \\
&\vdots.
\end{aligned} \tag{22}$$

Thus, all the findings are expressed as follows:

$$\begin{aligned}
u(x,t) &= u_0 + u_1 + u_2 \cdots, \\
u(x,t) &= \frac{1}{3}\left[1 - 4\text{sech}^2\left(\frac{x}{\sqrt{6}}\right)\right] - \frac{1}{27}\sqrt{\frac{2}{3}}\left[\sin h\left(\sqrt{\frac{3}{2}}x\right) + 25\sin h\left(\frac{x}{\sqrt{6}}\right)\right]\text{sech}^5\left(\frac{x}{\sqrt{6}}\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \cdots.
\end{aligned} \tag{23}$$

Finally, This series of solutions provide the smooth solitary wave solution for $\alpha = 1$.

$$u(x,t) = \frac{1}{3}\left[1 - 4\text{sech}^2\frac{1}{\sqrt{6}}\left(x - \frac{t}{3}\right)\right]. \tag{24}$$

It is noted that we calculate the results only up to two terms for obtaining the smooth solitary wave solution of equation (17) with initial condition (18). In Figure 1, we provide the graphical comparison between the obtained results of $\mathscr{L}$ T-HPM and the exact solution at $-5 \leq x \leq 5$ and $t = 1$. We see that only two term solutions by using $\mathscr{L}$ T-HPM are near with the exact solution at $\alpha = 1$. We also sketch a 2D plot of $\mathscr{L}$ T-HPM and the exact solution at $t = 0.05$ to show the graphical error in Figure 2. Hence we remark that the solutions with $\mathscr{L}$ T-HPM are in good agreement.

4.2. Example 2. Considering equation (17) with the initial condition,

$$u(x,0) = \frac{1}{3}\left[-3 + 4\cot h^2\left(\frac{x}{\sqrt{6}}\right)\right]. \tag{25}$$

Applying $\mathscr{L}$ T-HPM as described in equation (21) and equating the values of p, we obtain the following equation:

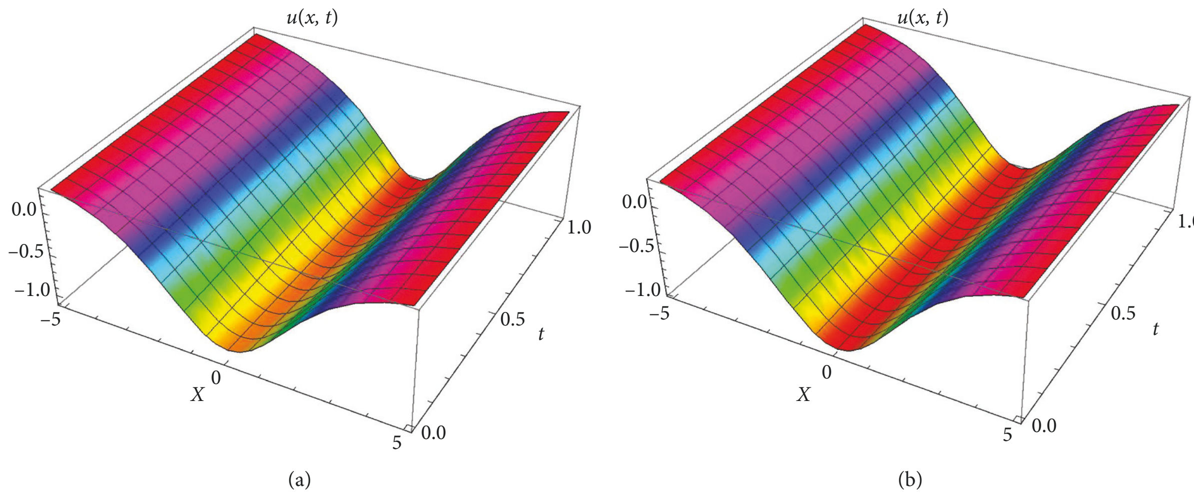

Figure 1: Surface solution between the proximate and the exact solutions w.r.t initial condition (18), when $\alpha = 1$. (a) Approximate solution. (b) Exact solution.

$$\begin{aligned} p^0 &: u_0 = u(x,0) \\ &= \frac{1}{3}\left[-3 + 4\cot h^2\left(\frac{x}{\sqrt{6}}\right)\right], \\ p^1 &: u_1 = \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left\{\frac{\partial}{\partial t}\left(\frac{\partial^2 u_0}{\partial x^2}\right) - 3u_0^2\frac{\partial u_0}{\partial x} + 2\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2} + u_0\frac{\partial^3 u_0}{\partial x^3}\right\}\right] \\ &= \frac{1}{27}\sqrt{\frac{2}{3}}\left[\cosh\left(\sqrt{\frac{3}{2}}x\right) - 25\cosh\left(\frac{x}{\sqrt{6}}\right)\right]\operatorname{csch}\left(\frac{x}{\sqrt{6}}\right)^5\frac{t^\alpha}{\Gamma(1+\alpha)}, \\ &\vdots . \end{aligned} \tag{26}$$

Thus, all the findings are expressed as follows: $u(x,t) = u_0 + u_1 + u_2 \cdots$,

$$\begin{aligned} u(x,t) &= \frac{1}{3}\left[-3 + 4\cot h^2\left(\frac{x}{\sqrt{6}}\right)\right] \\ &+ \frac{1}{27}\sqrt{\frac{2}{3}}\left[\cos h\left(\sqrt{\frac{3}{2}}x\right) - 25\cos h\left(\frac{x}{\sqrt{6}}\right)\right] \\ &\times \operatorname{csch}\left(\frac{x}{\sqrt{6}}\right)^5\frac{t^\alpha}{\Gamma(1+\alpha)} + \cdots . \end{aligned} \tag{27}$$

Finally, This series of solutions provide the singular wave solution for $\alpha = 1$.

$$u(x,t) = \frac{1}{3}\left[-3 + 4\cot h^2\frac{1}{\sqrt{6}}\left(x - \frac{t}{3}\right)\right]. \tag{28}$$

It is noted that we calculate the results only up to two terms for obtaining the smooth solitary wave solution of equation (17) with initial condition (25). In Figure 3, we provide the graphical comparison between the obtained results of $\mathscr{L}$ T-HPM and the exact solution at $-5 \leq x \leq 5$ and $t = 1$. We see that only two term solutions by using $\mathscr{L}$ T-HPM are near with the exact solution at $\alpha = 1$. We also sketch a 2D plot of $\mathscr{L}$ T-HPM and the exact solution at $t = 0.05$ to show the graphical error in Figure 4. Hence, we remark that the solutions with $\mathscr{L}$ T-HPM are in good agreement.

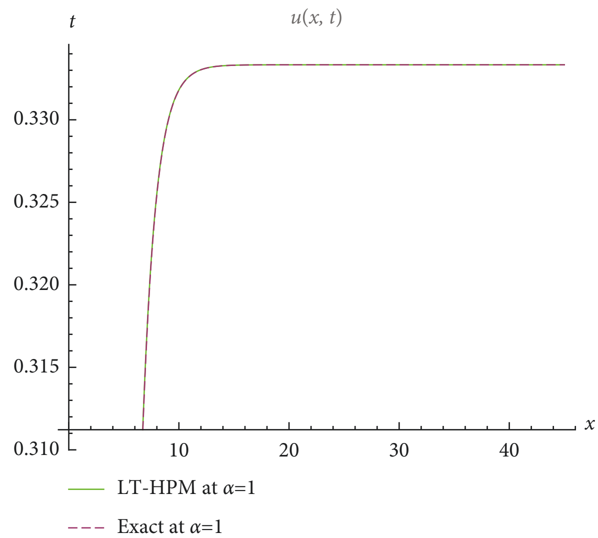

Figure 2: Error distribution between the approximate solution and the exact solution at $\alpha = 1$.

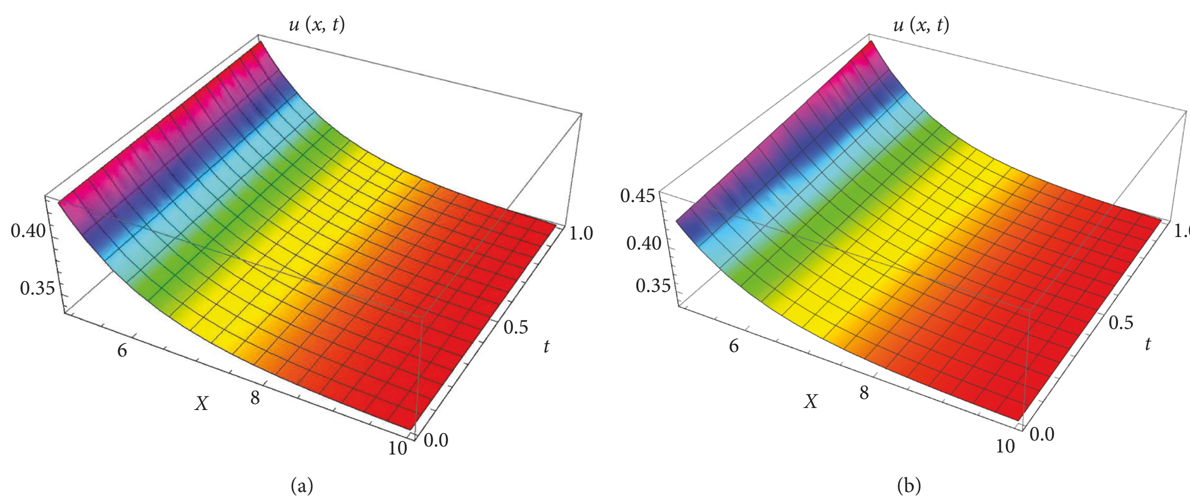

Figure 3: Surface solution between the proximate and the exact solutions w.r.t initial condition (25), when $\alpha = 1$. (a) The approximate solution. (b) The exact solution.

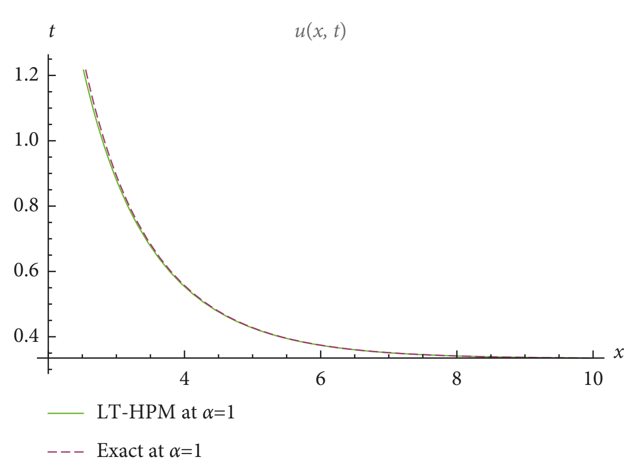

Figure 4: Error distribution between the approximate solution and the exact solution at.$\alpha = 1$.

5. Conclusion

In this study, we successfully applied $\mathscr{L}$ T-HPM to obtain the approximate solution of the mCH equation with fractional order. The most important benefit of this approach is that it does not consider any trivial perturbation and restrictions of variables for the solution of nonlinear problems with fractional order but also maintains an extreme authenticity of the solution. We observe that the obtained results are very close to the exact solution that confirms the accuracy and validity of this approach. We also present our solution results both in two-dimensional and three-dimensional graphs to show the accuracy of $\mathscr{L}$ T-HPM. On the other hand, $\mathscr{L}$ T-HPM plays a significant meaning in finding the simple solution process. This scheme can also be applied to other differential equations including fractal derivatives in our future applications.

Data Availability

The data used to support the study are included in the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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