Research Article

2-Prime Hyperideals of Multiplicative Hyperrings

Mahdi Anbarloei

Department of Mathematics, Faculty of Sciences, Imam Khomeini International University, Qazvin, Iran

Correspondence should be addressed to Mahdi Anbarloei; m.anbarloei@sci.ikiu.ac.ir

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Multiplicative hyperrings are an important class of algebraic hyperstructures which generalize rings further to allow multiple output values for the multiplication operation. Let $R$ be a commutative multiplicative hyperring. A proper hyperideal $I$ of $R$ is called 2-prime if $x \circ y \subseteq I$ for some $x, y \in R$, then, $x^2 \subseteq I$ or $y^2 \subseteq I$. 2-prime hyperideals are a generalization of prime hyperideals. In this paper, we aim to study 2-prime hyperideals and give some results. Moreover, we investigate $\delta$-2-primary hyperideals which are an expansion of 2-prime hyperideals.

1. Introduction

The theory of algebraic hyperstructures was first introduced by Marty [1]. He defined the hypergroups as a generalization of groups in 1934. Since then, algebraic hyperstructures have been investigated by many researchers with numerous applications in both pure and applied sciences [2–9]. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Similar to hypergroups, hyperrings are algebraic structures more general than rings, substituting both or only one of the binary operations of addition and multiplication by hyperoperations. The hyperrings were introduced and studied by many authors [10–13]. Krasner introduced a type of the hyperring where addition is a hyperoperation and multiplication is an ordinary binary operation. Such a hyperring is called a Krasner hyperring [14]. A well-known type of a hyperring, called the multiplicative hyperring. The hyperring was introduced by Rota in 1982 which the multiplication is a hyperoperation, while the addition is an operation [15]. There exists a general type of hyperrings that both the addition and multiplication are hyperoperations [16]. Ameri and Kordi have studied Von Neumann regularity in multiplicative hyperrings [17]. Moreover, they introduced the concept of clean multiplicative hyperrings and studied some topological concepts to realize clean elements of a multiplicative hyperring by clopen subsets of its Zariski topology [18]. The notions such as (weak) zero divisor, (weak) nilpotent and unit in an arbitrary commutative hyperrings were introduced in the study by Ameri and Norouzi [19]. Some equivalence relations—called fundamental relations—play important roles in the theory of algebraic hyperstructures. The fundamental relations are one of the most important and interesting concepts in algebraic hyperstructures that ordinary algebraic structures are derived from algebraic hyperstructures by them. For more details about hyperrings and fundamental relations on hyperrings, see [1, 16, 20–23]. Prime ideals and primary ideals are two of the most important structures in commutative algebra. The notion of primeness of hyperideal in a multiplicative hyperring was conceptualized by Procesi and Rota [24]. Dasgupta extended the prime and primary hyperideals in multiplicative hyperrings [25]. Beddani and Messirdi [26] introduced a generalization of prime ideals called 2-prime ideals, and this idea is further generalized by Ulucak and Çelikel [27]. In [28], Dongsheng defined a new notion which is called $\delta$-primary ideals in commutative rings. In [29], Ulucak introduced the concepts $\delta$-primary and 2-absorbing $\delta$-primary ideal over multiplicative hyperrings. In [30], we investigate $\delta$-primary hyperideals in a Krasner $(m, n)$-hyperring which unify prime hyperideals and primary hyperideals.
In this paper, we consider the class of multiplicative hyperring as a hyperstructure \((R, +, \cdot)\), where \((R, +)\) is an abelian group, \((R, \cdot)\) is a semihypergroup, and the hyperoperation “\(\cdot\)” is distributive with respect to the operation “+.” In this paper, we introduce and study the notion of 2-prime hyperideals of multiplicative hyperrings which are a generalization of prime hyperideals. Several properties of them are provided. Moreover, we investigate \(\delta\)-2-primary hyperideals which are an expansion of 2-prime hyperideals. An earlier version of the manuscript has been presented as a preprint in https://www.researchgate.net [31].

2. Preliminaries

In this section, we give some definitions and results which we need to develop our paper.

A hyperoperation “\(\ast\)” on a nonempty set \(G\) is a mapping of \(G \times G\) into the family of all nonempty subsets of \(G\). Assume that “\(\ast\)” is a hyperoperation on \(G\). Then, \((G, \ast)\) is called hypergroupoid. The hyperoperation on \(G\) can be extended to subsets of \(G\) as follows. Let \(X, Y\) be subsets of \(G\) and \(g \in G\), then

\[
X \ast Y = \bigcup_{x \in X, y \in Y} x \ast y, \quad X \ast g = X \ast \{g\}. \tag{1}
\]

A hypergroupoid \((G, \ast)\) is called a semihypergroup if for all \(x, y, z \in G\), \((x \ast y) \ast z = x \ast (y \ast z)\), which is associative. A semihypergroup is said to be a hypergroup if \(g \ast G = G = G \ast g\) for all \(g \in G\). A nonempty subset \(H\) of a semihypergroup \((G, \ast)\) is called a subhypergroup if for all \(x \in H\) we have \(x \ast H = H \ast x\). A commutative hypergroup \((G, \ast)\) is canonical if

(i) there exists \(e \in G\) with \(e \ast x = \{x\}\), for every \(x \in G\);

(ii) for every \(x \in G\) there exists a unique \(x^{-1} \in G\) with \(e \ast x \ast x^{-1} = e\); and

(iii) \(x \ast y \ast z\) implies \(y \in x \ast z^{-1}\).

A nonempty set \(R\) with two hyperoperations “\(+\)” and “\(\ast\)” is called a hyperring if \((R, +)\) is a canonical hypergroup, \((R, \ast)\) is a semihypergroup with \(r + 0 = 0 + r = r\) for all \(r \in R\) and the hyperoperation “\(\ast\)” is distributive with respect to “+,” i.e., \(x \ast (y + z) = x \ast y + x \ast z\) and \((x + y) \ast z = x \ast z + y \ast z\) for all \(x, y, z \in R\).

Definition 1. A multiplicative hyperring is an abelian group \((R, +)\) in which a hyperoperation “\(\cdot\)” is defined satisfying the following:

(i) for all \(a, b, c \in R\), we have \(a \ast (b \ast c) = (a \ast b) \ast c\);

(ii) for all \(a, b, c \in R\), we have \(a \ast (b + c) \subseteq a \ast b + a \ast c\) and \((b + c) \ast a \subseteq b \ast a + c \ast a\); and

(iii) for all \(a, b \in R\), we have \(a \ast (\neg b) = \neg (a \ast b)\).

If in (ii) the equality holds, then, we say that the multiplicative hyperring is strongly distributive. A nonempty subset \(I\) of a multiplicative hyperring \(R\) is a hyperideal,

(i) If \(a, b \in I\), then \(-a - b \in I\);

(ii) If \(x \in I\) and \(r \in R\), then \(r \ast x \subseteq I\).

Let \((\mathbb{Z}, +, \cdot)\) be the ring of integers. Corresponding to every subset \(A \subseteq P^*(\mathbb{Z})\) such that \(|A| \geq 2\), there exists a multiplicative hyperring \((\mathbb{Z}_A, +, \cdot)\) with \(\mathbb{Z}_A = \mathbb{Z}\) and for any \(a, b \in \mathbb{Z}_A\), \(a \ast b = \{a \cdot r \cdot b \mid r \in A\}\).

Definition 2 (see [25]). A proper hyperideal \(P\) of a multiplicative hyperring \(R\) is called a prime hyperideal if \(x \ast y \subseteq P\) for \(x, y \in R\) implies that \(x \in P\) or \(y \in P\). The intersection of all prime hyperideals of \(R\) containing \(I\) is called the prime radical of \(I\), being denoted by \(\sqrt{I}\). If the multiplicative hyperring \(R\) does not have any prime hyperideal containing \(I\), we define \(\sqrt{I} = R\).

Definition 3 (see [32]). A proper hyperideal \(I\) of a multiplicative hyperring \(R\) is maximal in \(R\) if for any hyperideal \(J\) of \(R\) with \(I \subseteq J \subseteq R\), then, \(J = R\). Also, we say that \(R\) is a local multiplicative hyperring, if it has just one maximal hyperideal.

Let 
\(C\) be the class of all finite products of elements of \(R\), i.e., \(C = \{r_1 \ast r_2 \ast \ldots \ast r_n \mid r_i \in R, n \in \mathbb{N}\} \subseteq P^*(R)\). A hyperideal \(I\) of \(R\) is said to be a \(C\)-hyperideal of \(R\) if, for any \(A \in C\), \(A \cap I \neq \emptyset\) implies \(A \subseteq I\). Let \(I\) be a hyperideal of \(R\). Then, \(D \subseteq \sqrt{I}\) where \(D = \{r \in R \mid r^n \subseteq I\} \text{ for some } n \in \mathbb{N}\). The equality holds when \(I\) is a \(C\)-hyperideal of \(R\) ([25], Proposition 1). In this paper, we assume that all hyperideals are \(C\)-hyperideal.

Definition 4 (see [25]). A nonzero proper hyperideal \(Q\) of a multiplicative hyperring \(R\) is called a primary hyperideal if \(x \ast y \subseteq Q\) for \(x, y \in R\) implies that \(x \in Q\) or \(y \in \sqrt{Q}\). Since 
\(\sqrt{Q} = P\) is a prime hyperideal of \(R\) by Proposition 3.6 in [25], \(Q\) is referred to as a \(P\)-primary hyperideal of \(R\).

Definition 5 (see [32]). Let \(R\) be a commutative multiplicative hyperring and \(e\) be an identity (i.e., for all \(a \in R\), \(a = a \ast e\)). An element \(x \in R\) is called unit, if there exists \(y \in R\), such that \(e \ast x \ast y\).

Definition 6. A hyperring \(R\) is called an integral hyperdomain, if for all \(x, y \in R\), \(0 \neq x \cdot y\) implies that \(x = 0\) or \(y = 0\).

Definition 7 (see [32]). Let \(R\) be a multiplicative hyperring. The element \(x \in R\) is an idempotent if \(x = x^2\).

Definition 8 (see [19]). An element \(a \in R\) is said to be zero divisor if there exists \(0 \neq b \in R\) such that \(0 = a \ast b\).

Definition 9. Let \((R_1, +_1, \cdot_1)\) and \((R_2, +_2, \cdot_2)\) be two multiplicative hyperrings. A mapping \(f\) from \(R_1\) into \(R_2\) is said to be a good homomorphism if for all \(x, y \in R_1\), 
\(f(x, y) = f(x +_1 y) = f(x) +_2 f(y)\), and 
\(f(x \cdot_1 y) = f(x) \cdot_2 f(y)\).

Definition 10 (see [33]). A function \(\delta\) is called a hyperideal expansion of \(R\) if it assigns to each hyperideal \(I\) of \(R\) a hyperideal \(\delta(I)\) such that it has the following properties:

(i) \(I \subseteq \delta(I)\).

(ii) If \(I \subseteq J\) for any hyperideals \(I, J\) of \(R\), then, \(\delta(I) \subseteq \delta(J)\).
For example, consider the hyperideal expansions $\delta_0$, $\delta_1$, $\delta_\sigma$, and $\delta_\varphi$ of $R$ defined with $\delta_0(I) = I$, $\delta_1(I) = \sqrt{I}$, $\delta_\sigma(I) = I + J$ (for some hyperideal $J$ of $R$), and $\delta_\varphi(I) = (I : K)$ (for some hyperideal $K$ of $R$) for all hyperideals $I$ of $R$, respectively. Also, let $\delta$ be a hyperideal expansion of $R$ and $I, J$ two hyperideals of $R$ such that $I \subseteq J$. Let $\delta_\gamma : (R/I) \to (R/J)$ be defined by $\delta_\gamma(J/I) = (\delta(J)/I)$. Then, $\delta_\gamma$ is a hyperideal expansion of $(R/I)$.

**Definition 11.** (See [29]) Let $\delta$ be a hyperideal expansion of $R$. A hyperideal $I$ of $R$ is called a $\delta$-primary hyperideal if

\[
x, y \in R \text{ and } x \cdot y \subseteq I \text{ imply either } x \in I \text{ or } y \in \delta(I).
\]

**Definition 12.** (See [29]) Let $f : R_1 \to R_2$ be a good hyperring homomorphism, $\delta$ and $\gamma$ hyperideal expansions of $R_1$ and $R_2$, respectively. Then, $f$ is called a $\delta\gamma$-homomorphism if $\delta(f^{-1}(I_{R_2})) = f^{-1}(\gamma(I_{R_2}))$ for each hyperideal $I_{R_2}$ of $R_2$.

Moreover, if $f$ is a $\delta\gamma$-epimorphism and $I$ is a hyperideal of $R$ with $\ker f \subseteq I$, then, $\gamma(f(I)) = f(\delta(I))$.

### 3. 2-Prime Hyperideals

**Definition 13.** Let $I$ be a proper hyperideal of a multiplicative hyperring $R$. We say that $I$ is $2$-prime if for all $x, y \in R$, $x \cdot y \subseteq I$ implies $x^2 \subseteq I$ or $y^2 \subseteq I$.

**Example 1.** Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers. In the multiplicative hyperring $(\mathbb{Z}_2, +, \cdot, \circ)$ with $A = \{2, 3\}$ and the hypoperation $a \circ b = \{a \cdot b \cdot r \mid r \in A\}$ for $a, b \in \mathbb{Z}_2$, the principal hyperideal $3\mathbb{Z}_2$ is $2$-prime.

**Example 2.** Consider the ring $(\mathbb{Z}_4, +, \cdot, \circ)$ that for each $x, y \in \mathbb{Z}_4$, $x + y$ and $x \cdot y$ are the remainder of $(x + y)/4$ and $(x \cdot y)/4$, respectively, which $+$ and $\cdot$ are ordinary addition and multiplication, and $x, y \in \mathbb{Z}_4$ (see Tables 1 and 2).

$(\mathbb{Z}_4, +, \cdot, \circ)$ is a multiplicative hyperring. In the hyperring, hyperideal $\{0, 2\}$ is a $2$-prime hyperideal of $\mathbb{Z}_4$.

Note that an intersection of the $2$-prime hyperideals may not be a $2$-prime hyperideal of $R$.

**Example 3.** Consider the ring of integers $\mathbb{Z}$. We define $a \circ b = \{2ab, 3ab\}$ for all $a, b \in \mathbb{Z}$. Then, $(\mathbb{Z}, +, \cdot, \circ)$ is a multiplicative hyperring. The hyperideals $5\mathbb{Z}$ and $7\mathbb{Z}$ are $2$-prime, but $5\mathbb{Z} \cap 7\mathbb{Z} = 35\mathbb{Z}$ since $5 \cdot 7 \subseteq 35\mathbb{Z}$ and $5, 7 \notin 35\mathbb{Z}$.

**Theorem 1.** Let $I$ be a hyperideal of a multiplicative hyperring $R$. Then,

1. If $I$ is a $2$-prime hyperideal, then $\sqrt{I}$ is a prime hyperideal.
2. If $P$ is a prime hyperideal of $R$, then $P^2$ is a $2$-prime hyperideal of $R$.
3. Let $R_1$ and $R_2$ be two multiplicative hyperrings such that $f : R_1 \to R_2$ is a good homomorphism. If $I_{R_2}$ is a $2$-prime hyperideal of $R_2$, then $f^{-1}(I_{R_2})$ is a $2$-prime hyperideal of $R_1$.
4. Let $R_1$ and $R_2$ be two multiplicative hyperrings such that $f : R_1 \to R_2$ is a good epimorphism. If $I_{R_1}$ is a $2$-prime hyperideal of $R_1$, then $f(I_{R_1})$ is a $2$-prime hyperideal of $R_2$. 

### Table 1: Addition operation.

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### Table 2: Multiplication hyperoperation.

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<td>$[0, 2]$</td>
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<tr>
<td>$3$</td>
<td>$0$</td>
<td>$Z_4$</td>
<td>$[0, 2]$</td>
<td>$Z_4$</td>
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</table>

**prime hyperideal of $R_1$ with $\ker f \subseteq I_1$, then, $f(I_1)$ is a $2$-prime hyperideal of $R_2$.

5. If $I_1$ is a $2$-prime hyperideal and $I_1, I_2$ are two subsets of $R$ with $I_1, I_2 \subseteq I_1$, then, $\cup_{x \in I, x^2 \subseteq I}$ or $\cup_{x \in I, x^2 \subseteq I}$.

**Proof**

1. Let the hyperideal $I$ be $2$-prime. If $x \cdot y \subseteq \sqrt{I}$ for $x, y \in R$, then, for some positive integer $n, x^{n} \cdot y^{n} \subseteq I$. Since hyperideal $I$ is $2$-prime, we have $x^{2n} \subseteq I$ or $y^{2n} \subseteq I$. Thus, we get $x \in \sqrt{I}$ or $y \in \sqrt{I}$ which means hyperideal $\sqrt{I}$ is prime.

2. Since $P^2 \subseteq P$, then, we are done.

3. Let $x \cdot y \subseteq f^{-1}(I_{R_2})$ for $x, y \in R_1$. Hence, $f(x \cdot y) \subseteq I_{R_2}$. Since the hyperideal $I_{R_2}$ is $2$-prime, then, $f^2(x) = f(x^2) \subseteq I_{R_2}$ or $f^2(y) = f(y^2) \subseteq I_{R_2}$. Thus, $x^2 \subseteq f^{-1}(I_{R_2})$ or $y^2 \subseteq f^{-1}(I_{R_2})$, i.e., the hyperideal $f^{-1}(I_{R_2})$ is $2$-prime.

4. Let for some elements $x_2, y_2 \in R_2, x_2 \cdot y_2 \subseteq f(I_1)$. Since $f$ is an epimorphism, there exist $x_1 \in R_1$ such that $x_1 \cdot y_1 = x_2 \cdot y_2$. If $x_1 \cdot y_1 \subseteq f(I_1)$, then, let $f(u) \cdot f(v) \subseteq I_{R_2}$ and so $f(u) = f(v)$ for some $u \in I_1$. This implies that $f(u - v) = 0$, that is, $u - v \in \ker f \subseteq I_1$ and so $u \in I_1$. Since $I_1$ is a $C$-hyperideal of $R$, then, we conclude that $x_1 \cdot y_1 \subseteq I_1$. Since $I_1$ is a $2$-prime hyperideal of $R_1$, then, $x_1^2 \subseteq I_1$ which implies $x_1^2 = f(x_1^2) \subseteq f(I_1)$ or $y_1^2 \subseteq I_1$ which implies $y_1^2 = f(y_1^2) \subseteq f(I_1)$. Consequently, $f(I_1)$ is a $2$-prime hyperideal of $R_2$.

5. Let $\cup_{x \in I, x^2 \subseteq I}$ or $\cup_{x \in I, x^2 \subseteq I}$. Hence, there exist $x_0 \in I_1$ and $y_0 \in I_2$ with $x_0^2 \subseteq I$ and $y_0^2 \subseteq I$. Since hyperideal $I_1$ is $2$-prime, then, $x_0 \cdot y_0 \subseteq I$ which is a contradiction.

Let $I$ be a $2$-prime hyperideal of $R$. Since $\sqrt{I} = P$ is a prime hyperideal of $R$ by Theorem 1. (1), $I$ is referred to as a $P$-2-prime hyperideal of $R$.

**Theorem 2.** Let $I$ be a $P$-2-prime hyperideal of a multiplicative hyperring $R$. Then, $(I : x^2)$ for all $x \in R - I$ is a $P$-2-prime hyperideal of $R$. 

\[ \square \]
Proof. Let \( y \in (I : x^2) \) for \( x \in R - P \). This means that \( y \circ x^2 \subseteq I \). Since the hyperideal \( I \) is \( P \)-prime and \( x \notin P = \sqrt{I} \), then, \( y^2 \subseteq I \) which means \( y \in P \). Therefore, we have \( (I : x^2) \subseteq P \). Since \( I \subseteq (I : x^2) \subseteq P \), then, \( \sqrt{I} \subseteq \sqrt{(I : x^2)} \subseteq \sqrt{P} \). Since \( \sqrt{I} = \sqrt{P} = P \), then, we have \( \sqrt{(I : x^2)} = P \). Assume that for \( u, w \in R, u \circ w \subseteq (I : x^2) \) such that \( w \circ x \subseteq I \). We have \( u \circ w \circ x^2 = (u \circ x) \circ (w \circ x) \subseteq I \). Since the hyperideal \( I \) is \( P \)-prime and \( (u \circ x)^2 \subseteq I \), then, \( (u \circ x)^2 = u^2 \circ x^2 \subseteq I \). This means \( u^2 \subseteq (I : x^2) \). Consequently, \( (I : x^2) \) is a \( P \)-prime hyperideal of \( R \).

In reference [17], recall that an element \( x \in R \) is called regular if there exists \( r \in R \) such that \( x \in x^2 + r \). So, we can define that \( R \) is regular multiplicative hyperring, if all of elements in \( R \) are regular elements.

\[ \Box \]

**Theorem 3.** Let \( R \) be a regular multiplicative hyperring and \( I \) be a \( 2 \)-prime hyperideal of \( R \). Then, \( I \) is prime.

**Proof.** Let \( I \) be a \( 2 \)-prime hyperideal of \( R \). Suppose that \( a \circ b \subseteq I \) with \( a \notin I \) for some \( a, b \in R \). Since \( R \) is regular, then, there exists \( r \in R \) such that \( a = a^2 + r \). Let \( a^2 \subseteq I \). Then, \( a \in a^2 \circ r \subseteq I \), which is a contradiction. Therefore, \( a^2 \not\subseteq I \). Since \( I \) is a \( 2 \)-prime hyperideal of \( R \), then, \( b^2 \subseteq I \). This means \( b \in \sqrt{I} = I \). Consequently, the hyperideal \( I \) is prime.

Recall that a proper hyperideal \( I \) of a multiplicative hyperring \( R \) is called semiprime if whenever \( x^k \circ y \subseteq I \) for some \( x, y \in R \) and \( k \in \mathbb{Z}^+ \), then \( x \circ y \subseteq I \). Note that if \( I \) is a semiprime hyperideal of \( R \) such that \( I^n \subseteq I \) for some hyperideal \( J \) of \( R \) and \( n \in \mathbb{N} \), then, \( J \subseteq I \) (Proposition 2 in [34]). Every prime hyperideal is a semiprime hyperideal, but, the converse is not true in general. For example, the hyperideal \( \{0\} \) of \( Z_{30} \) is semiprime, but it is not prime (see Example 2.3 in [34]).

\[ \Box \]

**Theorem 4.** Let \( I \) be a \( 2 \)-prime hyperideal of a multiplicative hyperring \( R \) and \( I \) be a \( 2 \)-prime hyperideal of \( R \). Then, \( I \) is a \( 2 \)-prime hyperideal of \( R \) if and only if \( 1 \) is a prime hyperideal of \( R \).

**Definition 14.** A proper hyperideal \( I \) of a multiplicative hyperring \( R \) is called semiprimary if for all \( x, y \in R \) such that \( x \circ y \subseteq I \), then, either \( x \) or \( y \) lies in \( \sqrt{I} \).

**Example 4.** Let \((\mathbb{Z}, +, \cdot)\) be the ring of integers. In the multiplicative hyperring \((\mathbb{Z}_A, +, \cdot)\) with \( A = \{2, 3\} \) and hyperoperation \( \circ = \{a \cdot r + b | r \in A\} \) for \( a, b \in \mathbb{Z}_A \), the hyperideal \( 11\mathbb{Z} \) is semiprimary.

\[ \begin{pmatrix} x \circ y & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \subseteq M_n(I), \]  

(2)

It is clear that

\[ \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \]  

(3)

**Theorem 5.** Every \( 2 \)-prime hyperideal of \( R \) is a semiprimary hyperideal of \( R \).

**Theorem 6.** Let \( Z_A \) be a multiplicative hyperring of integers and \( a (> 1) \) be a positive integer such that each element \( x \) of \( A \) and \( a \) are coprime and \( x^2 \in A \). Then, \( I = \langle a \rangle \) is a \( 2 \)-prime hyperideal of \( Z_A \) if and only if \( I = \langle p^n \rangle \) for some positive integer \( n \) and an irreducible element \( p \) of \( Z_A \) or \( p = 0 \).

**Proof.** \((\Rightarrow)\) Let \( I = \langle a \rangle \) be a \( 2 \)-prime hyperideal of \( Z_A \). Assume that \( a \) is not an irreducible element. Then, \( a = p_1^{m_1} p_2^{m_2} \ldots p_n^{m_n} \) is a representation of \( a \) as a product of distinct prime integers \( p_i \) such that \( m_i \) is a positive integer for \( 1 \leq i \leq n \). Put \( x = p_1^{m_1} \) and \( y = p_2^{m_2} \ldots p_n^{m_n} \) by Proposition 4.13 in [25]. For every \( c \in A, xcy \in \langle a \rangle \) which implies \( x \circ y \subseteq \langle a \rangle \). Since \( \langle a \rangle \) is \( 2 \)-prime, then, we have \( x^2 \subseteq \langle a \rangle \) or \( y^2 \subseteq \langle a \rangle \). If \( x^2 \subseteq \langle a \rangle \), then, for all \( d \in A \) we have \( x^2 d \subseteq \langle a \rangle \). Thus, we obtain \( x^2 d = p_1^{m_1} d = a a p_1^{m_1} \) which means \( p_1 \mid p_1 \) for some \( a \in Z_A \). Since each element of \( A \) and \( a \) is coprime, then, for some \( 2 \leq i \leq n \), we get \( p_i \mid p_i \) which is contradiction.

\((\Leftarrow)\) Let \( I = \langle p^n \rangle \) for some positive integer \( n \) and an irreducible element \( p \in Z_A \). Let \( x \circ y \subseteq I \) for some \( x, y \in I \). Then, for \( a \in A, xya \in I \) which implies \( x \circ y \subseteq I \). Since \( \langle a \rangle \) is \( 2 \)-prime, then, we have \( x^2 \subseteq \langle a \rangle \) or \( y^2 \subseteq \langle a \rangle \). If \( y^2 \subseteq \langle a \rangle \), then, for all \( e \in A, y^2 e \subseteq \langle a \rangle \) which implies \( y^2 e = (p_2^{m_2} \ldots p_n^{m_n})^e = \beta a = p_1^{m_1} p_2^{m_2} \ldots p_n^{m_n} \) for \( \beta \in Z \) which means \( p_1 | p_1 \) for some \( 2 \leq j \leq n \) which is a contradiction.

\( \Box \)

**Theorem 7.** Let \( R \) be a multiplicative hyperring. Then, we call \( M_n(R) \) as the set of all hypermatrices of \( R \). Also, for all \( A = (A_{ij})_{n \times n}, B = (B_{ij})_{n \times n} \in P^* (M_n(R)) \), \( A \subseteq B \) if and only if \( A_{ij} \subseteq B_{ij} \) [32].

**Proof.** Suppose that for \( x, y \in R \), \( x \circ y \subseteq I \). Then,

\[ \begin{pmatrix} x \circ y & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \subseteq M_n(I), \]  

(2)

It is clear that

\[ \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} y & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \]  

(3)
since $M_n(I)$ is a 2-prime hyperideal of $M_n(R)$, then,
\[
\begin{pmatrix}
x & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 
\end{pmatrix}^2 \begin{pmatrix} x^2 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 
\end{pmatrix} \subseteq M_n(I), \tag{4}
\]
which means $x^2 \subseteq I$ or
\[
\begin{pmatrix}
y & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 
\end{pmatrix}^2 \begin{pmatrix} y^2 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 
\end{pmatrix} \subseteq M_n(I), \tag{5}
\]
which means $y^2 \subseteq I$. Therefore, $I$ is a 2-prime hyperideal of $R$.

Lemma 1. \textit{Let $R$ be a local multiplicative hyperring with maximal hyperideal $M$ and $P$ be a prime hyperideal of $R$. Then, $P \circ M$ is a 2-prime hyperideal of $R$. Moreover, if $P \circ M$ is a prime hyperideal of $R$, then, $P \circ M = P$.}

Proof. Let $x \circ y \subseteq P \circ M \subseteq P$ for some $x, y \in R$. This means $x \in P$ or $y \in P$. Suppose that $x \in P$. Clearly, $x$ is not unit. Then, we have $x \in M$ which implies $x^2 \subseteq P \circ M$. Thus, the hyperideal $P \circ M$ is 2-prime.

For the second assertion, assume that the hyperideal $P \circ M$ is prime. Suppose that $x \in P$. Since $P \subseteq M$, then, $x^2 \subseteq P \circ M$ which implies $x \in P \circ M$ because $P \circ M$ is a prime hyperideal of $R$. This means $P \subseteq P \circ M$. Since $P \circ M \subseteq P$, then, we have $P \circ M = P$.

Theorem 8. \textit{Let $R$ be a multiplicative hyperring with scalar identity $1$. Then, a hyperideal $I$ of $R$ is 2-prime if and only if $I/\gamma^*$ is a 2-prime ideal of $R/\gamma^*$.}

Proof. ($\Rightarrow$) Let for $x, y \in R/\gamma^*$, $x \circ y \in I/\gamma^*$. Thus, there exist $a, b \in R$ such that $x = \gamma^*(a)$, $y = \gamma^*(b)$ and $x \circ y = \gamma^*(a) \circ \gamma^*(b) = \gamma^*(a + b)$. So, $\gamma^*(a) \circ \gamma^*(b) = \gamma^*(a + b) \in I/\gamma^*$, then, $a + b \subseteq I$. Since $I$ is 2-prime hyperideal, then, $a^2 \subseteq I$ or $b^2 \subseteq I$. Hence, $x^2 = \gamma^*(a)^2 \in I/\gamma^*$ or $y^2 = \gamma^*(b)^2 \in I/\gamma^*$. Thus, $I/\gamma^*$ is a 2-prime ideal of $R/\gamma^*$.

($\Leftarrow$) Suppose that $a + b \in I$ for $a, b \in R$, then, $\gamma^*(a) \circ \gamma^*(b) = \gamma^*(a + b) \in I/\gamma^*$. Since $I/\gamma^*$ is a 2-prime ideal of $R/\gamma^*$, then, we have $\gamma^*(a)^2 \in I/\gamma^*$ or $\gamma^*(b)^2 \in I/\gamma^*$. It means that $a^2 \subseteq I$ or $b^2 \subseteq I$. Hence, $I$ is a 2-prime hyperideal of $R$.

Theorem 9. \textit{Let $R$ be a local multiplicative hyperring with maximal hyperideal $M$. Then, every 2-prime hyperideal of $R$ is prime if and only if for each minimal prime hyperideal $I$ over an arbitrary 2-prime hyperideal $I$, $I \circ M = P$. Furthermore, if $P$ is a 2-prime hyperideal of $R$ is prime, then, $M$ is an idempotent hyperideal.}

Proof. ($\Rightarrow$) Assume that every 2-prime hyperideal of $R$ is prime. Since the hyperideal $I$ is 2-prime, then, $I = P$ is prime. Then, $I \circ M$ is a 2-prime hyperideal of $R$, by Lemma 1. This means $I \circ M$ is a prime hyperideal of $R$. Then, we conclude that $I \circ M = I$, by Lemma 1.

($\Leftarrow$) Suppose that $I$ is a 2-prime hyperideal of $R$. Then, we have $I \subseteq \sqrt{I} = P$, by Theorem 1 (1). By the assumption, we get $P = I \circ M \subseteq I \cap M = I$. This implies that $I = P$ is a prime hyperideal of $R$.

Theorem 10. \textit{Let every 2-prime hyperideal of a multiplicative hyperring $R$ be prime and $P$ be an arbitrary prime hyperideal of $R$. Then, $P^2 = P$.}

Proof. Assume that every 2-prime hyperideal of $R$ is prime and $P$ is an arbitrary prime hyperideal of $R$. Then, the hyperideal $P^2$ is 2-prime, by Theorem 1 (2). Since every 2-prime hyperideal of $R$ is prime, then, the hyperideal $P^2$ is prime. It is easy to see that $P^2$ is equal to $P$.

Definition 15. \textit{Let $I$ be a hyperideal of a multiplicative hyperring $R$. We say that a 2-prime hyperideal $P$ is minimal over $I$ if there is no a 2-prime hyperideal $P$ of $R$ with $I \subseteq P \subseteq P$. Note that $2$-$\text{Min}_R(I)$ denotes the set of minimal 2-prime hyperideals over $I$.}

Theorem 11. \textit{Let $R$ be a local multiplicative hyperring with maximal hyperideal $M$ and $P$ be a prime hyperideal of $R$. Let for each 2-prime hyperideal $I$ of $R$, $\left(\sqrt{I}\right)^2 I \subseteq I$. Then, the followings are equivalent:}

1. If $P \in \text{Min}_R(I)$ for each hyperideal $I \in 2 - \text{Min}_R(P^2)$, then, $I \circ M = P$.
2. If $I \subseteq P$ for each hyperideal $I \in 2 - \text{Min}_R(P^2)$, then, $I = P$.

Proof. (1) $\Rightarrow$ (2) Suppose that $I \subseteq P$ for some $I \subseteq 2 - \text{Min}_R(P^2)$. First, we show that the hyperideal $P$ is minimal over $I$. Assume that we have $I \subseteq P \subseteq P$ for some prime hyperideal $P'$ of $R$. By the assumption, we get $P' \subseteq I \subseteq P$. Let $a \in P$. Then, we obtain $a^2 \subseteq P^2$ which
implies \(a^2 \subseteq P'\). This means \(a \in P'\), since \(P'\) is a prime hyperideal of \(R\). Now, since \(I \circ M \subseteq I \subseteq P\) and \(I \circ M = P\), then, we get \(I = P\).

\((2) \Rightarrow (1)\) Let \(P \in \text{Min}_R(I)\) such that \(I \subseteq 2 - \text{Min}_R(P^2)\). We get \(\sqrt{I} = P\) since the hyperideal \(\sqrt{I}\) is prime. Clearly, \(\langle \sqrt{I} \rangle^2 = P^2 \subseteq I \subseteq P\). By Lemma 1, \(P \circ M\) is a 2-prime hyperideal of \(R\). Since \(P^2 \subseteq P \circ M \subseteq I\) and \(I = P\), then, \(P \circ M = I = P\). \(\square\)

**Corollary 1.** Let \(R\) be a local multiplicative hyperring with maximal hyperideal \(M\) and \(P\) be a prime hyperideal of \(R\). Let for each \(P\)-prime hyperideal \(I\), \(I \subseteq 2 - \text{Min}_R(P^2)\). Then, for each hyperideal \(I \subseteq 2 - \text{Min}_R(P^2)\) with \(I \subseteq P\), \(I = P\) if and only if every \(2\)-prime hyperideal of \(R\) is prime.

**Proof.** The claim follows by Theorems 9 and 11. \(\square\)

### 4. Expansion of 2-Prime Hyperideals

**Definition 16.** Let \(\delta\) be a hyperideal expansion of a multiplicative hyperring \(R\). A proper hyperideal \(I\) of \(R\) is called \(\delta\)-2-primary if for \(a, b \in R\), \(a \circ b \subseteq I\) implies either \(a^2 \subseteq I\) or \(b^2 \subseteq \delta(I)\).

**Example 5.** Suppose that \((Z, +, \cdot)\) is the ring of integers. For all \(a, b \in Z\), we define the hyperoperation \(a \circ b = \{x : x \cdot b \mid x \in \{11, 13\}\}\). Then, \((Z, +, \circ)\) is a multiplicative hyperring. Consider the hyperideal expansion \(\delta\) by \(\delta(I) = 3Z + I\). Since \(\delta(Z) = 3Z + 2Z = Z\), then, we conclude that \(I = 2Z\) is a \(\delta\)-2-primary hyperideal of \((Z, +, \circ)\).

Clearly, every prime hyperideal of a multiplicative hyperring \(R\) is a \(\delta\)-2-primary hyperideal but its inverse is not true in general.

**Example 6.** Let \(Z\) be the ring of integers, \(E\) be the set of all even integers of \(Z\) and \(A\) be the set of all positive even integers of \(Z\). In the multiplicative hyperring \(Z_A\) (see Example 3.5 in [25]), the hyperideal \(E\) is \(\delta\)-2-primary but is not prime.

**Example 7.** In the multiplicative hyperring \((Z, +, \circ)\) with trivial hyperoperation, i.e., \(a \circ b = \{ab\}\) for \(a, b \in Z\), the hyperideal \(2Z\) is \(\delta\)-2-primary.

We start the section with the following trivial result, and hence, we omit its proof.

**Theorem 12.** Let \(I\) be a proper of \(R\). Then,

1. \(I\) is a \(\delta\)-2-primary hyperideal if and only if \(I\) is a 2-prime hyperideal.
2. \(I\) is a primary hyperideal, then, \(I\) is a \(\delta\)-2-primary hyperideal.
3. \(I\) is a \(2\)-prime hyperideal, then, \(I\) is a \(\delta\)-2-primary hyperideal for every hyperideal expansion \(\delta\) of \(R\).
4. \(I\) is a \(\delta\)-primary hyperideal, then, \(I\) is a \(\delta\)-2-primary hyperideal for every hyperideal expansion \(\delta\) of \(R\).

\((5)\) If \(I\) is a \(\delta\)-2-primary hyperideal of \(R\) such that \(\delta(I) \subseteq \gamma(I)\) for some hyperideal expansion \(\gamma\) of \(R\), then, \(I\) is a \(\gamma\)-2-primary hyperideal of \(R\).

**Theorem 13.** Let \(I\) be a hyperideal of a multiplicative hyperring \(R\) and \(\delta\) a hyperideal expansion of \(R\). If \(I\) is a \(\delta\)-2-primary hyperideal of \(R\), then, for some idempotent element \(a \in R-I\), \((I: a)\) is a \(\delta\)-2-primary hyperideal of \(R\).

**Proof.** Let \(x \circ y \subseteq (I: a)\) such that \(x^2 \circ y \subseteq (I: a^2)\) for some \(x, y \in R\). This means that \(x^2 \circ y \subseteq (I: a^2)\) but \(x^2 \circ y \subseteq (I: a)\). Since \(I\) is a \(\delta\)-2-primary hyperideal of \(R\), we get \(y \circ y \subseteq \delta(I) \subseteq \delta(I: a)\). Thus, \((I: a)\) is a \(\delta\)-2-primary hyperideal of \(R\).

**Theorem 14.** Let \(I\) be a hyperideal of a multiplicative hyperring \(R\) such that for each \(a \in R-I\), \((I: a) = (I: a^2)\) and \(\delta\) a hyperideal expansion of \(R\). If the hyperideal \(I\) is irreducible, then, \(I\) is a \(\delta\)-2-primary hyperideal of \(R\).

**Proof.** We suppose that \(I\) is not a \(\delta\)-2-primary hyperideal of \(R\) and look for a contradiction. This means that there exist \(x, y \in R\) such that \(x \circ y \subseteq I\) but \(x^2 \circ y \subseteq \delta(I)\). Thus, we get \(x \circ y \subseteq I\) and \(y \notin \delta(I)\). Let \(t \in (I + \langle x \rangle) \cap (I + \langle y \rangle)\). Then, there are \(a_1, a_2 \in I\) and \(t_1, t_2 \in R\) such that for some \(t_1 \in r_1 \circ x\) and \(t_2 \in r_2 \circ y\) we have \(t = a_1 + t_1 = a_2 + t_2\). So, we get \(y \circ (a_2 + t_2) \subseteq y \circ a_2 + y \circ t_2\). Also, we have \(y \circ (a_1 + t_1) \subseteq y \circ a_1 + y \circ t_1\). Since \(y \circ a_1 + y \circ t_1 \subseteq I\), then, \(y \circ (a_2 + t_2) \subseteq I\). This implies that \((y \circ a_2 + y \circ t_2) \cap I = \emptyset\). Since \(I\) is a \(\delta\)-hyperideal of \(R\), then, \(y \circ T \subseteq I\) which implies \(r_2 \circ y \subseteq I\). Therefore, \(r_2 \in (I: y)\) which means \(r_2 = (I: y)\), by the assumption. So, \(r_2 \subseteq y \subseteq I\). Thus, we have \(t = a_2 + t_2 \subseteq a_2 + r_2 \circ y \subseteq I\). Then, \((I + \langle x \rangle) \cap (I + \langle y \rangle) \subseteq I\). Since \((I + \langle x \rangle) \cap (I + \langle y \rangle), then, we obtain \(I = (I + \langle x \rangle) \cap (I + \langle y \rangle)\). This is a contradiction since \(I\) is irreducible. Consequently, the hyperideal \(I\) of \(R\) is \(\delta\)-2-primary.

**Theorem 15.** Let \(\delta\) be a hyperideal expansion of a multiplicative hyperring \(R\). Let \(I\) and \(\delta(I)\) be semiprime hyperideals of \(R\). Then, \(I\) is a \(\delta\)-2-primary hyperideal of \(R\) if and only if \(I\) is \(\delta\)-primary.

**Theorem 16.** Let \(I\) be a hyperideal of a multiplicative hyperring \(R\) and \(\delta\) a hyperideal expansion of \(R\) such that \(\sqrt{\delta(I)} \subseteq \sqrt{\delta(I)}\). If \(I\) is a \(\delta\)-2-primary hyperideal of \(R\), then, \(\sqrt{\delta(I)}\) is a \(\delta\)-2-primary hyperideal of \(R\).

**Proof.** Let \(x \circ y \subseteq \sqrt{\delta(I)}\) for some \(x, y \in R\) such that \(x \notin \sqrt{\delta(I)}\). This means that we have \(x^n \circ y \subseteq I\) for some \(n \in \mathbb{N}\). Clearly, \(x^{2n} \subseteq \delta(I)\) which implies \(y \in \sqrt{\delta(I)} \subseteq \delta(\sqrt{\delta(I)})\). This means that \(\sqrt{\delta(I)}\) is a \(\delta\)-primary hyperideal of \(R\).

**Theorem 17.** Let \(I\) be a hyperideal of a multiplicative hyperring \(R\) and \(\delta\) a hyperideal expansion of \(R\) such that \(\sqrt{\delta(I)} = \delta(I)\). If the hyperideal \(I\) is primary, then, \(I\) is a \(\delta\)-2-primary hyperideal of \(R\).
Theorem 18. Let $\delta$ have the property of intersection preserving. If $I_i$ is a $\delta$-2-primary hyperideal of $R$ with $\delta(I_i) = P$ for all $1 \leq i \leq n$. Then, $I = \cap_1^n I_i$ is a $\delta$-2-primary hyperideal of $R$. 

Proof. Let $a, b \in I$ with $a + b \subseteq I$ and $a^2 \subseteq I$. Then, there exists some $1 \leq t \leq n$ such that $a^t \subseteq I_i$. Since $I_i$ is a $\delta$-2-primary hyperideal of $R$, then, $b^2 \subseteq \delta(I_i) = P$. Therefore, $b^2 \subseteq P = \cap_1^n \delta(I_i) = \delta(\cap_1^n I_i)$. Thus, we conclude that $I = \cap_1^n I_i$ is a $\delta$-2-primary hyperideal of $R$. 

Theorem 19. If $\{I_j | j \in \Lambda\}$ is a directed set of $\delta$-2-primary hyperideals of a multiplicative hyperring $R$, then, $\cup_{j \in \Lambda} I_j$ is a $\delta$-2-primary hyperideal of $R$. 

Proof. Let $x \circ y \subseteq \cup_{j \in \Lambda} I_j$ for some $x, y \in R$. Assume that $x^2 \subseteq \cup_{j \in \Lambda} I_j$. This implies that there exists $i \in \Lambda$ such that $x^2 \subseteq I_i$. Since $I_i$ is a $\delta$-2-primary hyperideal of $R$, then, $y^2 \subseteq \delta(I_i) \subseteq \delta(\cup_{j \in \Lambda} I_j)$. Thus, $\cup_{j \in \Lambda} I_j$ is a $\delta$-2-primary hyperideal of $R$. 

Theorem 20. Let $R$ be a regular multiplicative hyperring and $\delta$ be hyperideal expansion of $R$. If $I$ is a $\delta$-2-primary hyperideal of $R$, then, $I$ is $\delta$-primary. 

Proof. Let $I$ be a $\delta$-2-primary hyperideal of $R$. Assume that $a, b \in R$ and $a + b \subseteq I$ such that $a \notin I$. Since $R$ is regular, then, there exists $r \in R$ such that $a = ar + r$. Let $a^2 \subseteq I$. Then, $a = a^2 \circ r \subseteq I$ which is a contradiction. So, $a^2 \subseteq I$. Since $I$ is a $\delta$-2-primary hyperideal of $R$, then, $b^2 \subseteq \delta(I)$. Since $R$ is regular, then, there exists $r' \in R$ such that $b \subseteq b^2 \circ r' \subseteq \delta(I)$. Therefore, the hyperideal $I$ is $\delta$-primary. 

Theorem 21. Let $\delta$ and $\gamma$ be hyperideal expansions of $R_1$ and $R_2$, respectively, and $f: R_1 \rightarrow R_2$ a $\delta \gamma$-homomorphism. Then,

1. If $I_2$ is a $\gamma$-2-primary hyperideal of $R_2$, then, $f^{-1}(I_2)$ is a $\delta$-2-primary hyperideal of $R_1$.

2. Let $I_1$ be a hyperideal of $R_1$ and $f$ an epimorphism with $\ker f \subseteq I_1$. Then, $f(I_1)$ is a $\gamma$-2-primary hyperideal of $R_2$ if and only if $I_1$ is a $\delta$-2-primary hyperideal of $R_1$.

Proof. (1) Let $a \circ b \subseteq f^{-1}(I_2)$ for some $a, b \in R_1$. Then, we have $f(a \circ b) = f(a) \circ f(b) \subseteq I_2$. Since $I_2$ is a $\gamma$-2-primary hyperideal of $R_2$, we get $f(a) \subseteq I_2$ or $f(b) \subseteq \gamma(I_2)$ which implies $f(a) \subseteq I_2$ or $f(b) \subseteq \gamma(I_2)$. Then, $a^2 \subseteq f^{-1}(I_2)$ or $b^2 \subseteq f^{-1}(\gamma(I_2)) = \delta(f^{-1}(I_2))$ because $f$ is a $\delta$-homomorphism. Consequently, $f^{-1}(I_2)$ is a $\delta$-2-primary hyperideal of $R_1$. (2) (\Rightarrow) It is quite clear from (1).

(\Rightarrow) Let $a_1 \circ b_2 \subseteq f(I_1)$ for some $a_1 \in R_1$. Then, for some $a_1, b_1 \in R_1$ we have $f(a_1) = a_2$ and $f(b_1) = b_2$. So, $f(a_1) \circ f(b_1) = f(a_1 \circ b_1) \subseteq f(I_1)$. Now, take any $a \in a_1 \circ b_1$. Then, $f(a) \subseteq f(a_1 \circ b_1) \subseteq f(I_1)$ and so there exists $w \in I_1$ such that $f(w) = f(a)$. This means that $f(u - w) = 0$, that is, $u - w \in \ker f \subseteq I_1$ and then, $u \in I_1$. Since $I_1$ is a $\delta$-2-primary hyperideal of $R_1$, then, we get $a_1 \circ b_1 \subseteq I_1$. Since $I_1$ is a $\delta$-2-primary hyperideal of $R_1$, then, we obtain $a_1^2 \subseteq I_1$ or $b_1^2 \subseteq \delta(I_1)$. This implies that $f(a_1^2) = a_2^2 \subseteq f(I_1)$ or $f(b_1^2) = b_2^2 \subseteq f(\delta(I_1)) = \gamma(f(I_1))$. Thus, $f(I_1)$ is a $\gamma$-2-primary hyperideal of $R_2$. 

Corollary 2. Let $I$ and $J$ be two hyperideals of a multiplicative hyperring $R$ such that $I \subseteq J$. Then, $J$ is a $\delta$-primary hyperideal of $R$ if and only if $J/I$ is a $\delta$-primary hyperideal of $R/I$. 

Proof. The claim is verified from Theorem 21. 

Corollary 3. Let $I$ be a $\delta$-primary hyperideal of a multiplicative hyperring $R$ and $S$ a sub hyperring of $R$ with $S \subseteq I$. If $J$ is a $\delta$-primary hyperideal of $R$, then, $I \cap S$ is a $\delta$-primary hyperideal of $S$. 

Theorem 22. Let $\delta$ be a hyperideal expansion of $R$. Then, the followings are equivalent:

1. Every proper principal hyperideal of $R$ is $\delta$-2-primary.

2. Every proper hyperideal of $R$ is $\delta$-2-primary. 

Proof. (1) $\Rightarrow$ (2) Assume that $I$ is a proper hyperideal of $R$ such that $x \circ y \subseteq I$ for some $x, y \in R$. Take any $t \in x \circ y$. Clearly, $x \circ y \subseteq t$ and $t \subseteq I$. This implies that $x \circ y \subseteq t \subseteq I$. Since $t$ is a $\delta$-2-primary hyperideal of $R$, then, we get $x^2 \subseteq \delta(t) \subseteq I$ or $y^2 \subseteq \delta(t) \subseteq I$. Therefore, $I$ is a $\delta$-2-primary hyperideal of $R$. (2) $\Rightarrow$ (1) It is clear.

Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be two multiplicative hyperrings with nonzero identity [29]. Recall $(R_1 \times R_2, +, \circ)$ is a multiplicative hyperring with the operation $+ \circ$ and the hyperoperation $\circ$ are defined respectively as

$$(x_1, x_2) + (y_1, y_2) = (x_1 +_1 y_1, x_2 +_2 y_2)$$

and

$$(x_1, x_2) \circ (y_1, y_2) = \{ (x, y) \in R_1 \times R_2 | x \in x_1 \circ_1 y_1, y \in x_2 \circ_2 y_2 \}.$$ 

Assume that $\delta_1$ and $\delta_2$ are hyperideal expansions of $R_1$ and $R_2$, respectively. If $\delta_1 \cap \delta_2$ is a function of hyperideals of $R$ with $\delta_1 \cap \delta_2 (I_1 \times I_2) = \delta_1 (I_1) \cap \delta_2 (I_2)$ for every hyperideals $I_1$ and $I_2$ of $R_1$ and $R_2$, respectively, then, $\delta_1 \cap \delta_2$ is a hyperideal expansion of $R_1 \times R_2$. 

Theorem 23. Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be two multiplicative hyperrings with nonzero identity such that $\delta_1$ and $\delta_2$ be hyperideal expansions of $R_1$ and $R_2$, respectively. Let $I_1$ be a hyperideal of $R_1$. Then, the hyperideal $I_1 \times R_2$ is $\delta$-2-primary if and only if $I_1 \times R_2$ is a $\delta$-2-primary hyperideal of $R_1 \times R_2$. 

Proof. (\Rightarrow) Let $(x_1, x_2) \circ (y_1, y_2) \subseteq I_1 \times R_2$ for some $(x_1, x_2), (y_1, y_2) \in R_1 \times R_2$. This means $x_1 \circ_1 y_1 \subseteq I_1$. Since
\[ I_1 \text{ is a } \delta_1 \text{-2-primary hyperideal of } R_1, \text{ then we get } x_1^2 \subseteq I_1 \text{ or } y_1^2 \subseteq \delta_1 (I_1). \] This implies that \( (x_1, x_2)^2 = (x_1^2, x_2^2) \subseteq I_1 \times R_2 \) or \( (y_1, y_2)^2 = (y_1^2, y_2^2) \subseteq \delta_1 \times R_2 (I_1 \times R_2). \)

(\(\Leftarrow\)) Assume on the contrary that \( I_1 \) is not a \( \delta_1 \)-2-primary hyperideal of \( R_1 \). So, \( x_1 \circ_1 y_1 \subseteq I_1 \) with \( x_1, y_1 \in R_1 \) implies that \( x_1^2 \subseteq I_1 \) and \( y_1^2 \subseteq \delta_1 (I_1) \). It is clear that \( (x_1, 1_{R_2}) \subseteq I_1 \times R_2 \). Since \( I_1 \times R_2 \) is a \( \delta_1 \times R_2 \)-2-primary hyperideal of \( R_1 \times R_2 \), then we have \( (x_1, 1_{R_2})^2 = (x_1^2, 1_{R_2}) \subseteq I_1 \times R_2 \). Since \( (y_1, 1_{R_2})^2 = (y_1^2, 1_{R_2}) \subseteq \delta_1 \times R_2 (I_1 \times R_2) \). Hence, we get \( x_1^2 \subseteq I_1 \) or \( y_1^2 \subseteq \delta_1 (I_1) \) which is a contradiction. Consequently, \( I_1 \) is a \( \delta_1 \)-2-primary hyperideal of \( R_1 \).

**Theorem 24.** Let \((R_1, +_1, \circ_1)\) and \((R_2, +_2, \circ_2)\) be two multiplication hyperrings with nonzero identity such that \( \delta_1 \) and \( \delta_2 \) be hyperideal expansions of \( R_1 \) and \( R_2 \), respectively. Let \( I_1 \) and \( I_2 \) be some hyperideals of \( R_1 \) and \( R_2 \), respectively. Then, the following statements are equivalent:

1. \( I_1 \times I_2 \) is a \( \delta_1 \times \delta_2 \)-2-primary hyperideal of \( R_1 \times R_2 \).
2. \( I_1 = R_1 \) and \( I_2 \) is a \( \delta_1 \)-2-primary hyperideal of \( R_2 \), or \( I_2 = R_2 \) and \( I_1 \) is a \( \delta_2 \)-2-primary hyperideal of \( R_1 \).

**Proof.** (1) \(\Rightarrow\) (2) Assume that \( I_1 = R_1 \). Then, \( I_2 \) is a \( \delta_2 \)-2-primary hyperideal of \( R_2 \), by Theorem 23.

(2) \(\Rightarrow\) (1) This can be proved, by using Theorem 23. \(\square\)

**Example 8.** Suppose that \((Z, +, \cdot)\) is the ring of integers. Then, \((Z, +, \circ_1)\) is a multiplicative hyperoperation with a hyperoperation \( a \circ_1 b = \{ab, 7ab\} \). Also, \((Z, +, \circ_2)\) is a multiplicative hyperoperation with a hyperoperation \( a \circ_2 b = \{ab, 5ab\} \). Note that \((Z \times Z, +, \circ)\) is a multiplicative hyperoperation with a hyperoperation \( (a, b) \circ (c, d) = \{(x, y) \in Z \times Z | x \in a \circ_1 c, y \in b \circ_2 d\} \). Clearly, \( 7Z = \{7l | l \in Z\} \) and \( 5Z = \{5l | l \in Z\} \) are \( \delta_1 \)-2-primary of \((Z, +, \circ_1)\) and \((Z, +, \circ_2)\), respectively. Since \((5, 0)^2 \subseteq 7Z \times 5Z\), but \((5, 0)^2 \subseteq 7Z \times 5Z\) and \((5, 0)^2 \subseteq 7Z \times 5Z\), then \( 7Z \times 5Z \) is not a \( \delta_1 \times \delta_1 \)-2-primary hyperideal of \( Z \times Z \).

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**References**


