




## Research Article

# Inclusions Involving Interval-Valued Harmonically Co-Ordinated Convex Functions and Raina's Fractional Double Integrals

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The aim of this article is to obtain some new integral inclusions essentially using the interval-valued harmonically co-ordinated convex functions and  $\kappa$ -Raina's fractional double integrals. To show the validity of our theoretical results, we also give some numerical examples.

## 1. Introduction

A convex analysis is the branch of mathematics in which we study the properties of convex sets and convex functions. These classical concepts have a wide range of applications in both pure and applied sciences. For instance, every one is familiar with the role of convexity in the theory of optimization, operations research, mathematical economics, theory of means etc. In recent years the classical concepts of the convexity have been extended and generalized in different directions using novel and innovative ideas. For example, Dragomir [1] extended the notion of classical convex functions on the coordinates and introduced the class of co-ordinated convex functions. Iscan [2] introduced the notion of harmonically convex functions and observed that this class enjoys some nice properties which the convex functions have. Nikodem [3] introduced the class of interval-valued convex functions and discussed its properties. Zhao et al. [4] introduced the notion of interval-valued harmonically convex functions. For more details, interested readers are referred to the book [5].

Another charming aspect of the theory of convexity is its relation with the theory of inequalities. Many inequalities which are known to us are direct consequences of the

applications of the convexity property of the functions. In this regard, one of the most studied results is Hermite-Hadamard's inequality. This inequality provides us with a necessary and sufficient condition for a function to be convex. It reads as,

let  $\Theta: \mathcal{J} = [\wp_1, \wp_2] \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a convex function, then

$$\Theta\left(\frac{\wp_1 + \wp_2}{2}\right) \leq \frac{1}{\wp_2 - \wp_1} \int_{\wp_1}^{\wp_2} \Theta(x) dx \leq \frac{\Theta(\wp_1) + \Theta(\wp_2)}{2}. \quad (1)$$

This result is one of the most significant results pertaining to the convexity property of the functions which has been studied extensively as well as intensively. In recent years this result has been extended and generalized in different ways using novel and innovative ideas. For example, Dragomir [1] obtained a new version of Hermite-Hadamard's inequality by using the co-ordinated convexity property of the functions. Iscan [2] obtained Hermite-Hadamard's inequality using the class of harmonic convex functions. Zhao obtained a similar result by using the interval-valued harmonically convex functions. Sarikaya et al. [6] have utilized the concepts of fractional calculus and obtained fractional analogues of Hermite-Hadamard's inequality. For some more recent studies regarding Hermite-Hadamard's inequality and its applications, see [7].

On the other hand, the interval analysis, which is used in mathematics and computer models as one of the ways for resolving interval uncertainty, is an important material in mathematics. Despite the fact that this theory has a lengthy history dating back to Archimedes' estimate of the circumference of a circle, a substantial research in this topic was not published until the 1950s. In 1966, Moore, the pioneer of interval calculus, released the first book [8] on interval analysis. Following that, a slew of researchers delved into the theory and applications of interval analysis. Many authors have recently focused on integral inequalities derived from interval-valued functions. Sadowska [9] discovered the Hermite-Hadamard inequality for set-valued functions, which is a more general form of interval-valued mappings:

**Theorem 1** (see [9]). *Suppose that  $\Theta: [\wp_1, \wp_2] \rightarrow \mathbb{R}_I^+$  is interval-valued convex function such that  $F(t) = [\underline{\Theta}(t), \overline{F}(\Theta)]$ . Then, we have the following inequalities:*

$$\Theta\left(\frac{\wp_1 + \wp_2}{2}\right) \supseteq \frac{1}{\wp_2 - \wp_1} \int_{\wp_1}^{\wp_2} \Theta(x) dx \supseteq \frac{\Theta(\wp_1) + \Theta(\wp_2)}{2}. \quad (2)$$

Many papers have been devoted to generalizations of the inclusions (2). For example; Budak et al. [10] proved the Hermite-Hadamard inclusion by using Riemann-Liouville fractional integrals of interval-valued functions. Some papers focused on to generalization of (2) by utilizing some kinds of general convexities [4, 11–13]. Many authors also proved the corresponding Hermite-Hadamard inclusions for interval-valued functions with two variables [14–17]. For interested readers, we refer to the following articles [18–20]. The main objective of this paper is to derive new fractional refinements of Hermite–Hadamard’s inclusion using the class of interval-valued co-ordinated harmonically convex functions.

## 2. Preliminaries

In this section, we discuss some preliminary concepts and results.

A set  $\mathcal{C} \subseteq \mathbb{R}$  is said to be convex, if

$$(1-t)x + ty \in \mathcal{C}, \quad \forall x, y \in \mathcal{C}, t \in [0, 1]. \quad (3)$$

A function  $\Theta: \mathcal{C} \rightarrow \mathbb{R}$  is said to be convex, if

$$\Theta((1-t)x + ty) \leq (1-t)\Theta(x) + t\Theta(y), \quad \forall x, y \in \mathcal{C}, t \in [0, 1]. \quad (4)$$

Dragomir extended the notion of classical convexity and introduced the class of co-ordinated convex functions as follows.

**Definition 1** (see [1]). Let  $\Delta = [\wp_1, \wp_2] \times [\wp_3, \wp_4] \subseteq \mathbb{R} \times \mathbb{R}$  be a rectangle. A function  $\Theta: \Delta \rightarrow \mathbb{R}$  be a co-ordinated convex, if

$$\begin{aligned} \Theta((1-t) + ty, (1-r)u + rw) &\leq (1-t)(1-r)\Theta(x, u) \\ &\quad + r(1-t)\Theta(x, w) \\ &\quad + t(1-r)\Theta(y, u) \\ &\quad + tr\Theta(y, w), \end{aligned} \quad (5)$$

holds for all  $x, y, u, w \in \Delta, t, r \in [0, 1]$ .

The following result is the Hermite–Hadamard’s inequality pertaining to the co-ordinated convex functions.

**Theorem 2** (see [1]). *Suppose that  $\Theta: \Delta \rightarrow \mathbb{R}$  is co-ordinated convex function, then*

$$\begin{aligned} &\Theta\left(\frac{\wp_1 + \wp_2}{2}, \frac{\wp_3 + \wp_4}{2}\right) \\ &\leq \frac{1}{(\wp_2 - \wp_1)(\wp_4 - \wp_3)} \int_{\wp_1}^{\wp_2} \int_{\wp_3}^{\wp_4} \Theta(x, y) dx dy \\ &\leq \frac{\Theta(\wp_1, \wp_3) + \Theta(\wp_1, \wp_4) + \Theta(\wp_2, \wp_3) + \Theta(\wp_2, \wp_4)}{4}. \end{aligned} \quad (6)$$

Iscan [2] introduced the notion of harmonically convex functions as follows.

**Definition 2** (see [2]). A function  $\Theta: \mathcal{H} \subseteq (0, \infty) \rightarrow \mathbb{R}$  is said to be harmonically convex, if

$$\begin{aligned} \Theta\left(\frac{xy}{tx + (1-t)y}\right) &\leq (1-t)\Theta(x) + t\Theta(y), \\ \forall x, y \in \mathcal{H}, t &\in [0, 1]. \end{aligned} \quad (7)$$

**Proposition 1** (see [2])

- (1) If  $\Theta(x) \subset (0, \infty)$  is a nondecreasing function, then  $\Theta(x)$  is a harmonically convex function
- (2) If  $\Theta(x) \subset (0, \infty)$  is a nonincreasing harmonic convex function then  $\Theta(x)$  is a convex function

**Theorem 3** (see [2]). Let  $\Theta: \mathcal{F} = [\wp_1, \wp_2] \subseteq (0, \infty) \rightarrow \mathbb{R}$  be a harmonically convex function, then

$$\Theta\left(\frac{2\wp_1\wp_2}{\wp_1 + \wp_2}\right) \leq \frac{\wp_1\wp_2}{\wp_2 - \wp_1} \int_{\wp_1}^{\wp_2} \frac{\Theta(x)}{x^2} dx \leq \frac{\Theta(\wp_1) + \Theta(\wp_2)}{2}. \quad (8)$$

Noor et al. [21] introduced the class of co-ordinated harmonically convex functions and derived associated Hermite-Hadamard’s inequality.

**Definition 3** (see [21]). Consider a rectangle  $\Delta = [\wp_1, \wp_2] \times [\wp_3, \wp_4] \subseteq (0, \infty) \times (0, \infty)$ . A function  $\Theta: \Delta \rightarrow \mathbb{R}$  is said to be co-ordinated harmonically convex function on  $\Delta$ , if

$$\Theta\left(\frac{xy}{tx + (1-t)y}, \frac{uw}{ru + (1-r)w}\right) \leq (1-t)(1-r)\Theta(x, u) + t(1-r)\Theta(y, u) + (1-t)r\Theta(x, w) + tr\Theta(y, w), \tag{9}$$

whenever  $x, y \in [\wp_1, \wp_2], u, w \in [\wp_3, \wp_4]$  and  $t, r \in [0, 1]$ .

**Theorem 4** (see [21]). *Suppose that  $\Theta: \Delta \rightarrow \mathbb{R}$  is co-ordinated harmonically convex function, then*

$$\Theta\left(\frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4}\right) \leq \frac{abcd}{(\wp_2 - \wp_1)(\wp_4 - \wp_3)} \int_{\wp_1}^{\wp_2} \int_{\wp_3}^{\wp_4} \frac{\Theta(x, y)}{x^2 y^2} dx dy \leq \frac{\Theta(\wp_1, \wp_3) + \Theta(\wp_1, \wp_4) + \Theta(\wp_2, \wp_3) + \Theta(\wp_2, \wp_4)}{4}. \tag{10}$$

We now discuss some preliminaries from interval analysis. The idea behind the interval analysis is the observation that if you compute a number  $\mu$  and a bound  $M$  on the total error in  $\mu$ , as an approximation to some unknown number  $X$ , such that  $|X - \mu| \leq M$ , then no matter how you computed  $\mu$  and  $M$ , you will know that  $X$  lies in the interval  $[\mu - M, \mu + M]$ .

Let  $\mathcal{K}_{\wp_3}$  be a collection of all closed and bounded intervals of the form  $[w_*, w^*]$  for all  $w_*, w^* \in R$ . If  $w_*, w^* \geq 0$ , then  $[w_*, w^*]$  is called a positive interval. Furthermore,  $\mathcal{K}_{\wp_3}^+ = [w_*, w^*], \forall w_* \geq 0$  is known as the set of all positive intervals.

We now discuss about interval arithmetics and related properties Unlike the usual arithmetics the interval addition, subtraction product and division is defined as follows:

$$\begin{aligned} [u_*, u^*] + [w_*, w^*] &= [u_* + w_*, u^* + w^*], \\ [u_*, u^*] - [w_*, w^*] &= [u_* - w_*, u^* - w^*], \\ [u_*, u^*] \cdot [w_*, w^*] &= [r_*, r^*], \end{aligned} \tag{11}$$

where

$$\begin{aligned} r_* &= \min\{u_* w_*, u_* w^*, u^* w_*, u^* w^*\}, \\ r^* &= \max\{u_* w_*, u_* w^*, u^* w_*, u^* w^*\}. \end{aligned} \tag{12}$$

Also, if  $w$  is an interval without zero element, then

$$\frac{1}{w} = \left[ \frac{1}{w^*}, \frac{1}{w_*} \right], \tag{13}$$

so quotient of  $u$  and  $w$  is defined as follows:

$$\frac{u}{w} = \frac{u \cdot 1}{w}. \tag{14}$$

Note that, addition, subtraction, product, and quotient of intervals is again an interval.

Let  $\mathbb{R}_I, \mathbb{R}_{I^+}$ , and  $\mathbb{R}_{I^-}$  are set of closed intervals, set of positive closed intervals and set of negative closed intervals,

respectively, then following are some algebraic properties of intervals:

- (i)  $(u + v) + w = u + (v + w), \forall u, v, w \in \mathbb{R}_I$  (Associativity of addition)
- (ii)  $u + v = v + u, \forall u, v \in \mathbb{R}_I$  (Commutative property)
- (iii)  $u + 0 = u, \forall u \in \mathbb{R}_I$  (Additive element)
- (iv)  $u + w = v + w \Rightarrow u = v, \forall u, v, w \in \mathbb{R}_I$  (Cancellation law)

All of these properties also hold for multiplication. But distributive law does not hold for intervals. The following example illustrates this fact.

*Example 1*

$$\begin{aligned} u &= [-3, -2], \\ y &= [-2, 0], \\ w &= [0, 1], \end{aligned} \tag{15}$$

then

$$\begin{aligned} u \cdot (v + w) &= [-3, -2]([-2, 0] + [0, 1]) \\ &= [-3, -2]([-2, 1]) \\ &= [-3, 6]. \end{aligned} \tag{16}$$

But

$$\begin{aligned} u \cdot v + u \cdot w &= [-3, -2]([-2, 0] + [0, 1]) \\ &= [0.6] + [0.3] \\ &= [0.9]. \end{aligned} \tag{17}$$

For  $A_1 = [u_*, u^*], A_2 = [w_*, w^*] \in \mathcal{K}_{\wp_3}$ , the inclusion  $\subseteq$  is given by the following equation:

$$[u_*, u^*] \subseteq [w_*, w^*], \text{ iff } w_* \leq u_*, u^* \leq w^*. \tag{18}$$

Now, we rewrite the subsequent distance of intervals  $A_1$  and  $A_2$  is regarded as Hausdorff-Pompeiu distance as follows:

$$\begin{aligned} \wp_4(A_1, A_2) &= \wp_4([u_*, u^*], [w_*, w^*]) \\ &= \max\{|u_* - w_*|, |u^* - w^*|\}. \end{aligned} \tag{19}$$

We now discuss the integration of interval-valued functions.

If  $B([\wp_1, \wp_2])$  is set of all partitions of  $[\wp_1, \wp_2]$  and  $B(\rho, [\wp_1, \wp_2])$  be the set of all points  $P$  such that  $\text{mesh}(P) < \rho$ , then  $\Theta: [\wp_1, \wp_2] \rightarrow \mathbb{R}_I$  is called interval Riemann integrable on  $[\wp_1, \wp_2]$ , if there exist  $K \in \mathbb{R}_I$  and for each  $\epsilon > 0$  there exists  $\rho > 0$  such that

$$\wp_4(S(\Theta, P, \rho), k) < \epsilon, \tag{20}$$

where  $S(\Theta, P, \rho)$  is the Riemann sum of  $\Theta$  with respect to  $P \in B(\rho, [\wp_1, \wp_2])$ . (20) shows that  $K$  is the (IR)-integral of  $\Theta$  and given by the following equation:

$$K = (\mathbb{R}) \int_{\wp_1}^{\wp_2} \Theta(t)dt. \tag{21}$$

**Theorem 5.** Let  $\Theta: [\wp_1, \wp_2] \rightarrow \mathbb{R}$  be an interval-valued function such that  $\Theta(t) = [\Theta_*, \Theta^*], \Theta \in IR_{[\wp_1, \wp_2]}$  if and only if  $\Theta_*, \Theta^* \in R_{[\wp_1, \wp_2]}$  and

$$(\mathbb{R}) \int_{\wp_1}^{\wp_2} \Theta(t)dt = \left[ (R) \int_{\wp_1}^{\wp_2} \Theta_*(t)dt, (R) \int_{\wp_1}^{\wp_2} \Theta^*(t)dt \right]. \tag{22}$$

Now, we discuss interval-valued integration of double integrals and generalized Raina’s interval-valued integrals.

Suppose that the rectangle  $\Delta = [\wp_1, \wp_2] \times [\wp_3, \wp_4]$  where  $\wp_1 < \wp_2$  and  $\wp_3 < \wp_4$ . A group of numbers  $\{x_{i-1}, \mu_i, x_i\}_{i=1}^m$  is termed as a tagged partition  $P_1$  with respect to interval  $[\wp_1, \wp_2]$  if:  $P_1: \wp_1 = x_0 < x_1 < x_2 < \dots < x_m = \wp_2$  and if  $x_{i-1} \leq \mu_i \leq x_i, i = 1, 2, 3, \dots, m$ . Also, we consider  $\Delta x_i = x_i - x_{i-1}$ , and if  $\Delta x_i < \delta$  for every  $i$ , then partition  $P_1$  is known as  $\delta$ -fine. If  $\Lambda(\delta, [\wp_1, \wp_2])$  is collection of all  $\delta$ -partitions of interval  $[\wp_1, \wp_2]$ . Suppose that  $\{x_{i-1}, \mu_i, x_i\}_{i=1}^m$  belongs to  $\delta$ -fine  $P_1$  regarding  $[\wp_1, \wp_2]$  and  $\{y_{j-1}, \lambda_j, y_j\}_{j=1}^n$  is another  $\delta$ -fine  $P_2$  regarding  $[\wp_3, \wp_4]$ , then the series of rectangles  $\Lambda = [x_{i-1}, x_i] \times [y_j, y_j]$  are the partitions of  $\Delta$ , and the points  $(\mu_i, \lambda_j) \in \Lambda^\circ$ . Furthermore, letting  $\Lambda(\delta, \Lambda)$  be the set of all  $\delta$ -fine partitions of rectangle  $\Delta$  along with  $P_1 \times P_2$ , in which  $P_1 \in \Lambda(\delta, [\wp_1, \wp_2])$  along with  $P_2 \in \Lambda(\delta, [\wp_3, \wp_4])$ . We consider using the notation  $\Lambda A_{i,j}$  denote the area of the rectangle  $\delta_{i,j}$ . Within every rectangle  $\Lambda_{i,j}$ , for  $1 \leq i \leq m$  and  $1 \leq j \leq n$ , taking into account arbitrary  $\mu_i, \lambda_j$  and we have  $S(\Theta, P, \delta, \Delta) = \sum_{i=1}^m \sum_{j=1}^n \Theta(\mu_i, \lambda_j) \Lambda A_{i,j}$ , which is named as  $S(\Theta, P, \delta, \Delta)$  is the integral sum of  $\Theta$  in connection  $\Lambda(\delta, \Delta)$ .

**Theorem 6.** Assume that the interval-valued function  $\Theta: \Delta \rightarrow \mathbb{R}_I$ , then  $\Theta$  is called double integrable defined on rectangle  $\Delta$  with interval-valued double integral  $U = \int \int_{\Delta} \Theta(x, y)dA$ , if for every  $\epsilon > 0$  there exist  $\delta > 0$  satisfying that

$$\wp_4(S(\Theta, P, \delta, \Delta)) < \epsilon, \quad \text{for each } P \in P(\delta, \Delta). \tag{23}$$

**Theorem 7.** Assume that  $\Theta: \Delta \rightarrow \mathbb{R}_I$  is interval-valued double integrable defined on  $\Delta$ , then

$$\int \int_{\Delta} \Theta(x, y)dA = \int_{\wp_1}^{\wp_2} \int_{\wp_3}^{\wp_4} \Theta(x, y)dydx. \tag{24}$$

For more details, see [8, 22].

The class of interval-valued convex functions is defined as follows.

**Definition 4** (see [3]). A function  $\Theta: [\wp_1, \wp_2] \rightarrow \mathbb{R}_I^+$  is said to be interval-valued convex function, if the following inclusion holds:

$$\Theta((1-t)x + ty) \supseteq (1-t)\Theta(x) + t\Theta(y), \quad x, y \in [\wp_1, \wp_2], \tag{25}$$

where  $0 \leq \wp_1 < \wp_2$  and  $t \in [0, 1]$ .

Now, we recall the definition of interval-valued harmonically convexity, which is defined as follows.

**Definition 5** (see [4]). Let  $\Theta: [\wp_1, \wp_2] \rightarrow \mathbb{R}_I^+$  be interval-valued harmonically convex function if the following inclusion holds:

$$\Theta\left(\frac{\wp_1 \wp_2}{(1-t)x + ty}\right) \supseteq t\Theta(x) + (1-t)\Theta(y), \quad x, y \in [\wp_1, \wp_2], \tag{26}$$

where  $0 \leq \wp_1 < \wp_2$  and  $t \in [0, 1]$ .

The following is the definition of an interval-valued co-ordinated convex function.

**Definition 6** (see [17]). Suppose that  $\Theta: \Delta \rightarrow \mathbb{R}_I^+$  belongs to an interval-valued function regarding two variable form, in which  $\Delta = [\wp_1, \wp_2] \times [\wp_3, \wp_4]$  is a given rectangle. We say that  $\Theta$  is an interval-valued co-ordinated convex function, if

$$\begin{aligned} \Theta((1-t)x + ty, (1-r)u + rw) &\supseteq (1-t)(1-r)\Theta(x, u) \\ &\quad + r(1-t)\Theta(x, w) \\ &\quad + t(1-r)\Theta(y, u) \\ &\quad + tr\Theta(y, w), \end{aligned} \tag{27}$$

holds for every  $(x, y), (u, w) \in \Delta$  together with  $t, r \in [0, 1]$ .

**Proposition 2.** Let function  $\Theta: [\wp_1, \wp_2] \rightarrow \mathbb{R}_I^+$  be a interval-valued function with  $\Theta(x) = [\Theta_*(x), \Theta^*(x)]$ . Then,  $\Theta(x)$  will be interval-valued convex function if and only if  $\Theta_*(x)$  is a convex function and  $\Theta^*(x)$  is a concave function.

We now recall some concepts from fractional calculus. The classical Riemann-Liouville fractional integrals are defined as:

**Definition 7** (see [23]). Let  $\Theta \in L_1[\wp_1, \wp_2]$ . The Riemann-Liouville integrals  $J_{\wp_1^+}^\alpha \Theta$  and  $J_{\wp_2^-}^\alpha \Theta$  of order  $\alpha > 0$  are defined by the following equation:

$$\begin{aligned} J_{\wp_1^+}^\alpha \Theta(x) &= \frac{1}{\Gamma(\alpha)} \int_{\wp_1}^x (x-t)^{\alpha-1} \Theta(t)dt, \quad x > \wp_1, \\ J_{\wp_2^-}^\alpha \Theta(x) &= \frac{1}{\Gamma(\alpha)} \int_x^{\wp_2} (t-x)^{\alpha-1} \Theta(t)dt, \quad x < \wp_2. \end{aligned} \tag{28}$$

Sarikaya et al. [6] have obtained the following fractional analogue of Hermite–Hadamard’s inequality:

**Theorem 8.** Let  $\Theta: [\wp_1, \wp_2] \rightarrow \mathbb{R}$  be a positive function with  $0 \leq \wp_1 < \wp_2$  and  $\Theta: L_1[\wp_1, \wp_2]$ . If  $\Theta$  is a convex function on  $[\wp_1, \wp_2]$ , then

$$\begin{aligned} \Theta\left(\frac{\wp_1 + \wp_2}{2}\right) &\leq \frac{\Gamma(\alpha + 1)}{2(\wp_2 - \wp_1)^\alpha} \left[ J_{\wp_1^+}^\alpha \Theta(\wp_2) + J_{\wp_2^-}^\alpha \Theta(\wp_1) \right] \\ &\leq \frac{\Theta(\wp_1) + \Theta(\wp_2)}{2}. \end{aligned} \tag{29}$$

This motivated Iscan and Wu [24] and as result, they have obtained a fractional analogue of Hermite–Hadamard’s

inequality essentially using the class of harmonically convex functions.

Mubeen and Habibullah [25] introduced the notion of  $\kappa$ -Riemann-Liouville fractional integrals as follows.

*Definition 8.* Let  $\Theta \in L_1[\wp_1, \wp_2]$ . The  $\kappa$ -Riemann-Liouville integrals  $J_{\wp_1^+}^{\alpha, \kappa} \Theta$  and  $J_{\wp_2^-}^{\alpha, \kappa} \Theta$  of order  $\alpha > 0$  are defined by the following equation:

$$J_{\wp_1^+}^{\alpha, \kappa} \Theta(x) = \frac{1}{\kappa \Gamma_{\kappa}(\alpha)} \int_{\wp_1}^x (x - \tau)^{(\alpha/\kappa) - 1} \Theta(\tau) d\tau, \quad x > \wp_1,$$

$$J_{\wp_2^-}^{\alpha, \kappa} \Theta(x) = \frac{1}{\kappa \Gamma_{\kappa}(\alpha)} \int_x^{\wp_2} (\tau - x)^{(\alpha/\kappa) - 1} \Theta(\tau) d\tau, \quad x < \wp_2,$$

(30)

where  $\Gamma_{\kappa}(\alpha) = \int_0^{\infty} \tau^{\alpha-1} e^{-\tau/\kappa} d\tau$ ,  $\text{Re}(\alpha) > 0$ ,  $\kappa \in \mathbb{R}^+$  is the  $\kappa$ -gamma function which was introduced and studied in [26].

Now, we give generalized fractional integrals involving Raina's functions for interval-valued function, which is stated as follows.

*Definition 9.* Let  $\Theta \subseteq \mathbb{R} \rightarrow \mathbb{R}_I$  be an interval-valued function and  $F$  be a Riemann interval-valued integrable function satisfying the condition  $\Theta(x) = [\Theta_*(x), \Theta^*(x)]$ :

$$\rho, \alpha, w I_{\wp_1^+}^{\sigma, k} \Theta(\wp_2) = \int_{\wp_1}^{\wp_2} (\wp_2 - t)^{(\alpha/k) - 1} \mathcal{R}_{\rho, w, \alpha}^{\sigma, k} [w(\wp_2 - t)^{\rho}] f(t) dt, \quad \wp_2 > x,$$

$$\rho, \alpha, w I_{\wp_2^-}^{\sigma, k} \Theta(\wp_1) = \int_{\wp_1}^{\wp_2} (t - \wp_1)^{(\alpha/k) - 1} \mathcal{R}_{\rho, w, \alpha}^{\sigma, k} [w(t - \wp_1)^{\rho}] f(t) dt, \quad \wp_1 < x.$$

(31)

Using the definitions of  $\kappa$ -Riemann-Liouville fractional integrals Noor et al. [21] have obtained  $\kappa$ -fractional analogue of Hermite-Hadamard's inequality.

We now introduce the  $\kappa$ -Raina's fractional double integrals, which are defined as follows.

*Definition 10.* Let  $\Theta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}_I$  be a given interval valued functions in two variables, satisfying that  $\Theta(t, s) = [\Theta_*(t, s), \Theta^*(t, s)]$  and let  $\Theta$  be interval Riemann integrable on rectangle  $\Delta$ , then

$$\rho, \alpha, w I_{\wp_1^+, \wp_3^+}^{\sigma, k} \Theta(\wp_2, \wp_4) = \int_{\wp_1}^{\wp_2} \int_{\wp_3}^{\wp_4} (\wp_2 - t)^{(\alpha_1/k) - 1} (\wp_4 - s)^{(\alpha_2/k) - 1} \mathcal{R}_{\rho_1, w_1, \alpha_1}^{\sigma_1, k} [w_1(\wp_2 - t)^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2}^{\sigma_2, k} [w_2(\wp_4 - s)^{\rho_2}] \Theta(t, s) ds dt,$$

$$\rho, \alpha, w I_{\wp_1^+, \wp_4^-}^{\sigma, k} \Theta(\wp_2, \wp_3) = \int_{\wp_1}^{\wp_2} \int_{\wp_3}^{\wp_4} (\wp_2 - t)^{(\alpha_1/k) - 1} (s - \wp_3)^{(\alpha_2/k) - 1} \mathcal{R}_{\rho_1, w_1, \alpha_1}^{\sigma_1, k} [w_1(\wp_2 - t)^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2}^{\sigma_2, k} [w_2(s - \wp_3)^{\rho_2}] \Theta(t, s) ds dt,$$

$$\rho, \alpha, w I_{\wp_2^-, \wp_3^+}^{\sigma, k} \Theta(\wp_1, \wp_4) = \int_{\wp_1}^{\wp_2} \int_{\wp_3}^{\wp_4} (t - \wp_1)^{(\alpha_1/k) - 1} (\wp_4 - s)^{(\alpha_2/k) - 1} \mathcal{R}_{\rho_1, w_1, \alpha_1}^{\sigma_1, k} [w_1(t - \wp_1)^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2}^{\sigma_2, k} [w_2(\wp_4 - s)^{\rho_2}] \Theta(t, s) ds dt,$$

$$\rho, \alpha, w I_{\wp_2^-, \wp_4^-}^{\sigma, k} \Theta(\wp_1, \wp_3) = \int_{\wp_1}^{\wp_2} \int_{\wp_3}^{\wp_4} (t - \wp_1)^{(\alpha_1/k) - 1} (s - \wp_3)^{(\alpha_2/k) - 1} \mathcal{R}_{\rho_1, w_1, \alpha_1}^{\sigma_1, k} [w_1(t - \wp_1)^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2}^{\sigma_2, k} [w_2(s - \wp_3)^{\rho_2}] \Theta(t, s) ds dt.$$

(32)

Here,  $\rho = (\rho_1, \rho_2)$ ,  $\alpha = (\alpha_1, \alpha_2)$ ,  $w = (w_1, w_2)$  and  $\mathcal{R}_{\rho, \lambda}^{\sigma, k}(z)$  is the  $k$ -Raina's function which is defined as follows:

$$\mathcal{R}_{\rho, \lambda}^{\sigma, k}(z) = \mathcal{R}_{\rho, \lambda}^{\sigma(0), \sigma(1), \dots}(z) := \sum_{i=0}^{\infty} \frac{\sigma(i)}{k \Gamma_{\kappa}(\rho ki + \lambda)} z^i, \quad z \in \mathbb{C},$$

(33)

where  $\rho, \lambda > 0$ , with bounded modulus  $|z| < M$ , and  $\sigma = \{\sigma(0), \sigma(1), \dots, \sigma(i), \dots\}$  is a bounded sequence of positive real numbers. For details, see [27].

In the following example, we give the numerical verification of Definition (10).

*Example 2.* We choose  $[\wp_1, \wp_2] \times [\wp_3, \wp_4] = [0, 1] \times [1, 2]$  and  $\Theta(t, s) = [st, s + t]$ , then we have the following equation:

$$\rho, \alpha, w I_{0^+, 1^+}^{\sigma, k} \Theta(1, 2) = \int_0^1 \int_1^2 (1 - t)^{(\alpha_1/k) - 1} (2 - s)^{(\alpha_2/k) - 1} \mathcal{R}_{\rho_1, w_1, \alpha_1}^{\sigma_1, k} [w_1(1 - t)^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2}^{\sigma_2, k} [w_2(2 - s)^{\rho_2}] [st, s + t] ds dt.$$

(34)

Then, (34) can be written as follows:

$${}_{\rho, \alpha, w} I_{0^+, 1^+}^{\sigma, k} \Theta(1, 2) = [{}_{\rho, \alpha, w} I_{0^+, 1^+}^{\sigma, k} \Theta_*(1, 2), {}_{\rho, \alpha, w} I_{0^+, 1^+}^{\sigma, k} \Theta^*(1, 2)], \tag{35}$$

where

$${}_{\rho, \alpha, w} I_{0^+, 1^+}^{\sigma, k} \Theta_*(1, 2) = \int_0^1 \int_1^2 (1-t)^{\frac{\alpha_1}{k}-1} (2-s)^{\frac{\alpha_2}{k}-1} \mathcal{R}_{\rho_1, w_1, \alpha_1}^{\sigma_1, k} [w_1(1-t)^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2}^{\sigma_2, k} [w_2(2-s)^{\rho_2}] st ds dt \tag{36}$$

$$= k^2 \mathcal{R}_{\rho_1, w_1, \alpha_1+2k}^{\sigma_1, k} [w_1] \left[ k \mathcal{R}_{\rho_2, w_2, \alpha_2+k}^{\sigma_2, k} [w_2] + k^2 \mathcal{R}_{\rho_2, w_2, \alpha_2+2k}^{\sigma_2, k} [w_2] \right],$$

$${}_{\rho, \alpha, w} I_{0^+, 1^+}^{\sigma, k} \Theta^*(1, 2) = \int_0^1 \int_1^2 (1-t)^{(\alpha_1/k)-1} (2-s)^{(\alpha_2/k)-1} \mathcal{R}_{\rho_1, w_1, \alpha_1}^{\sigma_1, k} [w_1(1-t)^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2}^{\sigma_2, k} [w_2(2-s)^{\rho_2}] (s+t) ds dt \tag{37}$$

$$= k^2 \mathcal{R}_{\rho_1, w_1, \alpha_1+k}^{\sigma_1, k} [w_1] \left[ k \mathcal{R}_{\rho_2, w_2, \alpha_2+k}^{\sigma_2, k} [w_2] + k^2 \mathcal{R}_{\rho_2, w_2, \alpha_2+2k}^{\sigma_2, k} [w_2] \right] + k^3 \mathcal{R}_{\rho_1, w_1, \alpha_1+2k}^{\sigma_1, k} [w_1] \mathcal{R}_{\rho_2, w_2, \alpha_2+2k}^{\sigma_2, k} [w_2].$$

A combination of (35), (36), and (37) yields the required result.

The main objective of this article is to obtain some new integral inclusions essentially using the interval-valued harmonically co-ordinated convex functions and  $k$ -Raina's fractional double integrals. To show the validity of our theoretical results, we also give some numerical examples. We hope that the ideas and the techniques of this paper will inspire interested readers.

### 3. Results and Discussions

In this section, we discuss our main results. First of all, we introduce the class of interval-valued harmonically co-ordinated convex functions.

*Definition 11.* Let  $\Theta: [\wp_1, \wp_2] \times [\wp_3, \wp_4] \rightarrow \mathbb{R}_I^+$  be interval-valued harmonically co-ordinated convex function, if the following inclusion holds:

$$\Theta \left( \frac{xy}{(1-t)+ty}, \frac{uw}{(1-s)+sw} \right) \supseteq ts\Theta(x, u) + t(1-s)\Theta(x, w) + (1-t)s\Theta(y, u) + (1-t)(1-s)\Theta(y, w), \tag{38}$$

where  $(x, y), (u, w) \in \Delta$  and  $t, s \in [0, 1]$  and  $0 \leq \wp_1 < \wp_2, 0 \leq \wp_3 < \wp_4$ . By reversing the the above-given inclusion, we obtain the notion of harmonically interval-valued co-ordinated concave function.

**Lemma 1.** Let  $\Theta: [\wp_1, \wp_2] \times [\wp_3, \wp_4] \rightarrow \mathbb{R}_I^+$  be co-ordinated interval-valued function. Then,  $\Theta$  is co-ordinated interval-valued harmonically convex function if and only if there exist two harmonically convex function  $\Theta_y(x): [\wp_1, \wp_2] \rightarrow \mathbb{R}_I^+, \Theta_x(y) = \Theta(x, y)$  and  $\Theta_x(y): [\wp_3, \wp_4] \rightarrow \mathbb{R}_I^+, \Theta_y(x) = \Theta(x, y)$ .

*Proof.* Using the definition of co-ordinated harmonic convex functions the proof is straight forward.  $\square$

**Theorem 9.** Suppose that  $\Theta: [\wp_1, \wp_2] \times [\wp_3, \wp_4] \rightarrow \mathbb{R}_I^+$  is a given interval-valued harmonically convex function defined over a rectangle  $[\wp_1, \wp_2] \times [\wp_3, \wp_4]$  with  $0 \leq \wp_1 < \wp_2, 0 \leq \wp_3 < \wp_4$  and  $\Theta(x, y) = [\Theta_*(x, y), \Theta^*(x, y)]$ . Then, the following inclusion holds:

$$\Theta \left( \frac{2\wp_1\wp_2}{\wp_1 + \wp_2} \right) \supseteq \frac{1}{2k \mathcal{R}_{\rho, w, \alpha+k}^{\sigma, k} [w(\wp_2 - \wp_1/\wp_1\wp_2)^\rho]} \left( \frac{\wp_1\wp_2}{\wp_2 - \wp_1} \right)^{\alpha/k} \left[ {}_{\alpha, \rho, w} I_{(1/\wp_2)^+}^{\sigma, k} \Theta^\circ g \left( \frac{1}{\wp_1} \right) {}_{\alpha, \rho, w} I_{(1/\wp_1)^-}^{\sigma, k} \Theta^\circ g \left( \frac{1}{\wp_2} \right) \right] \tag{39}$$

$$\supseteq \frac{\Theta(\wp_1) + \Theta(\wp_2)}{2}.$$

If  $\Theta(x, y)$  is harmonically interval-valued concave function then we have the following equation:

$$\Theta\left(\frac{2\wp_1\wp_2}{\wp_1 + \wp_2}\right) \subseteq \frac{1}{2k\mathcal{R}_{\rho, w, \alpha+k}^{\sigma, k} [w(\wp_2 - \wp_1/\wp_1\wp_2)^\rho]} \left(\frac{\wp_1\wp_2}{\wp_2 - \wp_1}\right)^{\alpha/k} \left[ {}_{\alpha, \rho, w}I_{(1/\wp_2)^+}^{\sigma, k} \Theta^\circ g\left(\frac{1}{\wp_1}\right) {}_{\alpha, \rho, w}I_{(1/\wp_1)^-}^{\sigma, k} \Theta^\circ g\left(\frac{1}{\wp_2}\right) \right] \tag{40}$$

$$\subseteq \frac{\Theta(\wp_1) + \Theta(\wp_2)}{2}.$$

*Proof.* The proof is left for interested readers. □

*Example 3.* We set  $\rho = 0, \alpha = 1$  and  $k = 1$  with  $\sigma(0) = 1$  in Theorem 9. If  $\Theta(x) = [x^2, -6x^2 + 12x + 24]$  and  $[\wp_1, \wp_2] = [1, 2]$ , then

$$\Theta\left(\frac{2\wp_1\wp_2}{\wp_1 + \wp_2}\right) = [1.78, 29.33], \tag{41}$$

$$\frac{\wp_1\wp_2}{\wp_2 - \wp_1} \int_{\wp_1}^{\wp_2} \frac{\Theta(x)}{x^2} dx = [2, 28.63], \tag{42}$$

$$\frac{\Theta(\wp_1) + \Theta(\wp_2)}{2} = [2.5, 27]. \tag{43}$$

(41)–(43) implies that

$$[1.78, 29.33] \supseteq [2, 28.63] \supseteq [2.5, 27], \tag{44}$$

which gives the verification of Theorem 9.

**Theorem 10.** Suppose  $\Theta: [\wp_1, \wp_2] \rightarrow \mathbb{R}_I^+$  be an interval-valued harmonically convex functions with  $0 \leq \wp_1 < \wp_2$ , satisfying  $\Theta(t) = [\Theta_*(t), \Theta^*(t)]$ , then

$$\Theta\left(\frac{2\wp_1\wp_2}{\wp_1 + \wp_2}\right) \supseteq \frac{1}{2k\mathcal{R}_{\rho, w, \alpha+k}^{\sigma, k} [w(\wp_2 - \wp_1/2\wp_1\wp_2)^\rho]} \left(\frac{2\wp_1\wp_2}{\wp_2 - \wp_1}\right)^{\alpha_1/k} \left[ {}_{\alpha, \rho, w}I_{(\wp_1+\wp_2/2\wp_1\wp_2)^+}^{\sigma, k} \Theta^\circ g\left(\frac{1}{\wp_1}\right) + {}_{\alpha, \rho, w}I_{(\wp_1+\wp_2/2\wp_1\wp_2)^-}^{\sigma, k} \Theta^\circ g\left(\frac{1}{\wp_2}\right) \right] \tag{45}$$

$$\supseteq \frac{\Theta(\wp_1) + \Theta(\wp_2)}{2}.$$

If  $\Theta$  is interval-valued harmonically co-ordinated concave then the following relation holds:

$$\Theta\left(\frac{2\wp_1\wp_2}{\wp_1 + \wp_2}\right) \subseteq \frac{1}{2k\mathcal{R}_{\rho, w, \alpha+k}^{\sigma, k} [w(\wp_2 - \wp_1/2\wp_1\wp_2)^\rho]} \left(\frac{2\wp_1\wp_2}{\wp_2 - \wp_1}\right)^{\alpha_1/k} \left[ {}_{\alpha, \rho, w}I_{(\wp_1+\wp_2/2\wp_1\wp_2)^+}^{\sigma, k} \Theta^\circ g\left(\frac{1}{\wp_1}\right) + {}_{\alpha, \rho, w}I_{(\wp_1+\wp_2/2\wp_1\wp_2)^-}^{\sigma, k} \Theta^\circ g\left(\frac{1}{\wp_2}\right) \right] \tag{46}$$

$$\subseteq \frac{\Theta(\wp_1) + \Theta(\wp_2)}{2}.$$

*Proof.* The proof is obvious by definition and some computation as well. □

**Theorem 11.** Suppose that  $\Theta, G: [\wp_1, \wp_2] \rightarrow \mathbb{R}_I^+$  are two interval-valued harmonically convex functions with  $0 \leq \wp_1 < \wp_2$ , satisfying  $\Theta(t) = [\Theta_*(t), \Theta^*(t)]$  and  $G(t) = [G_*(t), G^*(t)]$ , then we have the following equation:

$$\begin{aligned} & \left( \frac{\wp_1 \wp_2}{\wp_2 - \wp_1} \right)^\alpha \left[ I_{(1/\wp_2)^+, \rho, w, \alpha}^{\sigma, k} G^\circ g \left( \frac{1}{\wp_1} \right) \Theta^\circ g \left( \frac{1}{\wp_1} \right) + I_{(1/\wp_1)^-, \rho, w, \alpha}^{\sigma, k} \Theta^\circ g \left( \frac{1}{\wp_2} \right) G^\circ g \left( \frac{1}{\wp_2} \right) \right] \\ & \geq H_1(\wp_1, \wp_2) \left[ 2 \sum_{i=0}^{\infty} \frac{\sigma(i) w^i (\wp_1 \wp_2 / \wp_2 - \wp_1)^{\rho i}}{(\rho k i + \alpha + 2k) \Gamma_k(\rho k i + \alpha)} - 2 \sum_{i=0}^{\infty} \frac{\sigma(i) w^i (\wp_1 \wp_2 / \wp_2 - \wp_1)^{\rho i}}{(\rho k i + \alpha + k) \Gamma_k(\rho k i + \alpha)} + \Theta_{\rho, w, \alpha + k}^{\sigma, k} \left[ w \left( \frac{\wp_1 \wp_2}{\wp_2 - \wp_1} \right)^\rho \right] \right] \\ & \quad + H_2(\wp_1, \wp_2) \left[ 2 \sum_{i=0}^{\infty} \frac{\sigma(i) w^i (\wp_1 \wp_2 / \wp_2 - \wp_1)^{\rho i}}{(\rho k i + \alpha + k) \Gamma_k(\rho k i + \alpha)} - 2 \sum_{i=0}^{\infty} \frac{\sigma(i) w^i (\wp_1 \wp_2 / \wp_2 - \wp_1)^{\rho i}}{(\rho k i + \alpha + 2k) \Gamma_k(\rho k i + \alpha)} \right], \end{aligned} \quad (47)$$

where

$$\begin{aligned} H_1(\wp_1, \wp_2) &= \Theta(\wp_1)G(\wp_1) + \Theta(\wp_2)G(\wp_2), \\ H_2(\wp_1, \wp_2) &= \Theta(\wp_1)G(\wp_2) + \Theta(\wp_2)G(\wp_1). \end{aligned} \quad (48)$$

*Proof 4.* From the given hypothesis and interval-valued harmonically convexity property, we have the following equation:

$$\begin{aligned} \Theta_* \left( \frac{\wp_1 \wp_2}{(1-t)\wp_1 + tb} \right) G_* \left( \frac{\wp_1 \wp_2}{(1-t)\wp_1 + tb} \right) &\leq t^2 \Theta_*(\wp_1) G_*(\wp_1) + t(1-t) [\Theta_*(\wp_1) G_*(\wp_2) + \Theta_*(\wp_2) G_*(\wp_1)] \\ &\quad + (1-t)^2 \Theta_*(\wp_2) G_*(\wp_2), \end{aligned} \quad (49)$$

$$\begin{aligned} \Theta_* \left( \frac{\wp_1 \wp_2}{t\wp_1 + (1-t)\wp_2} \right) G_* \left( \frac{\wp_1 \wp_2}{t\wp_1 + (1-t)\wp_2} \right) &\leq (1-t)^2 \Theta_*(\wp_1) G_*(\wp_1) + t(1-t) [\Theta_*(\wp_1) G_*(\wp_2) + \Theta_*(\wp_2) G_*(\wp_1)] \\ &\quad + t^2 \Theta_*(\wp_2) G_*(\wp_2). \end{aligned} \quad (50)$$

Adding (49), (50) and multiplying inequality by  $t^{(\alpha/k)-1} \mathcal{R}_{\rho, w, \alpha}^{\sigma, k} [w(\wp_1 \wp_2 / \wp_2 - \wp_1)^\rho t^\rho] p$ , we obtain the following equation:

$$\begin{aligned} & \left( \frac{\wp_1 \wp_2}{\wp_2 - \wp_1} \right)^\alpha \left[ I_{(1/\wp_2)^+, \rho, w, \alpha}^{\sigma, k} G_*^\circ g \left( \frac{1}{\wp_1} \right) \Theta_*^\circ g \left( \frac{1}{\wp_1} \right) + I_{(1/\wp_1)^-, \rho, w, \alpha}^{\sigma, k} \Theta_*^\circ g \left( \frac{1}{\wp_2} \right) G_*^\circ g \left( \frac{1}{\wp_2} \right) \right] \\ & \leq \left[ 2 \sum_{i=0}^{\infty} \frac{\sigma(i) w^i (\wp_1 \wp_2 / \wp_2 - \wp_1)^{\rho i}}{(\rho k i + \alpha + 2k) \Gamma_k(\rho k i + \alpha)} - 2 \sum_{i=0}^{\infty} \frac{\sigma(i) w^i (\wp_1 \wp_2 / \wp_2 - \wp_1)^{\rho i}}{(\rho k i + \alpha + k) \Gamma_k(\rho k i + \alpha)} + \Theta_{\rho, w, \alpha + k}^{\sigma, k} \left[ w \left( \frac{\wp_1 \wp_2}{\wp_2 - \wp_1} \right)^\rho \right] \right] \\ & \quad \times [\Theta_*(\wp_1) G_*(\wp_1) + \Theta_*(\wp_2) G_*(\wp_2)] + [\Theta_*(\wp_1) G_*(\wp_2) + \Theta_*(\wp_2) G_*(\wp_1)] \\ & \quad \cdot \left[ 2 \sum_{i=0}^{\infty} \frac{\sigma(i) w^i (\wp_1 \wp_2 / \wp_2 - \wp_1)^{\rho i}}{(\rho k i + \alpha + k) \Gamma_k(\rho k i + \alpha)} - 2 \sum_{i=0}^{\infty} \frac{\sigma(i) w^i (\wp_1 \wp_2 / \wp_2 - \wp_1)^{\rho i}}{(\rho k i + \alpha + 2k) \Gamma_k(\rho k i + \alpha)} \right]. \end{aligned} \quad (51)$$

Similarly, we can calculate, using interval-valued harmonically convexity,



$$\begin{aligned} & \left(\frac{\wp_1\wp_2}{\wp_2 - \wp_1}\right)^\alpha \left[ I_{(1/\wp_2)^+, \rho, w, \alpha}^{\sigma, k} \Theta^* \circ g\left(\frac{1}{\wp_1}\right) G^* \circ g\left(\frac{1}{\wp_1}\right) + I_{(1/\wp_1)^-, \rho, w, \alpha}^{\sigma, k} \Theta^* \circ g\left(\frac{1}{\wp_2}\right) G^* \circ g\left(\frac{1}{\wp_2}\right) \right] \\ & \geq \left[ 2 \sum_{i=0}^{\infty} \frac{\sigma(i)w^i (\wp_1\wp_2/\wp_2 - \wp_1)^{pi}}{(\rho ki + \alpha + 2k)\Gamma_k(\rho ki + \alpha)} - 2 \sum_{i=0}^{\infty} \frac{\sigma(i)w^i (\wp_1\wp_2/\wp_2 - \wp_1)^{pi}}{(\rho ki + \alpha + k)\Gamma_k(\rho ki + \alpha)} + \Theta_{\rho, w, \alpha + k}^{\sigma, k} \left[ w \left( \frac{\wp_1\wp_2}{\wp_2 - \wp_1} \right)^\rho \right] \right] \\ & \quad \times [\Theta^*(\wp_1)G^*(\wp_1) + \Theta^*(\wp_2)G^*(\wp_2)] + [\Theta^*(\wp_1)G^*(\wp_2) + \Theta^*(\wp_2)G^*(\wp_1)] \\ & \quad \cdot \left[ 2 \sum_{i=0}^{\infty} \frac{\sigma(i)w^i (\wp_1\wp_2/\wp_2 - \wp_1)^{pi}}{(\rho ki + \alpha + k)\Gamma_k(\rho ki + \alpha)} - 2 \sum_{i=0}^{\infty} \frac{\sigma(i)w^i (\wp_1\wp_2/\wp_2 - \wp_1)^{pi}}{(\rho ki + \alpha + k)\Gamma_k(\rho ki + \alpha)} \right]. \end{aligned} \tag{52}$$

Combination of (51) and (52) refers to relation (47). This completes the proof.  $\square$

**Theorem 12.** Suppose that  $\Theta: [\wp_1, \wp_2] \rightarrow \mathbb{R}_I^+$  is a given interval-valued harmonically co-ordinated convex function

with  $0 \leq \wp_1 < \wp_2$  and  $\Theta(x) = [\Theta_*(x), \Theta^*(x)]$ . If  $\phi: [\wp_1, \wp_2] \rightarrow \mathbb{R}$  be a non-negative, integrable, and symmetric function:  $\phi(x) = \phi(1/(1/\wp_1) + (1/\wp_2) - (1/x))$  then the following inclusion holds:

$$\begin{aligned} & \supseteq \left[ \alpha, \rho, w I_{(1/\wp_2)^+, \rho, w}^{\sigma, k} \Theta^* g\left(\frac{1}{\wp_1}\right) \phi^\circ g\left(\frac{1}{\wp_1}\right) + \alpha, \rho, w I_{(1/\wp_1)^-, \rho, w}^{\sigma, k} \Theta^* g\left(\frac{1}{\wp_2}\right) \phi^\circ g\left(\frac{1}{\wp_2}\right) \right] \Theta\left(\frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4}\right) \\ & \quad \cdot \left[ \alpha, \rho, w I_{(1/\wp_2)^+, \rho, w}^{\sigma, k} \phi^\circ g\left(\frac{1}{\wp_1}\right) + \alpha, \rho, w I_{(1/\wp_1)^-, \rho, w}^{\sigma, k} \phi^\circ g\left(\frac{1}{\wp_2}\right) \right] \\ & \supseteq \frac{\Theta(\wp_1) + \Theta(\wp_2)}{2} \left[ \alpha, \rho, w I_{(1/\wp_2)^+, \rho, w}^{\sigma, k} \phi^\circ g\left(\frac{1}{\wp_1}\right) + \alpha, \rho, w I_{(1/\wp_1)^-, \rho, w}^{\sigma, k} \phi^\circ g\left(\frac{1}{\wp_2}\right) \right]. \end{aligned} \tag{53}$$

*Proof.* The proof is left for interested readers. Firstly, we establish fractional Hermite-Hadamard inequality.  $\square$

**Theorem 13.** Suppose that  $\Theta: [\wp_1, \wp_2] \times [\wp_3, \wp_4] \rightarrow \mathbb{R}_I^+$  is a given interval-valued harmonically co-ordinated convex function defined over a rectangle  $[\wp_1, \wp_2] \times [\wp_3, \wp_4]$  with  $0 \leq \wp_1 < \wp_2, 0 \leq \wp_3 < \wp_4$ , and  $\Theta(x, y) = [\Theta_*(x, y), \Theta^*(x, y)]$ , then the following inclusion holds:

$$\begin{aligned} & \Theta\left(\frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4}\right) \supseteq \frac{1}{4k^2 \mathcal{R}_{\rho_1, w_1, \alpha_1 + k}^{\sigma_1, k} [w_1 (\wp_2 - \wp_1/\wp_1\wp_2)^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2 + k}^{\sigma_2, k} [w_2 (\wp_4 - \wp_3/\wp_3\wp_4)^{\rho_2}]} \left(\frac{\wp_1\wp_2}{\wp_2 - \wp_1}\right)^{\alpha_1/k} \left(\frac{\wp_3\wp_4}{\wp_4 - \wp_3}\right)^{\alpha_2/k} \\ & \quad \times \left[ \alpha, \rho, w I_{(1/\wp_2)^+, (1/\wp_4)^+, \rho, w}^{\sigma, k} \Theta^* g\left(\frac{1}{\wp_1}, \frac{1}{\wp_3}\right) + \alpha, \rho, w I_{(1/\wp_2)^+, (1/\wp_3)^-, \rho, w}^{\sigma, k} \Theta^* g\left(\frac{1}{\wp_1}, \frac{1}{\wp_4}\right) \right. \\ & \quad \left. + \alpha, \rho, w I_{(1/\wp_1)^-, (1/\wp_4)^+, \rho, w}^{\sigma, k} \Theta^* g\left(\frac{1}{\wp_2}, \frac{1}{\wp_3}\right) + \alpha, \rho, w I_{(1/\wp_1)^-, (1/\wp_3)^-, \rho, w}^{\sigma, k} \Theta^* g\left(\frac{1}{\wp_2}, \frac{1}{\wp_4}\right) \right] \\ & \supseteq \frac{\Theta(\wp_1, \wp_3) + \Theta(\wp_1, \wp_4) + \Theta(\wp_2, \wp_3) + \Theta(\wp_2, \wp_4)}{4}. \end{aligned} \tag{54}$$

If  $\Theta(x, y)$  is an interval-valued harmonically co-ordinated concave function then we have the following equation:

$$\begin{aligned} \Theta\left(\frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4}\right) &\subseteq \frac{1}{4k^2 \mathcal{R}_{\rho_1, w_1, \alpha_1+k}^{\sigma_1, k} [w_1(\wp_2 - \wp_1/\wp_1\wp_2)^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2+k}^{\sigma_2, k} [w_2(\wp_4 - \wp_3/\wp_3\wp_4)^{\rho_2}]} \left(\frac{\wp_1\wp_2}{\wp_2 - \wp_1}\right)^{\alpha_1/k} \left(\frac{\wp_3\wp_4}{\wp_4 - \wp_3}\right)^{\alpha_2/k} \\ &\times \left[ \alpha, \rho, w I_{(1/\wp_2)^+, (1/\wp_4)^+}^{\sigma, k} \Theta^\circ g\left(\frac{1}{\wp_1}, \frac{1}{\wp_3}\right) + \alpha, \rho, w I_{(1/\wp_2)^+, (1/\wp_3)^-}^{\sigma, k} \Theta^\circ g\left(\frac{1}{\wp_1}, \frac{1}{\wp_4}\right) \right. \\ &\left. + \alpha, \rho, w I_{(1/\wp_1)^-, (1/\wp_4)^+}^{\sigma, k} \Theta^\circ g\left(\frac{1}{\wp_2}, \frac{1}{\wp_3}\right) + \alpha, \rho, w I_{(1/\wp_1)^-, (1/\wp_3)^-}^{\sigma, k} \Theta^\circ g\left(\frac{1}{\wp_2}, \frac{1}{\wp_4}\right) \right] \\ &\subseteq \frac{\Theta(\wp_1, \wp_3) + \Theta(\wp_1, \wp_4) + \Theta(\wp_2, \wp_3) + \Theta(\wp_2, \wp_4)}{4}. \end{aligned} \tag{55}$$

*Proof.* From the definition of a harmonically co-ordinated convex function and according to a hypothesis, we have the following equation:

$$\begin{aligned} \Theta^*\left(\frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4}\right) &\leq \frac{1}{4} \left[ \Theta^*\left(\frac{\wp_1\wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3\wp_4}{s\wp_3 + (1-s)\wp_4}\right) \right. \\ &+ \Theta^*\left(\frac{\wp_1\wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + s d}\right) \Theta^*\left(\frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{s\wp_3 + (1-s)\wp_4}\right) \\ &\left. + \Theta^*\left(\frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + s d}\right) \right], \end{aligned} \tag{56}$$

$$\begin{aligned} \Theta^*\left(\frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4}\right) &\geq \frac{1}{4} \left[ \Theta^*\left(\frac{\wp_1\wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3\wp_4}{s\wp_3 + (1-s)\wp_4}\right) \right. \\ &+ \Theta^*\left(\frac{\wp_1\wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + s d}\right) \Theta^*\left(\frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{s\wp_3 + (1-s)\wp_4}\right) \\ &\left. + \Theta^*\left(\frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + s d}\right) \right]. \end{aligned} \tag{57}$$

Multiplying both sides of (56) by  $t^{(\alpha_1/k)-1} s^{(\alpha_2/k)-1} \mathcal{R}_{\rho_1, w_1, \alpha_1}^{\sigma_1, k} [w_1(\wp_2 - \wp_1/\wp_1\wp_2)^{\rho_1} t^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2}^{\sigma_2, k} [w_2(\wp_4 - \wp_3/\wp_3\wp_4)^{\rho_2} s^{\rho_2}]$  and integration with respect to  $(t, s)$  in  $[0, 1] \times [0, 1]$ , then we have the following equation:

$$\begin{aligned}
 & \Theta_* \left( \frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4} \right) \int_0^1 \int_0^1 t^{(\alpha_1/k)-1} s^{(\alpha_2/k)-1} \mathcal{R}_{\rho_1, \omega_1, \alpha_1}^{\sigma_1, k} \left[ w_1 \left( \frac{\wp_2 - \wp_1}{\wp_1\wp_2} \right)^{\rho_1} t^{\rho_1} \right] \mathcal{R}_{\rho_2, \omega_2, \alpha_2}^{\sigma_2, k} \left[ w_2 \left( \frac{\wp_4 - \wp_3}{\wp_3\wp_4} \right)^{\rho_2} s^{\rho_2} \right] ds dt \\
 & \leq \frac{1}{4} \int_0^1 \int_0^1 t^{(\alpha_1/k)-1} s^{(\alpha_2/k)-1} \mathcal{R}_{\rho_1, \omega_1, \alpha_1}^{\sigma_1, k} \left[ w_1 \left( \frac{\wp_2 - \wp_1}{\wp_1\wp_2} \right)^{\rho_1} t^{\rho_1} \right] \mathcal{R}_{\rho_2, \omega_2, \alpha_2}^{\sigma_2, k} \left[ w_2 \left( \frac{\wp_4 - \wp_3}{\wp_3\wp_4} \right)^{\rho_2} s^{\rho_2} \right] \\
 & \quad \cdot \left[ \Theta_* \left( \frac{\wp_1\wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3\wp_4}{s\wp_3 + (1-s)\wp_4} \right) + \Theta_* \left( \frac{\wp_1\wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + s\wp_4} \right) \Theta_* \left( \frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{s\wp_3 + (1-s)\wp_4} \right) \right. \\
 & \quad \left. + \Theta_* \left( \frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + s\wp_4} \right) \right] ds dt.
 \end{aligned} \tag{58}$$

After simple computation, we have the following equation:

$$\begin{aligned}
 \Theta_* \left( \frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4} \right) & \leq \frac{1}{4k^2 \mathcal{R}_{\rho_1, \omega_1, \alpha_1+k}^{\sigma_1, k} [w_1 (\wp_2 - \wp_1 / \wp_1\wp_2)^{\rho_1}] \mathcal{R}_{\rho_2, \omega_2, \alpha_2+k}^{\sigma_2, k} [w_2 (\wp_4 - \wp_3 / \wp_3\wp_4)^{\rho_2}]} \left( \frac{\wp_1\wp_2}{\wp_2 - \wp_1} \right)^{\alpha_1/k} \left( \frac{\wp_3\wp_4}{\wp_4 - \wp_3} \right)^{\alpha_2/k} \\
 & \times \left[ {}_{\alpha, \rho, w} I_{(1/\wp_2)^+, (1/\wp_4)^+}^{\sigma, k} \Theta_* \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_3} \right) + {}_{\alpha, \rho, w} I_{(1/\wp_2)^+, (1/\wp_3)^-}^{\sigma, k} \Theta_* \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_4} \right) \right. \\
 & \left. + {}_{\alpha, \rho, w} I_{(1/\wp_1)^-, (1/\wp_4)^+}^{\sigma, k} \Theta_* \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_3} \right) + {}_{\alpha, \rho, w} I_{(1/\wp_1)^-, (1/\wp_3)^-}^{\sigma, k} \Theta_* \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_4} \right) \right].
 \end{aligned} \tag{59}$$

From (57), we have the following equation:

$$\begin{aligned}
 \Theta_* \left( \frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4} \right) & \geq \frac{1}{4k^2 \mathcal{R}_{\rho_1, \omega_1, \alpha_1+k}^{\sigma_1, k} [w_1 (\wp_2 - \wp_1 / \wp_1\wp_2)^{\rho_1}] \mathcal{R}_{\rho_2, \omega_2, \alpha_2+k}^{\sigma_2, k} [w_2 (\wp_4 - \wp_3 / \wp_3\wp_4)^{\rho_2}]} \left( \frac{\wp_1\wp_2}{\wp_2 - \wp_1} \right)^{\alpha_1/k} \left( \frac{\wp_3\wp_4}{\wp_4 - \wp_3} \right)^{\alpha_2/k} \\
 & \times \left[ {}_{\alpha, \rho, w} I_{(1/\wp_2)^+, (1/\wp_4)^+}^{\sigma, k} \Theta_* \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_3} \right) + {}_{\alpha, \rho, w} I_{(1/\wp_2)^+, (1/\wp_3)^-}^{\sigma, k} \Theta_* \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_4} \right) \right. \\
 & \left. + {}_{\alpha, \rho, w} I_{(1/\wp_1)^-, (1/\wp_4)^+}^{\sigma, k} \Theta_* \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_3} \right) + {}_{\alpha, \rho, w} I_{(1/\wp_1)^-, (1/\wp_3)^-}^{\sigma, k} \Theta_* \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_4} \right) \right].
 \end{aligned} \tag{60}$$

Combination of (71) and (60), we obtain the first inclusion of (54).

To prove our second inclusion, we use a definition of harmonically co-ordinated convexity.

$$\Theta_* \left( \frac{\wp_1 \wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3 \wp_4}{(1-s)\wp_3 + s d} \right) \leq tsF_*(\wp_1, \wp_3) + t(1-s)\Theta_*(\wp_1, \wp_4) + (1-t)sF_*(\wp_2, \wp_3) + (1-t)(1-s)\Theta_*(\wp_2, \wp_4), \tag{61}$$

$$\Theta_* \left( \frac{\wp_1 \wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3 \wp_4}{s\wp_3 + (1-s)\wp_4} \right) \leq t(1-s)\Theta_*(\wp_1, \wp_3) + tsF_*(\wp_1, \wp_4) + (1-t)(1-s)\Theta_*(\wp_2, \wp_3) + (1-t)sF_*(\wp_2, \wp_4), \tag{62}$$

$$\Theta_* \left( \frac{\wp_1 \wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3 \wp_4}{(1-s)\wp_3 + s d} \right), \tag{63}$$

$$\Theta_* \left( \frac{\wp_1 \wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3 \wp_4}{(1-s)\wp_3 + s d} \right),$$

$$\Theta_* \left( \frac{\wp_1 \wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3 \wp_4}{s\wp_3 + (1-s)\wp_4} \right) \leq (1-t)(1-s)\Theta_*(\wp_1, \wp_3) + (1-t)sF_*(\wp_1, \wp_4) + t(1-s)\Theta_*(\wp_2, \wp_3) + tsF_*(\wp_2, \wp_4). \tag{64}$$

Adding (61)–(63) and (64), we have the following equation:

$$\Theta_* \left( \frac{\wp_1 \wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3 \wp_4}{(1-s)\wp_3 + s d} \right) + \Theta_* \left( \frac{\wp_1 \wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3 \wp_4}{s\wp_3 + (1-s)\wp_4} \right) + \Theta_* \left( \frac{\wp_1 \wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3 \wp_4}{(1-s)\wp_3 + s d} \right) + \Theta_* \left( \frac{\wp_1 \wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3 \wp_4}{s\wp_3 + (1-s)\wp_4} \right) \leq \Theta_*(\wp_1, \wp_3) + \Theta_*(\wp_1, \wp_4) + \Theta_*(\wp_2, \wp_3) + \Theta_*(\wp_2, \wp_4). \tag{65}$$

Multiplying both sides of (65) by  $t^{(\alpha_1/k)-1} s^{(\alpha_2/k)-1} \mathcal{R}_{\rho_1, w_1, \alpha_1}^{\sigma_1, k} [w_1 (\wp_2 - \wp_1 / \wp_1 \wp_2)^{\rho_1} t^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2}^{\sigma_2, k} [w_2 (\wp_4 - \wp_3 / \wp_3 \wp_4)^{\rho_2}]$  and integration with respect to  $(t, s)$  in  $[0, 1] \times [0, 1]$ , then we have the following equation:

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$$4k^2 \mathcal{R}_{\rho_1, w_1, \alpha_1 + k}^{\sigma_1, k} [w_1 (\wp_2 - \wp_1 / \wp_1 \wp_2)^{\rho_1}] \text{starts - with } (., ' \text{ hskip } ') \text{ hskip } ' \text{ substring - after (preceding - sibling :: comment } () \text{ [starts - with } ( \tag{66}$$

Using the harmonically co-ordinated concavity and similar argument as (66), we obtain the following inequality.

$$\begin{aligned} & \frac{1}{4k^2 \mathcal{R}_{\rho_1, w_1, \alpha_1+k}^{\sigma_1, k} [w_1 (\wp_2 - \wp_1 / \wp_1 \wp_2)^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2+k}^{\sigma_2, k} [w_2 (\wp_4 - \wp_3 / \wp_3 \wp_4)^{\rho_2}]} \left( \frac{\wp_1 \wp_2}{\wp_2 - \wp_1} \right)^{\alpha_1/k} \left( \frac{\wp_3 \wp_4}{\wp_4 - \wp_3} \right)^{\alpha_2/k} \\ & \times \left[ \alpha, \rho, w I_{(1/\wp_2)^+, (1/\wp_4)^+}^{\sigma, k} \ominus^* \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_3} \right) + \alpha, \rho, w I_{(1/\wp_2)^+, (1/\wp_3)^-}^{\sigma, k} \ominus^* \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_4} \right) + \alpha, \rho, w I_{(1/\wp_1)^-, (1/\wp_4)^+}^{\sigma, k} \ominus^* \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_3} \right) \right. \\ & \left. + \alpha, \rho, w I_{(1/\wp_1)^-, (1/\wp_3)^-}^{\sigma, k} \ominus^* \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_4} \right) \right] \\ & \geq \frac{\Theta^*(\wp_1, \wp_3) + \Theta^*(\wp_1, \wp_4) + \Theta^*(\wp_2, \wp_3) + \Theta^*(\wp_2, \wp_4)}{4}. \end{aligned} \tag{67}$$

Combining (66) and (67), we obtain the second part of our required result. This completes the proof.

Now, we develop the new midpoint Hermite-Hadamard's inequality in the sense of interval-valued harmonically co-ordinated convexity.  $\square$

**Theorem 14.** Suppose that  $\Theta: [\wp_1, \wp_2] \times [\wp_3, \wp_4] \rightarrow \mathbb{R}_I^+$  is a given interval-valued harmonically co-ordinated convex function defined over a rectangle  $[\wp_1, \wp_2] \times [\wp_3, \wp_4]$  with  $0 \leq \wp_1 < \wp_2, 0 \leq \wp_3 < \wp_4$  and  $\Theta(x, y) = [\Theta_*(x, y), \Theta^*(x, y)]$ , then the following inclusion holds:

$$\begin{aligned} \Theta \left( \frac{2\wp_1 \wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3 \wp_4}{\wp_3 + \wp_4} \right) & \supseteq \frac{1}{4k^2 \mathcal{R}_{\rho_1, w_1, \alpha_1+k}^{\sigma_1, k} [w_1 (\wp_2 - \wp_1 / 2\wp_1 \wp_2)^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2+k}^{\sigma_2, k} [w_2 (\wp_4 - \wp_3 / 2\wp_3 \wp_4)^{\rho_2}]} \left( \frac{2\wp_1 \wp_2}{\wp_2 - \wp_1} \right)^{\alpha_1/k} \left( \frac{2\wp_3 \wp_4}{\wp_4 - \wp_3} \right)^{\alpha_2/k} \\ & \times \left[ \alpha, \rho, w I_{(\wp_1+\wp_2/2\wp_1\wp_2)^+, (\wp_3+\wp_4/2\wp_3\wp_4)^+}^{\sigma, k} \ominus \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_3} \right) + \alpha, \rho, w I_{(\wp_1+\wp_2/2\wp_1\wp_2)^+, (\wp_3+\wp_4/2\wp_3\wp_4)^-}^{\sigma, k} \ominus \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_4} \right) \right. \\ & \left. + \alpha, \rho, w I_{(\wp_1+\wp_2/2\wp_1\wp_2)^-, (\wp_3+\wp_4/2\wp_3\wp_4)^+}^{\sigma, k} \ominus \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_3} \right) + \alpha, \rho, w I_{(\wp_1+\wp_2/2\wp_1\wp_2)^-, (\wp_3+\wp_4/2\wp_3\wp_4)^-}^{\sigma, k} \ominus \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_4} \right) \right] \\ & \supseteq \frac{\Theta(\wp_1, \wp_3) + \Theta(\wp_1, \wp_4) + \Theta(\wp_2, \wp_3) + \Theta(\wp_2, \wp_4)}{4}. \end{aligned} \tag{68}$$

If  $\Theta(x, y)$  is interval-valued harmonically co-ordinated concave function, then we have the following equation:

$$\begin{aligned} \Theta \left( \frac{2\wp_1 \wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3 \wp_4}{\wp_3 + \wp_4} \right) & \subseteq \frac{1}{4k^2 \mathcal{R}_{\rho_1, w_1, \alpha_1+k}^{\sigma_1, k} [w_1 (\wp_2 - \wp_1 / 2\wp_1 \wp_2)^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2+k}^{\sigma_2, k} [w_2 (\wp_4 - \wp_3 / 2\wp_3 \wp_4)^{\rho_2}]} \left( \frac{2\wp_1 \wp_2}{\wp_2 - \wp_1} \right)^{\alpha_1/k} \left( \frac{2\wp_3 \wp_4}{\wp_4 - \wp_3} \right)^{\alpha_2/k} \\ & \times \left[ \alpha, \rho, w I_{(\wp_1+\wp_2/2\wp_1\wp_2)^+, (\wp_3+\wp_4/2\wp_3\wp_4)^+}^{\sigma, k} \ominus \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_3} \right) + \alpha, \rho, w I_{(\wp_1+\wp_2/2\wp_1\wp_2)^+, (\wp_3+\wp_4/2\wp_3\wp_4)^-}^{\sigma, k} \ominus \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_4} \right) \right. \\ & \left. + \alpha, \rho, w I_{(\wp_1+\wp_2/2\wp_1\wp_2)^-, (\wp_3+\wp_4/2\wp_3\wp_4)^+}^{\sigma, k} \ominus \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_3} \right) + \alpha, \rho, w I_{(\wp_1+\wp_2/2\wp_1\wp_2)^-, (\wp_3+\wp_4/2\wp_3\wp_4)^-}^{\sigma, k} \ominus \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_4} \right) \right] \\ & \subseteq \frac{\Theta(\wp_1, \wp_3) + \Theta(\wp_1, \wp_4) + \Theta(\wp_2, \wp_3) + \Theta(\wp_2, \wp_4)}{4}. \end{aligned} \tag{69}$$

*Proof.* From the definition of a harmonically co-ordinated convex function, we have the following equation:

$$\begin{aligned} \Theta_* \left( \frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4} \right) &\leq \frac{1}{4} \left[ \Theta_* \left( \frac{2\wp_1\wp_2}{t\wp_1 + (2-t)\wp_2}, \frac{2\wp_3\wp_4}{s\wp_3 + (2-s)\wp_4} \right) \right. \\ &+ \Theta_* \left( \frac{2\wp_1\wp_2}{t\wp_1 + (2-t)\wp_2}, \frac{2\wp_3\wp_4}{(2-s)\wp_3 + s\wp_4} \right) \Theta_* \left( \frac{2\wp_1\wp_2}{(2-t)\wp_1 + t\wp_2}, \frac{2\wp_3\wp_4}{s\wp_3 + (2-s)\wp_4} \right) \\ &\left. + \Theta_* \left( \frac{2\wp_1\wp_2}{(2-t)\wp_1 + t\wp_2}, \frac{2\wp_3\wp_4}{(2-s)\wp_3 + s\wp_4} \right) \right]. \end{aligned} \tag{70}$$

Multiplying both sides of (70) by  $t^{(\alpha_1/k)-1} s^{\alpha_2/k-1} \mathcal{R}_{\rho_1, w_1, \alpha_1}^{\sigma_1, k} [w_1 (\wp_2 - \wp_1 / 2\wp_1\wp_2)^{\rho_1} t^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2}^{\sigma_2, k} [w_2 (\wp_4 - \wp_3 / 2\wp_3\wp_4)^{\rho_2} s^{\rho_2}]$  and integration with respect to  $(t, s)$  in  $[0, 1] \times [0, 1]$ , then we have the following equation:

$$\begin{aligned} &\Theta_* \left( \frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4} \right) \int_0^1 \int_0^1 t^{\alpha_1/k-1} s^{\alpha_2/k-1} \mathcal{R}_{\rho_1, w_1, \alpha_1}^{\sigma_1, k} \left[ w_1 \left( \frac{\wp_2 - \wp_1}{2\wp_1\wp_2} \right)^{\rho_1} t^{\rho_1} \right] \mathcal{R}_{\rho_2, w_2, \alpha_2}^{\sigma_2, k} \left[ w_2 \left( \frac{\wp_4 - \wp_3}{2\wp_3\wp_4} \right)^{\rho_2} s^{\rho_2} \right] ds dt \\ &\leq \frac{1}{4} \int_0^1 \int_0^1 t^{(\alpha_1/k)-1} s^{(\alpha_2/k)-1} \Theta_{\rho_1, w_1, \alpha_1}^{\sigma_1, k} \left[ w_1 \left( \frac{\wp_2 - \wp_1}{2\wp_1\wp_2} \right)^{\rho_1} t^{\rho_1} \right] \mathcal{R}_{\rho_2, w_2, \alpha_2}^{\sigma_2, k} \left[ w_2 \left( \frac{\wp_4 - \wp_3}{2\wp_3\wp_4} \right)^{\rho_2} s^{\rho_2} \right] \\ &\cdot \left[ \Theta_* \left( \frac{2\wp_1\wp_2}{t\wp_1 + (2-t)\wp_2}, \frac{2\wp_3\wp_4}{s\wp_3 + (2-s)\wp_4} \right) + \Theta_* \left( \frac{2\wp_1\wp_2}{t\wp_1 + (2-t)\wp_2}, \frac{2\wp_3\wp_4}{(2-s)\wp_3 + s\wp_4} \right) \Theta_* \left( \frac{2\wp_1\wp_2}{(2-t)\wp_1 + t\wp_2}, \frac{2\wp_3\wp_4}{s\wp_3 + (2-s)\wp_4} \right) \right. \\ &\left. + \Theta_* \left( \frac{2\wp_1\wp_2}{(2-t)\wp_1 + t\wp_2}, \frac{2\wp_3\wp_4}{(2-s)\wp_3 + s\wp_4} \right) \right] ds dt. \end{aligned} \tag{71}$$

After simple computation, we have the following equation:

$$\begin{aligned} \Theta_* \left( \frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4} \right) &\leq \frac{1}{4k^2 \mathcal{R}_{\rho_1, w_1, \alpha_1+k}^{\sigma_1, k} [w_1 (\wp_2 - \wp_1 / 2\wp_1\wp_2)^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2+k}^{\sigma_2, k} [w_2 (\wp_4 - \wp_3 / 2\wp_3\wp_4)^{\rho_2}]} \left( \frac{2\wp_1\wp_2}{\wp_2 - \wp_1} \right)^{\alpha_1/k} \\ &\cdot \left( \frac{2\wp_3\wp_4}{\wp_4 - \wp_3} \right)^{\alpha_2/k} \times \left[ {}_{\alpha, \rho, w} I_{(\wp_1 + \wp_2 / 2\wp_1\wp_2)^+, (\wp_3 + \wp_4 / 2\wp_3\wp_4)^+}^{\sigma, k} \Theta_* \circ g \left( \frac{\wp_1 + \wp_2}{2\wp_1\wp_2}, \frac{\wp_3 + \wp_4}{2\wp_3\wp_4} \right) \right. \\ &+ {}_{\alpha, \rho, w} I_{(\wp_1 + \wp_2 / 2\wp_1\wp_2)^+, (\wp_3 + \wp_4 / 2\wp_3\wp_4)^-}^{\sigma, k} \Theta_* \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_4} \right) + {}_{\alpha, \rho, w} I_{(\wp_1 + \wp_2 / 2\wp_1\wp_2)^-, (\wp_3 + \wp_4 / 2\wp_3\wp_4)^+}^{\sigma, k} \Theta_* \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_3} \right) \\ &\left. + {}_{\alpha, \rho, w} I_{(\wp_1 + \wp_2 / 2\wp_1\wp_2)^-, (\wp_3 + \wp_4 / 2\wp_3\wp_4)^-}^{\sigma, k} \Theta_* \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_4} \right) \right]. \end{aligned} \tag{72}$$

Similarly, we have the following equation:

$$\begin{aligned} \Theta^* \left( \frac{2\rho_1\rho_2}{\rho_1 + \rho_2}, \frac{2\rho_3\rho_4}{\rho_3 + \rho_4} \right) &\geq \frac{1}{4k^2 \mathcal{R}_{\rho_1, w_1, \alpha_1+k}^{\sigma_1, k} [w_1 (\rho_2 - \rho_1/2\rho_1\rho_2)^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2+k}^{\sigma_2, k} [w_2 (\rho_4 - \rho_3/2\rho_3\rho_4)^{\rho_2}]} \left( \frac{2\rho_1\rho_2}{\rho_2 - \rho_1} \right)^{\alpha_1/k} \left( \frac{2\rho_3\rho_4}{\rho_4 - \rho_3} \right)^{\alpha_2/k} \\ &\times \left[ {}_{\alpha, \rho, w} I^{\sigma, k}_{(\rho_1+\rho_2/2\rho_1\rho_2)^+, (\rho_3+\rho_4/2\rho_3\rho_4)^+} \Theta^* \circ g \left( \frac{\rho_1 + \rho_2}{2\rho_1\rho_2}, \frac{\rho_3 + \rho_4}{2\rho_3\rho_4} \right) + {}_{\alpha, \rho, w} I^{\sigma, k}_{(\rho_1+\rho_2/2\rho_1\rho_2)^+, (\rho_3+\rho_4/2\rho_3\rho_4)^-} \Theta^* \right. \\ &\circ g \left( \frac{1}{\rho_1}, \frac{1}{\rho_4} \right) + {}_{\alpha, \rho, w} I^{\sigma, k}_{(\rho_1+\rho_2/2\rho_1\rho_2)^-, (\rho_3+\rho_4/2\rho_3\rho_4)^+} \Theta^* \circ g \left( \frac{1}{\rho_2}, \frac{1}{\rho_3} \right) \\ &\left. + {}_{\alpha, \rho, w} I^{\sigma, k}_{(\rho_1+\rho_2/2\rho_1\rho_2)^-, (\rho_3+\rho_4/2\rho_3\rho_4)^-} \Theta^* \circ g \left( \frac{1}{\rho_2}, \frac{1}{\rho_4} \right) \right]. \end{aligned}$$

By the combination of (72) and (73), we obtain first inclusion of (68).

To prove our second inclusion, we use a definition of harmonically co-ordinated convexity.

$$\Theta^* \left( \frac{2\rho_1\rho_2}{(2-t)\rho_1 + tb}, \frac{2\rho_3\rho_4}{(2-s)\rho_3 + sd} \right) \tag{74}$$

$$\leq \frac{1}{4} [ts\Theta_*(\rho_1, \rho_3) + t(2-s)\Theta_*(\rho_1, \rho_4) + (2-t)sF_*(\rho_2, \rho_3) + (2-t)(2-s)\Theta_*(\rho_2, \rho_4)],$$

$$\Theta^* \left( \frac{2\rho_1\rho_2}{(2-t)\rho_1 + tb}, \frac{2\rho_3\rho_4}{s\rho_3 + (2-s)\rho_4} \right) \tag{75}$$

$$\leq \frac{1}{4} [t(2-s)\Theta_*(\rho_1, \rho_3) + tsF_*(\rho_1, \rho_4) + (2-t)(2-s)\Theta_*(\rho_2, \rho_3) + (2-t)sF_*(\rho_2, \rho_4)],$$

$$\Theta^* \left( \frac{2\rho_1\rho_2}{t\rho_1 + (2-t)\rho_2}, \frac{2\rho_3\rho_4}{(2-s)\rho_3 + sd} \right) \tag{76}$$

$$\leq \frac{1}{4} [(2-t)sF_*(\rho_1, \rho_3) + (2-t)(2-s)\Theta_*(\rho_1, \rho_4) + tsF_*(\rho_2, \rho_3) + t(2-s)\Theta_*(\rho_2, \rho_4)],$$

$$\Theta^* \left( \frac{2\rho_1\rho_2}{t\rho_1 + (2-t)\rho_2}, \frac{2\rho_3\rho_4}{s\rho_3 + (2-s)\rho_4} \right) \tag{77}$$

$$\leq \frac{1}{4} [(2-t)(2-s)\Theta_*(\rho_1, \rho_3) + (2-t)sF_*(\rho_1, \rho_4) + t(2-s)\Theta_*(\rho_2, \rho_3) + tsF_*(\rho_2, \rho_4)].$$

By adding (74)–(76) and (77), we have the following equation:

$$\begin{aligned} &\Theta^* \left( \frac{2\rho_1\rho_2}{(2-t)\rho_1 + tb}, \frac{2\rho_3\rho_4}{(2-s)\rho_3 + sd} \right) + \Theta^* \left( \frac{2\rho_1\rho_2}{(2-t)\rho_1 + tb}, \frac{\rho_3\rho_4}{s\rho_3 + (2-s)\rho_4} \right) \\ &+ \Theta^* \left( \frac{2\rho_1\rho_2}{t\rho_1 + (2-t)\rho_2}, \frac{2\rho_3\rho_4}{(2-s)\rho_3 + sd} \right) + \Theta^* \left( \frac{2\rho_1\rho_2}{t\rho_1 + (2-t)\rho_2}, \frac{2\rho_3\rho_4}{s\rho_3 + (2-s)\rho_4} \right) \\ &\leq \Theta_*(\rho_1, \rho_3) + \Theta_*(\rho_1, \rho_4) + \Theta_*(\rho_2, \rho_3) + \Theta_*(\rho_2, \rho_4). \end{aligned} \tag{78}$$

Multiplying both sides of (78) by  $t^{\alpha_1/k-1} s^{\alpha_2/k-1} \wp_3 \wp_4^{\rho_2} s^{\rho_2}$  and integration with respect to  $(t, s)$  in  $[0, 1] \times [0, 1]$ , then we have the following equation:

$$\begin{aligned} & \frac{1}{4k^2 \mathcal{R}_{\rho_1, \omega_1, \alpha_1+k}^{\sigma_1, k} [w_1 (\wp_2 - \wp_1 / \wp_1 \wp_2)^{\rho_1} t^{\rho_1}] \mathcal{R}_{\rho_2, \omega_2, \alpha_2+k}^{\sigma_2, k} [w_2 (\wp_4 - \wp_3 / 2\wp_3 \wp_4)^{\rho_2} s^{\rho_2}]} \left( \frac{2\wp_1 \wp_2}{\wp_2 - \wp_1} \right)^{\alpha_1/k} \left( \frac{2\wp_3 \wp_4}{\wp_4 - \wp_3} \right)^{\alpha_2/k} \\ & \times \left[ \alpha_{\rho, w} I_{(\wp_1 + \wp_2 / 2\wp_1 \wp_2)^+, (\wp_3 + \wp_4 / 2\wp_3 \wp_4)^+}^{\sigma, k} \circ g \left( \frac{\wp_1 + \wp_2}{2\wp_1 \wp_2}, \frac{\wp_3 + \wp_4}{2\wp_3 \wp_4} \right) + \alpha_{\rho, w} I_{(\wp_1 + \wp_2 / 2\wp_1 \wp_2)^+, (\wp_3 + \wp_4 / 2\wp_3 \wp_4)^-}^{\sigma, k} \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_4} \right) \right. \\ & \left. + \alpha_{\rho, w} I_{(\wp_1 + \wp_2 / 2\wp_1 \wp_2)^-, (\wp_3 + \wp_4 / 2\wp_3 \wp_4)^+}^{\sigma, k} \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_3} \right) + \alpha_{\rho, w} I_{(\wp_1 + \wp_2 / 2\wp_1 \wp_2)^-, (\wp_3 + \wp_4 / 2\wp_3 \wp_4)^-}^{\sigma, k} \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_4} \right) \right] \\ & \leq \frac{\Theta_*(\wp_1, \wp_3) + \Theta_*(\wp_1, \wp_4) + \Theta_*(\wp_2, \wp_3) + \Theta_*(\wp_2, \wp_4)}{4}. \end{aligned} \tag{79}$$

Using the harmonically co-ordinated concavity and similar argument as (79), we obtain the following inequality:

$$\begin{aligned} & \frac{1}{4k^2 \mathcal{R}_{\rho_1, \omega_1, \alpha_1+k}^{\sigma_1, k} [w_1 (\wp_2 - \wp_1 / 2\wp_1 \wp_2)^{\rho_1}] \mathcal{R}_{\rho_2, \omega_2, \alpha_2+k}^{\sigma_2, k} [w_2 (\wp_4 - \wp_3 / 2\wp_3 \wp_4)^{\rho_2}]} \left( \frac{2\wp_1 \wp_2}{\wp_2 - \wp_1} \right)^{\alpha_1/k} \left( \frac{2\wp_3 \wp_4}{\wp_4 - \wp_3} \right)^{\alpha_2/k} \\ & \times \left[ \alpha_{\rho, w} I_{(\wp_1 + \wp_2 / 2\wp_1 \wp_2)^+, (\wp_3 + \wp_4 / 2\wp_3 \wp_4)^+}^{\sigma, k} \circ g \left( \frac{\wp_1 + \wp_2}{2\wp_1 \wp_2}, \frac{\wp_3 + \wp_4}{2\wp_3 \wp_4} \right) + \alpha_{\rho, w} I_{(\wp_1 + \wp_2 / 2\wp_1 \wp_2)^+, (\wp_3 + \wp_4 / 2\wp_3 \wp_4)^-}^{\sigma, k} \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_4} \right) \right. \\ & \left. + \alpha_{\rho, w} I_{(\wp_1 + \wp_2 / 2\wp_1 \wp_2)^-, (\wp_3 + \wp_4 / 2\wp_3 \wp_4)^+}^{\sigma, k} \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_3} \right) + \alpha_{\rho, w} I_{(\wp_1 + \wp_2 / 2\wp_1 \wp_2)^-, (\wp_3 + \wp_4 / 2\wp_3 \wp_4)^-}^{\sigma, k} \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_4} \right) \right] \\ & \geq \frac{\Theta^*(\wp_1, \wp_3) + \Theta^*(\wp_1, \wp_4) + \Theta^*(\wp_2, \wp_3) + \Theta^*(\wp_2, \wp_4)}{4}. \end{aligned} \tag{80}$$

Combining (79) and (80), we obtain the second part of our required result.  $\square$

$$\left( \frac{\wp_1 \wp_2}{\wp_2 - \wp_1} \right) \left( \frac{\wp_3 \wp_4}{\wp_4 - \wp_3} \right) \int_{\wp_1}^{\wp_2} \int_{\wp_3}^{\wp_4} \frac{\Theta(x, y)}{x^2 y^2} dy dx = [8, 51.92], \tag{82}$$

*Example 4.* We set  $\rho_1 = 0 = \rho_2$ ,  $\alpha_1 = 1 = \alpha_2$  and  $k = 1$  with  $\sigma(0) = 1$  in Theorem 13. If  $\Theta(x, y) = [2x^2 y^2, (-4x^2 + 12x + 6)(-2y^2 + 4y + 2)]$  and  $[\wp_1, \wp_2] = [\wp_3, \wp_4] = [1, 2]$ , then

$$\begin{aligned} \Theta \left( \frac{2\wp_1 \wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3 \wp_4}{\wp_3 + \wp_4} \right) &= [5.71, 56.77], \\ \frac{\Theta(\wp_1, \wp_3) + \Theta(\wp_1, \wp_4) + \Theta(\wp_2, \wp_3) + \Theta(\wp_2, \wp_4)}{4} &= [12.5, 42]. \end{aligned} \tag{81}$$

$$\tag{83}$$



(81), (82), and (83) imply that

$$[5.71, 56.77] \supseteq [8, 51.92] \supseteq [12.5, 42]. \tag{84}$$

which provides the verification of Theorem 13.

Next, we establish some Hermite-Hadamard-Fejér type inequality as well.

**Theorem 15.** Suppose that  $\Theta: [\wp_1, \wp_2] \times [\wp_3, \wp_4] \longrightarrow \mathbb{R}_I^+$  is a given interval-valued harmonically co-ordinated convex function defined over a rectangle  $[\wp_1, \wp_2] \times [\wp_3, \wp_4]$  with  $0 \leq \wp_1 < \wp_2, 0 \leq \wp_3 < \wp_4$  and  $\Theta(x, y) = [\Theta_*(x, y), \Theta^*(x, y)]$ . If  $\phi: [\wp_1, \wp_2] \times [\wp_3, \wp_4] \longrightarrow \mathbb{R}$  be a non-negative, integrable, and symmetric function of two variables:

$$\phi(x, y) = \begin{cases} \phi\left(\frac{1}{1/\wp_1 + 1/\wp_2 - 1/x}, y\right), \\ \phi\left(x, \frac{1}{(1/\wp_1) + (\wp_3/\wp_4) - (1/y)}\right), \\ \phi\left(\frac{1}{1/\wp_1 + 1/\wp_2 - 1/x}, \frac{1}{(1/\wp_1) + (\wp_3/\wp_4) - (1/y)}\right), \end{cases} \tag{85}$$

then the following inclusion holds:

$$\begin{aligned} & \Theta\left(\frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4}\right) \left[ {}_{\alpha, \rho, w} I_{(1/\wp_2)^+, (1/\wp_4)^+}^{\sigma, k} \phi \circ g\left(\frac{1}{\wp_1}, \frac{1}{\wp_3}\right) + {}_{\alpha, \rho, w} I_{(1/\wp_2)^+, (1/\wp_3)^-}^{\sigma, k} \phi \circ g\left(\frac{1}{\wp_1}, \frac{1}{\wp_4}\right) \right. \\ & \quad \left. + {}_{\alpha, \rho, w} I_{(1/\wp_1)^-, (1/\wp_4)^+}^{\sigma, k} \phi \circ g\left(\frac{1}{\wp_2}, \frac{1}{\wp_3}\right) + {}_{\alpha, \rho, w} I_{(1/\wp_1)^-, (1/\wp_3)^-}^{\sigma, k} \phi \circ g\left(\frac{1}{\wp_2}, \frac{1}{\wp_4}\right) \right] \\ & \supseteq \left[ {}_{\alpha, \rho, w} I_{(1/\wp_2)^+, (1/\wp_4)^+}^{\sigma, k} \Theta \circ g\left(\frac{1}{\wp_1}, \frac{1}{\wp_3}\right) \phi \circ g\left(\frac{1}{\wp_1}, \frac{1}{\wp_3}\right) + {}_{\alpha, \rho, w} I_{(1/\wp_2)^+, (1/\wp_3)^-}^{\sigma, k} \Theta \circ g\left(\frac{1}{\wp_1}, \frac{1}{\wp_4}\right) \phi \circ g\left(\frac{1}{\wp_1}, \frac{1}{\wp_4}\right) \right. \\ & \quad \left. + {}_{\alpha, \rho, w} I_{(1/\wp_1)^-, (1/\wp_4)^+}^{\sigma, k} \Theta \circ g\left(\frac{1}{\wp_2}, \frac{1}{\wp_3}\right) \phi \circ g\left(\frac{1}{\wp_2}, \frac{1}{\wp_3}\right) + {}_{\alpha, \rho, w} I_{(1/\wp_1)^-, (1/\wp_3)^-}^{\sigma, k} \Theta \circ g\left(\frac{1}{\wp_2}, \frac{1}{\wp_4}\right) \phi \circ g\left(\frac{1}{\wp_2}, \frac{1}{\wp_4}\right) \right] \\ & \supseteq \frac{\Theta(\wp_1, \wp_3) + \Theta(\wp_1, \wp_4) + \Theta(\wp_2, \wp_3) + \Theta(\wp_2, \wp_4)}{4} \left[ {}_{\alpha, \rho, w} I_{(1/\wp_2)^+, (1/\wp_4)^+}^{\sigma, k} \phi \circ g\left(\frac{1}{\wp_1}, \frac{1}{\wp_3}\right) + {}_{\alpha, \rho, w} I_{(1/\wp_2)^+, (1/\wp_3)^-}^{\sigma, k} \phi \circ g\left(\frac{1}{\wp_1}, \frac{1}{\wp_4}\right) \right. \\ & \quad \left. + {}_{\alpha, \rho, w} I_{(1/\wp_1)^-, (1/\wp_4)^+}^{\sigma, k} \phi \circ g\left(\frac{1}{\wp_2}, \frac{1}{\wp_3}\right) + {}_{\alpha, \rho, w} I_{(1/\wp_1)^-, (1/\wp_3)^-}^{\sigma, k} \phi \circ g\left(\frac{1}{\wp_2}, \frac{1}{\wp_4}\right) \right]. \end{aligned} \tag{86}$$

If  $\Theta(x, y)$  is a interval-valued harmonically co-ordinated concave, then the above-given relation holds in a reverse direction.

*Proof.* We start our proof from relation (56) and multiplying both sides of it by  $t^{\alpha_1/k-1} s^{\alpha_2/k-1} \mathcal{R}_{\rho_1, w_1, \alpha_1}^{\sigma_1, k} [w_1(\wp_2 - \wp_1/$

$\wp_1\wp_2)^{\rho_1} t^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2}^{\sigma_2, k} [w_2(\wp_4 - \wp_3/ \wp_3\wp_4)^{\rho_2} s^{\rho_2}] \phi(\wp_1\wp_2/(1-t) \wp_1 + tb, \wp_3\wp_4/(1-s)\wp_3 + sd)$  and taking integration with respect to  $(t, s)$  on  $[0, 1] \times [0, 1]$ , we have the following equation:

$$\begin{aligned}
 & 4\Theta_* \left( \frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4} \right) \int_0^1 \int_0^1 t^{(\alpha_1/k)-1} s^{(\alpha_2/k)-1} \mathcal{R}_{\rho_1, \omega_1, \alpha_1}^{\sigma_1, k} \left[ w_1 \left( \frac{\wp_2 - \wp_1}{\wp_1\wp_2} \right)^{\rho_1} t^{\rho_1} \right] \mathcal{R}_{\rho_2, \omega_2, \alpha_2}^{\sigma_2, k} \\
 & \cdot \left[ w_2 \left( \frac{\wp_4 - \wp_3}{\wp_3\wp_4} \right)^{\rho_2} s^{\rho_2} \right] \phi \left( \frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + s d} \right) ds dt \\
 & \leq \int_0^1 \int_0^1 t^{(\alpha_1/k)-1} s^{(\alpha_2/k)-1} \mathcal{R}_{\rho_1, \omega_1, \alpha_1}^{\sigma_1, k} \left[ w_1 \left( \frac{\wp_2 - \wp_1}{\wp_1\wp_2} \right)^{\rho_1} t^{\rho_1} \right] \mathcal{R}_{\rho_2, \omega_2, \alpha_2}^{\sigma_2, k} \left[ w_2 \left( \frac{\wp_4 - \wp_3}{\wp_3\wp_4} \right)^{\rho_2} s^{\rho_2} \right] \phi \left( \frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + s d} \right) \\
 & \times \left[ \Theta_* \left( \frac{\wp_1\wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3\wp_4}{s\wp_3 + (1-s)\wp_4} \right) + \Theta_* \left( \frac{\wp_1\wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + s d} \right) \Theta_* \left( \frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{s\wp_3 + (1-s)\wp_4} \right) \right. \\
 & \left. + \Theta_* \left( \frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + s d} \right) \right] ds dt.
 \end{aligned} \tag{87}$$

After simplification and making use of symmetry of  $\phi(x, y)$ , we obtain the following equation:

$$\begin{aligned}
 & \Theta_* \left( \frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4} \right) \left[ {}_{\alpha, \rho, w} I_{(1/\wp_2)^+, (1/\wp_4)^+}^{\sigma, k} \phi \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_3} \right) + {}_{\alpha, \rho, w} I_{(1/\wp_2)^+, (1/\wp_3)^-}^{\sigma, k} \phi \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_4} \right) \right. \\
 & \left. + {}_{\alpha, \rho, w} I_{(1/\wp_1)^-, (1/\wp_4)^+}^{\sigma, k} \phi \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_3} \right) + {}_{\alpha, \rho, w} I_{(1/\wp_1)^-, (1/\wp_3)^-}^{\sigma, k} \phi \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_4} \right) \right] \\
 & \leq \left[ {}_{\alpha, \rho, w} I_{(1/\wp_2)^+, (1/\wp_4)^+}^{\sigma, k} \Theta_* \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_3} \right) \phi \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_3} \right) + {}_{\alpha, \rho, w} I_{(1/\wp_2)^+, (1/\wp_3)^-}^{\sigma, k} \Theta_* \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_4} \right) \phi \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_4} \right) \right. \\
 & \left. + {}_{\alpha, \rho, w} I_{(1/\wp_1)^-, (1/\wp_4)^+}^{\sigma, k} \Theta_* \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_3} \right) \phi \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_3} \right) + {}_{\alpha, \rho, w} I_{(1/\wp_1)^-, (1/\wp_3)^-}^{\sigma, k} \Theta_* \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_4} \right) \phi \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_4} \right) \right].
 \end{aligned} \tag{88}$$

Multiplying both sides of (57) by  $t^{\alpha_1/k-1} s^{\alpha_2/k-1} \mathcal{R}_{\rho_1, \omega_1, \alpha_1}^{\sigma_1, k} [w_1 (\wp_2 - \wp_1 / \wp_1 \wp_2)^{\rho_1} t^{\rho_1}] \mathcal{R}_{\rho_2, \omega_2, \alpha_2}^{\sigma_2, k} [w_2 (\wp_4 - \wp_3 / \wp_3 \wp_4)^{\rho_2} s^{\rho_2}] \phi (\wp_1 \wp_2 / (1-t)\wp_1 + tb, \wp_3 \wp_4 / (1-s)\wp_3 + s d)$  and

taking integration with respect to  $(t, s)$  on  $[0, 1] \times [0, 1]$ , we have the following equation:

$$\begin{aligned}
 & 4\Theta_* \left( \frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4} \right) \int_0^1 \int_0^1 t^{(\alpha_1/k)-1} s^{(\alpha_2/k)-1} \mathcal{R}_{\rho_1, \omega_1, \alpha_1}^{\sigma_1, k} \left[ w_1 \left( \frac{\wp_2 - \wp_1}{\wp_1\wp_2} \right)^{\rho_1} t^{\rho_1} \right] \mathcal{R}_{\rho_2, \omega_2, \alpha_2}^{\sigma_2, k} \\
 & \cdot \left[ w_2 \left( \frac{\wp_4 - \wp_3}{\wp_3\wp_4} \right)^{\rho_2} s^{\rho_2} \right] \phi \left( \frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + s d} \right) ds dt \\
 & \geq \int_0^1 \int_0^1 t^{(\alpha_1/k)-1} s^{(\alpha_2/k)-1} \mathcal{R}_{\rho_1, \omega_1, \alpha_1}^{\sigma_1, k} \left[ w_1 \left( \frac{\wp_2 - \wp_1}{\wp_1\wp_2} \right)^{\rho_1} t^{\rho_1} \right] \mathcal{R}_{\rho_2, \omega_2, \alpha_2}^{\sigma_2, k} \left[ w_2 \left( \frac{\wp_4 - \wp_3}{\wp_3\wp_4} \right)^{\rho_2} s^{\rho_2} \right] \phi \left( \frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + s d} \right) \\
 & \times \left[ \Theta_* \left( \frac{\wp_1\wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3\wp_4}{s\wp_3 + (1-s)\wp_4} \right) + \Theta_* \left( \frac{\wp_1\wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + s d} \right) \Theta_* \left( \frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{s\wp_3 + (1-s)\wp_4} \right) \right. \\
 & \left. + \Theta_* \left( \frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + s d} \right) \right] ds dt.
 \end{aligned} \tag{89}$$

After simplification, we obtain the following equation:

$$\begin{aligned}
 & \Theta^* \left( \frac{2\wp_1\wp_2}{\wp_1 + \wp_2}, \frac{2\wp_3\wp_4}{\wp_3 + \wp_4} \right) \left[ {}_{\alpha,\rho,w}I_{(1/\wp_2)^+, (1/\wp_4)^+}^{\sigma,k} \phi \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_3} \right) + {}_{\alpha,\rho,w}I_{(1/\wp_2)^+, (1/\wp_3)^-}^{\sigma,k} \phi \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_4} \right) \right. \\
 & \quad \left. + {}_{\alpha,\rho,w}I_{(1/\wp_1)^-, (1/\wp_4)^+}^{\sigma,k} \phi \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_3} \right) + {}_{\alpha,\rho,w}I_{(1/\wp_1)^-, (1/\wp_3)^-}^{\sigma,k} \phi \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_4} \right) \right] \\
 & \geq \frac{1}{k^2 \mathcal{R}_{\rho_1, w_1, \alpha_1+k}^{\sigma_1, k} [w_1 (\wp_2 - \wp_1/\wp_1\wp_2)^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2+k}^{\sigma_2, k} [w_2 (\wp_4 - \wp_3/\wp_3\wp_4)^{\rho_2}]} \left( \frac{\wp_1\wp_2}{\wp_2 - \wp_1} \right)^{\alpha_1/k} \left( \frac{\wp_3\wp_4}{\wp_4 - \wp_3} \right)^{\alpha_2/k} \\
 & \quad \times \left[ {}_{\alpha,\rho,w}I_{(1/\wp_2)^+, (1/\wp_4)^+}^{\sigma,k} \Theta^* \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_3} \right) \phi \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_3} \right) + {}_{\alpha,\rho,w}I_{(1/\wp_2)^+, (1/\wp_3)^-}^{\sigma,k} \Theta^* \right. \\
 & \quad \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_4} \right) \phi \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_4} \right) \phi \circ g \left( \frac{1}{\wp_1}, \frac{1}{\wp_4} \right) + {}_{\alpha,\rho,w}I_{(1/\wp_1)^-, (1/\wp_4)^+}^{\sigma,k} \Theta^* \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_3} \right) \phi \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_3} \right) \\
 & \quad \left. \left. + {}_{\alpha,\rho,w}I_{(1/\wp_1)^-, (1/\wp_3)^-}^{\sigma,k} \Theta^* \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_4} \right) \phi \circ g \left( \frac{1}{\wp_2}, \frac{1}{\wp_4} \right) \right]. \tag{90}
 \end{aligned}$$

In the context of (88) and (90), the first inclusion preserve.

The proof of the second relation proceeded from (78) and multiplying both sides by  $t^{(\alpha_1/k)-1} s^{(\alpha_2/k)-1} \mathcal{R}_{\rho_1, w_1, \alpha_1}^{\sigma_1, k}$

$[w_1 (\wp_2 - \wp_1/\wp_1\wp_2)^{\rho_1} t^{\rho_1}] \mathcal{R}_{\rho_2, w_2, \alpha_2}^{\sigma_2, k} [w_2 (\wp_4 - \wp_3/\wp_3\wp_4)^{\rho_2} s^{\rho_2}] \phi$   $(\wp_1\wp_2/(1-t)\wp_1 + tb, \wp_3\wp_4/(1-s)\wp_3 + sd)$  and taking integration with respect to  $(t, s)$  on  $[0, 1] \times [0, 1]$ , then we have the following equation:

$$\begin{aligned}
 & \int_0^1 \int_0^1 t^{(\alpha_1/k)-1} s^{(\alpha_2/k)-1} \mathcal{R}_{\rho_1, w_1, \alpha_1}^{\sigma_1, k} \left[ w_1 \left( \frac{\wp_2 - \wp_1}{\wp_1\wp_2} \right)^{\rho_1} t^{\rho_1} \right] \mathcal{R}_{\rho_2, w_2, \alpha_2}^{\sigma_2, k} \left[ w_2 \left( \frac{\wp_4 - \wp_3}{\wp_3\wp_4} \right)^{\rho_2} s^{\rho_2} \right] \phi \left( \frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + sd} \right) \\
 & \cdot \left[ \Theta_* \left( \frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + sd} \right) + \Theta_* \left( \frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{s\wp_3 + (1-s)\wp_4} \right) + \Theta_* \left( \frac{\wp_1\wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + sd} \right) \right. \\
 & \quad \left. + \Theta_* \left( \frac{\wp_1\wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{\wp_3\wp_4}{s\wp_3 + (1-s)\wp_4} \right) \right] ds dt \\
 & \leq [\Theta_* (\wp_1, \wp_3) + \Theta_* (\wp_1, \wp_4) + \Theta_* (\wp_2, \wp_3) + \Theta_* (\wp_2, \wp_4)] \times \int_0^1 \int_0^1 t^{(\alpha_1/k)-1} s^{(\alpha_2/k)-1} \mathcal{R}_{\rho_1, w_1, \alpha_1}^{\sigma_1, k} \left[ w_1 \left( \frac{\wp_2 - \wp_1}{\wp_1\wp_2} \right)^{\rho_1} t^{\rho_1} \right] \mathcal{R}_{\rho_2, w_2, \alpha_2}^{\sigma_2, k} \\
 & \quad \cdot \left[ w_2 \left( \frac{\wp_4 - \wp_3}{\wp_3\wp_4} \right)^{\rho_2} s^{\rho_2} \right] \times \phi \left( \frac{\wp_1\wp_2}{(1-t)\wp_1 + tb}, \frac{\wp_3\wp_4}{(1-s)\wp_3 + sd} \right) ds dt. \tag{91}
 \end{aligned}$$

From the above-given inequality, we infer

$$\begin{aligned}
& \left[ {}_{\alpha, \rho, w} I_{(1/\varrho_2)^+, (1/\varrho_4)^+}^{\sigma, k} \Theta_* \circ g\left(\frac{1}{\varrho_1}, \frac{1}{\varrho_3}\right) \phi \circ g\left(\frac{1}{\varrho_1}, \frac{1}{\varrho_3}\right) + {}_{\alpha, \rho, w} I_{(1/\varrho_2)^+, (1/\varrho_3)^-}^{\sigma, k} \Theta_* \circ g\left(\frac{1}{\varrho_1}, \frac{1}{\varrho_4}\right) \phi \circ g\left(\frac{1}{\varrho_1}, \frac{1}{\varrho_4}\right) \phi \circ g\left(\frac{1}{\varrho_1}, \frac{1}{\varrho_4}\right) \right. \\
& \left. + {}_{\alpha, \rho, w} I_{(1/\varrho_1)^-, (1/\varrho_4)^+}^{\sigma, k} \Theta_* \circ g\left(\frac{1}{\varrho_2}, \frac{1}{\varrho_3}\right) \phi \circ g\left(\frac{1}{\varrho_2}, \frac{1}{\varrho_3}\right) + {}_{\alpha, \rho, w} I_{(1/\varrho_1)^-, (1/\varrho_3)^-}^{\sigma, k} \Theta_* \circ g\left(\frac{1}{\varrho_2}, \frac{1}{\varrho_4}\right) \phi \circ g\left(\frac{1}{\varrho_2}, \frac{1}{\varrho_4}\right) \right] \\
& \leq [\Theta_*(\varrho_1, \varrho_3) + \Theta_*(\varrho_1, \varrho_4) + \Theta_*(\varrho_2, \varrho_3) + \Theta_*(\varrho_2, \varrho_4)] \\
& \cdot \left[ {}_{\alpha, \rho, w} I_{(1/\varrho_2)^+, (1/\varrho_4)^+}^{\sigma, k} \phi \circ g\left(\frac{1}{\varrho_1}, \frac{1}{\varrho_3}\right) + {}_{\alpha, \rho, w} I_{(1/\varrho_2)^+, (1/\varrho_3)^-}^{\sigma, k} \phi \circ g\left(\frac{1}{\varrho_1}, \frac{1}{\varrho_4}\right) + {}_{\alpha, \rho, w} I_{(1/\varrho_1)^-, (1/\varrho_4)^+}^{\sigma, k} \phi \circ g\left(\frac{1}{\varrho_2}, \frac{1}{\varrho_3}\right) \right. \\
& \left. + {}_{\alpha, \rho, w} I_{(1/\varrho_1)^-, (1/\varrho_3)^-}^{\sigma, k} \phi \circ g\left(\frac{1}{\varrho_2}, \frac{1}{\varrho_4}\right) \right].
\end{aligned} \tag{92}$$

Now, we calculate the following inequality essentially using harmonically co-ordinated concavity of  $\Theta^*$ .

$$\begin{aligned}
& \left[ {}_{\alpha, \rho, w} I_{(1/\varrho_2)^+, (1/\varrho_4)^+}^{\sigma, k} \Theta^* \circ g\left(\frac{1}{\varrho_1}, \frac{1}{\varrho_3}\right) \phi \circ g\left(\frac{1}{\varrho_1}, \frac{1}{\varrho_3}\right) + {}_{\alpha, \rho, w} I_{(1/\varrho_2)^+, (1/\varrho_3)^-}^{\sigma, k} \Theta^* \circ g\left(\frac{1}{\varrho_1}, \frac{1}{\varrho_4}\right) \phi \circ g\left(\frac{1}{\varrho_1}, \frac{1}{\varrho_4}\right) \phi \circ g\left(\frac{1}{\varrho_1}, \frac{1}{\varrho_4}\right) \right. \\
& \left. + {}_{\alpha, \rho, w} I_{(1/\varrho_1)^-, (1/\varrho_4)^+}^{\sigma, k} \Theta^* \circ g\left(\frac{1}{\varrho_2}, \frac{1}{\varrho_3}\right) \phi \circ g\left(\frac{1}{\varrho_2}, \frac{1}{\varrho_3}\right) + {}_{\alpha, \rho, w} I_{(1/\varrho_1)^-, (1/\varrho_3)^-}^{\sigma, k} \Theta^* \circ g\left(\frac{1}{\varrho_2}, \frac{1}{\varrho_4}\right) \phi \circ g\left(\frac{1}{\varrho_2}, \frac{1}{\varrho_4}\right) \right] \\
& \leq [\Theta^*(\varrho_1, \varrho_3) + \Theta^*(\varrho_1, \varrho_4) + \Theta^*(\varrho_2, \varrho_3) + \Theta^*(\varrho_2, \varrho_4)] \\
& \cdot \left[ {}_{\alpha, \rho, w} I_{(1/\varrho_2)^+, (1/\varrho_4)^+}^{\sigma, k} \phi \circ g\left(\frac{1}{\varrho_1}, \frac{1}{\varrho_3}\right) + {}_{\alpha, \rho, w} I_{(1/\varrho_2)^+, (1/\varrho_3)^-}^{\sigma, k} \phi \circ g\left(\frac{1}{\varrho_1}, \frac{1}{\varrho_4}\right) + {}_{\alpha, \rho, w} I_{(1/\varrho_1)^-, (1/\varrho_4)^+}^{\sigma, k} \phi \circ g\left(\frac{1}{\varrho_2}, \frac{1}{\varrho_3}\right) \right. \\
& \left. + {}_{\alpha, \rho, w} I_{(1/\varrho_1)^-, (1/\varrho_3)^-}^{\sigma, k} \phi \circ g\left(\frac{1}{\varrho_2}, \frac{1}{\varrho_4}\right) \right].
\end{aligned} \tag{93}$$

(92) and (93) yield the required relation.  $\square$

#### 4. Conclusion

In the recent study, we have interrogated the Hermite-Hadamard's and its Fejer type containments in connection with interval analysis. In order to investigate, we define the double integral Raina's fractional integrals in  $k$  form, which are demonstrated by an example. Furthermore, we introduced the notion of interval-valued harmonically co-ordinated convex functions. Initially, we obtained generalized fractional inclusions in association with harmonically interval-valued convex functions and their numeric verifications, then we established some two dimensional Hermite-Hadamard's and Hermite-Hadamard-Fejer like inclusions involving co-ordinated harmonically interval-valued convexity. As the theory of convexity and interval analysis has great applications in optimization and error analysis. I hope the technique of the paper will inspire the interested readers and will stimulate the further research in the following direction.

#### Data Availability

The data used to support the study are included in the paper.

#### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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