Hindawi
Journal of Mathematics
Volume 2022, Article ID 5830590, 10 pages
https://doi.org/10.1155/2022/5830590

Research Article

A Novel Concept of Fuzzy Subsemigroup (Ideal)

Yu-Hong Che [1] and Qi Liu [2]

[1]School of Mathematics and Statistics, Weinan Normal University, Shaanxi 714099, China
[2]School of Mathematics and Statistics, Hubei Minzu University, Enshi 445000, China

Correspondence should be addressed to Yu-Hong Che; hongmeigui_2022@163.com

Received 28 May 2022; Revised 16 August 2022; Accepted 8 September 2022; Published 30 September 2022

Academic Editor: Feng Feng

This paper focuses on a generalized definition of fuzzy subsemigroup (ideal) on semigroup. Let L be a completely distributive lattice; we introduce the definition of L-fuzzy ideal and also the novel concept of subsemigroup (ideal) on semigroup. Then, we discuss the necessary and sufficient conditions of L-fuzzy subsemigroup (ideal) measure using the four level cuts of an L-fuzzy set. Moreover, we study the properties of L-fuzzy subsemigroup (ideal) measure. As an application of L-fuzzy subsemigroup (ideal) measure, we obtain the L-fuzzy convexities on a semigroup and bijective semigroup homomorphic mapping is an L -fuzzy isomorphism.

1. Introduction

Zadeh introduced fuzzy set theory [1]. Then, Van de Vel [2] gave a systematic study of the theory of convexity in 1993. Rosa [3] extended the concept of convexity to the fuzzy situation in 1994, which is said to be an L-convex structure later, and the relevant results are shown in [4–8]. In addition, Maruyama [9] studied the concept of convexity on a completely distributive lattice in 2009. As a matter of fact, there is a convex space in many other mathematical structures [10–13].

Many academics studied on generalized semigroups (ideal) [14–19]. Zhao [19] introduced the concept of L -fuzzy subsemigroup. To be worth mentioning, Shi and Xin opened up a prospect of inducing the L-fuzzy convexity [20]. In recent years, mathematicians have constructed the L -fuzzy convexities on a series of algebraic structures. Li and Shi [21] constructed an L-fuzzy convexity according to L -convex fuzzy sublattice measure. Furtherly, they obtained (L, M)-fuzzy convexity spaces. In particular, an (L, M) -fuzzy convexity became known simply as L-fuzzy convexity if $L = M$, where M is a completely distributive lattice [22]. In 2017, Wen et al. [23] defined an L-fuzzy convexity by the L -convex measure on vector spaces. Two years later, Han and Shi [24] introduced an L-fuzzy convexity according to L -convex ideal measure on a lattice. At the same year, Zhong et al. [25] constructed an L-fuzzy convexity according to L -fuzzy convex subgroup measure on an ordered group. The study by Mehmood et al. [26] constructed an L-fuzzy convexity on a ring in 2020. In 2021, Zeng et al. (preprint [27]) constructed an L-fuzzy convexity according to an L -fuzzy subfield degree on a field.

As a continuation of [19], we will extend fuzzy subsemigroup (ideal) on semigroup to general cases on the basis of the above researches. It recalled some necessary concepts and theorems of general semigroup, fuzzy semigroup, and L -fuzzy convexity. Following that, the L-fuzzy ideal is studied and its characterizations are given. Then, an L-fuzzy subsemigroup (ideal) measure is presented, and at the same time, the characterizations and properties are studied. Then, an L -fuzzy convexity is generated by an L-fuzzy subsemigroup (ideal) measure on a semigroup. What is more, the related L-fuzzy convex preserving mappings, L-fuzzy convex to convex mappings, and L-fuzzy isomorphism mappings are discussed. The findings show that a semigroup homomorphic mapping can be considered as L-fuzzy convexity preserving

(L-fuzzy convex-to-convex) mapping and bijective semigroup homomorphic mapping is an L-fuzzy isomorphism.

2. Preliminaries

To begin with, we give some useful materials for necessary. From [28], we know the operation of AC is defined as $AC = \{xy \in S : x \in A \text{ and } y \in C\}$, where S is a semigroup and $A, C \subseteq S$.

Definition 1 (see [28]). Let S be a semigroup. A nonempty-subset C of S is said to be a subsemigroup (ideal) of S if

$$CC \subseteq C(SC \subseteq C \text{ and } CS \subseteq C). \tag{1}$$

Moreover, the intersection of any collection of subsemigroup of S is a subsemigroup of S [28]. Beside, a semigroup S is said to be a commutative semigroup if the operation of S satisfies the commutative law.

Mordeson et al. extended Definition 1 to the fuzzy situation.

Definition 2 (see [28]). Let S be a semigroup. An L-fuzzy subset $\nu \in L^S$ is called to be

(1) a fuzzy subsemigroup of S if $\forall t_1, t_2 \in S$ such that $\nu(t_1 t_2) \geq \nu(t_1) \wedge \mu(t_2)$

(2) a fuzzy left (right) ideal of S if $\forall t_1, t_2 \in S$ such that $\nu(t_1 t_2) \geq \nu(t_2)(\nu(t_1 t_2) \geq \nu(t_1))$

(3) a fuzzy ideal of S if it is a fuzzy left ideal and it is a fuzzy right ideal of S

Throughout this paper, $\perp$ and T denote the minimum element and the maximum element in L, respectively. We denote A as a subset of S by $A \subseteq S$, the set of all L-fuzzy subsets on S by L^S, and an L-fuzzy subset ν of S by $\nu \in L^S$. L^S is a completely distributive lattice if it maintains the lattice L via defining $\leq$ point wisely, naturally. In addition, symbols $\chi_\varnothing$ and χ_S denote the minimum element and the maximum element in L^S, respectively. Moreover, $\wedge\varnothing = T$ and $\vee\varnothing = \perp$. Let x, y, and z be three elements in L. Then, x is said to be a coprime if x is less than or equal to $y \vee z$ leading to x less than or equal to$z$ or x less than or equal to y. Moreover x is said to be prime if $y \wedge z$ is less than or equal to x leading to either y less than or equal to x or z less than or equal to x. In addition, $P(L)$ and $J(L)$ denote the collection of all nonunit prime and nonzero coprime elements, respectively. $\forall x, y \in L$ and $D \subseteq L$, y less than or equal to $\vee D$, $x \prec y$ means that x is less than or equal to d for some $d \in D$. Then, $\beta(y)$ is equal to $\{x \in L : x \prec y\}$ and $\beta^*(y)$ is equal to $\beta(y) \cap J(L)$ mainly based on [29]. Moreover, $\forall x, y \in L$, $y \prec^{op} x$, in another way, there exists d in D, by $D \subseteq L$, $\wedge D$ less than or equal to y; we can have d which is less than or equal to x; then, we say $\alpha(x)$ is less than or equal to $\{y \in M : x \prec^{op} y\}$ and $\alpha^*(y)$ is less than or equal to $\alpha(y) \cap P(L)$ [29]. Moreover, $\forall x, y \in L$ and $\forall s_1 \in P(L)(s_1 \in J(L))$, it means $x \leq y \Leftrightarrow x \nleq s_1 (s_1 \leq x) \Rightarrow y \nleq s_1 (s_1 \leq y)$ [23].

The four cut sets play an important part in fuzzy theory. Supposing $\nu \in L^S$ and $\forall x, y \in L$, according to the definition of cut sets and the fact that an implication operator $\longrightarrow$ in correspondence to $\wedge$ in [23], we infer $x \longrightarrow y$ is equal to $\vee\{z \in L : x \wedge z \leq y\}$ and [23]

$$\begin{aligned} \nu_{[x]} &= \{s_1 \in S : x \leq \mu(s_1)\}, & \nu^{(x)} &= \{s_1 \in S : \nu(s_1) \nleq x\}, \\ \nu_{(x)} &= \{s_1 \in S : x \in \beta(\mu(s_1))\}, & \nu^{[x]} &= \{s_1 \in S : x \notin \alpha(\nu(s_1))\}. \end{aligned} \tag{2}$$

We denote L-fuzzy convexity and L-fuzzy convexity preserving mapping (L-fuzzy convex-to-convex mapping) with LFC and LFCPM(LFCCM), respectively.

From the definition of (L, M)-fuzzy convexity [22], we know the following.

Definition 3 (see ([23], [28]). A mapping $\mathcal{S} : L^S \longrightarrow L$ is called to be an LFC on S if the following conditions are true:

(1) $\mathcal{S}(\chi_\varnothing) = \mathcal{S}(\chi_S) = T$

(2) $\bigwedge_{i\in\Omega} \mathcal{S}(\nu_i) \leq \mathcal{S}(\bigwedge_{i\in\Omega} \nu_i)$ if $\{\nu_i \mid i \in \Omega\} \subseteq L^S$ is nonempty

(3) $\bigwedge_{i\in\Omega} \mathcal{S}(\nu_i) \leq \mathcal{S}(\bigvee_{i\in\Omega} \nu_i)$ if $\{\nu_i \mid i \in \Omega\} \subseteq L^S$ is nonempty and totally ordered

From [23], we know the following.

Definition 4 (see ([23]). Let $(S, \mathscr{C})$ and $(S_1, \mathscr{D})$ be LFC spaces. A mapping $f : S \longrightarrow S_1$ is called to be an LFCPM (LFCCM) if

$$\mathscr{D}(E) \leq \mathscr{C}(f_L^{\leftarrow}(E)) \quad \text{for all } E \in L^{S_1} \left(\mathscr{C}(G) \leq \mathscr{D}(f_L^{\rightarrow}(G)) \text{ for all } G \in L^S\right). \tag{3}$$

Then, $f : X \longrightarrow Y$ is called to be an L-fuzzy isomorphism if f is bijective, LFCPM, and LFCCM.

In this paper, we aim at a novel definition of fuzzy subsemigroup (ideal) on semigroup. According to it, any fuzzy set can be considered as a fuzzy subsemigroup (ideal) generalized. This paper is arranged as follows: first, we will treat the novel definition of fuzzy subsemigroup (ideal) on the basis of the notion of an L-fuzzy subsemigroup (ideal) on semigroup, their properties, and their equivalent characterizations. Then, we will obtain L-fuzzy convexities induced by L-fuzzy subsemigroup (ideal) measure and a conclusion that bijective semigroup homomorphic mapping is an L-fuzzy isomorphism and so on.

3. A Novel Definition of an L-Fuzzy Subsemigroup (Ideal)

In this section, the novel definitions of L-fuzzy subsemigroup (ideal) are introduced and their characterizations are given. First, we will extend Definition 2 to the L-fuzzy situation.

(1) Let $A = \{l, n\}$ and $B = \{l, p\}$. We define $\beta, \alpha \in L^S$ as follows:

$$\beta(l) = \beta(n) = 0.7,\ \beta(p) = 0.6, \beta(q) = 0.5, \tag{4}$$

$$\alpha(l) = \alpha(n) = 0.7, \alpha(p) = 0.8, \alpha(q) = 0.9 \tag{5}$$

Definition 5. Let S be a semigroup, $\nu \in L^S$; then, ν is called to be

(1) an L-fuzzy subsemigroup of S, if $\nu(t_1t_2) \geq \nu(t_1) \wedge \nu(t_2) \forall t_1, t_2 \in S$ [19]

(2) an L-fuzzy left (right) ideal of S, if $\nu(t_1t_2) \geq \nu(t_2) (\nu(t_1t_2) \geq \nu(t_1)) \forall t_1, t_2 \in S$

(3) an L-fuzzy ideal of S, if it is an L-fuzzy left ideal and it is an L-fuzzy right ideal of S

Example 1. Let $S = \{l, n, p, q\}$ be a semigroup with the following table.

Table 1: Values of the semigroup operator $\circ$.

s_1	s_2 $s_1 \circ s_2$: l	n	p	q
l	l	l	l	l
n	n	l	l	l
p	l	l	n	l
q	l	l	l	l

From Definition 6 (1), we can infer that β is an L-fuzzy subsemigroup of S, but α is not.

(2) Then, we define $\mu \in L^S$ as follows:

$$\begin{aligned} \mu(l) &= \mu(n) = 0.7, \\ \mu(p) &= 0.5, \\ \mu(q) &= \perp_L, \end{aligned} \tag{6}$$

and $\nu \in L^S$ as follows:

$$\nu(l) = \nu(n) = 0.7, \nu(p) = 0.8, \nu(q) = 0.9 \tag{7}$$

From Definition 5 (2), we get μ is an L-fuzzy ideal of S, but ν is not.

According to the definition of an L-fuzzy subsemigroup in [19], an L-subset is either an L-fuzzy subsemigroup or not. We discuss the measure to which an L-subset is an L-fuzzy subsemigroup using the implication operator of L. Moreover, we define an L-fuzzy ideal and study the measure to which an L-subset is an L-fuzzy ideal.

Definition 6. Let S be a semigroup, $\nu \in L^S$; then,

(1) $\mathcal{S}(\nu)$ is called to be the subsemigroup measure of ν as

$$\mathcal{S}(\nu) = \bigwedge_{t_1,t_2 \in S} ((\nu(t_1) \wedge \nu(t_2)) \mapsto \nu(t_1t_2)) \tag{8}$$

(2) $\mathcal{L}(\nu)$ is called to be the ideal measure of ν as

$$\mathcal{L}(\nu) = \bigwedge_{t_1,t_2 \in S} ((\nu(t_1) \mapsto \nu(t_1t_2)) \wedge (\nu(t_2) \mapsto \nu(t_1t_2))) \tag{9}$$

Obviously, ν is an L-fuzzy subsemigroup (ideal) of S if and only if $\mathcal{S}(\nu) = T (\mathcal{L}(\nu) = T)$, and ν always is an L-fuzzy subsemigroup (ideal) to the measure $\mathcal{S}(\nu) (\mathcal{L}(\nu))$.

From Example 1, we know $\beta(\mu)$ is an L-fuzzy subsemigroup (ideal) of S, but $\alpha(\nu)$ is not. We will explain L-fuzzy subsemigroup (ideal) measure according to the following example, which indicates that any L-subset can be regarded as an L-fuzzy subsemigroup (ideal) to some measure.

Example 2. On the basis of Example 1, we define L-fuzzy subsemigroup (ideal) measure of $\alpha(\nu)$ as

$$\begin{aligned} \mathcal{S}(\alpha) &= \bigwedge_{t_1,t_2 \in S} ((\alpha(t_1) \wedge \alpha(t_2)) \mapsto \alpha(t_1t_2)) \\ &= (0.7 \mapsto 0.7) \wedge (0.8 \mapsto 0.7) \wedge (0.9 \mapsto 0.7). = 0.7. \end{aligned} \tag{10}$$

According to Definition 6 (1), this means that α is an L-fuzzy subsemigroup to the measure $\mathcal{S}(\alpha) = 0.7$. In addition,

$$\begin{aligned} \mathcal{L}(\nu) &= \bigwedge_{t_1,t_2 \in S} ((\nu(t_1) \mapsto \nu(t_1t_2)) \wedge (\nu(t_2) \mapsto \nu(t_1t_2)) \\ &= (0.7 \mapsto 0.7) \wedge (0.8 \mapsto 0.7) \wedge (0.9 \mapsto 0.7) = 0.7. \end{aligned} \tag{11}$$

From Definition 6 (2), we can say that ν is an L-fuzzy ideal to the measure $\mathcal{L}(\nu) = 0.7$.

By Definition 6, we can infer the following conclusion.

Theorem 7. *Let S be a semigroup and $\nu \in L^S$, then $\mathcal{L}(\nu) \leq \mathcal{S}(\nu)$.*

Inspired by the conclusion $z \leq x \longrightarrow y$ if and only if $x \wedge z \leq y$ [23] $\forall x, y, z \in L$, the following lemma is clear.

Lemma 8. *Let S be a semigroup and $\nu \in L^S$; then,*

(1) $\mathcal{S}(\nu) \leq x$ if and only if $\forall t_1, t_2 \in S, \nu(t_1) \wedge \nu(t_2) \wedge x \leq \nu(t_1t_2) \forall x \in L$

(2) $\mathcal{L}(\nu) \le x$ *if and only if* $\forall t_1, t_2 \in S, \nu(t_1)\wedge x \le \nu(t_1t_2)$, *and* $\nu(t_2)\wedge x \le \nu(t_1t_2) \forall x \in L$

By Lemma 8, we can easily obtain the other characterizations of the subsemigroup measure of an L-fuzzy subset as follows.

Theorem 9. *Let S be a semigroup and $\nu \in L^S$; then,*

(1) $\mathcal{S}(\nu) = \vee\{x \in L | \nu(t_1)\wedge\nu(t_2)\wedge x \le \nu(t_1t_2), \forall t_1, t_2 \in S\}$

(2) $\mathcal{L}(\nu) = \vee\{x \in L | \nu(t_1)\wedge x \le \nu(t_1t_2), \nu(t_2)\wedge x \le \nu(t_1t_2), \forall t_1, t_2 \in S\}$

Next, we characterize the subgroup (ideal) measure of an L-fuzzy set by means of its four levels in the following theorem.

Theorem 10. *Let S be a semigroup, $\nu \in L^S$; then,*

(1) $\mathcal{S}(\nu) = \vee\{x \in L : \forall y \le x, \nu_{[y]}$ *is a subsemigroup*$\}$

(2) $\mathcal{S}(\nu) = \vee\{x \in L : \forall y \notin \alpha(x), \nu^{[y]}$ *is a subsemigroup*$\}$

(3) $\mathcal{S}(\nu) = \vee\{x \in L : \forall y \in P(L), x \le y, \nu^{(y)}$ *is a subsemigroup*$\}$

(4) $\mathcal{S}(\nu) = \vee\{x \in L : \forall y \in \beta(x), \nu_{(y)}$ *is a subsemigroup*$\}$

(5) $\mathcal{L}(\nu) = \vee\{x \in L : \forall y \le x, \nu_{[y]}$ *is an ideal*$\}$

(6) $\mathcal{L}(\nu) = \vee\{x \in L : \forall y \notin \alpha(x), \nu^{[y]}$ *is an ideal*$\}$

(7) $\mathcal{L}(\nu) = \vee\{x \in L : \forall y \in P(L), x \le y, \nu^{(y)}$ *is an ideal*$\}$

(8) $\mathcal{L}(\nu) = \vee\{x \in L : \forall y \in \beta(x), \nu_{(y)}$ *is an ideal*$\}$

Proof.

(1) First, we prove

$$\mathcal{S}(\nu) \le \vee\left\{x \in L : \forall y \le x, \nu_{[y]} \text{ is a subsemigroup}\right\} \tag{12}$$

$\forall t_1, t_2 \in S$, let $\nu(t_1)\wedge\nu(t_2)\wedge x \le \nu(t_1t_2)$; then, $\forall y \le x$, $\forall t_1, t_2 \in \nu_{[y]}$; we have

$$\nu(t_1t_2) \ge \nu(t_1)\wedge\nu(t_2)\wedge x \ge \nu(t_1)\wedge\nu(t_2)\wedge y = y. \tag{13}$$

This shows $t_1t_2 \in \nu_{[y]}$; therefore, $\nu_{[y]}$ is a subsemigroup of S. Thus,

$$\mathcal{S}(\nu) \le \vee\left\{x \in L : \forall y \le x, \nu_{[y]} \text{ is a subsemigroup}\right\}. \tag{14}$$

Next, we prove

$$\vee\left\{x \in L : \forall y \le x, \nu_{[y]} \text{ is a subsemigroup}\right\} \le \mathcal{S}(\nu). \tag{15}$$

Let $\nu_{[y]}$ be a subsemigroup of S, $\forall y \le x, x \in L$. We want to show

$$\forall t_1, t_2 \in S, \nu(t_1)\wedge\nu(t_2)\wedge x \le \nu(t_1t_2). \tag{16}$$

Suppose $y = \nu(t_1)\wedge\nu(t_2)\wedge x$, we can infer $y \le x$ and $t_1, t_2 \in \nu_{[y]}$, then $t_1t_2 \in \nu_{[y]}$ if and only if $\nu(t_1t_2) \ge y = \nu(t_1)\wedge\nu(t_2)\wedge x$. Thus,

$$\vee\left\{x \in L : \forall y \le x, \nu_{[y]} \text{ is a subsemigroup}\right\} \le \mathcal{S}(\nu). \tag{17}$$

Now, we get

$$\vee\left\{x \in L : \forall y \le x, \nu_{[y]} \text{ is a subsemigroup}\right\} = \mathcal{S}(\nu). \tag{18}$$

(2) First, we prove

$$\mathcal{S}(\nu) \le \vee\left\{x \in L : \forall y \notin \alpha(x), \nu^{[y]} \text{ is a subsemigroup}\right\} \tag{19}$$

We take any $t_1, t_2 \in S$, if $\nu(t_1)\wedge\nu(t_2)\wedge x \le \nu(t_1t_2)$, then $\forall t_1, t_2 \in \nu^{[y]}$, $\forall y \notin \alpha(x)$ if and only if $y \notin \alpha(\nu(t_1)), y \notin \alpha(\nu(t_2))$; we can obtain

$$y \notin \alpha(\nu(t_1)) \cup \alpha(\nu(t_2)) \cup \alpha(x) = \alpha(\nu(t_1)\wedge\nu(t_2)\wedge x). \tag{20}$$

By $\nu(t_1)\wedge\nu(t_2)\wedge x \le \mu(t_1t_2)$, then

$$\alpha(\nu(t_1t_2)) \subseteq \alpha(\nu(t_1)\wedge\nu(t_2)\wedge x), \tag{21}$$

we can infer $y \notin \alpha(\nu(t_1t_2))$ if and only if $t_1t_2 \in \nu^{[y]}$. This means $\nu^{[y]}$ is a subsemigroup of S, and

$$x \in \left\{x \in L : \forall y \notin \alpha(x), \mu^{[y]} \text{ is a subsemigroup}\right\}. \tag{22}$$

This demonstrates

$$\begin{aligned}\mathcal{S}(\nu) &= \vee\{x \in L : \nu(t_1)\wedge\nu(t_2)\wedge x \le \nu(t_1t_2), \forall t_1, t_2 \in S\} \\ &\le \vee\left\{x \in L | \forall y \notin \alpha(x), \nu^{[y]} \text{ is a subsemigroup}\right\}.\end{aligned} \tag{23}$$

Next, we prove

$$\vee\left\{x \in L : \forall y \notin \alpha(x), \nu^{[y]} \text{ is a subsemigroup}\right\} \le \mathcal{S}(\nu). \tag{24}$$

Suppose $\nu^{[y]}$ is a subsemigroup of S for any $y \notin \alpha(x), x$

$\in L$. We want to show

$$\nu(t_1)\wedge\nu(t_2)\wedge x \leq \nu(t_1 t_2) \quad \text{for any } t_1, t_2 \in S. \tag{25}$$

Suppose $y \notin \alpha(\nu(t_1)\wedge\nu(t_2)\wedge x) = \alpha(\nu(t_1)) \cup \alpha(\nu(t_2)) \cup \alpha(x)$, then $y \notin \alpha(x)$ and $t_1, t_2 \in \nu^{[y]}$; thus, $t_1 t_2 \in \nu^{[y]}$ if and only if $y \notin \alpha(\nu(t_1 t_2))$. This shows $\nu(t_1)\wedge\nu(t_2)\wedge x \leq \nu(t_1 t_2)$. It is proven that

$$\begin{aligned}\mathcal{S}(\nu) &= \vee\{x \in L : \nu(t_1)\wedge\nu(t_2)\wedge x \leq \nu(t_1 t_2), \forall t_1, t_2 \in S\} \\ &\geq \vee\left\{x \in L : \forall y \notin \alpha(x), \nu^{[y]} \text{ is a subsemigroup}\right\}.\end{aligned} \tag{26}$$

Thus, (2) is true.

(3) First, we prove

$$\mathcal{S}(\nu) \leq \vee\left\{x \in L : \forall x \nleq y, \nu^{(y)} \text{ is a subsemigroup}\right\} \tag{27}$$

Supposing $x \in L$, it satisfies $\nu(t_1)\wedge\nu(t_2)\wedge x \leq \nu(t_1 t_2) \forall t_1, t_2 \in S$. Let $y \in P(L)$, $x \nleq y$, and $t_1, t_2 \in \nu^{(y)}$; now we prove $t_1 t_2 \in \nu^{(y)}$. Assume that $t_1 t_2 \notin \nu^{(y)}$. According to the fact $t_1 t_2 \notin \nu^{(y)}$ if and only if $\nu(t_1 t_2) \leq y$, we can infer $\nu(t_1)\wedge\nu(t_2)\wedge x \leq \nu(t_1 t_2) \leq y$. By $y \in P(L)$ and $t_1, t_2 \in \nu^{(y)}$, now, we can get $x \leq y$, which contradicts $x \nleq y$. Hence, $t_1 t_2 \in \nu^{(y)}$. This shows that $\nu^{(y)}$ is a subsemigroup of S. Therefore,

$$\begin{aligned}\mathcal{S}(\nu) &= \vee\{x \in L : \nu(t_1)\wedge\nu(t_2)\wedge x \leq \nu(t_1 t_2), \forall t_1, t_2 \in S\} \\ &\leq \vee\left\{x \in L : \forall x \nleq y, \nu^{(y)} \text{ is a subsemigroup}\right\}.\end{aligned} \tag{28}$$

Next, we prove

$$\vee\left\{x \in L : \forall x \nleq y, \nu^{(y)} \text{ is a subsemigroup}\right\} \leq \mathcal{S}(\nu). \tag{29}$$

Supposing $x \in L$, it satisfies $\nu^{(y)}$is a subsemigroup of S$\}\forall x \nleq y$. Now, we want to prove

$$\nu(t_1)\wedge\nu(t_2)\wedge x \leq \nu(t_1 t_2), \quad \forall t_1, t_2 \in S. \tag{30}$$

Suppose $y \in P(L)$ and $\nu(t_1)\wedge\nu(t_2)\wedge x \nleq y$, then $\nu(t_1) \nleq y$, $\nu(t_2) \nleq y$, and $x \nleq y$ if and only if $t_1, t_2 \in \nu^{(y)}$. For a subsemigroup $\nu^{(y)}$, it means $t_1 t_2 \in \nu^{(y)}$ if and only if $\nu(t_1 t_2) \nleq y$. This shows that $\nu(t_1)\wedge\nu(t_2)\wedge x \leq \nu(t_1 t_2)$. It is proved that $\forall t_1, t_2 \in S$,

$$\begin{aligned}\mathcal{S}(\nu) &= \vee\left\{x \in L : \forall x \nleq y, \nu^{(y)} \text{ is a subsemigroup}\right\} \\ &\geq \vee\{x \in L : \nu(t_1)\wedge\nu(t_2)\wedge x \leq \nu(t_1 t_2)\}.\end{aligned} \tag{31}$$

It is now obvious that (3) holds.

(4) First, we prove

$$\mathcal{S}(\nu) \leq \vee\left\{x \in L : \nu_{(y)} \text{ is a subsemigroup}\right\} \tag{32}$$

We assume $\forall t_1, t_2 \in S$, $x \in \{x \in L : \nu(t_1)\wedge\nu(t_2)\wedge x \leq \nu(t_1 t_2)\}$, then $\forall y \in \beta(x)$, $t_1, s_2 \in \nu_{(y)}$; it holds that

$$y \in \beta(\nu(t_1)) \cap \beta(\nu(t_2)) \cap \beta(x) = \beta(\nu(t_1)\wedge\nu(t_2)\wedge x) \subseteq \beta(\nu(t_1 t_2)). \tag{33}$$

Then, $t_1 t_2 \in \nu_{(y)}$. This shows that $\nu_{(y)}$ is a subsemigroup of S. This means that

$$\begin{aligned}\mathcal{S}(\nu) &= \vee\{x \in L : \nu(t_1)\wedge\nu(t_2)\wedge x \leq \nu(t_1 t_2), \forall t_1, t_2 \in S\} \\ &\leq \vee\left\{x \in L : \forall y \in \beta(x), \nu_{(y)} \text{ is a subsemigroup}\right\}.\end{aligned} \tag{34}$$

Next, we prove

$$\vee\left\{x \in L : , \nu_{(y)} \text{ is a subsemigroup}\right\} \leq \mathcal{S}(\nu). \tag{35}$$

Supposing $x \in L$, it satisfies $\nu_{(y)}$ is a subsemigroup $\forall y \in \beta(x)$. Now, we prove that

$$\nu(t_1)\wedge\nu(t_2)\wedge x \leq \nu(t_1 t_2), \quad \forall t_1, t_2 \in S. \tag{36}$$

Let $y \in \beta(\nu(t_1)\wedge\nu(t_2)\wedge x)$. By $\beta(\nu(t_1)\wedge\nu(t_2)\wedge x) = \beta(\nu(t_1)) \cap \beta(\nu(t_2)) \cap \beta(x)$, we know that $t_1, t_2 \in \nu_{(y)}$, and $y \in \beta(x)$. Since $\nu_{(y)}$ is a subsemigroup of S, it holds that $t_1 t_2 \in \nu_{(y)}$ if and only if $y \in \beta(\nu(t_1 t_2))$. This shows that $\nu(t_1)\wedge\nu(t_2)\wedge x \leq \nu(t_1 t_2)$. Therefore,

$$\begin{aligned}\mathcal{S}(\nu) &= \vee\{x \in L : \nu(t_1)\wedge\nu(t_2)\wedge x \leq \nu(t_1 t_2), \forall t_1, t_2 \in S\} \\ &\geq \vee\left\{x \in L : \forall y \in \beta(x), \nu_{(y)} \text{ is a subsemigroup}\right\}.\end{aligned} \tag{37}$$

It is now obvious that (4) holds.

(5)-(7) By the fact that $\nu_{[x]}$, $\nu^{[x]}$, and $\nu^{(x)}$ are the ideals of S for any $x \in L$ and $\nu \in L^S$, then the statements are easy to be proved with Lemma 8 (2) and are omitted

(8) Suppose $\nu(t_1)\wedge x \leq \nu(t_1 t_2)$ and $\nu(t_2)\wedge x \leq \nu(t_1 t_2) \forall t_1, t_2 \in S$. It is a fact that $y \in \beta(\nu(t_2)) \cap \beta(x) = \beta(\nu(t_2)\wedge x) \subseteq \beta(\nu(t_1 t_2))$ if and only if $t_1 t_2 \in \nu_{(y)} \forall y \in \beta(x)$ and $t_2 \in \nu_{(y)}$. In addition, this shows that $\nu_{(y)}$ is an ideal of S. This means that $\forall t_1, t_2 \in S$ for any $t_1 \in \nu_{(y)}$, $t_1 t_2 \in \nu_{(y)}$,

$$\begin{aligned}\mathcal{L}(\nu) &= \vee\{x \in L \mid \nu(t_1)\wedge x \leq \nu(t_1 t_2) \text{ and } \nu(t_2)\wedge x \leq \nu(t_1 t_2), \forall t_1, t_2 \in S\} \\ &\leq \vee\left\{x \in L : \forall y \in \beta(x), \nu_{(y)} \text{ is an ideal}\right\}\end{aligned} \tag{38}$$

Conversely, suppose $\nu_{(y)}$ is an ideal of $S \forall y \in \beta(x)$. Now, we prove that

$$\nu(t_1)\wedge x \le \nu(t_1t_2),\quad \nu(t_2)\wedge x \le \nu(t_1t_2), \forall t_1, t_2 \in S. \tag{39}$$

Let $y\in\beta(\nu(t_2)\wedge x)$; by $\beta(\nu(t_2)\wedge x = \beta(\nu(t_2))\cap\beta(x)$, we know that $t_2\in\nu_{(y)}$ and $y\in\beta(x)$. For an ideal $\nu_{(y)}$ of S, it holds that $t_1t_2\in\nu_{(y)}$ if and only if $y\in\beta(\nu(t_1t_2))$; it shows that $\nu(t_2)\wedge x\le\nu(t_1t_2)$. We can obtain $\nu(t_1)\wedge x\le\nu(t_1t_2)$ in a similar way. Therefore,

$$\mathscr{L}(\nu)\ge\vee\left\{x\in L:\forall y\in\beta(x), \nu_{(y)}\text{ is an ideal}\right\}. \tag{40}$$

It is now obvious that (8) holds. □

Theorem 11. *Let $\{\nu_i\}_{i\in\Omega}$ be a family of L-fuzzy subsets in a semigroup S. Then,*

(1) $\underset{i\in\Omega}{\wedge}\mathcal{S}(\nu_i)\le\mathcal{S}(\underset{i\in\Omega}{\wedge}\nu_i)$

(2) $\underset{i\in\Omega}{\wedge}\mathscr{L}(\nu_i)\le\mathscr{L}(\underset{i\in\Omega}{\wedge}\nu_i)$

Proof. Suppose $\{\nu_i\}_{i\in\Omega}$ is a family of L-fuzzy subsets in a semigroup $S\forall t_1, t_2\in S, x\in L$.

(1) Suppose $x\le\underset{i\in\Omega}{\wedge}\mathcal{S}(\nu_i)$, then $\underset{i\in\Omega}{\wedge}\nu_i(t_1)\wedge\underset{i\in\Omega}{\wedge}\nu_i(t_2)\wedge x\le\underset{i\in\Omega}{\wedge}\nu_i(t_1t_2)$. We can infer $x\le\mathcal{S}(\underset{i\in\Omega}{\wedge}\nu_i)$

$$\begin{aligned}\mathcal{S}(f_L^{\rightarrow}(\nu)) &= \vee\left\{x\in L: f_L^{\rightarrow}(\nu)\left(t_1'\right)\wedge f_L^{\rightarrow}(\nu)\left(t_2'\right)\wedge x\le f_L^{\rightarrow}(\nu)\left(t_1't_2'\right),\forall t_1', t_2'\in S\right\}.\\ &= \vee\left\{x\in L:\bigvee_{f(t_1)=t_1'}\nu(t_1)\wedge\bigvee_{f(t_2)=t_2'}\nu(t_2)\wedge x\le\bigvee_{f(z)=t_1't_2'}\nu(z),\forall t_1', t_2'\in S\right\}\\ &\ge\vee\{x\in L:\nu(t_1)\wedge\nu(t_2)\wedge x\le\nu(t_1t_2),\forall t_1, t_2\in S\}=\mathcal{S}(\nu)\end{aligned} \tag{43}$$

(2) Suppose $x\le\underset{i\in\Omega}{\wedge}\mathscr{L}(\nu_i)$, then $\nu_i(t_2)\wedge x\le\nu_i(t_1t_2)$; it is easy to obtain $\underset{i\in\Omega}{\wedge}\nu_i(t_2)\wedge x\le\underset{i\in\Omega}{\wedge}\nu_i(t_1t_2)$, and by $\nu_i(t_1)\wedge x\le\nu_i(t_1t_2)$, we can obtain

$$\underset{i\in\Omega}{\wedge}\nu_i(t_1)\wedge x\le\underset{i\in\Omega}{\wedge}\nu_i(t_1t_2) \tag{41}$$

We can infer $x\le\mathscr{L}(\underset{i\in\Omega}{\wedge}\nu_i)$. □

If f is injective, the above $\le$ can be replaced by $=$, i.e., $\mathcal{S}(\nu)=\mathcal{S}(f_L^{\rightarrow}(\nu))$. Now, (1) is true.

(2) We complete the proof by combining the fact

The following result is obvious.

Let $f: S_0\longrightarrow S_1$ be a mapping and ν and η be two L-fuzzy subsets in S_0 and S_1, respectively. Then, we have

(1) $f_{\mathcal{S}}^{\rightarrow}(f_{\mathcal{S}}^{\leftarrow}(\eta))=\eta(f_{\mathscr{L}}^{\rightarrow}(f_{\mathscr{L}}^{\leftarrow}(\eta))=\eta)$ if f is surjective

(2) $f_{\mathcal{S}}^{\leftarrow}(f_{\mathcal{S}}^{\rightarrow}(\nu))=\nu(f_{\mathscr{L}}^{\leftarrow}(f_{\mathscr{L}}^{\rightarrow}(\nu))=\nu)$ if f is injective

where $f_{\mathcal{S}}^{\rightarrow}: L^{S_0}\longrightarrow L^{S_1}(f_{\mathscr{L}}^{\rightarrow}: L^{S_0}\longrightarrow L^{S_1})$ and $f_{\mathcal{S}}^{\leftarrow}: L^{S_1}\longleftarrow L^{S_0}(f_{\mathscr{L}}^{\leftarrow}: L^{S_1}\longleftarrow L^{S_0})$ are defined by

$$\begin{aligned}&f_{\mathcal{S}}^{\rightarrow}(\nu)\left(s'\right)=\vee\left\{\nu(s): f(s)=s'\right\},\left(f_{\mathscr{L}}^{\rightarrow}(\nu)\left(s'\right)=\vee\left\{\nu(s): f(s)=s'\right\}\right),\\ &f_{\mathcal{S}}^{\leftarrow}(\eta)=(\eta\circ f),(f_{\mathscr{L}}^{\leftarrow}(\eta)=(\eta\circ f)).\end{aligned} \tag{42}$$

Now, we investigate the subsemigroup (ideal) measures of homomorphic image and preimage of L-fuzzy subset.

Theorem 12. *Suppose $f: S\longrightarrow S'$ is a semigroup homomorphism. Then,*

(1) if $\nu\in L^S$, then $\mathcal{S}(f_S^{\rightarrow}(\nu))\ge\mathcal{S}(\nu)$. And if f is injective, then $\mathcal{S}(f_S^{\rightarrow}(\nu))=\mathcal{S}(\nu)$

(2) if $\eta\in L^{S'}$, then $\mathcal{S}(f_S^{\leftarrow}(\eta))\ge\mathcal{S}(\eta)$. And if f is surjective, then $\mathcal{S}(f_S^{\leftarrow}(\eta))=\mathcal{S}(\eta)$

(3) if $\nu\in L^S$, then $\mathscr{L}(f_L^{\rightarrow}(\nu))\ge\mathscr{L}(\nu)$. And if f is injective, then $\mathscr{L}(f_L^{\rightarrow}(\nu))=\mathscr{L}(\nu)$

(4) if $\eta\in L^{S'}$, then $\mathscr{L}(f_L^{\leftarrow}(\eta))\ge\mathscr{L}(\eta)$. And if f is surjective, then $\mathscr{L}(f_L^{\leftarrow}(\eta))=\mathscr{L}(\eta)$

Proof.

(1) We complete the proof by combining the fact

$$\begin{aligned}\mathcal{S}(f_L^{\leftarrow}(\eta)) &= \bigwedge_{t_1,t_2\in S}((f_L^{\leftarrow}(\eta)(t_1)\wedge f_L^{\leftarrow}(\eta)(t_2))\mapsto f_L^{\leftarrow}(\eta)(t_1t_2))\\ &= \bigwedge_{t_1,t_2\in S}((\eta(f(t_1)))\wedge(\eta(f(t_2))))\mapsto\eta(f(t_1)f(t_2)))\\ &\ge\bigwedge_{t_1',t_2'\in S'}\left(\left(\eta\left(t_1'\right)\right)\wedge\left(\eta\left(t_2'\right)\right)\right)\mapsto\left(\eta\left(t_1't_2'\right)\right)\right)=\mathcal{S}(\eta)\end{aligned} \tag{44}$$

If f is surjective, the above $\leq$ can be replaced by =, i.e., $\mathcal{S}(\nu) = \mathcal{S}(f_L^{\leftarrow}(\eta))$. Now, (2) is true.

(3) We complete the proof by combining the fact

For any $t_1, t_2 \in S, t_1', t_2' \in S'$,

$$\begin{aligned}\mathcal{L}(f_L^{\rightarrow}(\nu)) &= \vee\left\{x \in L \middle| \begin{array}{c} f_L^{\rightarrow}(\nu)\left(t_1'\right)\wedge x \leq f_L^{\rightarrow}(\nu)\left(t_1't_2'\right) \\ f_L^{\rightarrow}(\nu)\left(t_2'\right)\wedge x \leq f_L^{\rightarrow}(\nu)\left(t_1't_2'\right) \\ \forall t_1', t_2' \in S' \end{array}\right\} \\ &= \vee\left\{x \in L \middle| \begin{array}{c} \vee_{f(t_1)=t_1'} \nu(t_1)\wedge x \leq \vee_{f(z)=t_1't_2'} \nu(z) \\ \vee_{f(t_2)=t_2'} \nu(t_2)\wedge x \leq \vee_{f(z)=t_1't_2'} \nu(z) \\ \forall t_1', t_2' \in S' \end{array}\right\} \\ &\geq \vee\left\{x \in L \middle| \begin{array}{l} \nu(t_1)\wedge x \leq \nu(t_1t_2), \\ \nu(t_2)\wedge x \leq \nu(t_1t_2), \end{array} \quad \forall t_1, t_2 \in S\right\} = \mathcal{L}(\nu).\end{aligned} \tag{45}$$

If f is injective, the above $\leq$ can be replaced by =, i.e., $\mathcal{L}(\nu) = \mathcal{L}(f_L^{\rightarrow}(\nu))$. Now, (3) is true.

(4) We complete the proof by combining the fact

$$\begin{aligned}\mathcal{L}(f_L^{\leftarrow}(\eta)) &= \wedge_{t_1,t_2\in S}(f_L^{\leftarrow}(\eta)(t_1) \mapsto f_L^{\leftarrow}(\eta)(t_1t_2))\wedge(f_L^{\leftarrow}(\eta)(t_2) \mapsto f_L^{\leftarrow}(\eta)(t_1t_2)) \\ &= \wedge_{t_1,t_2\in S}(\eta(f(t_1)) \mapsto \eta(f(t_1)f(t_2)))\wedge(\eta(f(t_2)) \mapsto \eta(f(t_1)f(t_2))) \\ &\geq \wedge_{t_1',t_2'\in S'}\left(\eta\left(t_1'\right)\wedge\eta\left(t_2'\right)\right) \mapsto \eta\left(t_1't_2'\right) = \mathcal{L}(\eta)\end{aligned} \tag{46}$$

If f is surjective, the above $\leq$ can be replaced by =, i.e., $\mathcal{L}(\eta) = \mathcal{L}(f_L^{\leftarrow}(\eta))$. It is now obvious that the theorem holds. □

On the basis of the definition of the Cartesian product of the L-fuzzy subset [20], we discuss the subsemigroup (ideal) measure of the product of L-fuzzy subsets of a semigroup.

(2) The proof is simple, and we omit it

Theorem 13. *Let $\{S_i\}_{i=1}^n$ be a family of semigroup and $\prod_{i=1}^n \nu_i$ be the product of $\{\nu_i\}_{i=1}^n$, where $\nu_i \in L^{S_i}$. Then,*

(1) $\mathcal{S}(\prod_{i=1}^n \nu_i) \leq \wedge_{i=1}^n \mathcal{S}(\nu_i)$

(2) $\mathcal{L}(\prod_{i=1}^n \nu_i) \leq \wedge_{i=1}^n \mathcal{L}(\nu_i)$

Proof.

(1) For $\prod_{i=1}^n \nu_i = \wedge_{i=1}^n P_i^{-1}(\nu_i)$, let $P_i : \prod_{i=1}^n S_i \longrightarrow S_i$ be projections $(i = 1, 2, \cdots, n)$; then, P_i is semigroup homomorphic mappings; we can infer

$$\mathcal{S}\left(P_i^{-1}(\nu_i)\right) \geq \mathcal{S}(\nu_i) \tag{47}$$

Then,

$$\mathcal{S}\left(\prod_{i=1}^n \nu_i\right) = \mathcal{S}\left(\wedge_{i=1}^n P_i^{-1}(\nu_i)\right) \geq \wedge_{i=1}^n \mathcal{S}\left(P_i^{-1}(\nu_i)\right) \geq \wedge_{i=1}^n \mathcal{S}(\nu_i). \tag{48}$$

□

Definition 14. Let S be a semigroup, $x \in P(L)$, and $\nu, \eta \in L^S$, then $\nu \circ \eta \in L^S$ defined by

$$(\nu \circ \eta)(z) = \vee_{z=t_1t_2}(\nu(t_1)\wedge\eta(t_2)). \tag{49}$$

It is obvious that $(\nu \circ \eta)^{(x)} = \nu^{(x)} \circ \eta^{(x)}$.

Example 3. Let $S = \{l, n, p, q\}$ be a semigroup with the same operation as Example 1. We define $\nu, \eta \in L^S$ as follows:

$$\begin{aligned}&\nu(l) = 0.4, \nu(n) = 0.7, \nu(p) = 0.6, \nu(q) = 0.5, \\ &\eta(l) = 0.3, \eta(n) = 0.7, \eta(p) = 0.8, \eta(q) = 0.9.\end{aligned} \tag{50}$$

From Definition 14, we get

$$\begin{aligned}(\nu \circ \eta)(l) &= \vee_{l=t_1t_2}(\nu(t_1)\wedge\eta(t_2)) = 0.7, \\ (\nu \circ \eta)(n) &= \vee_{n=t_1t_2}(\nu(t_1)\wedge\eta(t_2)) = 0.6.\end{aligned} \tag{51}$$

Theorem 15. *Let S be a semigroup, $\nu, \eta \in L^S$. If $\nu \circ \eta = \eta \circ \nu$, then*

$$\mathcal{S}(\nu \circ \eta) \geq \mathcal{S}(\nu)\wedge\mathcal{S}(\eta). \tag{52}$$

Proof. . Let $\mathcal{S}(\nu)\wedge\mathcal{S}(\eta) \geq x$ for any $x \in P(L)$, then $\mathcal{S}(\nu) \geq x$ and $\mathcal{S}(\eta) \geq x$. Then, $\forall z \ngeq x$, by Theorem 10 (3), $\nu^{(z)}$ and $\eta^{(z)}$ are subsemigroups, respectively. By $(\nu \circ \eta)^{(z)} = \nu^{(z)} \circ \eta^{(z)}$, then $(\nu \circ \eta)^{(z)}$ is a subsemigroup of S. We can infer $\mathcal{S}(\nu \circ \eta) \geq z$. It is now obvious that the theorem holds. □

Theorem 16. *Let S be a commutative semigroup, ν, $\eta \in L^S$, then*

$$\mathcal{L}(\nu \circ \eta) \geq \mathcal{L}(\nu)\wedge\mathcal{L}(\eta). \tag{53}$$

Proof. Suppose $\mathcal{L}(\nu)\wedge\mathcal{L}(\eta) \geq x$ for any $x \in P(L)$; we can infer $\mathcal{L}(\nu) \geq x$ and $\mathcal{L}(\eta) \geq x$. Then, $\forall z \ngeq x$; by Theorem 10 (7), $\nu^{(z)}$ and $\eta^{(z)}$ are an ideal, respectively. By $(\nu \circ \eta)^{(z)} = \mu^{(z)} \circ \eta^{(z)}$, then $(\nu \circ \eta)^{(z)}$ is an ideal of S. We can infer $\mathcal{L}(\nu \circ \eta) \geq x$. It is now obvious that the theorem holds. □

4. The L-Fuzzy Convexities and Their Properties

In the following, we will use the L-fuzzy subsemigroup (ideal) measures, we will construct an L-fuzzy convexity on a semigroup. After that, we will study the relationship between LFC and LFCPM (LFCCP).

$\forall \nu \in L^S$, $\mathcal{S}(\nu)$ $(\mathcal{L}(\nu))$ can be reasonably considered as a mapping $\mathcal{S}$: $L^S \longrightarrow L(\mathcal{L}$: $L^S \longrightarrow L)$ specified by $\nu \mapsto \mathcal{S}(\nu)($ $\nu \mapsto \mathcal{L}(\nu))$. It will indicate that $\mathcal{S}(\mathcal{L})$ is still an LFC on a semigroup S.

The following convention is that the null set is a subsemigroup (ideal) of any semigroup. It is necessary to discuss the next theorem.

Theorem 17. *Let S be a semigroup, $\nu \in L^S$. Then,*

(1) the mapping $\mathcal{S}$: $L^S \longrightarrow L$ specified by $\nu \mapsto \mathcal{S}(\nu)$ is an LFC induced via L-fuzzy subsemigroup measure on S

(2) the mapping $\mathcal{L}$: $L^S \longrightarrow L$ specified by $\nu \mapsto \mathcal{L}(\nu)$ is an LFC induced via L-fuzzy ideal measure on S

Proof. (1) We will prove (1) in accordance with the three axioms of Definition 3 as follows:

(i) It indicates $\mathcal{S}(\chi_\varnothing) = \mathcal{S}(\chi_S) = T$

(ii) Theorem 11 has proven $\bigwedge_{i\in\Omega} \mathcal{S}(\nu_i) \le \mathcal{S}(\bigwedge_{i\in\Omega} \nu_i)$ if $\{\nu_i\}_{i\in\Omega}$, a family of L-fuzzy subsets in a semigroup S, is nonempty

(iii) Suppose $\{\nu_i : i \in \Omega\} \neq \varnothing$, $\{\nu_i : i \in \Omega\} \subseteq L^S$ is a chain. We want to prove

$$\bigwedge_{i\in\Omega} \mathcal{S}(\nu_i) \le \mathcal{S}\Big(\bigvee_{i\in\Omega} \nu_i\Big) \tag{54}$$

It needs to show that

$$x \le \mathcal{S}\Big(\bigvee_{i\in\Omega} \nu_i\Big) \forall x \le \bigwedge_{i\in\Omega} \mathcal{S}(\nu_i). \tag{55}$$

By Lemma 8(2), $\forall i \in \Omega, t_1, t_2 \in S$, we know

$$\nu_i(t_1) \wedge \nu_i(t_2) \wedge x \le \nu_i(t_1 t_2). \tag{56}$$

Assume $y \in J(L)$; it satisfies

$$y \prec \Big(\bigvee_{i\in\Omega} \nu_i(t_1)\Big) \wedge \Big(\bigvee_{i\in\Omega} \nu_i(t_2)\Big) \wedge x. \tag{57}$$

Then, we have

$$\begin{aligned} y &\prec \bigvee_{i\in\Omega} \nu_i(s_1), \\ y &\prec \bigvee_{i\in\Omega} \nu_i(t_2), \\ y &\le x. \end{aligned} \tag{58}$$

So there exists $i, j \in \Omega$; it satisfies

$$\begin{aligned} y &\le \nu_i(t_1), \\ y &\le \nu_j(t_2), \\ y &\le x. \end{aligned} \tag{59}$$

For $\{\nu_i : i \in \Omega\}$, suppose $\nu_j \le A_i$; it follows that $y \le \nu_i($ $t_1) \wedge \nu_i(t_2) \wedge x$. By $\nu_i(t_1) \wedge \nu_i(t_2) \wedge x \le \nu_i(t_1 t_2)$, we obtain $y \le \nu_i$ $(t_1 t_2)$. Hence, $y \le \bigvee_{i\in\Omega} \nu_i(t_1 t_2)$. Then, we can infer

$$\Big(\bigvee_{i\in\Omega} \nu_i(t_1)\Big) \wedge \Big(\bigvee_{i\in\Omega} \nu_i(t_2)\Big) \wedge x \le \bigvee_{i\in\Omega} \nu_i(t_1 t_2). \tag{60}$$

Combining Lemma 8 (2), we have $x \le \mathcal{S}(\bigvee_{i\in\Omega} \nu_i)$. Then, we obtain $\bigwedge_{i\in\Omega} \mathcal{S}(\nu_i) \le \mathcal{S}(\bigvee_{i\in\Omega} \nu_i)$. Therefore, it is proved.

(2) There is something in common shown in Theorem 17 (1), and it is omitted here. □

In the rest of this paper, we will study the properties of LFC on semigroup. We denote L-convexity with LC. From [22, 23], we get the equivalent characterization between an LC and LFC. It is found that $\mathcal{S}(\mathcal{L})$ is an LFC if and only if $\mathcal{S}_{[x]}(\mathcal{L}_{[x]})$ is an LC for any $x \in S(L) \setminus \{\bot\}$ or $\mathcal{S}^{[x]}(\mathcal{L}^{[x]})$ is an LC for any $x \in \alpha(\bot)$. Then, we get the next theorem.

Theorem 18. *Let S be a semigroup; then, the following characterizations are equivalent:*

(1) $S(L)$ is the LFC induced by $\mathcal{S}(\nu)(\mathcal{L}(\nu))$, where $\nu \in L^S$

(2) $\forall x \in J(L)$, $\mathcal{S}_{[x]}(\mathcal{L}_{[x]})$ is an LC on S

(3) $\forall x \in \alpha(\bot)$, $\mathcal{S}^{[x]}(\mathcal{L}^{[x]})$ is an LC on S

Combining Definition 3 and Theorem 12, we can get the next theorem.

Theorem 19. *Let $f : S_0 \longrightarrow S_1$ be a semigroup homomorphic mapping and $\mathcal{S}_{S_0}(\mathcal{L}_{S_0})$ and $\mathcal{S}_{S_1}(\mathcal{L}_{S_1})$ be the L-fuzzy convexities induced by L-fuzzy subsemigroup (ideal) measures on S_0 and S_1, respectively. Then,*

(1) $f : (S_0, \mathcal{S}_{S_0}) \longrightarrow (S_1, \mathcal{S}_{S_1})$ is an LFCPM

(2) $f : (S_0, \mathcal{L}_{S_0}) \longrightarrow (S_1, \mathcal{L}_{S_1})$ is an LFCPM

(3) $f : (S_0, \mathcal{S}_{S_0}) \longrightarrow (S_1, \mathcal{S}_{S_1})$ is an LFCCM

(4) $f : (S_0, \mathcal{L}_{S_0}) \longrightarrow (S_1, \mathcal{L}_{S_1})$ is an LFCCM

Proof. To prove (1) and (2) hold, combining Theorem 12 (2) and (4) with the conditions $f : S_0 \longrightarrow S_1$ is a semigroup homomorphic mapping and $\mathcal{S}_{S_0}(\mathcal{L}_{S_0})$ and $\mathcal{S}_{S_1}(\mathcal{L}_{S_1})$ are the L-fuzzy convexities induced by L-fuzzy subsemigroup (ideal) measures on S_0 and S_1, we can get

$$\mathcal{S}_{S_0}(f_S^{\longleftarrow}(\nu)) \geq \mathcal{S}_{S_1}(\nu)(\mathcal{L}(f_L^{\longleftarrow}(\nu)) \geq \mathcal{L}(\nu)) \quad \text{for any } \eta \in L^{S'}. \tag{61}$$

Then, we can infer (1) and (2) hold according to Definition 4.

Similarly, we can prove (3) and (4) hold. □

Combining Theorem 13 and Theorem 19 with Definition 4, we can get the next theorem in a similar way of Theorem 19.

Theorem 20. *Let $\{S_i\}_{i=1}^n$ be a family of semigroup and $\{P_i\}_{i=1}^n$ be a family of projection by $P_i : \prod_{i=1}^n S_i \longrightarrow S_i (i = 1, 2, \cdots, n)$. Then,*

(1) $P_i : (\prod_{i=1}^n S_i, \mathcal{S}_{\prod_{i=1}^n S_i}) \longrightarrow (S_i, \mathcal{S}_{S_i})$ $(P_i : (\prod_{i=1}^n S_i, \mathcal{L}_{\prod_{i=1}^n S_i}) \longrightarrow (S_i, \mathcal{L}_{S_i}))$ is an LFCPM

(2) $P_i : (\prod_{i=1}^n S_i, \mathcal{S}_{\prod_{i=1}^n S_i}) \longrightarrow (S_i, \mathcal{S}_{S_i})$ $(P_i : (\prod_{i=1}^n S_i, \mathcal{L}_{\prod_{i=1}^n S_i}) \longrightarrow (S_i, \mathcal{L}_{S_i}))$ is an LFCCM

From Definition 4 and Theorem 19 again, we can infer the following theorem.

Theorem 21. *Let f be a semigroup homomorphic mapping and a bijective by $f : S_0 \longrightarrow S_1$. Then*

(1) $f : (S_0, \mathcal{S}_{S_0}) \longrightarrow (S_1, \mathcal{S}_{S_1})$ is an L-fuzzy isomorphism

(2) $f : (S_0, \mathcal{L}_{S_0}) \longrightarrow (S_1, \mathcal{L}_{S_1})$ is an L-fuzzy isomorphism

5. Conclusions

It is presented in this paper that any L-fuzzy subsets can be considered as an L-fuzzy subsemigroup (ideal) on semigroup to some measure. By this, we obtained L-fuzzy convexities and studied their properties. Next, we will research the related definitions and properties on order semigroup.

Data Availability

All data supporting the findings of this study are included in the submitted article.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

Y.-H.C. and Q.L. were responsible for conceptualization, software, and formal analysis; Q.L. was responsible for the methodology, supervision, project administration, and funding acquisition; Y.-H.C. was responsible for writing the original draft preparation, writing, review, and editing. All authors have read and agreed to the published version of the manuscript.

Acknowledgments

The authors would like to thank Professor Fu-Gui Shi for his helps and suggestions. The project is funded by the National Natural Science Foundation of China (12101210) and the Scientific Research Project of Hubei Provincial Department of Education (Q20201903).

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] M. Van de Vel, *Theory of Convex Structures*, North-Holland: New York, 1993.

[3] M. V. Rosa, "On fuzzy topology fuzzy convexity spaces and fuzzy local convexity," *Fuzzy Sets and Systems*, vol. 62, no. 1, pp. 97–100, 1994.

[4] Y. Y. Dong and J. Li, "Fuzzy convex structure and prime fuzzy ideal space on residuated lattices," *Journal of Nonlinear and Convex Analysis*, vol. 21, no. 12, pp. 2683–2692, 2020.

[5] F. Mehmood, F. G. Shi, K. Hayat, and X. P. Yang, "The homomorphism theorems of M-hazy rings and their induced fuzzifying convexities," *Mathematics*, vol. 8, no. 3, p. 411, 2020.

[6] X. W. Wei and B. Wang, "Fuzzy (restricted) hull operators and fuzzy convex structures on L-sets," *Journal of Nonlinear and Convex Analysis*, vol. 21, no. 12, pp. 2805–2815, 2020.

[7] H. Yang and E. Q. Li, "A new approach to interval operators in L-convex spaces," *Journal of Nonlinear and Convex Analysis*, vol. 21, no. 12, pp. 2705–2714, 2020.

[8] Y. Zhong, S. J. Yang, and F. G. Shi, "The relationship between L -subuniverses and L-convexities," *Journal of Intelligent & Fuzzy Systems*, vol. 33, no. 6, pp. 3363–3372, 2017.

[9] Y. Maruyama, "Lattice-valued fuzzy convex geometry," *Rims Kokyuroku*, vol. 1641, pp. 22–37, 2009.

[10] Q. Liu and F. G. Shi, "M-fuzzifying median algebras and its induced convexities," *Journal of Intelligent & Fuzzy Systems*, vol. 36, no. 1, pp. 1–9, 2018.

[11] V. P. Soltan, "D-convexity in graphs," *Soviet Mathematics Doklady*, vol. 28, no. 28, pp. 419–421, 1983.

[12] X. L. Xin and Y. L. Fu, "Some results of convex fuzzy sublattices," *Journal of Intelligent & Fuzzy Systems*, vol. 27, no. 1, pp. 287–298, 2014.

[13] Y. Shi and F. G. Shi, "Lattice-valued betweenness relations and its induced lattice-valued convex structures," *Journal of Intelligent & Fuzzy Systems*, vol. 37, no. 6, pp. 8523–8533, 2019.

[14] Y. J. Tan, "A note on the law of large numbers for fuzzy numbers," *Fuzzy Systems and Mathematics*, vol. 64, no. 1, pp. 59–61, 1994.

[15] J. Shan, "New characterizations of L-fuzzy ideals," *Fuzzy Systems and Mathematics*, vol. 16, no. 3, pp. 22–26, 2002.

[16] G. Muhiuddin, "Some new concepts on Int-soft ideals in ordered semigroups," *New Mathematics and Natural Computation*, vol. 17, no. 2, pp. 267–279, 2021.

[17] G. Muhiuddin, A. Mahboob, N. M. Khan, and D. Al-Kadi, "New types of fuzzy (m, n)-ideals in ordered semigroups,"

Journal of Intelligent & Fuzzy Systems, vol. 41, no. 6, pp. 6561–6574, 2021.

[18] J. M. Zhan and Y. B. Jun, “Generalized fuzzy interior ideals of semigroups,” *Neural Computing & Applications*, vol. 19, no. 4, pp. 515–519, 2010.

[19] L. J. Zhao, “L-fuzzy bi-ideals and L-fuzzy interior ideals on L-fuzzy semigroups,” *Journal of Capital Normal University*, vol. 2, pp. 5–8, 2002.

[20] F. G. Shi and X. Xin, “L-fuzzy subgroup degrees and L-fuzzy normal subgroup degrees,” *Journal of Advanced Research in Pure Mathematics*, vol. 3, no. 4, pp. 92–108, 2011.

[21] J. Li and F. G. Shi, “L-fuzzy convexity induced by L-convex fuzzy sublattice degree,” *Iranian Journal of Fuzzy Systems*, vol. 14, no. 5, pp. 83–102, 2017.

[22] F. G. Shi and Z. Y. Xiu, “(L, M)-fuzzy convex structures,” *Journal of Nonlinear Sciences and Applications*, vol. 10, no. 7, pp. 3655–3669, 2017.

[23] Y. F. Wen, Y. Zhong, and F. G. Shi, “L-fuzzy convexity induced by L-convex degree on vector spaces,” *Journal of Intelligent & Fuzzy Systems*, vol. 33, no. 6, pp. 4031–4041, 2017.

[24] Y. L. Han and F. G. Shi, “L-fuzzy convexity induced by L-convex fuzzy ideal degree,” *Journal of Intelligent & Fuzzy Systems*, vol. 36, no. 2, pp. 1705–1714, 2019.

[25] Y. Zhong, F. H. Chen, and F. G. Shi, “A new approach of describing L-fuzzy convexity,” *Journal of Intelligent & Fuzzy Systems*, vol. 37, no. 2, pp. 2253–2264, 2019.

[26] F. Mehmood, F. G. Shi, and K. Hayat, “A new approach to the fuzzification of rings,” *Journal of Nonlinear and Convex Analysis*, vol. 21, no. 12, pp. 2637–2646, 2020.

[27] M. Y. Zeng, F. G. Shi, and L. Wang, “A novel approach to the fuzzification of fields,” *Symmetry*, vol. 14, no. 6, p. 1190, 2022.

[28] J. N. Mordeson, D. S. Malik, and N. Kuroki, *Fuzzy Semigroups*, Springer-Verlag, Berlin, Heidelberg, Germany, 2003.

[29] G. J. Wang, “Theory of topological molecular lattices,” *Fuzzy Sets and Systems*, vol. 47, no. 3, pp. 351–376, 1992.