Research Article

Some Results on Pixley–Roy Hyperspaces

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In this paper, we prove that if a space X has a point-countable cn -network, then the Pixley–Roy hyperspace $\mathbf{PR}[X]$ also has a point-countable cn -network. If X is a regular space with a point-countable ck -network, then so does the Pixley–Roy hyperspace $\mathbf{PR}[X]$. Moreover, if X has a point-countable sp -network (resp., strict Pytkeev network), then the Pixley–Roy hyperspace $\mathbf{PR}_2[X]$ also has a point-countable sp -network (resp., strict Pytkeev network). On the other hand, we show that if the Pixley–Roy hyperspace $\mathbf{PR}[X]$ has a countable cn -network (resp., sp -network and strict Pytkeev network), then so does X . By these results, we obtain that if the Pixley–Roy hyperspace $\mathbf{PR}[X]$ is a cosmic space (resp., $\mathfrak{P}_0$ -space, strict $\mathfrak{P}_0$ -space, and strict $\mathfrak{P}_0$ -space), then so is X . Furthermore, the Pixley–Roy hyperspace $\mathbf{PR}_n[S_2]$ is not a k -space for each $n \geq 2$.

1. Introduction and Preliminaries

The topological and generalized metric properties on Pixley–Roy hyperspaces have been studied by many authors from different points of view, as shown in [1–14]. Especially, they studied the relation between a space X satisfying such a property and its Pixley–Roy hyperspaces satisfying the same property.

In this paper, we study relationships between some generalized metric properties of a space X and generalized metric properties of Pixley–Roy hyperspaces over X . We prove that if X has a point-countable cn -network, then $\mathbf{PR}[X]$ also has a point-countable cn -network. If X is regular with a point-countable ck -network, then so does $\mathbf{PR}[X]$. Furthermore, if $\mathbf{PR}[X]$ has a countable cn -network, then X also has a countable cn -network. Next, we study the notion of strict Pytkeev network which was introduced by T. Banach in 2015 [15], and the notion of sp -network that

was introduced by X. Liu, C. Liu, and S. Lin in 2019 [16] on Pixley–Roy hyperspaces and obtained the following results:

- (1) If X has a point-countable sp -network (resp., strict Pytkeev network), then so does $\mathbf{PR}_2[X]$
- (2) If $\mathbf{PR}[X]$ has a countable sp -network (resp., strict Pytkeev network), then so does X

We also obtain that if X is regular and $\mathbf{PR}[X]$ is a cosmic space (resp., $\mathfrak{P}_0$ -space, strict $\mathfrak{P}_0$ -space, and strict $\mathfrak{P}_0$ -space), then X is also a cosmic space (resp., $\mathfrak{P}_0$ -space, strict $\mathfrak{P}_0$ -space, and strict $\mathfrak{P}_0$ -space). Moreover, we show that there exists a space X with a countable sp -network, but $\mathbf{PR}[X]$ and $\mathbf{PR}_n[X]$ with $n \geq 3$ do not have point-countable sp -networks.

On the other hand, in [9], M. Sakai proved that for each $n \in \mathbb{N}$, $\mathbf{PR}_n[S_\omega]$ is sequential but $\mathbf{PR}_3[S_\omega]$ is not Fréchet–Urysohn. Moreover, neither $\mathbf{PR}_2[X]$ nor $\mathbf{PR}[S_\omega]$ is

a k -space. However, he does not know whether or not $\mathbf{PR}_n[S_2]$ is a k -space for each $n \in \mathbb{N}$. By the abovementioned results, we show that $\mathbf{PR}_1[S_2]$ is second-countable but $\mathbf{PR}_n[S_2]$ is not a k -space for each $n \geq 2$.

Throughout this paper, all spaces are assumed to be at least T_1 . $\mathbb{N}$ denotes the set of all positive integers, and the first infinite ordinal is denoted by ω . Moreover, if $\mathcal{P}$ is a family of subsets of a space X and $G \subset X$, then we set $(\mathcal{P})_G = \{P \in \mathcal{P} : P \cap G \neq \emptyset\}$.

The Pixley–Roy hyperspace $\mathbf{PR}[X]$ over a space X , defined by C. Pixley and P. Roy in [6], is the set of all nonempty finite subsets of X with the topology generated by the sets of the form,

$$[F, V] = \{G \in \mathbf{PR}[X] : F \subset G \subset V\}, \quad (1)$$

where $F \in \mathbf{PR}[X]$, and V is an open subset in X containing F . It is known that $\mathbf{PR}[X]$ is always zero-dimensional and completely regular [3].

For each $n \in \mathbb{N}$, let

$$\mathbf{PR}_n[X] = \{F \in \mathbf{PR}[X] : |F| \leq n\}. \quad (2)$$

Remark 1 (see [12]). For each $n \in \mathbb{N}$, $\mathbf{PR}_n[X]$ is a closed subspace of $\mathbf{PR}[X]$ and, in particular, $\mathbf{PR}_1[X]$ is a closed discrete subspace of $\mathbf{PR}[X]$.

By Remark 1, we have the following remark.

Remark 2. Every $\mathbf{PR}_m[X]$ is a closed subspace of $\mathbf{PR}_n[X]$ for each $m, n \in \mathbb{N}$ with $m < n$.

For each $F \in \mathbf{PR}[X]$ and $A \subset X$, denote

$$[F, A] = \{H \in \mathbf{PR}[X] : F \subset H \subset A\}. \quad (3)$$

Moreover, if $\mathcal{A} \subset \mathbf{PR}[X]$, then we denote the closure of $\mathcal{A}$ in $\mathbf{PR}[X]$ by $Cl(\mathcal{A})$.

Definition 1. Let $\mathcal{P}$ be a family of subsets of a space X .

- (1) $\mathcal{P}$ is a network [17] for X , if for any neighborhood U of a point $x \in X$, there exists a set $P \in \mathcal{P}$ such that $x \in P \subset U$
- (2) $\mathcal{P}$ is a k -network [18] for X , if whenever K is a compact subset of an open set U in X , there exists a finite subfamily $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \bigcup \mathcal{F} \subset U$
- (3) $\mathcal{P}$ is a cn -network [19] for X , if for any neighborhood U of a point $x \in X$, the set $\bigcup\{P \in \mathcal{P} : x \in P \subset U\}$ is a neighborhood of x
- (4) $\mathcal{P}$ is a ck -network [19] for X , if for any neighborhood U of a point $x \in X$, there is a neighborhood U_x of x such that for each compact subset $K \subset U_x$, there exists a finite subfamily $\mathcal{F} \subset \mathcal{P}$ satisfying $x \in \bigcap \mathcal{F}$ and $K \subset \bigcup \mathcal{F} \subset U$
- (5) $\mathcal{P}$ is a cp -network [19] for X , if for each $x \in X$ if either x is an isolated point of X and $\{x\} \in \mathcal{P}$, or for

each subset A of X with $x \in \overline{A} \setminus A$ and for each neighborhood U of x there exists a set $P \in \mathcal{P}$ such that $P \cap A$ is infinite and $x \in P \subset U$

- (6) $\mathcal{P}$ is a cs^* -network [15] in X if for each $x \in X$, each neighborhood U of x and any sequence $(x_n) \subset X$ converging to x there is $P \in \mathcal{P}$ such that $P \subset U$ and P contains infinitely many members of the sequence (x_n)
- (7) $\mathcal{P}$ is a Pytkeev network [20–22] (resp., strict Pytkeev network) [15] for X , if $\mathcal{P}$ is a network for X , and for each neighborhood U of a point x in X and each subset A of X accumulating at x , there exists a set $P \in \mathcal{P}$ such that $P \cap A$ is infinite and $P \subset U$ (resp., $x \in P \subset U$)
- (8) $\mathcal{P}$ is an sp -network [16] for X , if for each $x \in U \cap \overline{A}$ with U open and A subset in X , there is a set $P \in \mathcal{P}$ such that $x \in P \subset U$ and $x \in \overline{P \cap A}$
- (9) $\mathcal{P}$ is called point-countable [17], if the family $\{P \in \mathcal{P} : x \in P\}$ is countable for each $x \in X$

Remark 3 (see [19]). Bases $\Rightarrow ck$ -networks $\Rightarrow cn$ -networks $\Rightarrow$ networks.

Remark 4 (see [16]).

- (1) In [15, p. 152], it was said that a subset A of a topological space X accumulates at a point $x \in X$ if each neighborhood of x contains infinitely many points of the set A . It is obvious that for a T_1 -space X , a point $x \in X$ is an accumulation point of a set $A \subset X$ if and only if $x \in \overline{A \setminus \{x\}}$.
- (2) If X is a T_1 -space, then $\mathcal{P}$ is a strict Pytkeev network if and only if it is a cp -network for X .
- (3) Every base for X is an sp -network; every sp -network for X is a strict Pytkeev network (i.e., a cp -network), and the converse is also true for a Fréchet–Urysohn space; every strict Pytkeev network is a Pytkeev network.

Definition 2. Let X be a regular space.

- (1) X is said to be a cosmic space [15], if X has a countable network
- (2) X is said to be a (strict) $\mathfrak{P}_0$ -space [15], if X has a countable (strict) Pytkeev network
- (3) X is said to be a stric $\mathfrak{P}_0$ -space [16], if X has a countable sp -network

Remark 5 (see [15, 19]).

- (1) A regular space X is cosmic if and only if X has a countable cn -network
- (2) X is a strict $\mathfrak{P}_0$ -space if and only if X is a $\mathfrak{P}_0$ -space

Definition 3 (see [17]).

- (1) A space X is called a k -space if X is a Hausdorff space and X is an image of a locally compact space under a quotient mapping
- (2) A space X is called a sequential space if a set $A \subset X$ is closed if and only if together with any sequence it contains all its limits
- (3) A space X is called a Fréchet–Urysohn space if for each $A \subset X$ and each $x \in \bar{A}$, there is a sequence in A converging to the point x in X

Remark 6 (see [17]).

- (1) Every Fréchet–Urysohn space is a sequential space
- (2) Every sequential Hausdorff space is a k -space

Definition 4 (see [16]). Let

$$X = \{\infty\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_n(m) : m, n \in \mathbb{N}\}, \quad (4)$$

where every x_n , $x_n(m)$, and ∞ are different from each other. The set X endowed with the following topology is called the Arens space and denoted briefly as S_2 : each $x_n(m)$ is isolated; a basic neighborhood of x_n has the form $\{x_n\} \cup \{x_n(m) : m > k\}$ for some $k \in \mathbb{N}$; a basic neighborhood of ∞ has the form $\{\infty\} \cup (\bigcup \{V_n : n \geq k\})$ for some $k \in \mathbb{N}$, where each V_n is a neighborhood of x_n .

Definition 5 (see [16]). A topological space X is called the sequential fan, which is denoted briefly as S_ω if X is the quotient space by identifying all the limit points of ω many nontrivial convergent sequences.

2. Main Results

Let $\mathcal{P}$ be a family of subsets of a space X . If we put

$$\mathfrak{P} = \left\{ \left[F, \bigcup \mathcal{F} \right] : F \in \mathbf{PR}[X] \text{ and } \mathcal{F} \text{ is a finite subfamily of } (\mathcal{P})_F \right\}, \quad (5)$$

then $\mathfrak{P}$ is a family of subsets of $\mathbf{PR}[X]$.

We prove now a few technical lemmas which will be used in the sequel.

Lemma 1. Let X be a space. If $\mathcal{U}$ is an open subset of $\mathbf{PR}[X]$, then $\bigcup \mathcal{U}$ is an open subset of X .

Proof. Let $x \in \bigcup \mathcal{U}$. Then, there exists $F \in \mathcal{U}$ such that $x \in F$. Since $F \in \mathcal{U}$, there exists V open in X such that $F \in [F, V] \subset \mathcal{U}$. Moreover, since $x \in V$, we only need to prove that $V \subset \bigcup \mathcal{U}$. Given a point $z \in V$. Then, $F \subset \{z\} \cup F \subset V$. This implies that

$$\{z\} \cup F \in [F, V] \subset \mathcal{U}. \quad (6)$$

Therefore, $z \in \{z\} \cup F \subset \bigcup \mathcal{U}$. It shows that $V \subset \bigcup \mathcal{U}$. $\square$

Lemma 2. Let X be a space. If $\mathcal{C}$ is a compact subset of $\mathbf{PR}[X]$, then $\bigcup \mathcal{C}$ is a compact subset of X .

Proof. Let $\mathcal{U}$ be an open cover of $\bigcup \mathcal{C}$ in X . Then, for each $F \in \mathcal{C}$, $F \subset \bigcup \mathcal{U} \subset \bigcup \mathcal{U}$. Thus, for each $x \in F$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Since $F \in [F, \bigcup_{x \in F} U_x]$ for each $F \in \mathcal{C}$, we claim that

$$\mathcal{U} = \left\{ \left[F, \bigcup_{x \in F} U_x \right] : F \in \mathcal{C} \right\}, \quad (7)$$

is an open cover of $\mathcal{C}$ in $\mathbf{PR}[X]$. On the other hand, since $\mathcal{C}$ is compact, there exists a finite subfamily $\mathfrak{B}$ of $\mathcal{U}$ such that $\mathcal{C} \subset \bigcup \mathfrak{B}$. This implies that there exist $F_1, \dots, F_m \in \mathcal{C}$ such that

$$\mathfrak{B} = \left\{ \left[F_1, \bigcup_{x \in F_1} U_x \right], \dots, \left[F_m, \bigcup_{x \in F_m} U_x \right] \right\}. \quad (8)$$

Put $\mathcal{V} = \{U_x : x \in F_i, i \leq m\}$. Because each set F_i is finite, $\mathcal{V}$ is a finite subfamily of $\mathcal{U}$. Hence, we only need to prove that $\bigcup \mathcal{C} \subset \bigcup \mathcal{V}$. Take any point $z \in \bigcup \mathcal{C}$, then $z \in A$ for some $A \in \mathcal{C}$. This implies that there exists $i \leq m$ such that $A \in [F_i, \bigcup_{x \in F_i} U_x]$, hence $A \subset \bigcup_{x \in F_i} U_x$. Therefore, $z \in U_x \subset \bigcup \mathcal{V}$ for some $x \in F_i$. $\square$

Lemma 3. If $\mathcal{P}$ is point-countable for X , then $\mathfrak{P}$ is point-countable for $\mathbf{PR}[X]$.

Proof. Let $F \in \mathbf{PR}[X]$ and $\mathcal{W} \in \mathfrak{P}$ such that $F \in \mathcal{W}$. Then, there exist $G \in \mathbf{PR}[X]$ and a finite subfamily $\mathcal{F} \subset (\mathcal{P})_G$ such that $\mathcal{W} = [G, \bigcup \mathcal{F}]$ and $G \subset F \subset \bigcup \mathcal{F}$. Since $\mathcal{F} \subset (\mathcal{P})_G$, we have $\mathcal{F} \subset (\mathcal{P})_F$. This implies that

$$\{\mathcal{W} \in \mathfrak{P} : F \in \mathcal{W}\} \subset \left\{ \left[G, \bigcup \mathcal{F} \right] : G \subset F, \mathcal{F} \text{ is a finite subfamily of } (\mathcal{P})_F \right\}. \quad (9)$$

Because $\mathcal{P}$ is point-countable for X and the set F is finite, $(\mathcal{P})_F$ is countable and the set $\{G : G \subset F\}$ is finite. Thus,

$$\left\{ \left[G, \bigcup \mathcal{F} \right] : G \subset F, \mathcal{F} \text{ is a finite subfamily of } (\mathcal{P})_F \right\}, \quad (10)$$

is countable. Therefore, $\{\mathcal{W} \in \mathfrak{P} : F \in \mathcal{W}\}$ is countable. This proves that $\mathfrak{P}$ is point-countable for $\mathbf{PR}[X]$. $\square$

Lemma 4

- (1) If $\mathcal{P}$ is a cn -network for X , then $\mathfrak{P}$ is a cn -network for $\mathbf{PR}[X]$
- (2) If X is regular and $\mathcal{P}$ is a ck -network for X , then $\mathfrak{P}$ is a ck -network for $\mathbf{PR}[X]$

Proof. Let $F \in \mathbf{PR}[X]$ and $\mathcal{U}$ be an open neighborhood of F in $\mathbf{PR}[X]$. Then, there exists V open in X such that $F \in [F, V] \subset \mathcal{U}$.

(1) Let $\mathcal{P}$ be a cn -network for X . Put

$$\mathcal{H} = \{P \in (\mathcal{P})_F : P \subset V\}. \quad (11)$$

Then, $\bigcup \mathcal{H}$ is a neighborhood of F in X . This implies that there exists W open in X such that $F \subset W \subset \bigcup \mathcal{H}$. Thus, $F \in [F, W] \subset [F, \bigcup \mathcal{H}]$. Moreover, we claim that

$$[F, \bigcup \mathcal{H}] \subset \bigcup \{\mathcal{W} \in \mathfrak{P} : F \in \mathcal{W} \subset \mathcal{U}\}. \quad (12)$$

Indeed, take any $H \in [F, \bigcup \mathcal{H}]$, then $F \subset H \subset \bigcup \mathcal{H}$. Because the set H is finite, there is a finite subfamily $\mathcal{G} \subset \mathcal{H}$ such that $H \subset \bigcup \mathcal{G}$. On the other hand, since $\bigcup \mathcal{G} \subset \bigcup \mathcal{H} \subset V$, $[F, \bigcup \mathcal{G}] \subset [F, V] \subset \mathcal{U}$. Furthermore, since $\mathcal{G} \subset \mathcal{H} \subset (\mathcal{P})_F$, we have

$$[F, \bigcup \mathcal{G}] \in \{\mathcal{W} \in \mathfrak{P} : F \in \mathcal{W} \subset \mathcal{U}\}. \quad (13)$$

Because $H \in [F, \bigcup \mathcal{G}]$, this implies that $H \in \bigcup \{\mathcal{W} \in \mathfrak{P} : F \in \mathcal{W} \subset \mathcal{U}\}$.

Therefore,

$$F \in [F, W] \subset [F, \bigcup \mathcal{H}] \subset \bigcup \{\mathcal{W} \in \mathfrak{P} : F \in \mathcal{W} \subset \mathcal{U}\}. \quad (14)$$

Since the set $[F, W]$ is open in $\mathbf{PR}[X]$, $\bigcup \{\mathcal{W} \in \mathfrak{P} : F \in \mathcal{W} \subset \mathcal{U}\}$ is a neighborhood of F in $\mathbf{PR}[X]$. This shows that $\mathfrak{P}$ is a cn -network for $\mathbf{PR}[X]$.

(2) Suppose that $\mathcal{P}$ is a ck -network for X . Then, for each $x \in F$, there exists a neighborhood V_x of x such that $V_x \subset V$ and for each compact subset $A_x \subset V_x$, there exists a finite subfamily $\mathcal{A}_x$ of $\mathcal{P}$ satisfying $x \in \bigcap \mathcal{A}_x$ and $A_x \subset \bigcup \mathcal{A}_x \subset V$. For each $x \in F$, since X is regular, there is an open set W_x in X such that $x \in W_x \subset \overline{W_x} \subset V_x$. Put $\mathcal{V}_F = [F, \bigcup_{x \in F} W_x]$, then for each compact subset $\mathcal{K} \subset \mathcal{V}_F$, we have

$$\bigcup \mathcal{K} \subset \bigcup_{x \in F} \overline{W_x}. \quad (15)$$

(i) By Lemma 2, $\bigcup \mathcal{K}$ is compact in X . Thus, $K_x = (\bigcup \mathcal{K}) \cap \overline{W_x}$ is compact in X and $K_x \subset V_x$ for each $x \in F$. This implies that there is a finite subfamily $\mathcal{F}_x \subset \mathcal{P}$ such that $x \in \bigcap \mathcal{F}_x$ and $K_x \subset \bigcup \mathcal{F}_x \subset V$. Put

$$\begin{aligned} \mathcal{H} &= \bigcup_{x \in F} \mathcal{F}_x \text{ and} \\ \mathcal{H} &= \bigcup_{x \in F} \mathcal{F}_x, \text{ and } \mathfrak{F} = \{[F, \bigcup \mathcal{G}] \neq \emptyset : \mathcal{G} \subset \mathcal{H}\}. \end{aligned} \quad (16)$$

Then, $F \in \bigcap \mathfrak{F}$ and $\bigcup \mathfrak{F} \subset [F, V]$. On the other hand, since the set F is finite and $\mathcal{H} \subset (\mathcal{P})_F$, the family $\mathfrak{F}$ is a finite subfamily of $\mathfrak{P}$. Furthermore, we have $\mathcal{K} \subset \bigcup \mathfrak{F}$. In fact, take any $H \in \mathcal{K}$, then $H \subset \bigcup \mathcal{K}$. For each $y \in H$, since $\bigcup \mathcal{K} = \bigcup_{x \in F} K_x$, there exists

$x_y \in F$ such that $y \in K_{x_y} \subset \bigcup \mathcal{F}_{x_y}$. Put $\mathcal{G} = \bigcup_{y \in H} \mathcal{F}_{x_y}$. Then, since $H \in \mathcal{K} \subset \mathcal{V}_F$, $F \subset H$. This implies that $H \in [F, \bigcup \mathcal{G}] \in \mathfrak{F}$. Hence, $H \in \bigcup \mathfrak{F}$. It shows that $\mathcal{K} \subset \bigcup \mathfrak{F}$. Thus, $\mathcal{K} \subset \bigcup \mathfrak{F} \subset [F, V] \subset \mathcal{U}$. Therefore, $\mathfrak{P}$ is a ck -network for $\mathbf{PR}[X]$.

By Lemmas 3 and 4, we obtain the following theorem. $\square$

Theorem 1. *Let X be a space.*

- (1) *If X has a point-countable cn -network, then so does $\mathbf{PR}[X]$*
- (2) *If X is a regular space with a point-countable ck -network, then so does $\mathbf{PR}[X]$*

Question 1. Let X be a space. If $\mathbf{PR}[X]$ has a point-countable cn -network (resp., ck -network), then does X have a point-countable cn -network (resp., ck -network)?

Theorem 2. *Let X be a space. If $\mathbf{PR}[X]$ has a countable cn -network, then so does X .*

Proof. Assume that $\mathfrak{P}$ is a countable cn -network for $\mathbf{PR}[X]$. Put

$$\mathcal{P} = \{\bigcup \mathcal{W} : \mathcal{W} \in \mathfrak{P}\}. \quad (17)$$

It is clear that $\mathcal{P}$ is countable. Let $x \in U$ with U open in X . Then, $\{x\}, U$ is an open neighborhood of $\{x\}$ in $\mathbf{PR}[X]$. Because $\mathfrak{P}$ is a cn -network for $\mathbf{PR}[X]$, we have that

$$\bigcup \{\mathcal{W} \in \mathfrak{P} : \{x\} \in \mathcal{W} \subset [\{x\}, U]\}, \quad (18)$$

is a neighborhood of $\{x\}$ in $\mathbf{PR}[X]$. This implies that there is an open set $\mathcal{U}$ in $\mathbf{PR}[X]$ such that

$$\{x\} \in \mathcal{U} \subset \bigcup \{\mathcal{W} \in \mathfrak{P} : \{x\} \in \mathcal{W} \subset [\{x\}, U]\}. \quad (19)$$

Therefore,

$$x \in \bigcup \mathcal{U} \subset \bigcup \{\bigcup \mathcal{W} : x \in \bigcup \mathcal{W} \subset U\}. \quad (20)$$

Hence, $\bigcup \{\bigcup \mathcal{W} : x \in \bigcup \mathcal{W} \subset U\}$ is a neighborhood of x in X . It shows that $\mathcal{P}$ is a cn -network for X . $\square$

Question 2. Let X be a space. If $\mathbf{PR}[X]$ has a countable ck -network, then does X have a countable ck -network?

Remark 7. Let X be a space. Then, $\mathbf{PR}_1[X]$ has a point-countable base that is defined by

$$\mathfrak{P}_1 = \{\{\{x\}\} : x \in X\}. \quad (21)$$

In particular, if X is countable, then $\mathfrak{P}_1$ is a countable base for $\mathbf{PR}_1[X]$.

Lemma 5. *Let X be a space and $n \in \mathbb{N}$. Then, $F = \{x_1, \dots, x_n\}$ is not an accumulation point of any subset of $\mathbf{PR}_n[X]$.*

Proof. Otherwise, suppose that there is a subset $\mathcal{A}$ of $\mathbf{PR}_n[X]$ that accumulates at F . Then, $F \in Cl(\mathcal{A} \setminus \{F\}) \cap \mathbf{PR}_n[X]$. This implies that

$$[F, X] \cap (\mathcal{A} \setminus \{F\}) \neq \emptyset. \quad (22)$$

Hence, there exists $G \in \mathcal{A} \setminus \{F\}$ such that $F \subset G$. Since $F = \{x_1, \dots, x_n\}$ and $G \in \mathbf{PR}_n[X]$, we conclude that $G = F$, which is a contradiction. $\square$

$$\mathfrak{G} = \{[F, \bigcup \mathcal{F}] \cap \mathbf{PR}_2[X] : F \in \mathbf{PR}_2[X] \text{ and } \mathcal{F} \text{ is a finite subfamily of } (\mathcal{P})_F\}. \quad (23)$$

Since the point-countable property is hereditary, the family $\mathfrak{G}$ is point-countable for $\mathbf{PR}_2[X]$ by Lemma 3.

(i) $\mathcal{P}$ is an sp -network for X .

Given a point $F \in \mathcal{U} \cap Cl(\mathcal{A}) \cap \mathbf{PR}_2[X]$ with $\mathcal{U}$ open and $\mathcal{A}$ subset in $\mathbf{PR}_2[X]$. Then, there exists an open neighborhood V of F in X such that

$$F \in [F, V] \cap \mathbf{PR}_2[X] \subset \mathcal{U}. \quad (24)$$

Case 1: $F = \{x\}$. In this case, if $\{x\} \in \mathcal{A}$, then since $\mathcal{P}$ is a network for X and $x \in V$, there exists $P \in \mathcal{P}$ such that $x \in P \subset V$. Clearly, $[\{x\}, P] \cap \mathbf{PR}_2[X] \in \mathfrak{G}$ satisfies the conditions of an sp -network. Otherwise, we put

$$A = \{x\} \cup \{y : y \neq x, \{x, y\} \in \mathcal{A}\}. \quad (25)$$

Because $\{x\} \in Cl(\mathcal{A}) \cap \mathbf{PR}_2[X]$, $[\{x\}, V] \cap \mathcal{A} \neq \emptyset$. On the other hand, since $\{x\} \notin \mathcal{A}$, there exists $y \in V \setminus \{x\}$ such that $\{x, y\} \in \mathcal{A}$. This implies that $A \neq \{x\}$. Take any open neighborhood U of x in X . Similarly, we can show that there is a point $y \in \overline{U \setminus \{x\}}$ such that $y \in U \cap (A \setminus \{x\})$. Therefore, $x \in \overline{A \setminus \{x\}}$.

Next, since $x \in V \cap \overline{A \setminus \{x\}}$ and $\mathcal{P}$ is an sp -network for X , there is a set $P \in \mathcal{P}$ satisfying $x \in P \subset V$ and $x \in \overline{P \cap (A \setminus \{x\})}$.

Then, $[\{x\}, P] \cap \mathbf{PR}_2[X] \in \mathfrak{G}$ and

$$\{x\} \in [\{x\}, P] \cap \mathbf{PR}_2[X] \subset [\{x\}, V] \cap \mathbf{PR}_2[X] \subset \mathcal{U}. \quad (26)$$

Furthermore, we claim that

$$\{x\} \in Cl([\{x\}, P] \cap \mathcal{A}) \cap \mathbf{PR}_2[X]. \quad (27)$$

Indeed, let $\mathcal{W}$ be any open neighborhood of $\{x\}$ in $\mathbf{PR}[X]$. Then, there exists an open neighborhood W of x in X such that $\{x\} \in [\{x\}, W] \subset \mathcal{W}$. Because $x \in \overline{P \cap (A \setminus \{x\})}$, we have $W \cap P \cap (A \setminus \{x\}) \neq \emptyset$. Thus, there exists $z \in W \cap P \cap (A \setminus \{x\})$. This implies that

$$\{x, z\} \in [\{x\}, W] \cap [\{x\}, P] \cap ([\{x\}, A] \setminus \{\{x\}\}). \quad (28)$$

Theorem 3. Let X be a space. If X has a point-countable sp -network (resp., strict Pytkeev network), then so does $\mathbf{PR}_2[X]$.

Proof. Let $\mathcal{P}$ be a point-countable sp -network (resp., strict Pytkeev network) for X . Put

Therefore, $\{x, z\} \in \mathcal{W} \cap [\{x\}, P] \cap \mathcal{A}$, hence $\mathcal{W} \cap [\{x\}, P] \cap \mathcal{A} \neq \emptyset$. This shows that $\{x\} \in Cl([\{x\}, P] \cap \mathcal{A}) \cap \mathbf{PR}_2[X]$.

Case 2: $F = \{x_1, x_2\}$. In this case, it is easy to check that $\{x_1, x_2\} \in \mathcal{A}$. Then, since $x_1, x_2 \in V$ and $\mathcal{P}$ is a network for X , there exist $P_1, P_2 \in \mathcal{P}$ such that $x_1 \in P_1 \subset V$ and $x_2 \in P_2 \subset V$. Observe that $[\{x_1, x_2\}, P_1 \cup P_2] \cap \mathbf{PR}_2[X] \in \mathfrak{G}$ satisfies the conditions of an sp -network.

Therefore, we conclude that $\mathfrak{G}$ is an sp -network for $\mathbf{PR}_2[X]$.

(2) $\mathcal{P}$ is a strict Pytkeev network for X .

First, we will prove that $\mathfrak{G}$ is a network for $\mathbf{PR}_2[X]$. Indeed, assume that $F \in \mathbf{PR}_2[X]$ and $\mathcal{U}$ is an open neighborhood of F in $\mathbf{PR}_2[X]$. Then, there is an open set V in X satisfying,

$$F \in [F, V] \cap \mathbf{PR}_2[X] \subset \mathcal{U}. \quad (29)$$

Because V is an open neighborhood of F in X and $\mathcal{P}$ is a network for X , for each $x \in F$, there is a set $P_x \in \mathcal{P}$ such that $x \in P_x \subset V$. This implies that

$$F \in \left[F, \bigcup_{x \in F} P_x \right] \cap \mathbf{PR}_2[X] \subset [F, V] \cap \mathbf{PR}_2[X] \subset \mathcal{U}. \quad (30)$$

Thus, $\mathfrak{G}$ is a network for $\mathbf{PR}_2[X]$.

Next, by Lemma 5, $\{x_1, x_2\} \in \mathbf{PR}_2[X]$ is not an accumulation point of any subset $\mathcal{A}$ of $\mathbf{PR}_2[X]$. Thus, suppose that $\mathcal{A} \subset \mathbf{PR}_2[X]$ accumulates at $\{x\}$ and $\mathcal{U}$ is an open neighborhood of $\{x\}$ in $\mathbf{PR}_2[X]$. Then, $\{x\} \in Cl(\mathcal{A} \setminus \{\{x\}\}) \cap \mathbf{PR}_2[X]$, and there exists an open neighborhood V of x in X such that

$$\{x\} \in [\{x\}, V] \cap \mathbf{PR}_2[X] \subset \mathcal{U}. \quad (31)$$

This implies that

$$[\{x\}, V] \cap (\mathcal{A} \setminus \{\{x\}\}) \neq \emptyset. \quad (32)$$

Hence, there is a point $y \in V \setminus \{x\}$ such that $\{x, y\} \in \mathcal{A}$. Put

$$A = \{x\} \cup \{y : y \neq x, \{x, y\} \in \mathcal{A}\}. \quad (33)$$

Let U be any open neighborhood of x in X . It is easy to see that there exists $y \in U \cap (A \setminus \{x\})$. Thus, $x \in \overline{A \setminus \{x\}}$.

Finally, because $\mathcal{P}$ is a strict Pytkeev network for X , there exists a set $P \in \mathcal{P}$ such that $P \cap A$ is infinite and $x \in P \subset V$. Then, the set $\{\{x\}, P\} \cap \mathbf{PR}_2[X] \in \mathfrak{P}$ satisfies that

$$\{\{x\}, P\} \cap \mathbf{PR}_2[X] \cap \{\{x\}, A\} = \{\{x\}, P \cap A\} \cap \mathbf{PR}_2[X], \quad (34)$$

is infinite. This implies that $\{\{x\}, P\} \cap \mathbf{PR}_2[X] \cap \mathcal{A}$ is infinite. Furthermore,

$$\{x\} \in \{\{x\}, P\} \cap \mathbf{PR}_2[X] \subset \{\{x\}, V\} \cap \mathbf{PR}_2[X] \subset \mathcal{U}. \quad (35)$$

Therefore, $\mathfrak{P}$ is a strict Pytkeev network for $\mathbf{PR}_2[X]$. $\square$

Lemma 6 (see [16, Theorem 4.2]). *The following conditions are equivalent for a Hausdorff space X .*

- (1) X is a k -space with a point-countable sp -network
- (2) X is a Fréchet-Urysohn space with a point-countable cs^* -network

Example 1. There exists a space X with a countable sp -network but $\mathbf{PR}_n[X]$ with $n \geq 3$ and $\mathbf{PR}[X]$ do not have point-countable sp -networks.

Proof. It follows from [16, example 4.3] that the sequential fan S_ω has a countable sp -network. By [9, Theorem 3.9] and Remark 6, $\mathbf{PR}_3[S_\omega]$ is a k -space. However, $\mathbf{PR}_3[S_\omega]$ does not have a point-countable sp -network. Otherwise, $\mathbf{PR}_3[S_\omega]$ is a Fréchet-Urysohn space by Lemma 6. This is a contradiction, as shown in [9, p. 313]. It is easy to check that the point-countable sp -network property is hereditary. Therefore, $\mathbf{PR}_n[S_\omega]$ with $n \geq 3$ and $\mathbf{PR}[S_\omega]$ do not have point-countable sp -networks. $\square$

Question 3. Let X be a space and $n \geq 3$. If X has a point-countable strict Pytkeev network, then does $\mathbf{PR}_n[X]$ have a point-countable strict Pytkeev network?

Question 4. Let X be a space. If $\mathbf{PR}[X]$ has a point-countable sp -network (resp., strict Pytkeev network), then does X have a point-countable sp -network (resp., strict Pytkeev network)?

Theorem 4. *Let X be a space. If $\mathbf{PR}[X]$ has a countable sp -network (resp., strict Pytkeev network), then so does X .*

Proof. Suppose that $\mathfrak{P}$ is a countable sp -network (resp., strict Pytkeev network) for $\mathbf{PR}[X]$. Put

$$\mathcal{P} = \{\bigcup \mathcal{W} : \mathcal{W} \in \mathfrak{P}\}, \quad (36)$$

and observe that $\mathcal{P}$ is countable.

- (1) $\mathfrak{P}$ is an sp -network for $\mathbf{PR}[X]$. Then, let $x \in U \cap \overline{A}$ with U open and A subset in X .

Case 1. $x \in A$. Then, since $\mathfrak{P}$ is a network for $\mathbf{PR}[X]$, there exists $\mathcal{W} \in \mathfrak{P}$ such that $\{x\} \in \mathcal{W} \subset \{\{x\}, U\}$. Moreover, we have $\{x\} \in \mathcal{W} \cap \{\{x\}, A\}$. Therefore,

$$\begin{aligned} x &\in \bigcup \mathcal{W} \subset \bigcup \{\{x\}, U\} = U, \\ x &\in (\bigcup \mathcal{W}) \cap A \subset \overline{(\bigcup \mathcal{W}) \cap A}. \end{aligned} \quad (37)$$

Case 2. $x \notin A$. First, we prove that

$$\{x\} \in \mathbf{Cl}(\{\{x\}, A \cup \{x\}\} \setminus \{\{x\}\}). \quad (38)$$

Take any open neighborhood $\mathcal{V}$ of $\{x\}$ in $\mathbf{PR}[X]$. Then, there is an open neighborhood V of x in X such that $\{x\} \in \{\{x\}, V\} \subset \mathcal{V}$. Since $x \in \overline{A}$, $V \cap A \neq \emptyset$. On the other hand, because $x \notin A$, there exists $y \neq x$ such that $y \in V \cap A$. This implies that

$$\begin{aligned} \{x, y\} &\in \{\{x\}, V\} \cap (\{\{x\}, A \cup \{x\}\} \setminus \{\{x\}\}) \subset \mathcal{V} \\ &\cap (\{\{x\}, A \cup \{x\}\} \setminus \{\{x\}\}). \end{aligned} \quad (39)$$

Hence, (38) is proved.

Next, since $\{x\} \in \{\{x\}, U\}$ and by (1), we have

$$\{x\} \in \{\{x\}, U\} \cap \mathbf{Cl}(\{\{x\}, A \cup \{x\}\} \setminus \{\{x\}\}). \quad (40)$$

On the other hand, because $\mathfrak{P}$ is an sp -network for $\mathbf{PR}[X]$, there is a set $\mathcal{W} \in \mathfrak{P}$ satisfying

$$\begin{aligned} \{x\} &\in \mathcal{W} \subset \{\{x\}, U\}, \\ \{x\} &\in \mathbf{Cl}(\mathcal{W} \cap (\{\{x\}, A \cup \{x\}\} \setminus \{\{x\}\})). \end{aligned} \quad (41)$$

Clearly, $x \in \bigcup \mathcal{W} \subset U$.

Finally, we will prove that $x \in \overline{(\bigcup \mathcal{W}) \cap A}$. Let W be an open neighborhood of x in X . Then, $\{\{x\}, W\}$ is an open neighborhood of $\{x\}$ in $\mathbf{PR}[X]$. Since

$$\{x\} \in \mathbf{Cl}(\mathcal{W} \cap (\{\{x\}, A \cup \{x\}\} \setminus \{\{x\}\})), \quad (42)$$

we conclude that

$$\{\{x\}, W\} \cap \mathcal{W} \cap (\{\{x\}, A \cup \{x\}\} \setminus \{\{x\}\}) \neq \emptyset. \quad (43)$$

Hence, there exist $M \subset W \cap A$ and $x \notin M$ such that

$$\{x\} \cup M \in \{\{x\}, W\} \cap \mathcal{W} \cap (\{\{x\}, A \cup \{x\}\} \setminus \{\{x\}\}). \quad (44)$$

This implies that $M \subset W \cap (\bigcup \mathcal{W}) \cap A$. Thus, $x \in \overline{(\bigcup \mathcal{W}) \cap A}$. Therefore, $\mathcal{P}$ is an sp -network for X .

- (2) $\mathfrak{P}$ is a strict Pytkeev network for $\mathbf{PR}[X]$.

First, we will prove that $\mathcal{P}$ is a network for X . Take any $x \in U$ with U open in X . Then, $\{\{x\}, U\}$ is an open neighborhood of $\{x\}$ in $\mathbf{PR}[X]$. Since $\mathfrak{P}$ is a network for $\mathbf{PR}[X]$, there exists $\mathcal{W} \in \mathfrak{P}$ such that $\{x\} \in \mathcal{W} \subset \{\{x\}, U\}$. This implies that

$$x \in \bigcup \mathcal{W} \subset \bigcup \{\{x\}, U\} = U. \quad (45)$$

Thus, $\mathcal{P}$ is a network for X .

Now, assume that $A \subset X$ accumulates at x and U is an open neighborhood of x in X .

Case 1. $x \in A$. In this case, we show that

$$\{x\} \in \mathcal{C}1([\{x\}, A] \setminus \{\{x\}\}). \quad (46)$$

Indeed, take any open neighborhood $\mathcal{V}$ of $\{x\}$ in $\mathbf{PR}[X]$. Then, there is an open neighborhood V of x in X satisfying $\{x\} \in [\{x\}, V] \subset \mathcal{V}$. Since $x \in A \setminus \{x\}$, we have that $V \cap (A \setminus \{x\}) \neq \emptyset$. Thus, there exists $y \neq x$ such that $y \in V \cap A$. This implies that

$$\{x, y\} \in [\{x\}, V] \cap ([\{x\}, A] \setminus \{\{x\}\}) \subset \mathcal{V} \cap ([\{x\}, A] \setminus \{\{x\}\}). \quad (47)$$

Hence, (2) is true.

Because $\{x\} \in [\{x\}, U]$ and $\mathfrak{P}$ is a strict Pytkeev network for $\mathbf{PR}[X]$, by (46), there exists a set $\mathcal{W} \in \mathfrak{P}$ such that $\mathcal{W} \cap [\{x\}, A]$ is infinite and $\{x\} \in \mathcal{W} \subset [\{x\}, U]$. This implies that $(\bigcup \mathcal{W}) \cap A$ is infinite and $x \in \bigcup \mathcal{W} \subset U$.

Case 2. $x \notin A$. Similar to the proof of Case 1, we claim that

$$\{x\} \in \mathcal{C}1([\{x\}, A \cup \{x\}] \setminus \{\{x\}\}). \quad (48)$$

Thus, we also find a set $\mathcal{W} \in \mathfrak{P}$ such that $\mathcal{W} \cap [\{x\}, A \cup \{x\}]$ is infinite and $\{x\} \in \mathcal{W} \subset [\{x\}, U]$. This shows that $(\bigcup \mathcal{W}) \cap A$ is infinite and $x \in \bigcup \mathcal{W} \subset U$.

Therefore, we claim that $\mathcal{P}$ is a strict Pytkeev network for X . $\square$

By Definition 2, Remark 5, Theorem 2, and Theorem 4, we have the following corollary. $\square$

Corollary 1. *Let X be a regular space. Then, if $\mathbf{PR}[X]$ is a cosmic space (resp., $\mathfrak{P}_0$ -space, strict $\mathfrak{P}_0$ -space, stric $\mathfrak{P}_0$ -space), then so is X .*

Lemma 7

- (1) *Every point-countable Pytkeev network for a Hausdorff space is a k -network [18, Theorem 3.1].*
- (2) *Every k -space with a point-countable k -network is a sequential space [23, Corollary 3.4].*

Corollary 2. *For each $n \geq 2$, $\mathbf{PR}_n[S_2]$ is not a k -space.*

Proof. Since each closed subspace of a k -space is a k -space and for each $m < n$, $\mathbf{PR}_m[S_2]$ is closed in $\mathbf{PR}_n[S_2]$ by Remark 2, we only need to prove that $\mathbf{PR}_2[S_2]$ is not a k -space. Otherwise, $\mathbf{PR}_2[S_2]$ is a k -space. By [16, example 3.7], the Arens space S_2 is a regular sequential space with a countable strict Pytkeev network. It follows from Theorem 3 that $\mathbf{PR}_2[S_2]$ has a point-countable strict Pytkeev network. By Remark 4 (3) and Lemma 7, we conclude that $\mathbf{PR}_2[S_2]$ is a sequential space. This implies that $\mathbf{PR}_2[S_2]$ is a Fréchet–Urysohn space by [9, Lemma 4.1].

Because $\mathbf{PR}_2[S_2]$ is a Fréchet–Urysohn space with a point-countable strict Pytkeev network, $\mathbf{PR}_2[S_2]$ has a point-countable sp -network by Remark 4 (3). On the other hand,

because $\mathbf{PR}_2[S_2]$ is countable, it is separable. Furthermore, observe that $\mathbf{PR}_2[S_2]$ is regular. It follows from [16, Theorem 4.11] that $\mathbf{PR}_2[S_2]$ has a countable sp -network. Similar to the proof of Theorem 4, we claim that S_2 has a countable sp -network, which is a contradiction, as shown in [16, example 3.7]. Therefore, $\mathbf{PR}_2[S_2]$ is not a k -space. $\square$

3. Conclusion

In this paper, we demonstrated that a few topological properties of a space X are preserved by passing to the Pixley–Roy hyperspaces over X . On the other hand, a space X has some generalized metric properties whenever the Pixley–Roy hyperspace over X has such properties. In particular, there are nice relations between Pytkeev-type properties of a space X and its Pixley–Roy hyperspace $\mathbf{PR}[X]$. We mention that these properties have been intensively studied in recent years.

It would be interesting to do a similar research for the other known hyperspace topologies (for example for the Vietoris and Fell topologies on the spaces of closed subsets of a space X).

Data Availability

The data used to support the findings of this study are cited at relevant places within the text as references and are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

References

- [1] A. Bella and M. Sakai, “Compactifications of a pixley-roy hyperspace,” *Topology and Its Applications*, vol. 196, pp. 173–182, 2015.
- [2] P. Daniels, “Pixley-Roy spaces over subsets of the reals,” *Topology and Its Applications*, vol. 29, no. 1, pp. 93–106, 1988.
- [3] E. K. van Douwen, “The Pixley-Roy topology on spaces of subsets,” in *Set-Theoretic Topology*, G. M. Reed, Ed., Academic Press, New York, NY, USA, pp. 111–134, 1977.
- [4] Lj.D. R. Kočinac, “The Pixley-Roy topology and selection principles,” *Questions and Answers in General Topology*, vol. 19, no. 2, pp. 210–225, 2001.
- [5] D. J. Lutzer, “Pixley-Roy topology,” *Topology Proceedings*, vol. 3, pp. 139–158, 1978.
- [6] C. Pixley and P. Roy, “Uncompletable moore spaces,” in *Proceedings of the Auburn University Conference*, W. R. R. Transue, Ed., pp. 76–86pp. 76–, Auburn, Alabama, 1969.
- [7] T. Przymusiński, “Normality and paracompactness of Pixley-Roy hyperspaces,” *Fundamenta Mathematicae*, vol. 113, no. 3, pp. 201–215, 1981.
- [8] M. E. Rudin, “Pixley-roy and the souslin line,” *Proceedings of the American Mathematical Society*, vol. 74, no. 1, pp. 128–134, 1979.
- [9] M. Sakai, “The Fréchet-Urysohn property of Pixley-Roy hyperspaces,” *Topology and Its Applications*, vol. 159, no. 1, pp. 308–314, 2012.

- [10] M. Scheepers, "Combinatorics of open covers (V): Pixley-Roy spaces of sets of reals, and ω -covers," *Topology and Its Applications*, vol. 102, no. 1, pp. 13–31, 2000.
- [11] H. Tanaka, "Paracompactness of pixley-roy hyperspaces. I," *Proceedings of the American Mathematical Society*, vol. 85, no. 1, pp. 108–112, 1982.
- [12] H. Tanaka, "Metrizability of pixley-roy hyperspaces," *Tsukuba Journal of Mathematics*, vol. 7, no. 2, pp. 299–315, 1983.
- [13] H. Tanaka, "Normality and hereditary countable paracompactness of Pixley-Roy hyperspaces," *Fundamenta Mathematicae*, vol. 126, no. 3, pp. 201–208, 1986.
- [14] H. Tanaka, "Countable paracompactness of Pixley-Roy hyperspaces," *Proceedings of the American Mathematical Society*, vol. 108, no. 4, pp. 1115–1120, 1990.
- [15] T. Banach, "P0-spaces," *Topology and Its Applications*, vol. 195, pp. 151–173, 2015.
- [16] X. Liu, C. Liu, and S. Lin, "Strict Pytkeev networks with sensors and their applications in topological groups," *Topology and Its Applications*, vol. 258, pp. 58–78, 2019.
- [17] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, Germany, 1989.
- [18] X. Liu and S. Lin, "On spaces defined by Pytkeev networks," *Filomat*, vol. 32, no. 17, pp. 6115–6129, 2018.
- [19] S. S. Gabrielyan and J. Kakol, "On P-spaces and related concepts," *Topology and Its Applications*, vol. 191, pp. 178–198, 2015.
- [20] E. G. Pytkeev, "On maximally resolvable spaces," *Proceedings of the Steklov Institute of Mathematics*, vol. 154, pp. 209–213, 1983, Trudy Matematicheskogo Instituta im. V.A. Steklova.
- [21] Lj.D. R. Kočinac, "The Reznichenko property and the Pytkeev property in hyperspaces," *Acta Mathematica Hungarica*, vol. 107, no. 3, pp. 225–233, 2005.
- [22] Lj.D. R. Kočinac and R. Sen, "On the reznichenko and pytkeev properties in hyperspaces," *Acta Mathematica Academiae Scientiarum Hungaricae*, vol. 107, no. 3, pp. 225–233, 2021.
- [23] G. Gruenhage, E. Michael, and Y. Tanaka, "Spaces determined by point-countable covers," *Pacific Journal of Mathematics*, vol. 113, no. 2, pp. 303–332, 1984.