Research Article

Variable \(\lambda\)-Central Morrey Space Estimates for the Fractional Hardy Operators and Commutators

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This paper aims to show that the fractional Hardy operator and its adjoint operator are bounded on central Morrey space with variable exponent. Similar results for their commutators are obtained when the symbol functions belong to \(\lambda\)-central bounded mean oscillation (\(\lambda\)-central BMO) space with variable exponent.

1. Introduction

The boundedness of operators on function spaces is one of the core issues in harmonic analysis [1–3]. It is mainly because many problems in the theory of partial differential equations, in their simplified form, are reduced to the boundedness of operators on function spaces. It stimulates the research community to embark on such problems in this field. In this paper, we mainly obtain the boundedness of fractional Hardy operators [4]:

\[
H_\beta g(z) = \frac{1}{|z|^{n-\beta}} \int_{|t|<|z|} g(t) \, dt,
\]

\[
H_\beta^* g(z) = \int_{|t|>|z|} \frac{g(t)}{|t|^{n-\beta}} \, dt, \quad 0 < \beta < n, \, z \in \mathbb{R}^n \setminus \{0\},
\]

(1)
on variable exponent central Morrey spaces. In addition, commutators of these operators

\[
[b, H_\beta] g = b H_\beta g - H_\beta (bg),
\]

\[
[b, H_\beta^*] g = b H_\beta^* g - H_\beta^* (bg),
\]

(2)

with symbol functions \(b\) in variable \(\lambda\)-central BMO spaces are shown bounded on central Morrey spaces with variable exponent. However, before stating our main results, we need to introduce the reader to some basic definitions and preliminary results regarding variable exponent function spaces.

Notably, the function spaces with variable exponents have considerable importance in Harmonic analysis as well. Back in 1931, Orlicz [5] started the theory of variable exponent Lebesgue space. Musielak Orlicz spaces were defined and studied in [6]. The study of Sobolev and Lebesgue spaces with variable exponents in [7–11] further stimulated the subject. In the meantime, \(\lambda\)-central Morrey space, central BMO space, and associated function spaces have attractive applications by exploring estimates for operators along with their commutators [12–20]. Mizuta et al. defined the variable exponent nonhomogeneous \(\lambda\)-central Morrey space in [21]. The central BMO space first appeared in [22]. Meanwhile, the authors in [23] gave the definition of variable exponent central Morrey and \(\lambda\)-central BMO space along with some important results regarding the estimation of some operators. Recently, some publications [24–26] discussing the continuity of multilinear integral operators on these function spaces have added substantially to the existing literature on this topic.

The one-dimensional Hardy operator was firstly defined by Hardy in [27] and is considered a classical operator in operator theory. Its mathematical form can be obtained from
(1) by taking \( n = 1 \) and \( \beta = 0 \). Later on, different authors extended the definition of the one-dimensional Hardy operator to multidimensions in [28, 29]. As stated earlier, the fractional Hardy operator and its adjoint operator were introduced first in [4]. Following these publications, a flux of new results emerged discussing the boundedness of Hardy-type operators and their commutators on different function spaces [30–35]. The commutator operator also enjoyed a lot of attention from different zones of the globe [4, 20, 36–40]. However, the continuity of Hardy-type operators and their commutators on variable exponent function spaces took less attention by the research community worldwide [41–44]. The same is the case with central Morrey space with variable exponent. The present article aims to fill this gap by proving the boundedness of the fractional Hardy operator and its adjoint operator in this space. In Section 2 of this article, we denote by \( H^{0} \) throughout this article, we denote by \( H \) the mutators generated by \( H^{0} \) (or \( H^{0} ) \) and the \( \lambda \)-central BMO function \( b \) on the variable central Morrey space.

Let us describe the framework of this paper. In Section 2, we will remind some lemmas and propositions related to variable exponent function spaces. In Section 3 of this article, we will demonstrate the boundedness for Hardy operators and their commutators on central Morrey space with variable exponent. In Section 4, we shall investigate the similar estimates for the adjoint fractional Hardy operator and its commutators.

2. Function Spaces with Variable Exponents

In this section, we are going to introduce some notations and definitions related to the variable exponent function spaces. Throughout this article, we denote by \( |B| \) and \( \chi_{B} \) the Lebesgue measure and characteristic function of a measurable set \( B \in \mathbb{R}^{n} \), respectively. Also, \( B_{j} = B(0, 2^{j}) = \{x \in \mathbb{R}^{n}: |x| \leq 2 \} \) with \( A_{j} = \{x \in \mathbb{R}^{n}: 2^{j-1} < |x| \leq 2^{j} \} \) and \( \chi_{j} = \chi_{A_{j}} \) for \( j \in \mathbb{Z} \). The notation \( g = f \) implies that there exist two positive constants \( C_{1} \) and \( C_{2} \) such that \( C_{1} \leq g \leq C_{2} \) \( f \). Furthermore, \( E \subseteq \mathbb{R}^{n} \) represents an open set and \( p(\cdot): E \rightarrow [1, \infty) \) is a measurable function, and \( p' \) denotes the conjugate exponent of \( p(\cdot) \) which satisfies

\[
\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1.
\]

The set \( P(E) \) consists of all \( p(\cdot) \) and \( p'(\cdot) \) such that

\[
1 < p^{\ast} = \text{essinf}\{p(x): x \in E \} < p^{\ast \ast} = \text{esssup}\{p(x): x \in E \} < \infty.
\]

The space \( L^{p(\cdot)} \) is a set of all measurable function \( f \) on the open set \( E \), in such a way that, for positive \( \eta \),

\[
\int_{E} \left( \frac{|f(x)|}{\eta} \right)^{p(x)} \, dx < \infty,
\]

which becomes a Banach function space when equipped with the Luxemburg-norm

\[
\|f\|_{L^{p(\cdot)}(E)} = \inf \left\{ \eta > 0 : \int_{E} \left( \frac{|f(x)|}{\eta} \right)^{p(x)} \, dx \leq 1 \right\}.
\]

Local version of variable exponent Lebesgue space is denoted by \( L^{p(\cdot)}_{loc}(E) \) and is defined by

\[
L^{p(\cdot)}_{loc}(E) = \{f: f \in L^{p(\cdot)}(F) \forall \text{compact subset } F \subseteq E \}.
\]

We use \( \mathfrak{B}(\mathbb{R}^{n}) \) to denote a set containing \( p(\cdot) \in P(\mathbb{R}^{n}) \) satisfying the condition that the Hardy-Littlewood maximal operator \( M \):

\[
Mf = \sup_{r>0} \frac{1}{|B_{r}|} \int_{B_{r}} |f| \, dy,
\]

where \( B_{r} = \{y \in \mathbb{R}^{n}: |x - y| < r \} \) is bounded on \( L^{p(\cdot)}(\mathbb{R}^{n}) \).

Proposition 1 (see [8, 45]). Let \( E \) denote an open set and \( p(\cdot) \in P(E) \) fulfill the following inequalities:

\[
|p(x) - p(y)| \leq \frac{C}{\log((|x| + e)^{\gamma})} \leq |x - y|, \quad (9)
\]

\[
|p(x) - p(y)| \leq \frac{C}{\log(|x| + e)} \leq |x| \leq |y|, \quad (10)
\]

then \( p(\cdot) \in \mathfrak{B}(E) \), where \( C \) is a positive constant independent of \( x \) and \( y \).

Lemma 1 (see [7]) (generalized Hölder inequality). Let \( p(\cdot), p_{1}(\cdot), p_{2}(\cdot) \in P(E) \).

(a) If \( g \in L^{p(\cdot)}(E) \) and \( f \in L^{p'(\cdot)}(E) \), then we have

\[
\int_{E} |g(x)f(x)| \leq r_{p} \|g\|_{L^{p(\cdot)}(E)} \|f\|_{L^{p'(\cdot)}(E)},
\]

where \( r_{p} = 1 + 1/p_{1} - 1/p_{2} \).

(b) If \( g \in L^{p(\cdot)}(E) \), \( f \in L^{p(\cdot)}(E) \), and \( 1/p(\cdot) = 1/p_{1}(\cdot) + 1/p_{2}(\cdot) \), then we have

\[
\|gf\|_{L^{p(\cdot)}(E)} \leq r_{p,p_{1}} \|g\|_{L^{p(\cdot)}(E)} \|f\|_{L^{p(\cdot)}(E)},
\]

where \( r_{p,p_{1}} = (1 + 1/(p_{1} - p_{1}))^{1/p_{1}} \).

Lemma 2 (see [46]). If \( p(\cdot) \in \mathfrak{B}(\mathbb{R}^{n}) \), then there exist constants \( 0 < \delta < 1 \) and a positive constant \( C \) such that for all balls \( B \in \mathbb{R}^{n} \) and all measurable subsets \( S \subseteq B \),

\[
\begin{align*}
\|\chi_{S}\|_{L^{p(\cdot)}(\mathbb{R}^{n})} & \leq C \frac{|S|}{|B|}, \\
\|\chi_{\partial B}\|_{L^{p(\cdot)}(\mathbb{R}^{n})} & \leq C \left( \frac{|S|}{|B|} \right)^{\delta}.
\end{align*}
\]

Remark 1. Let \( p(\cdot) \in P(\mathbb{R}^{n}) \) and meet conditions (9) and (10) in Proposition 1, then so does \( p'(\cdot) \). This implies that...
Lemma 3 (see [46]). Assuming that \( p(\cdot) \in \mathfrak{B}(\mathbb{R}^n) \), for all balls \( B \subset \mathbb{R}^n \) and for a positive constant \( C \), the following inequality holds:

\[
C^{-1} \lesssim \frac{1}{|B|} \| \mathbf{x} \|_{L^p(\mathbb{R}^n)} \| \mathbf{x} \|_{L^p(\mathbb{R}^n)} \lesssim C.
\]

Definition 1 (see [47]). Let \( f \in L_{loc}^1(\mathbb{R}^n) \), set

\[
\| b \|_{\text{BMO}(\mathbb{R}^n)} = \sup_{B \text{ ball}} \frac{1}{|B|} \int_B |b(x) - b_B| dx,
\]

where supremum is taken over all the ball \( B \subset \mathbb{R}^n \) and \( b_B = |B|^{-1} \int_B b(y) dy \). The function \( b \) is known as bounded mean oscillation if \( \| b \|_{\text{BMO}(\mathbb{R}^n)} < \infty \) and \( \text{BMO}(\mathbb{R}^n) \) consist of all \( \| b \|_{\text{BMO}(\mathbb{R}^n)} < \infty \).

Lemma 4 (see [48]). Let \( p(\cdot) \in P(\mathbb{R}^n) \), then for all \( b \in \text{BMO}(\mathbb{R}^n) \) and all \( i, l \in \mathbb{Z} \) with \( l > i \), we have

\[
C^{-1} \| b \|_{\text{BMO}(\mathbb{R}^n)} \lesssim \sup_{B \text{ ball}} \frac{1}{|B|} \| \mathbf{x} \|_{L^p(\mathbb{R}^n)} \| (b - b_B) \mathbf{x} \|_{L^p(\mathbb{R}^n)} \lesssim C \| b \|_{\text{BMO}(\mathbb{R}^n)},
\]

\[
\| (b - b_B) \mathbf{x} \|_{L^p(\mathbb{R}^n)} \lesssim C |l - i| \| b \|_{\text{BMO}(\mathbb{R}^n)} \| \mathbf{x} \|_{L^p(\mathbb{R}^n)}.
\]

Definition 2 (see [23]). Let \( p(\cdot) \in P(\mathbb{R}^n) \) and \( \lambda \in \mathbb{R} \). Then, the variable exponent central Morrey space \( \mathfrak{B}^{p(\cdot),\lambda}(\mathbb{R}^n) \) is defined as

\[
\mathfrak{B}^{p(\cdot),\lambda}(\mathbb{R}^n) = \left\{ f \in L_{loc}^{p(\cdot)}(\mathbb{R}^n) : \| f \|_{\mathfrak{B}^{p(\cdot),\lambda}(\mathbb{R}^n)} < \infty \right\},
\]

where

\[
\| f \|_{\mathfrak{B}^{p(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{R > 0} \left\{ \frac{\| f \mathbf{1}_{(0,R)} \|_{L^p(\mathbb{R}^n)}}{|B(0,R)|^{1/p(\cdot)} \| \mathbf{1}_{(0,R)} \|_{L^p(\mathbb{R}^n)}} \right\}.
\]
Proof. By definition of the fractional Hardy operator and Lemma 1, it is easy to see that
\[
\left| H_\beta f(x) \chi_k(x) \right| \leq \frac{1}{|\chi_k|^{m-\beta}} \int_{B_k} |f(t)| dt \chi_k(x)
\]
\[
\leq C2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \| f \|_{L^p(R^n)} \| \chi_k \|_{L^r(R^n)}
\]
(29)

Taking the $L^p(R^n)$ norm on both sides, we have
\[
\| H_\beta f \chi_k \|_{L^p(R^n)} \leq C2^{\beta} \sum_{j=-\infty}^{k} \| f \|_{L^p(R^n)} \| \chi_k \|_{L^r(R^n)} \| B_k \|_{L^p(R^n)}^{-1}
\]
(30)
\[
\leq C2^{\beta} \sum_{j=-\infty}^{k} \| f \|_{L^p(R^n)} \| \chi_k \|_{L^r(R^n)} \| B_k \|_{L^p(R^n)}^{-1}
\]
(31)
\[
\leq C2^{\beta} \sum_{j=-\infty}^{k} 2^{\frac{\delta j}{2}(j-k)} \| f \|_{L^p(R^n)} \| \chi_k \|_{L^r(R^n)} \| B_k \|_{L^p(R^n)}^{-1}
\]
(32)

In view of the condition $1/p_j(x) = 1/p_j(x) - \beta/n$ and Lemma 5, the last inequality reduces to the following inequality:
\[
\| H_\beta f \chi_k \|_{L^p(R^n)} \leq C\| f \|_{B^{\beta}_{p_j}(R^n)} \sum_{j=-\infty}^{k} 2^{\frac{\delta j}{2}(j-k)} |B_j|^{\delta/n} \| \chi_k \|_{L^r(R^n)}
\]
(33)

Since
\[
\| \chi_k \|_{L^r(R^n)} \approx |B|^{1/p_j} \approx |B|^{1/p_j + \beta/n} \approx |B|^{\beta/n} \| \chi_k \|_{L^r(R^n)}
\]
(34)

therefore, from the inequality (32), we infer that
\[
\| H_\beta f \chi_k \|_{L^p(R^n)} \leq C\| f \|_{B^{\beta}_{p_j}(R^n)} \sum_{j=-\infty}^{k} 2^{\frac{\delta j}{2}(j-k)} |B_j|^{\delta/n} \| \chi_k \|_{L^r(R^n)}
\]
(35)

Finally, inequality (14) helps us to have
\[
\| H_\beta f \|_{B^{\beta}_{p_j}(R^n)} \leq C\| f \|_{B^{\beta}_{p_j}(R^n)} \sum_{j=-\infty}^{k} 2^{\delta j} \left( \delta_j + \lambda_j + \lambda_j \right)
\]
(36)

Theorem 2. Let $0 < \beta < n$ and let $p(\cdot), q(\cdot), r(\cdot) \in P(R^n)$ and satisfying conditions (9) and (10) in Proposition 1 with $p(\cdot) < n/\beta, p'(\cdot) < r(\cdot)$ and
\[ \frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{r(\cdot)} - \frac{\beta}{n} \quad (37) \]

Let \( 0 < \nu < 1/n \) and \(-1/q_+ < \mu \). If \( \mu = \nu + \lambda + \beta/n \), with 
\[ \max(-\nu + 1, -(\delta_1 + \delta_3 + \beta/n)) < \lambda, \] 
where \( \delta_1, \delta_3 \) are the same constants as appeared in inequalities (14) and (15), and 
\( b \in \|b\|_{CBMO^{(\lambda)}} \), then

\[ \left\| [b, H_\beta] f \right\|_{l^1(\mu)} \leq C \|b\|_{CBMO^{(\lambda)}} \|f\|_{l^1(\mu)}. \] 

\[ \left\| [b, H_\beta] f (x) \cdot \chi_B(x) \right\| \leq \frac{1}{|x|^{n-\beta}} \int_{|y| \leq |x|} |(b(x) - b(y)) f(y)| \, dy \cdot \chi_B(x) \]

\[ \leq \frac{1}{|x|^{n-\beta}} \int_{|y| \leq |x|} |(b(x) - b_B) f(y)| \, dy \cdot \chi_B(x) \]

\[ + \frac{1}{|x|^{n-\beta}} \int_{|y| \leq |x|} (b(y) - b_B) f(y) \, dy \cdot \chi_B(x) \]

\[ = A_1 + A_2. \]

**Proof.** We decompose the integral appearing in the commutator operator as

Let us first estimate \( A_1 \). By taking the variable Lebesgue space norm on both sides, we get

\[ \left\| A_1 \right\|_{L^1(\mu)} = \left\| (b(\cdot) - b_B) H_\beta f(\cdot) \chi_B(\cdot) \right\|_{L^1(\mu)}. \] 

Taking into consideration the condition
\( 1/q(\cdot) = 1/s(\cdot) + 1/r(\cdot) \) and \( 1/p(\cdot) = 1/\beta(\cdot) - \beta/n \), the generalized Hölder inequality gives us the following estimation of \( A_1 \):

\[ \left\| A_1 \right\|_{L^1(\mu)} \leq \left\| (b(\cdot) - b_B) \chi_B \right\|_{L^1(\mu)} \left\| H_\beta f \right\|_{L^1(\mu)} \]

\[ = C \|b\|_{CBMO^{(\lambda)}} \|B\|^{1/q_+} \left\| \chi_B \right\|_{L^1(\mu)} \left\| f \right\|_{l^1(\mu)}. \]

Therefore, on account of the condition \( \mu = \nu + \sigma \) for \( A_1 \), we have

\[ \left\| A_1 \right\|_{L^1(\mu)} \leq C \|B\|^{1/q_+} \left\| \chi_B \right\|_{L^1(\mu)} \left\| f \right\|_{l^1(\mu)}. \]

Next, we consider \( A_2 \) for approximation:

\[ A_2 = \frac{1}{|x|^{n-\beta}} \int_{|y| \leq |x|} |(b(y) - b_B) f(y)| \, dy \cdot \chi_B(x), \]

which can be decomposed further as

\[ A_2 = \sum_{k=\infty}^{0} \frac{1}{|x|^{n-\beta}} \sum_{j=\infty}^{k} \int_{2^j B_{-2^j} \cdot 2^j B} |(b(y) - b_B) f(y)| \, dy \cdot \chi_{2^j B_{-2^j} \cdot 2^j B}(x) \]

\[ \leq \sum_{k=\infty}^{0} \frac{1}{|x|^{n-\beta}} \sum_{j=\infty}^{k} \int_{2^j B_{-2^j} \cdot 2^j B} |(b(y) - b_{2^j B}) f(y)| \, dy \cdot \chi_{2^j B_{-2^j} \cdot 2^j B}(x) \]

\[ + \sum_{k=\infty}^{0} \frac{1}{|x|^{n-\beta}} \sum_{j=\infty}^{k} \int_{2^j B_{-2^j} \cdot 2^j B} |(b_B - b_{2^j B}) f(y)| \, dy \cdot \chi_{2^j B_{-2^j} \cdot 2^j B}(x) \]

\[ = A_{21} + A_{22}. \]
where

\[
A_{21} = \sum_{k = -\infty}^{0} 2^k B^\beta \sum_{j = -\infty}^{k} \int_{2^j B - 2^{j+1} B} [(b(y) - b_{2^j B}) f(y)] dy \cdot \chi_{2^j B - 2^{j+1} B}(x).
\]

We define a new variable \( t(\cdot) \) such that \( 1/t(\cdot) = 1/\ell'(\cdot) = 1/\ell(\cdot) \), then by the generalized Hölder inequality, we have

\[
A_{21} \leq C \sum_{k = -\infty}^{0} 2^k B^\beta \sum_{j = -\infty}^{k} \| (b(y) - b_{2^j B}) \chi_{2^j B - 2^{j+1} B}(x) \|_{L^\ell(\cdot)} \| \chi_{2^j B} \|_{L^{\ell(\cdot)}} \| \chi_{2^{j+1} B} \|_{L^{\ell(\cdot)}}
\]

\[
= C \sum_{k = -\infty}^{0} 2^k B^\beta \sum_{j = -\infty}^{k} \| (b(y) - b_{2^j B}) \chi_{2^j B - 2^{j+1} B}(x) \|_{L^\ell(\cdot)} \| \chi_{2^j B} \|_{L^{\ell(\cdot)}} \| \chi_{2^{j+1} B} \|_{L^{\ell(\cdot)}}
\]

\[
\leq C \sum_{k = -\infty}^{0} 2^k B^\beta \sum_{j = -\infty}^{k} \| (b(y) - b_{2^j B}) \chi_{2^j B - 2^{j+1} B}(x) \|_{L^\ell(\cdot)} \| b_{2^j B} \|_{L^{\ell(\cdot)}} \| b_{2^{j+1} B} \|_{L^{\ell(\cdot)}}
\]

\[
= C \| b \|_{CBMO(\ell(\cdot))} \| f \|_{g^\alpha(\ell(\cdot))} \| \chi_{2^j B} \|_{L^{\ell(\cdot)}} \| \chi_{2^{j+1} B} \|_{L^{\ell(\cdot)}}
\]

With the Lebesgue space with variable exponent norm on both sides, the above inequality takes the following form:

\[
\| A_{21} \|_{L^\ell(\cdot)} \leq C \| b \|_{CBMO(\ell(\cdot))} \| f \|_{g^\alpha(\ell(\cdot))} \| \chi_{2^j B} \|_{L^{\ell(\cdot)}} \| \chi_{2^{j+1} B} \|_{L^{\ell(\cdot)}}
\]

\[
= C \| b \|_{CBMO(\ell(\cdot))} \| f \|_{g^\alpha(\ell(\cdot))} \| b_{2^j B} \|_{L^{\ell(\cdot)}} \| b_{2^{j+1} B} \|_{L^{\ell(\cdot)}}
\]

\[
= C \| b \|_{CBMO(\ell(\cdot))} \| f \|_{g^\alpha(\ell(\cdot))} \| \chi_{2^j B} \|_{L^{\ell(\cdot)}} \| \chi_{2^{j+1} B} \|_{L^{\ell(\cdot)}}
\]

Finally, consider

\[
A_{22} = \sum_{k = -\infty}^{0} 2^k B^\beta \sum_{j = -\infty}^{k} \int_{2^j B - 2^{j+1} B} [(b(y) - b_{2^j B}) f(y)] dy \cdot \chi_{2^j B - 2^{j+1} B}(x).
\]
In turn, $A_{22}$ satisfies the below inequality:

$$
A_{22} \leq C \sum_{k=0}^{\infty} X^{2B-2-1}_2\mathcal{B}(x) \left\| b^k 2^{k(n-1)} \sum_{j=0}^{\infty} 2^{j+1} B \right\| f \mathcal{B}_{\mathcal{B}}(x) \left\| f \mathcal{B}_{\mathcal{B}}(x) \right\| \text{L}^{p(\cdot)}(\mathbb{R}^n)
$$

4. Adjoint Fractional Hardy Operator and Commutator

In this last section, we first establish the boundedness of adjoint fractional Hardy operator and then use it to prove the boundedness of commutator generated by this operator and λ-central BMO function $b$. The first result is as follows.

**Theorem 3.** Let $p_1(\cdot) \in P(\mathbb{R}^n)$ and satisfying conditions (9) and (10) in Proposition 1, define the variable exponent $p_2(\cdot)$ by

$$
\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\beta}{n}
$$

If $\lambda_2 = \lambda_1 + \beta/n$ and $\lambda_1 < (\delta_2 - 1) - \beta/n$, where $\delta_2$ is the same constant as appeared in inequality (15), then

$$
\left\| H_\beta^* f \right\|_{p_2(\cdot)} \leq C \left\| f \right\|_{p_1(\cdot)}.
$$

**Proof.** Since
\[ |H_\beta f(x)\chi(x)| \leq \sum_{j=k+1}^{\infty} 2^{(j-k)\beta-n} \|f\|_{L^1(R^n)}\|\chi_j\|_{L^1(R^n)} \chi_j(x), \]

from which we infer that
\[ \|H_\beta^* f(x)\chi\|_{L^p(R^n)} \leq C \sum_{j=k+1}^{\infty} 2^{(j-k)\beta-n} \]
\[ \cdot \|f\|_{L^1(R^n)}\|\chi_j\|_{L^1(R^n)} \chi_j\|_{L^2(R^n)} \chi_j, \]

(60)

where we made use of inequality (15) in the last step of the above result. Hence, we obtain
\[ \|H_\beta^* f\|_{\mathcal{B}^{(1/2)}(R^n)} \leq C \|f\|_{\mathcal{B}^{(1/2)}(R^n)} \]
\[ \sum_{j=k+1}^{\infty} 2^{(j-k)(\beta-n\beta)} \|\chi_j\|_{L^2(R^n)} \]
\[ \|B_k\|_{L^2(R^n)}, \]

(63)

Utilizing the condition \( \lambda_2 = \lambda_1 + \beta / n \), a result similar to (33) and Lemma 2, we obtain
\[ \|H_\beta^* f\|_{\mathcal{B}^{(1/4)}(R^n)} \leq C \|f\|_{\mathcal{B}^{(1/4)}(R^n)} \sum_{j=k+1}^{\infty} 2^{(j-k)(\beta/n)} \|\chi_j\|_{L^2(R^n)} \chi_j, \]

(64)

Finally, the series is convergent due to the fact that \( \lambda_1 < (\delta_2 - 1) - \beta / n \), and hence the result.

Keeping in view the analysis made in the previous section, we only outline the proof of the following theorem without going into many details.

**Theorem 4.** Let \( p(\cdot), q(\cdot), r(\cdot) \in P(R^n) \) and satisfying conditions (9) and (10) in Proposition 1 with \( p(\cdot) < n/\beta, p'(\cdot) < r(\cdot) \) and
\[ \frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{r(\cdot)} - \frac{\beta}{n}, \]

(65)

Lemma 3 guides us to have the following inequality:
\[ \|H_\beta^* f(x)\chi\|_{L^q(R^n)} \leq C \sum_{j=k+1}^{\infty} 2^{j\beta} \]
\[ \|f\|_{L^1(R^n)}\|\chi_j\|_{L^1(R^n)} \chi_j\|_{L^2(R^n)} \chi_j, \]

(61)

which by Lemma 5 reduces to the following one:
\[ \|H_\beta^* f(x)\chi\|_{L^q(R^n)} \leq C \sum_{j=k+1}^{\infty} 2^{j\beta} \]
\[ \|f\|_{L^1(R^n)}\|\chi_j\|_{L^1(R^n)} \chi_j\|_{L^2(R^n)} \chi_j, \]

(62)

Let \( 0 < \nu < 1/n \) and \( -1/q, \mu \leq 0 \). If \( \mu = \nu + \lambda + \beta/n \), then
\[ \|b_{1/2} H_\beta^* f\|_{\mathcal{B}^{(1/2)}(R^n)} \leq C \|b_{1/2} H_\beta f\|_{\mathcal{B}^{(1/2)}(R^n)}. \]

(66)

Proof. As in the previous section, we start from decomposing the integral:
\[ \|b_{1/2} H_\beta^* f\|_{\mathcal{B}^{(1/2)}(R^n)} \leq \int \left| \frac{[b_{1/2} f(x)] f(x) - b_{1/2} f(x)}{[y]^{\nu+\beta}} \right| \]
\[ \chi_b \|\chi_{b_{1/2}}\|_{L^2(R^n)} \chi_{b_{1/2}} \|\chi_{b_{1/2}}\|_{L^2(R^n)} \chi_{b_{1/2}}, \]

(67)

Following the steps taken to approximate \( A_1 \) in Theorem 2, we directly estimate \( D_1 \) as follows:
\[ \|D_1\|_{L^q(R^n)} \leq C \|b_{1/2} f\|_{L^q(R^n)} H_\beta^* f \chi_b \|\chi_{b_{1/2}}\|_{L^2(R^n)} |B| \|b_{1/2} f\|_{L^q(R^n)} B. \]

(68)
where $\sigma = \lambda + \beta/n$. Now, the result (43) and Theorem 3 help us to write

$$\|D_1\|_{L^p(\mathbb{R}^n)} \leq C\|b\|_{\text{CBMO}^{(\lambda,\beta)}} \|X_2\|_{L^p(\mathbb{R}^n)} \|f\|_{B^{p,1}}.$$  \hfill (69)

Next, comparing $D_2$ with $A_2$ of Theorem 2, we arrive at

\begin{align*}
D_2 &\leq \sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} 2^{j}\| (b(y) - b_{2j}) x_j \|_{L^{\infty}} \int \frac{|(b(y) - b_{2j}) f(y)|}{2^{j}\| (b(y) - b_{2j}) x_j \|_{L^{\infty}}} \, dy \\
&\quad + \sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} 2^{j}\| (b(y) - b_{2j}) x_j \|_{L^{\infty}} \int \frac{|(b(y) - b_{2j}) f(y)|}{2^{j}\| (b(y) - b_{2j}) x_j \|_{L^{\infty}}} \, dy \\
&= D_{21} + D_{22}.
\end{align*}

and for the approximation of $D_{21}$, we follow a procedure similar to the one followed in the estimation of $A_{21}$. Hence, we get

\begin{align*}
D_{21} &\leq C\sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} 2^{j}\| (b(y) - b_{2j}) x_j \|_{L^{\infty}} \int \frac{|(b(y) - b_{2j}) f(y)|}{2^{j}\| (b(y) - b_{2j}) x_j \|_{L^{\infty}}} \, dy \\
&\quad \leq C\sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} 2^{j}\| (b(y) - b_{2j}) x_j \|_{L^{\infty}} \int \frac{|(b(y) - b_{2j}) f(y)|}{2^{j}\| (b(y) - b_{2j}) x_j \|_{L^{\infty}}} \, dy \\
&\quad \leq C\|b\|_{\text{CBMO}^{(\lambda,\beta)}} \|f\|_{B^{p,1}} \sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} 2^{j}\| (b(y) - b_{2j}) x_j \|_{L^{\infty}} \\
&\quad \leq C\|b\|_{\text{CBMO}^{(\lambda,\beta)}} \|f\|_{B^{p,1}} \sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} 2^{j}\| (b(y) - b_{2j}) x_j \|_{L^{\infty}} \\
&\quad \leq C\|b\|_{\text{CBMO}^{(\lambda,\beta)}} \|f\|_{B^{p,1}} \sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} 2^{j}\| (b(y) - b_{2j}) x_j \|_{L^{\infty}}.
\end{align*}

Eventually, it is easy to see that

\begin{align*}
\|D_{21}\|_{L^p(\mathbb{R}^n)} &\leq C\|b\|_{\text{CBMO}^{(\lambda,\beta)}} \|f\|_{B^{p,1}} \sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} 2^{j}\| (b(y) - b_{2j}) x_j \|_{L^{\infty}} \\
&\quad \leq C\|b\|_{\text{CBMO}^{(\lambda,\beta)}} \|f\|_{B^{p,1}} \sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} 2^{j}\| (b(y) - b_{2j}) x_j \|_{L^{\infty}} \\
&\quad \leq C\|b\|_{\text{CBMO}^{(\lambda,\beta)}} \|f\|_{B^{p,1}} \sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} 2^{j}\| (b(y) - b_{2j}) x_j \|_{L^{\infty}} \\
&\quad \leq C\|b\|_{\text{CBMO}^{(\lambda,\beta)}} \|f\|_{B^{p,1}} \sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} 2^{j}\| (b(y) - b_{2j}) x_j \|_{L^{\infty}}.
\end{align*}

\hfill (72)
Next, by virtue of inequality (53), \( D_{22} \) satisfies

\[
D_{22} \leq C \sum_{k = -\infty}^{0} \chi_{2^k B_{2^{-1} B}}(x) \sum_{j = k+1}^{\infty} |2^j B|^\beta/(n-1) \|b\|_{\text{CBMO}^{(\alpha, \beta)}} \|f\|_{L^p(\cdot)} \|f_X^{2^j B}\| \|X^{2^j B}_{2^{-1} B}\|_{L^p(\cdot)}
\]

\[
\leq C\|b\|_{\text{CBMO}^{(\alpha, \beta)}} \sum_{k = -\infty}^{0} \chi_{2^k B_{2^{-1} B}}(x) \sum_{j = k+1}^{\infty} |2^j B|^\beta/(n-1) \|f\|_{L^p(\cdot)} \|f_X^{2^j B}\| \|X^{2^j B}_{2^{-1} B}\|_{L^p(\cdot)}
\]

\[
\leq C\|b\|_{\text{CBMO}^{(\alpha, \beta)}} \sum_{k = -\infty}^{0} \chi_{2^k B_{2^{-1} B}}(x) \sum_{j = k+1}^{\infty} |2^j B|^\beta/(n-1) \|f\|_{L^p(\cdot)} \|f_X^{2^j B}\| \|X^{2^j B}_{2^{-1} B}\|_{L^p(\cdot)}
\]

\[
\leq C\|b\|_{\text{CBMO}^{(\alpha, \beta)}} \|f\|_{L^p(\cdot)} \sum_{k = -\infty}^{0} \chi_{2^k B_{2^{-1} B}}(x) \sum_{j = k+1}^{\infty} |2^j B|^\beta/(n-1) \|f\|_{L^p(\cdot)} \|f_X^{2^j B}\| \|X^{2^j B}_{2^{-1} B}\|_{L^p(\cdot)}
\]

To finish the estimation, we take norm on both sides of the above inequality to obtain

\[
\|D_{22}\|_{L^p(\cdot)} \leq C\|b\|_{\text{CBMO}^{(\alpha, \beta)}} \|f\|_{L^p(\cdot)} \sum_{k = -\infty}^{0} |k + 1| \|2^{k+1} B\|^\mu \|X^{2^k B}_{2^{-1} B}\|_{L^p(\cdot)}
\]

\[
\leq C\|b\|_{\text{CBMO}^{(\alpha, \beta)}} \|f\|_{L^p(\cdot)} \sum_{k = -\infty}^{0} |k + 1| \|2^{k+1} B\|^\mu \|2^k B\|^{1/\lambda}(\cdot)
\]

\[
\leq C\|b\|_{\text{CBMO}^{(\alpha, \beta)}} \|f\|_{L^p(\cdot)} \sum_{k = -\infty}^{0} |k + 1| \|2^{k+1} B\|^\mu \|B\|^{1/\lambda}(\cdot)
\]

(74)

In the end, combining all the estimates of \( D_1, D_2, D_{21}, D_{22} \), we arrive at the following conclusive inequality:

\[
\| [b, H^\alpha_p] f_X \|_{L^p(\cdot)(\mathbb{R}^n)} \leq C\|b\|_{\text{CBMO}^{(\alpha, \beta)}} \|f\|_{L^p(\cdot)} \|B\| \|f_X^{2^j B}\| \|X^{2^j B}_{2^{-1} B}\|_{L^p(\cdot)}
\]

(75)

which is as desired.

**Data Availability**

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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