Research Article

A New Approach to Evaluate Regular Semirings in terms of Bipolar Fuzzy k-Ideals Using k-Products

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Received 28 October 2021; Revised 28 January 2022; Accepted 2 March 2022; Published 16 June 2022

Academic Editor: Ching-Feng Wen

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In this paper, we provide a generalized form of ideals that is \( k \)-ideals of semirings with the combination of a bipolar fuzzy set (BFS). The BFS is a generalization of fuzzy set (FS) that deals with uncertain problems in both positive and negative aspects. The main theme of this paper is to present the idea of \((\alpha, \beta)\)-bipolar fuzzy \( k \)-subsemiring (\( k \)-BFSS), \((\alpha, \beta)\)-bipolar fuzzy \( k \)-ideals (\( k \)-BFIs), and \((\alpha, \beta)\)-bipolar fuzzy \( k \)-bi-ideals (\( k \)-BFbIs) in semirings by applying belongingness (\( \epsilon \)) and quasi-coincidence (\( q \)) of the bipolar fuzzy (BF) point. After that, upper and lower parts of \( k \)-product of BF subsets of semirings are introduced. Lastly, the notions of \( k \)-regular and \( k \)-intraregular semirings in terms of \((\epsilon, \epsilon \vee q)\)-\( k \)-BFIs and \((\epsilon, \epsilon \vee q)\)-\( k \)-BFbIs are characterized.

1. Introduction and Motivation

In engineering, decision making theory, management science, and medical science, we may face uncertainty and vagueness in the data. In classical mathematics, all the formulae and methods consist on crisp set that cannot tackle the vague problems. After an extensive struggle, many theories are invented to tackle such problems. In 1965, Zadeh [1] gave the idea of fuzzy set theory to handle such complicated problems. This concept is applied on theory of rings, theory of groups, real analysis, logics, and topological space. In FS, membership degree is limited to \([0,1]\], but there was a difficulty to deliberate the irrelevancy of data. To resolve this difficulty, Zhang [2] offered the idea of BFS in which the membership degree is \([-1,1]\). Since the BFS theory is a development of the FS theory, thus a BFS in semirings is also useful.

Bipolarity is a significant theory which is mostly used in real life. It is noticed that people may have different responses at a time for the same qualities of an item or a plan. Someone may have negative thoughts and the other one may have positive thoughts. For example, is a big amount for a needy person but at the same phase, this amount is negligible for a rich man. BFS is also useful in database query, psychology, image processing, multicriteria decision making, argumentation, and in human reasoning problems. Recently, Riaz et al. [3] have worked on BFS with the combination of picture fuzzy set and discussed its application on pattern recognition. Facial recognition or face detection is an artificial intelligence-based computer technology used to find and identify human faces in digital images; see [4–15] for more examples and results which are relevant to BFS.

The role of semirings as an algebraic tool is very important in theoretical computer science ([16, 17]). Semirings are broadly used in formal languages, automata theory, optimization theory, graph theory, coding theory, and theory of discrete event dynamical systems. In the structural theory, ideals play a central role and are beneficial for many other purposes. In general, ideals of semiring do not coincide with ideals of the ring. Many results in rings apparently are not equivalent to results in semirings using only ideals. Henriksen [18] introduced a most generalized form of ideals in semiring, which is \( k \)-ideal, with the characterization that if the semiring \( R \) is a ring, then a complex in \( R \) is a \( k \)-ideal equivalent to a ring ideal. The \( k \)-ideal is the most restricted form of an ideal. Every \( k \)-ideal is ideal but the converse does not hold. The set of whole numbers \( \mathbb{W} \) is a semiring. Let
I = \{0, 2\} \cup \{3, 4, 5, \ldots\} then \( I \) is an ideal of \( W \) while it is not a \( k \)-ideal of \( W \) because 2, \( 2 + 1 \in I \) but 1 \( \notin I \) if we take a subset \( I = \{0, 3, 6, 9, \ldots\} \) of \( W \), then \( I \) is a \( k \)-ideal of \( W \). For more details; see [19].

1.1. Related Works. Zadeh’s fuzzy set is a much innovative, crucial, and useful set due to its significance in multiple research dimensions. The fuzzy set addresses the ill-defined conclusions and future plans. The list of acronyms used here is given in Table 1.

2. Preliminaries

Basic definitions, results, and some back ground material are given. Here, a subset means a nonempty subset.

The \( k \)-closure of a subset \( K \) of a semiring \( R \) is \( K = \{ u \in R | \exists c, d \in K \text{ such that } u + c = d \} \). A subsemiring \( K \) of \( R \) is called \( k \)-subsemiring of \( R \) if and only if (iff) \( K = K \). Thus, an ideal (left, right) \( K \) of \( R \) is a \( k \)-ideal (left, right) if \( K = K \). A subsemiring \( K \) is called a \( k \)-bi-ideal of \( R \) if \( K \subseteq K \subseteq K \). A semiring \( R \) is known as \( k \)-regular if for all, \( u \in R \), there are elements \( c, c' \in R \) such that \( u + u'c = uc' \). A semiring \( R \) is known as \( k \)-intragular if for all \( u \in R \), there are elements \( a, a', b, b' \in R \) such that \( u + \sum a'ab = a'ab \).

Lemma 1 (see [40]). A semiring \( R \) is \( k \)-regular iff \( K \cap I = K \cap P \) for every right \( k \)-ideal \( K \) and \( k \)-ideal \( P \) of \( R \).

Lemma 2 (see [40]). A semiring \( R \) is \( k \)-regular iff \( P = \overline{P} \) for every \( k \)-bi-ideal \( P \) of \( R \).

Lemma 3 (see [40]). A semiring \( R \) is \( k \)-intragular iff \( K \cap P = K \cap P \) for every \( k \)-ideal \( K \) and \( k \)-ideal \( P \) of \( R \).

Now, we will discuss some essential concepts related to BFSs. A semiring subset \( \Psi = (R; \Psi_n, \Psi_p) \) of a semiring \( R \), where \( \Psi_n, R \rightarrow [-1, 0] \) gives the satisfaction degree for somewhat opposite property of \( \Psi \) and \( \Psi_p : R \rightarrow [0, 1] \) gives the degree of satisfaction of the corresponding property of \( \Psi \). While in a fuzzy set, the membership function maps to \([0,1]\). The difference between a fuzzy set and bipolar fuzzy set is shown by the following example.

Let \( A = \{u, v, w, x, y, z\} \) be a set of workers of a company. Define a fuzzy set on \( A \) with fuzzy property “honesty.” The workers \( x, y \) having property “honesty” mapped to \([0,1]\) as shown by a bar graph in Figure 1. While other workers have no membership degree in range \([0,1]\) as they are not honest. In the fuzzy set, we can cover only a positive aspect of any situation. We cannot deal with negative aspects of the situation. To facilitate, we deal such problems with a bipolar fuzzy set. The property “dishonesty” is against “honesty.” The workers \( u, w \) and \( z \) are mapped to \([-1,0]\) with property “dishonesty.” In such a way, the bipolar fuzzy set gives information about all elements as shown in Figure 2.

For \( u \in R \), if \( \Psi_p(u) = 0 \), \( \Psi_n(u) \neq 0 \) then \( u \) does not hold the property of \( \Psi \) and if \( \Psi_p(u) \neq 0 \), \( \Psi_n(u) = 0 \) then \( u \) holds the property of \( \Psi \). If \( \Psi_p(u) \neq 0 \) and \( \Psi_n(u) \neq 0 \), then its opposite property and membership function intersect. A BF
Assume Proposition 1.

Proof. Let $u \in R$ then $(\Psi \circ_k \mu_n)(u) = \vee_{i=1}^{m} a_i \circ_j \mu_n(b_i)$.

For $u, a_i, b_i \in R$.

**Proposition 1.** Assume $\Psi, \mu, \nu, \sigma$ are BF subsets of $R$. At that time, $\Psi \leq \nu$ and $\mu \leq \sigma$; this implies that $\Psi \circ_k \mu \leq \nu \circ_k \sigma$.

And, $(\Psi \circ_k \mu_n)(u) = \vee_{i=1}^{m} a_i \circ_j \mu_n(b_i)$.

As $\Psi \circ_k \mu_n \geq \nu_n$ implies $\Psi \circ_k \mu_n \geq \nu_n$.

Moreover, $\mu_n \geq \sigma_n$, means $\mu_n(b_i) \geq \sigma_n(b_i)$, $\mu_n(b_i) \geq \sigma_n(b_i)$.

So, $\vee_{i=1}^{m} a_i \circ_j \mu_n(b_i)$.

Hence, $\Psi \circ_k \mu \leq \nu \circ_k \sigma$.

**Definition 2.** If $K$ is a $k$-subset of $R$ Then, the bipolar characteristic function of $K$ is denoted by $C_K = (R; C_{nk}, C_{nk})$.
For all $u \in R$, if $K = R$, then we have a BF subset $R = (R; R_n, R_p)$ defined as $R_n(x) = -1$ and $R_p(x) = 1$ for all $u \in R$

Definition 3. A BF subset $\Psi = (R; \Psi_n, \Psi_p)$ of $R$ is called $k$-BFSS of $R$ if it satisfies the following:

(i) $\Psi_n(0) \leq \Psi_n(c)$ and $\Psi_p(0) \geq \Psi_p(c)$;
(ii) $\Psi_n(c + d) \leq \max\{\Psi_n(c), \Psi_n(d)\}$ and $\Psi_p(c + d) \geq \min\{\Psi_p(c), \Psi_p(d)\}$;
(iii) $\Psi_n(cd) \leq \Psi_n(c)$ and $\Psi_p(cd) \geq \Psi_p(c)$;
(iv) If $u + c = d$ then $\Psi_n(u) \leq \max\{\Psi_n(c), \Psi_n(d)\}$ and $\Psi_p(u) \geq \min\{\Psi_p(c), \Psi_p(d)\}$ for all $u, c, d \in R$.

Definition 4. A BF subset $\Psi = (R; \Psi_n, \Psi_p)$ of $R$ is said to be $k$-BF (right resp. left) if it satisfies the following:

(i) $\Psi_n(c + d) \leq \max\{\Psi_n(c), \Psi_n(d)\}$ and $\Psi_p(c + d) \geq \min\{\Psi_p(c), \Psi_p(d)\}$;
(ii) $\Psi_n(cd) \leq \Psi_n(c)$ and $\Psi_p(cd) \geq \Psi_p(c)$, (resp. $\Psi_n(c) \leq \Psi_n(d)$ and $\Psi_p(c) \geq \Psi_p(d)$);
(iii) If $u + c = d$ then $\Psi_n(u) \leq \max\{\Psi_n(c), \Psi_n(d)\}$ and $\Psi_p(u) \geq \min\{\Psi_p(c), \Psi_p(d)\}$ for all $u, c, d \in R$.

Example 1. A BF subset $\Psi$ of a semiring $\square = \{0, 1, 2, 3, \ldots\}$ is defined as follows:

\[
\Psi_n(u) = \begin{cases} 
0, & \text{if } u = 1, \\
-0.5, & \text{otherwise},
\end{cases}
\Psi_p(u) = \begin{cases} 
0, & \text{if } u = 1, \\
0.5, & \text{otherwise},
\end{cases}
\]

Then $\Psi$ is a BFI of $\square$ but not a $k$-BFI because if we take $1 + 2 = 3$ then $\Psi_n(1) = 0 - 0.5 = \max\{\Psi_n(2), \Psi_n(3)\}$ and $\Psi_p(1) = 0 \geq 0.5 = \min\{\Psi_p(2), \Psi_p(3)\}$.

Example 2. Let a BF subset $\Psi$ of a semiring $\square = \{0, 1, 2, 3, \ldots\}$ be defined as follows:

\[
\Psi_n(u) = \begin{cases} 
-0.1, & \text{if } u \text{ is odd}, \\
-0.5, & \text{if } u \text{ is even},
\end{cases}
\Psi_p(u) = \begin{cases} 
0.1, & \text{if } u \text{ is odd}, \\
0.5, & \text{if } u \text{ is even},
\end{cases}
\]

Then, it is easy to check that $\Psi$ is a $k$-BFI of $\square$.

Definition 5. A BF subset $\Psi = (R; \Psi_n, \Psi_p)$ of $R$ is said to be bipolar fuzzy $k$-BFBl of $R$ if it satisfies the following:

(i) $\Psi_n(0) \leq \Psi_n(c)$ and $\Psi_p(0) \geq \Psi_p(c)$;
(ii) $\Psi_n(c + d) \leq \max\{\Psi_n(c), \Psi_n(d)\}$ and $\Psi_p(c + d) \geq \min\{\Psi_p(c), \Psi_p(d)\}$;
(iii) $\Psi_n(cd) \leq \max\{\Psi_n(c), \Psi_n(d)\}$ and $\Psi_p(cd) \geq \min\{\Psi_p(c), \Psi_p(d)\}$;
(iv) $\Psi_n(c \ x \ d) \leq \max\{\Psi_n(c), \Psi_n(d)\}$ and $\Psi_p(c \ x \ d) \geq \min\{\Psi_p(c), \Psi_p(d)\}$;
(v) If $u + c = d$ then $\Psi_n(u) \leq \max\{\Psi_n(c), \Psi_n(d)\}$ and $\Psi_p(u) \geq \min\{\Psi_p(c), \Psi_p(d)\}$ for all $u, c, d, e \in R$.

Lemma 4. If $K$ is $k$-ideal (left, right) of $R$, then $C_K = (R, C_{nk}, C_{pk})$ is $k$-BFI (left, right) of $R$.

Proof. We prove it for right $k$-ideal. For this, we have to verify the below three inequalities, for each element $c, d, u, v \in R$.

(i) $C_{nk}(c + d) \leq \max\{C_{nk}(c), C_{nk}(d)\}$ and $C_{pk}(c + d) \geq \min\{C_{pk}(c), C_{pk}(d)\}$;
(ii) $C_{nk}(c) \leq C_{nk}(c)$ and $C_{pk}(c) \leq C_{pk}(c)$;
(iii) $u + c = d \implies C_{nk}(u) \leq \max\{C_{nk}(c), C_{nk}(d)\}$ and $C_{pk}(u) \geq \min\{C_{pk}(c), C_{pk}(d)\}$.

For the proofs of (i) and (ii), see [32]. Here, we just prove part (iii). For this, we discuss the following cases.

Case 1. Suppose $u, c, d \in K$

Then, $C_{nk}(u) = -1 = C_{nk}(c) = C_{nk}(d)$ and $C_{pk}(u) = 1 = C_{pk}(c) = C_{pk}(d)$

Since $K$ is a $k$-ideal of $R$, for $u, c, d \in K$, $C_{nk}(u) = -1 \leq \max\{C_{nk}(c), C_{nk}(d)\}$ and $C_{pk}(u) = 1 \geq \min\{C_{pk}(c), C_{pk}(d)\}$.

Case 2. If $u, c, d \notin K$

Then, $C_{nk}(u) = 0 = C_{nk}(c) = C_{nk}(d)$ and $C_{pk}(u) = 0 = C_{pk}(c) = C_{pk}(d)$.

Case 3. If one of the $c$ or $d$ does not belong to $K$, say $c \notin K$

Then, $C_{nk}(c) = 0$ and $C_{pk}(c) = 0$, $C_{nk}(u) \leq \max\{C_{nk}(c), C_{nk}(d)\}$ and $C_{pk}(u) \geq \min\{C_{pk}(c), C_{pk}(d)\}$

So, $C_K$ is a $k$-BFI.

Lemma 5. If $K$ is a $k$-bi-ideal of $R$, then $C_K = (R, C_{nk}, C_{pk})$ is $k$-BFBl of $R$.

Proof. Straightforward.

3. $(\alpha, \beta)$ Bipolar Fuzzy $k$-Ideals

Here, ideas of $(\alpha, \beta)$-$k$-BFSS, $(\alpha, \beta)$-$k$-BFI, and $(\alpha, \beta)$-$k$-BFBl of semirings are introduced and their related properties are discussed.

Definition 6 (see [32,33]). Let $(c, d) \in [-1, 0) \times (0, 1]$, then $u/(c, d)$ is called a BF point in $R$ if the BF subset $\Psi$ of $R$ is given in the form of the following:

\[
\Psi_p(a) = \begin{cases} 
1, & \text{if } a = u, \\
0, & \text{if } a \neq u,
\end{cases}
\Psi_n(a) = \begin{cases} 
c, & \text{if } a = u, \\
0, & \text{if } a \neq u.
\end{cases}
\]

for all $a, u \in R$. 

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Definition 7 (see [32, 33]). For a BF subset $\Psi$ of $R$ and a BF point $u(c,d)$, we say that $u(c,d)$ belongs to (quasi-coincident to resp.) the $\Psi$ is written as $(u(c,d)) \in \Psi$, $(u(c,d)) \in \Psi$ resp. if it satisfies the following:

(i) $(u(c,d)) \in \Psi$ if $\Psi_n(u) \leq c$ and $\Psi_p(u) \geq d$.
(ii) $(u(c,d)) \in \Psi$ if $\Psi_n(u) + c < -1$ and $\Psi(p,u) + d > 1$.
(iii) $(u(c,d)) \in \Psi$ if $(u(c,d)) \in \Psi \text{ or } (u(c,d)) \in \Psi$.
(iv) $(u(c,d)) \in \Psi$ if $(u(c,d)) \in \Psi$ and $(u(c,d)) \in \Psi$.

Here, we consider $a, b \in \{a, b, c, d\}$ and $a \neq b$. We consider a BF subset $\Psi = (R; \Psi_n, \Psi_p)$ such that $\Psi_n(u) \geq 0.5$ and $\Psi_p(u) \geq 0.5$ for all $u \in R$ and $(u(c,d)) \in \Psi$. Then, $\Psi_n(u) \leq c$, $\Psi_p(u) \geq d$, $\Psi_n(u) + c < -1$ and $\Psi_p(u) + d > 1$. It follows that $\Psi_n(u) \leq 0.5$, thus a contradiction arises. So, $(u(c,d)) \in \Psi$ and hence, $a \neq b$.

Definition 8. A BF subset $\Psi = (R; \Psi_n, \Psi_p)$ of $R$ is called an $(a, b)\beta$-k-BFSS of $R$ if for all $t, u \in R$ and $(c_1, d_1), (c_2, d_2) \in [-1, 0) \times (0, 1)$, it satisfies the following:

(i) $(t(c_1, d_1)) \in \Psi$ and $(u(c_2, d_2)) \in \Psi \rightarrow (t + u + c_2 - d_2) \beta$.
(ii) $(t(c_1, d_1)) \in \Psi$ and $(u(c_2, d_2)) \in \Psi \rightarrow (t + u + c_2 - d_2) \beta$.
(iii) Suppose $u, c, d \in R$ such as $u + c = d$, then $(c_1, d_1)) \in \Psi$ and $(d_1, d_2) \in \Psi$, here, $a \neq b$.

Definition 9. A BF subset $\Psi = (R; \Psi_n, \Psi_p)$ of $R$ is called an $(a, b)\beta$-k-BBF of $R$, if for $(c_1, d_1), (c_2, d_2) \in [-1, 0) \times (0, 1)$, and for all $t, u \in R$, it satisfies the following:

(i) $(t(c_1, d_1)) \in \Psi$ and $(u(c_2, d_2)) \in \Psi \rightarrow (t + u + c_2 - d_2) \beta$.
(ii) $(t(c_1, d_1)) \in \Psi$ and $(u(c_2, d_2)) \in \Psi \rightarrow (t + u + c_2 - d_2) \beta$.
(iii) Suppose $u, c, d \in R$ such as $u + c = d$, then $(c_1, d_1)) \in \Psi$ and $(d_1, d_2) \in \Psi$, here, $a \neq b$.

Definition 10. A BF subset $\Psi = (R; \Psi_n, \Psi_p)$ of $R$ is called an $(a, b)\beta$-k-BBF of $R$, if for $(c_1, d_1), (c_2, d_2) \in [-1, 0) \times (0, 1)$, and for every $u, c, d \in R$ it satisfies the following:

(i) $(t(c_1, d_1)) \in \Psi$ and $(u(c_2, d_2)) \in \Psi \rightarrow (t + u + c_2 - d_2) \beta$.
(ii) $(t(c_1, d_1)) \in \Psi$ and $(u(c_2, d_2)) \in \Psi \rightarrow (t + u + c_2 - d_2) \beta$.
(iii) Suppose $u, c, d \in R$ such as $u + c = d$, then $(c_1, d_1)) \in \Psi$ and $(d_1, d_2) \in \Psi$, here, $a \neq b$.

Theorem 1. If a BF subset $\Psi = (R; \Psi_n, \Psi_p)$ of $R$ is $(\epsilon, \varphi)$-k-BFSS of $R$, then it fulfills the following conditions for each $c, d, u \in R$.

(i) $\Psi_n(c + d) \geq \max\{\Psi_n(c), \Psi_n(d), -0.5\}$ and $\Psi_p(c + d) \geq \min\{\Psi_p(c), \Psi_p(d), 0.5\}$
(ii) $\Psi_n(c + d) \geq \max\{\Psi_n(c), \Psi_n(d), -0.5\}$ and $\Psi_p(c + d) \geq \min\{\Psi_p(c), \Psi_p(d), 0.5\}
(iii) For $u + c = d$, $\Psi_n(u) \leq \max\{\Psi_n(c), \Psi_n(d), -0.5\}$ and $\Psi_p(u) \geq \min\{\Psi_p(c), \Psi_p(d), 0.5\}$

Proof. For the proofs of part (i) and part (ii), see [33]; here, we just prove part (iii).

For $c, d \in R$, four following cases arise:

(i) $\max\{\Psi_n(c), \Psi_n(d)\} \leq -0.5$ and $\min\{\Psi_p(c), \Psi_p(d)\} \geq 0.5$
(ii) $\max\{\Psi_n(c), \Psi_n(d)\} \leq -0.5$ and $\min\{\Psi_p(c), \Psi_p(d)\} < 0.5$
(iii) $\max\{\Psi_n(c), \Psi_n(d)\} \leq 0.5$ and $\min\{\Psi_p(c), \Psi_p(d)\} < 0.5$
(iv) $\max\{\Psi_n(c), \Psi_n(d)\} > 0.5$ and $\min\{\Psi_p(c), \Psi_p(d)\} < 0.5$

We will prove with contrary supposition. Then, assume for $c, d \in R$, $\Psi_n(u) > \max\{\Psi_n(c), \Psi_n(d), -0.5\}$ or $\Psi_p(u) < \min\{\Psi_p(c), \Psi_p(d), 0.5\}$.

Case 1. $\max\{\Psi_n(c), \Psi_n(d)\} \leq -0.5$ and $\min\{\Psi_p(c), \Psi_p(d)\} \geq 0.5$.
Then, $\Psi_n(u) > 0.5$ or $\Psi_p(u) < 0.5$ implies $(u(0.5), 0.5)$ is $\Psi$. Now, again consider $\Psi_n(c), \Psi_n(d) \leq -0.5$, then $\Psi_p(u) \leq 0.5$, $\Psi_n(u) \leq 0.5$, and $\min\{\Psi_p(c), \Psi_p(d)\} > 0.5$ and $\Psi_p(u) < 0.5$. Therefore, $(c_1, d_1) \in \Psi$, $(d_1, 0.5)$ is $\Psi$ but for $u \in R$, $(u(0.5), 1)$ is $\Psi$. Similarly, $(u(0.5), 0.5)$ is $\Psi$ because $\Psi_n(u) < 0.5$. We will prove with contrary supposition. Then, assume for $c, d \in R$, $\Psi_n(u) > \max\{\Psi_n(c), \Psi_n(d), -0.5\}$ or $\Psi_p(u) < \min\{\Psi_p(c), \Psi_p(d), 0.5\}$.

Case 2. $\max\{\Psi_n(c), \Psi_n(d)\} \leq 0.5$ and $\min\{\Psi_p(c), \Psi_p(d)\} < 0.5$.
Then, $\Psi_n(u) > 0.5$ or $\Psi_p(u) < 0.5$ implies $(u(0.5), 0.5)$ is $\Psi$. Now, again consider $\Psi_n(c), \Psi_n(d) \leq -0.5$, then $\Psi_n(u) \leq 0.5$ and $\Psi_n(d) \leq 0.5$. Also, $\min\{\Psi_p(c), \Psi_p(d)\} > 0.5$. Then, we get $(c_1, 0.5)$ is $\Psi$ and $(d(0.5), 0.5)$ is $\Psi$ but for $u \in R$, $(u(0.5), 0.5)$ is $\Psi$. Similarly, $(u(0.5), 0.5)$ is $\Psi$ because $\Psi_p(u) < 0.5$.

Case 3. $\max\{\Psi_n(c), \Psi_n(d)\} > 0.5$ and $\min\{\Psi_p(c), \Psi_p(d)\} \geq 0.5$.
Then, $\Psi_n(u) > 0.5$ or $\Psi_p(u) < 0.5$. So, for $s \in (-0.5, 0)$, $\Psi_n(u) > s$ implies $\Psi_n(c), \Psi_n(d) \leq s$. Also, $\min\{\Psi_p(c), \Psi_p(d)\} \geq 0.5$ indicates $\Psi_p(c) \geq 0.5$ and $\Psi_p(d) \geq 0.5$. Then, we get $(c_1, 0.5)$ is $\Psi$, $(d(0.5), 0.5)$ is $\Psi$ but for $u \in R$, $(u(0.5), 0.5)$ is $\Psi$. Similarly, $(u(0.5), 0.5)$ is $\Psi$ because $\Psi_p(u) < 0.5$. We will prove with contrary supposition. Then, assume for $c, d \in R$, $\Psi_n(u) > \max\{\Psi_n(c), \Psi_n(d), -0.5\}$ or $\Psi_p(u) < \min\{\Psi_p(c), \Psi_p(d), 0.5\}$.
Case 4. \(\max\{\Psi_n(c), \Psi_n(d)\} > -0.5\) and \(\min\{\Psi_p(c), \Psi_p(d)\} < 0.5\), then \(\Psi_n(u) > \max\{\Psi_n(c), \Psi_n(d)\}\) or \(\Psi_p(u) < \min\{\Psi_p(c), \Psi_p(d)\}\). Then for \(s \in (-0.5, 0)\) and \(t \in (0, 0.5)\), \(\Psi_n(u) > s = \max\{\Psi_n(c), \Psi_n(d)\}\) and \(\Psi_p(u) < t = \min\{\Psi_p(c), \Psi_p(d)\}\) then \((u(s, t)) \in \Psi\).

Also, \(\max\{\Psi_n(c), \Psi_n(d)\} = s\) implies \(\Psi_n(c) \leq s\) and \(\Psi_n(d) \leq s\). Moreover, \(\min\{\Psi_p(c), \Psi_p(d)\} = t\) implies \(\Psi_p(c) \leq t\) and \(\Psi_p(d) \geq t\). Now, combining all these, we get \((s, t) \in \Psi\) \& \((u(s, t)) \in \Psi\).

However, for \(u \in R\), \((u(s, t)) \in \Psi\). Similarly, \((u(s, t)) \in \Psi\), \(\Psi\) because \(\Psi_n(u) > s = s + s = -1\) or \(\Psi_p(u) > t + t = 1\). Thus, \((u(s, t)) \in \Psi\) \& \(\Psi\) which is a contradiction.

**Example 3.** Let \(R = \{\alpha, \beta, \delta\}\) be a semiring under operations defined as below (see Tables 2 and 3)

Let \(\Psi = (R; \Psi_n, \Psi_p)\) be a BF subset defined by \(\Psi_n(\alpha) = -0.48\), \(\Psi_n(\beta) = -0.3\), \(\Psi_n(\delta) = -0.2\), and \(\Psi_p(\alpha) = 0.55\), \(\Psi_p(\beta) = 0.9\), \(\Psi_p(\delta) = 0.72\).

Then, for \(u + c = d\), \(\Psi_n(u) \leq \max\{\Psi_n(c), \Psi_n(d), -0.5\}\) and \(\Psi_p(u) \geq \min\{\Psi_p(c), \Psi_p(d), 0.5\}\) for each \(c, d, u \in R\). The reverse of this theorem does not hold in general as shown by the following example.

**Theorem 2.** If a BF subset \(\Psi = (R; \Psi_n, \Psi_p)\) of \(R\) is \((\varepsilon, \wedge, \vee)\)-k-BF Bl of \(R\), then it fulfills the following conditions for each \(c, d, u \in R\):

(i) \(\Psi_n(c + d) \leq \max\{\Psi_n(c), \Psi_n(d), -0.5\}\) and \(\Psi_p(c + d) \geq \min\{\Psi_p(c), \Psi_p(d), 0.5\}\);

(ii) \(\Psi_n(c d) \leq \max\{\Psi_n(c), \Psi_n(d), -0.5\}\) and \(\Psi_p(c d) \geq \min\{\Psi_p(c), \Psi_p(d), 0.5\}\);

(iii) \(\Psi_n(c d) \leq \max\{\Psi_n(c), \Psi_n(d), -0.5\}\) and \(\Psi_p(c d) \geq \min\{\Psi_p(c), \Psi_p(d), 0.5\}\);

(iv) \(\Psi_n(u) \geq \max\{\Psi_n(c), \Psi_n(d), -0.5\}\) and \(\Psi_p(u) \geq \min\{\Psi_p(c), \Psi_p(d), 0.5\}\).

**Proof.** Proof follows from Theorem 1.

**Theorem 3.** If a BF subset \(\Psi = (R; \Psi_n, \Psi_p)\) of \(R\) is an \((\varepsilon, \wedge, \vee)\)-kBF Bl (left, resp. right) of \(R\), then it fulfills the following statements for each \(c, d, u \in R\):

(i) \(\Psi_n(c + d) \leq \max\{\Psi_n(c), \Psi_n(d), -0.5\}\) and \(\Psi_p(c + d) \geq \min\{\Psi_p(c), \Psi_p(d), 0.5\}\);

(ii) \(\Psi_n(c d) \leq \max\{\Psi_n(c), \Psi_n(d), -0.5\}\) and \(\Psi_p(c d) \geq \min\{\Psi_p(c), \Psi_p(d), 0.5\}\);

(iii) \(\Psi_n(c d) \leq \max\{\Psi_n(c), \Psi_n(d), -0.5\}\) and \(\Psi_p(c d) \geq \min\{\Psi_p(c), \Psi_p(d), 0.5\}\);

(iv) \(\Psi_n(u) \geq \max\{\Psi_n(c), \Psi_n(d), -0.5\}\) and \(\Psi_p(u) \geq \min\{\Psi_p(c), \Psi_p(d), 0.5\}\).

**Proof.** Proof follows from Theorem 1.
\[(\Psi_n \circ k\mu_n)^- (u) = (\Psi_n \circ k\mu_n) (u) \lor 0.5 \]

\[= \bigwedge u^+ \sum_{a_i \in \mathbb{A}_L} \bigg[ \bigvee_{j=1}^n (\Psi_n(a_j)) \bigg] \bigg[ \bigvee_{j=1}^n (\mu_n(b_j)) \bigg] \bigg] \lor 0.5 \]

\[= \bigwedge u^+ \sum_{a_i \in \mathbb{A}_L} \bigg[ \bigvee_{j=1}^n (\Psi_n(a_j)) \bigg] \bigg[ \bigvee_{j=1}^n (\mu_n(b_j)) \bigg] \bigg] \lor 0.5 \]

\[\geq \bigwedge u^+ \sum_{a_i \in \mathbb{A}_L} \bigg[ \bigvee_{j=1}^n (\Psi_n(a_j)) \bigg] \bigg[ \bigvee_{j=1}^n (\mu_n(b_j)) \bigg] \bigg] \lor 0.5 \]

\[\leq \bigwedge u^+ \sum_{a_i \in \mathbb{A}_L} \bigg[ \bigvee_{j=1}^n (\Psi_n(a_j)) \bigg] \bigg[ \bigvee_{j=1}^n (\mu_n(b_j)) \bigg] \bigg] \lor 0.5 \]

\[\leq \bigwedge u^+ \sum_{a_i \in \mathbb{A}_L} \bigg[ \bigvee_{j=1}^n (\Psi_n(a_j)) \bigg] \bigg[ \bigvee_{j=1}^n (\mu_n(b_j)) \bigg] \bigg] \lor 0.5 \]

\[= (\Psi_n \lor \mu_n)(u) \lor 0.5 \]

\[= (\Psi_n \lor \mu_n)^- (u). \]

So, \((\Psi_n \circ k\mu_n)^- (u) \geq (\Psi_n \lor \mu_n)^- (u)\). Similarly, we can prove \((\Psi_p \circ k\mu_p)^- (u) \leq (\Psi_n \lor \mu_n)^- (u)\).

Hence, \((\Psi_n \circ k\mu_n)^- \leq (\Psi \lor \mu)^-\).

**Definition 12.** If \(L\) is a \(k\)-subset of \(R\). The upper part \(C^L_{nL} = (R; C^+_n, CPL)\) of the bipolar characteristic function \(C_L = (R; C_{nL}, CPL)\) of \(L\) is defined as

\[C^L_{nL}(x) = \begin{cases} 1 & \text{if } x \in L \\ -0.5 & \text{if } x \notin L \end{cases} \]

\[C^L_{PL}(x) = \begin{cases} 1 & \text{if } x \in L \\ 0 & \text{if } x \notin L \end{cases} \]

Similarly, the lower part \(C^L_{nL} = (R; C^CL, CPL)\) of the bipolar characteristic function \(C_L = (R; C_{nL}, CPL)\) of \(L\) is defined as

\[C^L_{nL}(x) = \begin{cases} -0.5 & \text{if } x \in L \\ 0 & \text{if } x \notin L \end{cases} \]

\[C^L_{PL}(x) = \begin{cases} 1 & \text{if } x \in L \\ 0 & \text{if } x \notin L \end{cases} \]

\[(C_{np} \circ k\mu_Q)^- (x) = (C_{np} \circ k\mu_Q)(x) \lor 0.5 \]

\[= x^+ \sum_{a_i \in \mathbb{A}_L} \bigg[ \bigvee_{j=1}^n C_{np}(a_j) \bigg] \bigg[ \bigvee_{j=1}^n C_{nQ}(b_j) \bigg] \bigg] \lor 0.5 \]

\[= x^+ \sum_{a_i \in \mathbb{A}_L} \bigg[ \bigvee_{j=1}^n C_{np}(a_j) \bigg] \bigg[ \bigvee_{j=1}^n C_{nQ}(b_j) \bigg] \bigg] \lor 0.5 \]

\[= x^+ \sum_{a_i \in \mathbb{A}_L} \bigg[ \bigvee_{j=1}^n C_{np}(a_j) \bigg] \bigg[ \bigvee_{j=1}^n C_{nQ}(b_j) \bigg] \bigg] \lor 0.5 \]

**Lemma 9.** Let \(P\) and \(Q\) be two nonempty \(k\)-subsets of \(R\), then

(i) \((C_p \land C_Q)^- = C_{p \land Q}^-\)

(ii) \((C_p \lor C_Q)^- = C_{p \lor Q}^-\)

(iii) \((C_p \circ k\mu_n)^- = C_{p \circ k\mu_n}^-\)

**Proof.** The proofs of part (i) and (ii) are quite clear. (iii) Let \(x \in P\). Then, \(C_{n\mu_n}^{-}(x) = -0.5\) and \(C_{n\mu_n}^{-}(x) = 0.5\).

Given \(x \in P\), so there are some elements \(a_i, a'_i \in P\) and \(b_i, b'_i \in Q\) such that \(x + \sum_{i=1}^n a_i b_i = \sum_{i=1}^m a'_i b'_i\); thus, we have \(C_{np}(a_i) = -1 = C_{np}(a'_i) = C_{nQ}(b_i) = C_{nQ}(b'_i)\) and \(C_{pQ}(a_i) = -1 = C_{pQ}(a'_i) = C_{pQ}(b_i) = C_{pQ}(b'_i)\).

Now,

\[(C_{np} \circ k\mu_Q)^- (x) = (C_{np} \circ k\mu_Q)(x) \lor 0.5 \]

\[= x^+ \sum_{a_i \in \mathbb{A}_L} \bigg[ \bigvee_{j=1}^n C_{np}(a_j) \bigg] \bigg[ \bigvee_{j=1}^n C_{nQ}(b_j) \bigg] \bigg] \lor 0.5 \]
\[(C_p \ast k C_{pQ})^{-1}(x) = (C_p \ast k C_{pQ})(x) \land 0.5\]

\[= \bigvee_{x^*} \left( \sum_{i=1}^n a_i b_i = \sum_{j=1}^n d_j d_j \right) \left[ \bigwedge_{i=1}^n C_p(a_i) \bigwedge_{j=1}^n C_p(b_j) \right] \land 0.5\]

So, \((C_p \ast k C_Q)^{-1} = C_{pQ}^{-1}\). If \(x \notin PQ\), then \(C_{pQ}^{-1}(x) = 0\) and \(C_{pQ}^{-1}(x) = 0\).

Then, \((C_{np} \ast k C_{nQ})^{-1}(x) = (C_{np} \ast k C_{nQ})(x) \lor 0.5 = 0 \lor 0.5 = 0 \land 0.5 = C_{npPQ}(x)\).

Moreover, \((C_p \ast k C_{pQ})^{-1}(x) = (C_{np} \ast k C_{nQ})(x) \land 0.5 = 0 \lor 0.5 = C_{pQ}(x)\).

Hence, \((C_p \ast k C_Q)^{-1} = C_{pQ}^{-1}\).

\[\]

**Lemma 10.** If \(P\) is \(k\)-ideal (left, resp. right) of \(R\), then \(C_{p}^{-1} = (R, C_{np}, C_{pP})\) is \((\varepsilon, \varepsilon \land \varepsilon)-BFIL\) (left, resp. right) of \(R\).

**Proof.** Straightforward. \(\square\)

**Lemma 11.** If \(P\) is \(k\)-bi-ideal of \(R\), then \(C_{p}^{-1} = (R, C_{np}, C_{pP})\) is \((\varepsilon, \varepsilon \land \varepsilon)-BFbl\) of \(R\).

**Proof.** Straightforward. \(\square\)

\[\]

\[(\Psi_n \ast k \mu_n)^{-1}(u) = (\Psi_n \ast k \mu_n)(u) \lor 0.5\]

\[= \bigwedge_{u^*} \sum_{i=1}^n a_i b_i = \sum_{j=1}^n d_j d_j \left[ \bigvee_{i=1}^n \Psi_n(a_i) \bigvee_{j=1}^n \mu_n(b_j) \right] \lor 0.5\]

\[\leq \left\{ \Psi_n(u) \lor \mu_n(u) \lor (\Psi_n(u) \lor 0.5) \lor \mu_n(u) \lor 0.5 \right\} \lor 0.5\]

\[\leq \left\{ (\Psi_n(u) \lor 0.5) \lor \mu_n(u) \lor (\Psi_n(u) \lor 0.5) \lor \mu_n(u) \lor 0.5 \right\} \lor 0.5\]

\[= (\Psi_n(u) \lor \mu_n(u) \lor 0.5) \lor 0.5 = (\Psi_n \lor \mu_n)^{-1}(u).\]

Similarly, \((\Psi_n \ast k \mu_n)^{-1}(u) \geq (\Psi_n \lor \mu_n)^{-1}(u)\). Consequently, \((\varepsilon \lor \mu) \leq (\varepsilon \lor \mu)^{-1}\). By Lemma 8, we have \((\varepsilon \lor \mu)^{-1} \leq (\varepsilon \lor \mu)^{-1}\). Hence, \((\varepsilon \lor \mu)^{-1} = (\varepsilon \lor \mu)^{-1}\).

Conversely, suppose \((\varepsilon \lor \mu)^{-1} = (\varepsilon \lor \mu)^{-1}\) for every \((\varepsilon, \varepsilon \lor \varepsilon)-k-BFI_R\), \(\Psi = (R, \Psi_n, \Psi_p)\), also for every \((\varepsilon, \varepsilon \lor \varepsilon)-k-BFI_R\mu = (R, \mu_n, \mu_p)\) of \(R\). Suppose \(G\) and \(H\) are right \(k\)-ideal and left \(k\)-ideal of \(R\), respectively. The lower components of \(\Phi\) characteristic function \(C_G = (R, C_{ng}, C_{pc})\) is an \((\varepsilon, \varepsilon \lor \varepsilon)-k-BFI_R\) and \(C_H = (R, C_{nh}, C_{ph})\) is an \((\varepsilon, \varepsilon \lor \varepsilon)-k-BFI_L\) of \(R\). From our supposition, \((C_G \lor C_H)^{-1} = C_{G \lor H}^{-1}\). This indicates that \(C_{G \lor H}^{-1} = C_{G \lor H}^{-1}\) implies \(G \lor H = G \lor H\). Thus, by Lemma 1, \(R\) is \(k\)-regular. \(\square\)

**Theorem 5.** \(R\) is \(k\)-regular iff \((\varepsilon \lor \mu \lor \varepsilon)^{-1} \leq (\varepsilon \lor \mu \lor \varepsilon)^{-1}\) for every \((\varepsilon, \varepsilon \lor \varepsilon)-k-BFI_R\Psi = (R, \Psi_n, \Psi_p)\), for every \((\varepsilon, \varepsilon \lor \varepsilon)-k-BFI_R\mu = (R, \mu_n, \mu_p)\) of \(R\). and for all \((\varepsilon, \varepsilon \lor \varepsilon)-k-BFI_L\Psi = (R, \Psi_n, \Psi_p)\) of \(R\).

**Proof.** Assume \(R\) is \(k\)-regular semiring then for all \(u \in R\), there are elements \(c, c' \in R\) such as \(u + uc = uc'u\). Moreover, we have the following:
\((\Psi_n \circ k \mu_n \circ k \gamma_n)^{-1}(u) = (\Psi_n \circ k \mu_n \circ k \gamma_n)(u) \vee 0.5\)

\[
\bigwedge_{i=1}^n a_i \bigwedge_{j=1}^m b_j \left[ \bigvee_{i=1}^n \left( \Psi_n \circ k \mu_n \right)(a_i) \bigwedge \left( \bigvee_{i=1}^n \gamma_n(b_i) \right) \right] \bigvee 0.5
\]

\[
\leq \left\{ \left( \Psi_n \circ k \mu_n \right)(u) \vee \gamma_n(cu) \vee \left( \Psi_n \circ k \mu_n \right)(u) \vee \gamma_n(cu) \right\} \bigvee 0.5
\]

\[
= \left\{ \left( \Psi_n \circ k \mu_n \right)(u) \vee \gamma_n(cu) \vee \left( \Psi_n \circ k \mu_n \right)(u) \right\} \bigvee 0.5
\]

\[
\leq \left\{ \left( \Psi_n \circ k \mu_n \right)(u) \vee 0.5 \right\} \bigvee \left\{ \left( \Psi_n \circ k \mu_n \right)(u) \vee 0.5 \right\} \bigvee \left\{ \left( \Psi_n \circ k \mu_n \right)(u) \vee 0.5 \right\} \bigvee \left\{ \left( \Psi_n \circ k \mu_n \right)(u) \right\} \bigvee 0.5
\]

\[
= \left( \Psi_n \circ k \mu_n \right)(u) \vee 0.5 = (\Psi_n \circ k \mu_n \circ k \gamma_n)^{-1}(u)
\]

Similarly, \((\Psi_p \circ k \mu_p \circ k \gamma_p)^{-1}(u) \geq (\Psi_p \circ k \mu_p \circ k \gamma_p)^{-1}(u)\). Consequently, \((\Psi \wedge \mu \wedge \nu)^{-1} \leq (\Psi \circ k \mu \circ k \gamma)^{-1}\).

Conversely, let \(\Psi = (R, \Psi_n, \Psi_p)\) be an \((\epsilon, \epsilon \vee \eta)\)-k-BFL\(R\) and \(\gamma = (R, \gamma_n, \gamma_p)\) be an \((\epsilon, \epsilon \vee \eta)\)-k-BFL\(R\) of \(R\).

Then, \((\Psi \wedge \mu \wedge \nu)^{-1} \leq (\Psi_p \circ k \mu_p \circ k \gamma_p)^{-1}\) implies \((\Psi \wedge \mu \wedge \nu)^{-1} \leq (\Psi_p \circ k \gamma_p)^{-1}\) but \((\Psi \circ k \gamma_p)^{-1}\) \(\geq (\Psi_p \circ k \gamma_p)^{-1}\) by Lemma 8. Hence, \((\Psi \wedge \mu \wedge \nu)^{-1} = (\Psi_p \circ k \gamma_p)^{-1}\). Similarly, \((\Psi \wedge \mu \wedge \nu)^{-1} = (\Psi_p \circ k \gamma_p)^{-1}\). Hence, \(R\) is \(k\)-regular semiring by Theorem 4.

\(\square\)

**Theorem 7.** \(R\) is \(k\)-regular if \((\Psi \wedge \mu)^{-1} \leq (\Psi \circ k \mu \circ k \gamma)^{-1}\) to each \((\epsilon, \epsilon \vee \eta)\)-k-BFL\(I\) \(\Psi = (R, \Psi_n, \Psi_p)\), and for every \((\epsilon, \epsilon \vee \eta)\)-k-BFL \(\mu = (R, \mu_n, \mu_p)\) of \(R\).

**Proof.** Assume \(R\) is \(k\)-regular, also to each \(u \in R\), there are elements \(c, c' \in R\) such that \(u + uc = uc'\). Thus, we have as follows:

\[
\begin{align*}
\left( (\Psi_n \circ k \mu_n \circ k \gamma_n) \right)^{-1}(u) &= (\Psi_n \circ k \mu_n \circ k \gamma_n)(u) \vee 0.5 \\
&= \bigwedge_{i=1}^n \bigwedge_{j=1}^m \left[ \bigvee_{i=1}^n \left( \Psi_n \circ k \mu_n \right)(a_i) \bigwedge \left( \bigvee_{i=1}^n \gamma_n(b_i) \right) \right] \bigvee 0.5 \\
&\leq \left\{ \left( \Psi_n \circ k \mu_n \right)(u) \vee \gamma_n(cu) \vee \left( \Psi_n \circ k \mu_n \right)(u) \right\} \bigvee 0.5 \\
&= \left\{ \left( \Psi_n \circ k \mu_n \right)(u) \vee \gamma_n(cu) \vee \left( \Psi_n \circ k \mu_n \right)(u) \right\} \bigvee 0.5 \\
&\leq \left\{ \left( \Psi_n \circ k \mu_n \right)(u) \vee 0.5 \right\} \bigvee \left\{ \left( \Psi_n \circ k \mu_n \right)(u) \vee 0.5 \right\} \bigvee \left\{ \left( \Psi_n \circ k \mu_n \right)(u) \right\} \bigvee 0.5 \\
&= \left( \Psi_n \circ k \mu_n \right)(u) \vee 0.5 = (\Psi_n \circ k \mu_n \circ k \gamma_n)^{-1}(u)
\end{align*}
\]

Theorem 6. \(R\) is \(k\)-regular if for each \((\epsilon, \epsilon \vee \eta)\)-k-BFL\(I\) \(\Psi = (R, \Psi_n, \Psi_p)\) of \(R\), we have \(\Psi \leq (\Psi_p \circ k \gamma_p)^{-1}\).
Since \( u + ucu = uc'\), so
\[
\mu (uc')\leq \left\{ \Psi_n(u)\wedge \mu_n(ceuc)\vee \Psi_n(u)\wedge \mu_n('ceuc')\right\} \vee \Psi_n(u)\wedge \mu_n('ceuc') \leq \left\{ \Psi_n(u)\wedge (\mu(u)-0.5)\right\} \vee -0.5 = \left\{ \Psi_n(u)/\mu_n(u)\right\} \vee -0.5 = \left( \Psi_n/\mu_n\right)^- (u).
\]
which implies \( \Psi^- \leq \left( \Psi \circ R \circ \Psi \right)^- \). So, by Theorem 6, \( R \) is \( k \)-regular. \( \square \)

**Theorem 8.** For a semiring \( R \), the given conditions are equivalent.

(i) \( R \) is \( k \)-regular semiring,
(ii) \( (\Psi \wedge \mu)^- \leq (\Psi \circ k \circ \Psi)^- \) for every \( (\epsilon, \epsilon \vee q) \)-BFIL \( \Psi = (R, \Psi^\mu, \Psi^\mu) \) and for every \( (\epsilon, \epsilon \vee q) \)-\( k \)-BFIL \( \mu = (R, \mu^\mu, \mu^\mu) \) of \( R \).
(iii) \( (\Psi \wedge \mu)^- \leq (\Psi \circ k \circ \Psi)^- \) for each \( (\epsilon, \epsilon \vee q) \)-\( k \)-BFIL \( \Psi = (R, \Psi^\mu, \Psi^\mu) \), and for all \( (\epsilon, \epsilon \vee q) \)-\( k \)-BFIL \( \mu = (R, \mu^\mu, \mu^\mu) \) of \( R \).

**Proof.** (i) \( \rightarrow \) (ii)
Suppose \( R \) is \( k \)-regular, so for \( u \in R \), there are elements \( c, c' \in R \) such as \( u + ucu = uc' \). Then, \( (\Psi_n/\circ k \circ \Psi)^- (u) = (\Psi_n/\circ k \circ \Psi)(u) \vee -0.5 \)
\[
= \wedge u, \sum a_i b_i = \sum_{i=1}^n a_i b_i \left[ \begin{array}{c c}
\vee \Psi_n(a_i) \\
\wedge \mu_n(b_i)
\end{array} \right] \vee -0.5 \leq \left\{ \Psi_n(u)\wedge \mu_n(uc')\right\} \vee -0.5 \leq \left\{ \Psi_n(u)\wedge (\mu(u)-0.5)\right\} \vee -0.5 = \left( \Psi_n/\wedge \mu_n\right)^- (u).
\]

**Theorem 9.** For a semiring \( R \), the given statements are equivalent:

(i) \( R \) is \( k \)-intra-regular.
(ii) \( (\Psi \wedge \mu)^- \leq (\Psi \circ k \circ \Psi)^- \) for every \( (\epsilon, \epsilon \vee q) \)-BFIL \( \Psi = (R, \Psi^\mu, \Psi^\mu) \) and \( (\epsilon, \epsilon \vee q) \)-\( k \)-BFIL \( \mu = (R, \mu^\mu, \mu^\mu) \) of \( R \).

**Proof.** Suppose \( R \) is a \( k \)-intra-regular, so for each \( u \in R \), there are elements \( a_i, a'_i, b_i, b'_i \in R \) such as \( u + \sum a_i b_i = \sum a'_i b'_i \). That is, \( u + \sum (a_i b_i) = \sum (a'_i b'_i) \)
Then, $(\Psi_n \circ_k \mu_H)^\sim(u) = (\Psi_n \circ_k \mu_H)(u) \lor 0.5$

$$= \land \sum_{a \in A} \sum_{b \in B} \left[ \sum_{i=1}^{m} \psi_n(a_i) \lor \sum_{j=1}^{n} \mu_n(b_j) \right] \lor 0.5 \leq \left\{ \psi_n(a \lor b) \lor \psi_n(a \lor b) \lor \mu_n(a \lor b) \right\} \lor 0.5 \leq \left\{ (\psi_n(u) \lor 0.5) \lor (\mu_n(u) \lor 0.5) \lor (\psi_n(u) \lor 0.5) \lor (\mu_n(u) \lor 0.5) \right\} \lor 0.5 = (\psi_n \lor \mu_n)(u).$$

(14)

Similarly, $(\Psi_p \circ_k \mu_H)^\sim(u) \geq (\Psi_p \lor \mu_p)^\sim(u)$. Consequently, $(\Psi \lor \mu)^\sim \leq (\Psi \lor \mu)^\sim$.

Conversely, let $G$ and $H$ are left and right $k$-ideals of $R$, respectively. The lower components of the BF characteristic function $C_G = (R, C_G, C_H)$ is an $(\varepsilon, \epsilon \lor \eta)$-$k$-BFIS and $C_H = (R, C_H, C_H)$ is an $(\varepsilon, \epsilon \lor \eta)$-$k$-BFIS of $R$. From our supposition, $C_G \land C_H \leq C_G \lor C_H$. This indicates that $C_G \land C_H \leq C_G \lor C_H$ implies $G \cap H \subseteq GH$. Thus $R$ is $k$-intra regular by Lemma 3.

6. Comparative Study and Discussion

In [33], Shabir et al. characterized regular and intra regular semirings with the help of $(\alpha, \beta)$-BFIS. We established the results of [33] in the framework of $(\alpha, \beta)$-$k$-BFIS. We have characterized $k$-regular and $k$-intra regular semiring by using $(\alpha, \beta)$-$k$-BFIS. As $k$-ideal is a more restricted form of the ideal in semiring, so our enlargement has expanded in the applications than the access deliberated in [33].

7. Conclusion

BFS is an important theory of mathematics to tackle the fuzziness in positive and negative features of an object. In this paper, we have studied $(\alpha, \beta)$-$k$-BFIS. Basic definitions, concepts, operations, and related properties with respect to $(\alpha, \beta)$-$k$-BFIS and $(\alpha, \beta)$-$k$-BFSSS are purposed. Many examples are illuminated to show the importance of $k$-ideals in semiring. Generally, we have proved with an example that if a BF subset $\psi$ of $R$ is an $(\varepsilon, \epsilon \lor \eta)$-$k$-BFSS of $R$, then it satisfies three particular conditions, but the reverse may not hold true. In addition, we have studied the lower and upper parts of $(\varepsilon, \epsilon \lor \eta)$-$k$-BFIS of semiring. Also, we have characterized $k$-regular and $k$-intra regular semiring in terms of $(\varepsilon, \epsilon \lor \eta)$-$k$-BFIS of semiring.

A bipolar fuzzy set has membership degree ranges from $-1$ to $1$, but we often face such critical situations in real life which cannot be handled by bipolar fuzzy sets due to their membership degree $[-1, 1]$. To control these critical situations, Pythagorean fuzzy sets are much more effective because of sum of their membership degree and non-membership degree which can be greater than $1$.

In future, we will extend this work for hyperstructures, LA-semigroups, near-rings, etc. In addition, we will study its real life applications in medical science, computer science, management science, and many other fields.

Data Availability

No data were used for this research work.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

Conceptualization was performed by S.B. and R.M. Methodology was developed by T.K. Software was provided by S.B. Validation was performed by A.N.A.-K. Formal analysis was performed by T.K. Investigation was done by R.M. Resources were provided by A.N.A.-K. Data curation was performed by R.M. Original draft was written by T.K. Reviewing and editing were done by S.B. and R.M. Visualization was performed by R.M. Supervision was done by S.B. Project administration was performed by T.K. Funding acquisition was done by A.N.A.-K. All authors have read and agreed to the published version of the manuscript.

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