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Research Article

A Variational Problem and Classification Theorem for Extremal Hypersurfaces in the Space Form

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The classification of the isoparametric extremal hypersurfaces in the space form is obtained in this paper. We also derived the Euler-Lagrange equation for extremal hypersurface and obtained Simons' type integral inequality. When the integral equality holds, we can obtain the characteristics of isoparametric extremal hypersurfaces by using this classification theorem. The classification of isoparametric extremal hypersurfaces in the space form is the first study that has not been seen in the previous literature.

1. Introduction

Let $N^{n+1}(c)$ be a space form with constant sectional curvature c, we choose $c = -1$, $c = 0$, and $c = 1$ in this paper to represent the hyperbolic space $\mathbb{H}^{n+1}$, Euclidean space $\mathbb{R}^{n+1}$, and unit sphere $\mathbb{S}^{n+1}$, respectively. Let x: $M^n \longrightarrow N^{n+1}(c)$ be an n-dimensional hypersurface, choose an orthonormal frame field $\{e_1, e_2, \ldots, e_n, e_{n+1}\}$ in $N^{n+1}(c)$ such that $\{e_1, e_2, \ldots, e_n\}$ is tangent to M, $\{e_{n+1}\}$ is normal to M, and $\{\theta_1, \theta_2, \ldots, \theta_n, \theta_{n+1}\}$ is represented as the dual frame. We use the following convention on the range of indices:

$$\begin{aligned} &1 \le A, B, C, \ldots \le n+1,\\ &1 \le i, j, k, \ldots \le n. \end{aligned} \tag{1}$$

The first fundamental form is the inner product of tangent vectors $I(v,w) = \langle v, w\rangle$ and often written by the metric tensor (g_{ij}). The second fundamental form is the symmetric bilinear form on the tangent space $h(v,w) = \langle S_p(v), w\rangle$, where S_p is the shape operator (see [1]). Let $h_{ij} = \langle S_p(e_i), e_j\rangle$; then, the second fundamental form can be denoted by

$$h = \sum_{i,j} h_{ij}\theta_i\theta_j. \tag{2}$$

The mean curvature is defined by $H = (1/n)\sum_i h_{ii}$. We define a tensor as

$$h^0 = \sum_{i,j} \widetilde{h}_{ij}\theta_i\theta_j, \tag{3}$$

where $\widetilde{h}_{ij} = h_{ij} - H\delta_{ij}$. Let S and ρ^2 be the square length of h and h^0, respectively. Then,

$$\begin{aligned} S &= \sum_{i,j} h_{ij}^2,\\ \rho^2 &= \sum_{i,j} \widetilde{h}_{ij}^2,\\ \rho^2 &= S - nH^2. \end{aligned} \tag{4}$$

If M is an n-dimensional Riemannian manifold in $(n+p)$-dimensional Riemannian manifold N^{n+P}, the Willmore functional is well-known (see [2, 3]; also refer to [4–6] in $\mathbb{S}^{n+P}$)

$$W_{n/2}(x) = \int_M \left(S - nH^2\right)^{n/2} \mathrm{d}\mu = \int_M \rho^n \mathrm{d}\mu, \tag{5}$$

where $S = \sum_{\alpha,i,j}(h_{ij}^{\alpha})^2$, $\alpha = n+1, \ldots, n+p$. A Willmore submanifold is a critical point of $W_{n/2}(x)$. Consider $p = 1$, the Willmore functional of hypersurface is the same as before with $S=\sum_{i,j}h_{ij}^2$, and a critical point of $W_{n/2}(x)$ is called a Willmore hypersurface.

In the research on holographic entanglement entropy of physics, Willmore functional is a newer tool [7]. Recently, Deng and Li consider the Willmore functional of spacelike hypersurface in Lorentzian space $\mathbb{R}_1^{n+1}$; its critical point is called a Willmore spacelike hypersurface [8]. In [9], the following nonnegative functional $F(x)$ was studied.

$$F(x)=\int_M\left(S-nH^2\right)\mathrm{d}\mu=\int_M\rho^2\mathrm{d}\mu. \tag{6}$$

When $n=2$, the functional $F(x)$ is consistent with the Willmore functional.

Definition 1 (see [9]). If M is a hypersurface in the space form $N^{n+1}(c)$, then M is called extremal if it is a critical point of

$$F(x)=\int_M\rho^2\mathrm{d}\mu. \tag{7}$$

In this paper, we shall study the classification and variation problems for extremal hypersurface in the space form $N^{n+1}(c)$. Firstly, we shall derive the Euler-Lagrange equation for extremal hypersurface in $N^{n+1}(c)$ and then classify the isoparametric extremal hypersurfaces in the space form. Finally, we shall consider a compact extremal hypersurface in $N^{n+1}(c)$ and obtain an integral inequality for this compact extremal hypersurfaces. More precisely, we have the following three theorems (see Sections 3–5, respectively).

Theorem 1. *If $x\colon M^n\longrightarrow N^{n+1}(c)$ is an n-dimensional compact hypersurface without boundary in the space form $N^{n+1}(c)$, then M is an extremal hypersurface of $F(x)$ given by (6) if and only if*

$$(n-1)\Delta H+\sum_{i,j,k}\widetilde{h}_{ij}\widetilde{h}_{ik}\widetilde{h}_{kj}-\frac{(n-4)}{2}\rho^2H=0, \tag{8}$$

where $\Delta H=\sum_iH_{ii}$.

Theorem 2. *Suppose that $x\colon M^n\longrightarrow N^{n+1}(c)$ is an isoparametric extremal hypersurface with distinct principal curvatures $\kappa_1>\kappa_2>\cdots>\kappa_s$. The multiplicity of κ_i is denoted by m_i, $i=1,2,\ldots,s$. Then the following properties hold.*

(1) If $s=1$, then $\rho^2=0$. In other words, M is total umbilical.

(2) If $s=2$, assume that the two distinct principal curvatures $\kappa_1=\lambda$ and $\kappa_2=\mu$ with multiplicity k and $n-k$, respectively, $1\le k\le n-1$, and then either $\rho^2=0$, or $n=2k$, or

$$\rho^2=\frac{nk(n-k)(n-4)^2}{4(n-2k)^2}H^2. \tag{9}$$

(a) If $n=2k=4$, then M is the hypersurface $\mathbb{S}^2(\coth^2\theta-1)\times\mathbb{H}^2(\tanh^2\theta-1)$ for $c=-1$, or $\mathbb{S}^2(r)\times\mathbb{R}^2$ for $c=0$, or $\mathbb{S}^2(r)\times\mathbb{S}^2(\sqrt{1-r^2})$ for $c=1$, where $\widetilde{\lambda}=\lambda-H$, $\widetilde{\mu}=\mu-H$, and $\theta>0$.

(b) If $n=2k$ and $n\neq4$, then $N^{n+1}(c)=\mathbb{S}^{n+1}$, $M=\mathbb{S}^k(\sqrt{1/2})\times\mathbb{S}^k(\sqrt{1/2})$, and $\rho^2=n$.

(c) If $n\neq4$, then $H=0$ if and only if $n=2k$.

(3) If $s>2$, then the space form $N^{n+1}(c)$ is a sphere $\mathbb{S}^{n+1}(c)$ and $s=3,4$ or 6.

(a) If $s=3$, then either $n=6$ and M has distinct principal curvatures

$$\begin{aligned}\kappa_1&=\lambda,\\ \kappa_2&=\frac{\lambda-\sqrt{3}}{1+\sqrt{3}\lambda},\\ \kappa_3&=\frac{\lambda+\sqrt{3}}{1-\sqrt{3}\lambda},\end{aligned} \tag{10}$$

with multiplicities $m_1=m_2=m_3=2$, or M is minimal and has distinct principal curvatures

$$\begin{aligned}\kappa_1&=\sqrt{3},\\ \kappa_2&=0,\\ \kappa_3&=-\sqrt{3},\end{aligned} \tag{11}$$

with multiplicities $m_1=m_2=m_3=1,4$ or 8 ($n=3,12$, or 24).

(b) If $s=4$, then $m_1=m_3$, $m_2=m_4$,

$$\begin{aligned}\kappa_1&=\lambda,\\ \kappa_2&=\frac{\lambda-1}{\lambda+1},\\ \kappa_3&=-\frac{1}{\lambda},\\ \kappa_4&=-\frac{\lambda+1}{\lambda-1},\end{aligned} \tag{12}$$

and $A=\lambda-(1/\lambda)$ is the positive solution of the equation

$$\begin{aligned}&m_1\left[-5m_1^3-m_1^2(7m_2-10)-2m_1m_2(m_2-3)+4m_2^2\right]A^6\\ &+4m_1\left[-m_1^3+m_2^2(m_2+4)+2m_1m_2(5m_2-9)+2m_1^2(4m_2+1)\right]A^4\\ &-16m_2\left[m_1^3-(m_2-2)m_2^2+2m_1m_2(4m_2-9)+2m_1^2(5m_2+2)\right]A^2\\ &+64m_2\left[2m_1^2(m_2-2)+5(m_2-2)m_2^2+m_1m_2(7m_2-6)\right]=0,\end{aligned} \tag{13}$$

with multiplicities $(m_1, m_2) = (2, 2)$, *or* $(m_1, m_2) = (4, 5)$, *or* $2^{\varphi(m_1-1)}k = m_1 + m_2 + 1$ *for some integer k. Here* $\varphi(x)$ *is a number of integers* y *with* $1 \le y \le x$ *and* $y \equiv 0, 1, 2, 4$ *mod 8.*

(c) *If* $s = 6$, *then either* $n = 12$ *and M has distinct principal curvatures*

$$\begin{aligned}\kappa_1 &= \lambda,\\ \kappa_2 &= \frac{\sqrt{3}\lambda - 1}{\lambda + \sqrt{3}},\\ \kappa_3 &= \frac{\lambda - \sqrt{3}}{1 + \sqrt{3}\lambda},\\ \kappa_4 &= -\frac{1}{\lambda},\\ \kappa_5 &= \frac{\lambda + \sqrt{3}}{1 - \sqrt{3}\lambda},\\ \kappa_6 &= \frac{1 + \sqrt{3}\lambda}{\sqrt{3} - \lambda},\end{aligned} \tag{14}$$

with multiplicities $m_1 = \cdots = m_6 = 2$, *or* $n = 6$ *and M is minimal and has distinct principal curvatures*

$$\begin{aligned}\kappa_1 &= 2 + \sqrt{3},\\ \kappa_2 &= 1,\\ \kappa_3 &= 2 - \sqrt{3},\\ \kappa_4 &= -2 + \sqrt{3},\\ \kappa_5 &= -1,\\ \kappa_6 &= -(2 + \sqrt{3}),\end{aligned} \tag{15}$$

with multiplicities $m_1 = \cdots = m_6 = 1$.

Theorem 3. *If* $x\colon M^n \longrightarrow N^{n+1}(c)$ *is a compact extremal hypersurface without boundary in the space form* $N^{n+1}(c)$, *then*

$$\int_M \rho^2 \left[n\left(c + \frac{n-2}{2} H^2 \right) - \rho^2 \right] \mathrm{d}\mu \le 0. \tag{16}$$

In particular, if

$$0 \le \rho^2 \le n\left(c + \frac{n-2}{2} H^2 \right), \tag{17}$$

then either $\rho^2 \equiv 0$ *or* $\rho^2 \equiv n(c + ((n-2)/2)H^2)$. *If* $\rho^2 \equiv n(c + ((n-2)/2)H^2)$, *then either* $n = 2k$ *or*

$$k = \frac{n}{2} \pm \frac{1}{2}\sqrt{\frac{n^2 H^2 (n-4)^2}{n^2 H^2 + 16c}}. \tag{18}$$

If $n = 2k = 4$ *and* $H \ne 0$, *then M is the hypersurface* $\mathbb{S}^2(\coth^2\theta - 1) \times \mathbb{H}^2(\tanh^2\theta - 1)$ *for* $c = -1$, *or* $\mathbb{S}^2(r) \times \mathbb{R}^2$ *for* $c = 0$, *or* $\mathbb{S}^2(r) \times \mathbb{S}^2(\sqrt{1 - r^2})$ *for* $c = 1$. *If* $n = 2k$ *and* $n \ne 4$, *or* $n = 2k = 4$ *and* $H = 0$, *we have that* $N^{n+1}(c) = \mathbb{S}^{n+1}$ *and M is the hypersurface* $\mathbb{S}^k(\sqrt{1/2}) \times \mathbb{S}^k(\sqrt{1/2})$.

2. Preliminaries

If $x\colon M^n \longrightarrow N^{n+1}(c)$ is a hypersurface in $N^{n+1}(c)$, choose a unit basis $\{e_1, \ldots, e_n, e_{n+1}\}$ such that $\{e_1, \ldots, e_n\}$ forms a local orthonormal frame tangent to M with dual frame $\{\omega_1, \ldots, \omega_n\}$ and e_{n+1} is normal to M. Let d be the differential operator in M and the structure equations as follows:

$$\begin{aligned}\mathrm{d}x &= \sum_i \theta_i e_i,\\ \mathrm{d}e_i &= \sum_j \theta_{ij} e_j + \sum_j h_{ij}\theta_j e_{n+1} - \theta_i x,\\ \mathrm{d}e_{n+1} &= -\sum_{i,j} h_{ij}\theta_j e_i,\end{aligned} \tag{19}$$

where we use the ranges of indices

$$\begin{aligned}&1 \le A, B, C, \ldots \le n+1,\\ &1 \le i, j, k, \ldots \le n.\end{aligned} \tag{20}$$

The covariant derivatives of h_{ij} are defined by

$$\begin{aligned}\sum_k h_{ijk}\theta_k &= d_M h_{ij} + \sum_k h_{kj}\theta_{ik} + \sum_k h_{ik}\theta_{jk},\\ \sum_l h_{ijkl}\theta_l &= d_M h_{ijk} + \sum_l h_{ljk}\theta_{il} + \sum_l h_{ilk}\theta_{jl} + \sum_l h_{ijl}\theta_{kl}.\end{aligned} \tag{21}$$

We need the following lemmas. Related lemmas and calculations can also refer to [5, 6, 9–11].

Lemma 1 (see [11]). *If* $x\colon M^n \longrightarrow N^{n+1}(c)$, *we have the following Codazzi equation and Gauss equation:*

$$\begin{aligned}h_{ijk} &= h_{ikj},\\ R_{ijkl} &= c\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) + \left(h_{ik}h_{jl} - h_{il}h_{jk}\right).\end{aligned} \tag{22}$$

Proof. Refer to Lemma 2.1 in [11] with $p = 1$. □

Lemma 2 (see [11]). *If* $x\colon M^n \longrightarrow N^{n+1}(c)$, *the following Ricci identities hold:*

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{23}$$

More precisely,

$$\begin{aligned}h_{ijkl} - h_{ijlk} &= c\left(\delta_{il}h_{kj} - \delta_{ik}h_{lj}\right) + c\left(\delta_{jl}h_{ik} - \delta_{jk}h_{li}\right)\\ &\quad + \sum_m \left(h_{mj}h_{mk}h_{il} - h_{mj}h_{ml}h_{ik}\right)\\ &\quad + \sum_m \left(h_{im}h_{mk}h_{jl} - h_{im}h_{ml}h_{jk}\right).\end{aligned} \tag{24}$$

Proof. Refer to Lemma 2.2 in [11] with $p = 1$. □

Lemma 3 (see [12]). *Suppose that $x: M^n \longrightarrow N^{n+1}(c)$ is a hypersurface in the space form $N^{n+1}(c)$. Consider a one-parameter family $x_t = X(\cdot, t): M^n \times (-\varepsilon, \varepsilon) \longrightarrow N^{n+1}(c)$ with $x_0 = x$. Denoted by ξ and e_{n+1} the deformation vector field $(dx_t/dt)|_{t=0}$ and the unit normal field in $N^{n+1}(c)$, respectively. Set $\eta = \langle \xi, e_{n+1}\rangle$, where $\langle , \rangle$ is the metric on $N^{n+1}(c)$. For any smooth function f of n variables,*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\rfloor_{t=0}\int_M f(S_1,\ldots,S_n)\mathrm{d}\mu = \int_M \eta\left\{-S_1 f(S_1,\ldots,S_n) + \sum_{r=1}^{n}(S_rS_1-(r+1)S_{r+1})D_r f(S_1,\ldots,S_n) + \sum_{i,j,r=1}^{n}(D_r f(S_1,\ldots,S_n))_{,ij}T_{r-1}^{ij} + \sum_{r=1}^{n} D_r f(S_1,\ldots,S_n)(n-r+1)S_{r-1}\right\}\mathrm{d}\mu, \tag{25}$$

where S_r is the r-th elementary symmetric function of the eigenvalues $\kappa_1,\ldots,\kappa_n$ of $(h_{ij}) = (\kappa_i\delta_{ij})$, $D_r f(S_1,\ldots,S_n) = (\partial f/\partial S_r)$, $(D_r f(S_1,\ldots,S_n))_{,ij}$ is the covariant derivative, and T_r^{ij}, $r = 0,1,\ldots,n-1$, are defined by

$$\begin{aligned} T_0^{ij} &= \delta_{ij},\\ T_{r+1}^{ij} &= S_{r+1}\delta_{ij} - \sum_k T_r^{ik}h_{kj}. \end{aligned} \tag{26}$$

More precisely,

$$S_0 = 1, S_1 = \kappa_1 + \cdots + \kappa_n, \ldots, S_n = \kappa_1\cdots\kappa_n. \tag{27}$$

Proof. Refer to Theorem A in [12]. □

3. The Euler-Lagrange Equation for Extremal Hypersurface

In this section, we will use Lemmas 1–4 to derive the Euler-Lagrange equation for extremal hypersurface in the space form $N^{n+1}(c)$. More precisely, we have the following theorem.

Theorem 4. *If $x: M^n \longrightarrow N^{n+1}(c)$ is an n-dimensional compact hypersurface without boundary in the space form $N^{n+1}(c)$, then M is an extremal hypersurface of $F(x)$ given by (6) if and only if*

$$(n-1)\Delta H + \sum_{i,j,k}\widetilde{h}_{ij}\widetilde{h}_{ik}\widetilde{h}_{kj} - \frac{(n-4)}{2}\rho^2 H = 0, \tag{28}$$

where $\Delta H = \sum_i H_{ii}$.

Proof. Choose $f(S_1,\ldots,S_n) = (n-1)S_1^2 - 2nS_2$ in Lemma 3. By a direct calculation, we can easily obtain that

$$\begin{aligned} S_1 &= nH,\\ S_2 &= \frac{1}{2}\left(S_1^2 - S\right),\\ S_3 &= \frac{1}{3}\left(\sum_i \kappa_i^3 - S_1S + S_1S_2\right). \end{aligned} \tag{29}$$

Since $S_1^2 = (\kappa_1 + \cdots + \kappa_n)^2 = n^2H^2$ and $S_1^2 - 2S_2 = \kappa_1^2 + \cdots + \kappa_n^2 = S$, we have

$$\begin{aligned} f(S_1,\ldots,S_n) &= n\left(S_1^2 - 2S_2\right) - S_1^2\\ &= n\left(S - nH^2\right) = n\rho^2,\\ D_1 f &= 2(n-1)S_1,\\ D_2 f &= -2n,\\ D_r f &= 0, \quad r \geq 3. \end{aligned} \tag{30}$$

From (25), we can obtain

$$\begin{aligned} n\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\rfloor_{t=0}\int_M \rho^2\mathrm{d}\mu &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\rfloor_{t=0}\int_M (n-1)S_1^2 - 2nS_2\mathrm{d}\mu\\ &= \int_M \eta\left\{(n-1)S_1^3 - 4(n-1)S_1S_2 + 6nS_3 + 2(n-1)\Delta S_1\right\}\mathrm{d}\mu. \end{aligned} \tag{31}$$

Then, we have the Euler-Lagrange equation of $F(x)$ as follows:

$$(n-1)S_1^3 - 4(n-1)S_1S_2 + 6nS_3 + 2(n-1)\Delta S_1 = 0. \tag{32}$$

By using (29) and $S = \rho^2 + nH^2$, we can conclude that

$$-n(n+2)\rho^2H - 2n^2H^3 + 2n\sum_i \kappa_i^3 + 2n(n-1)\Delta H = 0. \tag{33}$$

This implies that

$$(n-1)\Delta H - \frac{(n+2)}{2}\rho^2 H - nH^3 + \sum_i \kappa_i^3 = 0. \tag{34}$$

Since $(\widetilde{h}_{ij}) = (\widetilde{\kappa}_i\delta_{ij})$ and $\widetilde{h}_{ij} = h_{ij} - H\delta_{ij}$, we have $\sum_i\widetilde{\kappa}_i^3 = \sum_i\kappa_i^3 - 3\sum_i\kappa_i^2 H + 3\sum_i\kappa_i H^2 - nH^3$. By using $\sum_i\kappa_i^2 = S = \rho^2 + nH^2$ and $\sum_i\kappa_i = nH$, we can obtain $\sum_i\kappa_i^3 = \sum_i\widetilde{\kappa}_i^3 + 3\rho^2 H + nH^3$. Then the Euler-Lagrange equation of $F(x)$ can be written by

$$(n-1)\Delta H - \frac{(n-4)}{2}\rho^2 H + \sum_i \widetilde{\kappa}_i^3 = 0. \tag{35}$$

Now, by using $\sum_{i,j,k}\widetilde{h}_{ij}\widetilde{h}_{ik}\widetilde{h}_{kj} = \sum_i\widetilde{\kappa}_i^3$, we have that

$$(n-1)\Delta H - \frac{(n-4)}{2}\rho^2 H + \sum_{i,j,k} \widetilde{h}_{ij}\widetilde{h}_{ik}\widetilde{h}_{kj} = 0. \tag{36}$$

In other words, M is an extremal hypersurface of $F(x)$ if and only if

$$(n-1)\Delta H - \frac{(n-4)}{2}\rho^2 H + \sum_{i,j,k} \widetilde{h}_{ij}\widetilde{h}_{ik}\widetilde{h}_{kj} = 0. \tag{37}$$

Hence, Theorem 4 holds. □

4. The Classification of Isoparametric Extremal Hypersurfaces in the Space Form

In this section we shall classify the isoparametric extrema hypersurfaces of $F(x)$ in the space form $N^{n+1}(c)$. We need Lemma 4; Lemma 4 is a comprehensive arrangement of many theorems.

Lemma 4 (see [5, 13–19]). *Suppose that $N^{n+1}(c)$ is a space form and x: $M^n \longrightarrow N^{n+1}(c)$ is an isoparametric hypersurface. Assume that M has distinct principal curvatures $\kappa_1 > \kappa_2 > \cdots > \kappa_s$ with multiplicities $m_1, m_2, \ldots, m_s$.*

(1) If $N^{n+1}(c)$ is the Euclidean space $\mathbb{R}^{n+1}$, then $s \le 2$ (see [13, 14]).

(2) If $N^{n+1}(c)$ is a hyperbolic space $\mathbb{H}^{n+1}(c)$, then $s \le 2$ (see [15]).

(3) If $N^{n+1}(c)$ is a sphere $\mathbb{S}^{n+1}(c)$, then $s = 1, 2, 3, 4$, or 6 and the following holds (see [5, 16–19]):

(a) If $s = 1$, then $\rho^2 = 0$.

(b) If $s = 2$, then $M = \mathbb{S}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1-r^2})$.

(c) If $s = 3$, then the three multiplicities m_i are the same and $m_i = 2^k$, $k = 0, 1, 2, 3$.

(d) If $s = 4$, then $m_1 = m_3$ and $m_2 = m_4$. Moreover, either $(m_1, m_2) = (2, 2)$ or $(4, 5)$, or $2^{\varphi(m_1 - 1)}k = m_1 + m_2 + 1$ for some integer k. Here $\varphi(x)$ is a number of integers y with $1 \le y \le x$ and $y \equiv 0, 1, 2, 4$ mod 8.

(e) If $s = 6$, the six multiplicities m_i are the same and $m_i = 1$ or 2.

(f) There exists an angle θ, $0 < \theta < (\pi/s)$, such that $\kappa_t = \cot(\theta + ((t-1)/s)\pi)$, $t = 1, \ldots, s$.

Proof. Refer to [5, 13–19]. □

Lemma 5. *If x: $M^n \longrightarrow N^{n+1}(c)$ is isoparametric and extrema, then*

$$\sum_i \widetilde{\kappa}_i^3 - \frac{(n-4)}{2}\rho^2 H = 0, \tag{38}$$

where $\widetilde{\kappa}_i = \widetilde{h}_{ij}\delta_{ij}$.

Proof. If M is isoparametric, then H is constant. Equation (38) is an immediate consequence of Theorem 4 and the fact H is constant. □

Lemma 6 (see [20]). *If x: $M^n \longrightarrow N^{n+1}(c)$ is an n-dimensional complete connected hypersurface in $N^{n+1}(c)$ and M has two distinct principal curvatures, assume that $n \ge 3$, ρ^2 is constant, and the principal curvatures λ and μ of M have multiplicities k and $n-k$, respectively.*

(1) If $1 < k < n-1$, then M is isometric to one of the following: $\mathbb{S}^k(\coth^2\theta - 1) \times \mathbb{H}^{n-k}(\tanh^2\theta - 1)$ for $c = -1$, or $\mathbb{S}^k(r) \times \mathbb{R}^{n-k}$ for $c = 0$, or $\mathbb{S}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1-r^2})$ for $c = 1$, where $\theta > 0$ and

$$\begin{aligned} \lambda_1 &= \cdots = \lambda_k = \coth\theta, \\ \lambda_{k+1} &= \cdots = \lambda_n = \tanh\theta. \end{aligned} \tag{39}$$

(2) If one of the two distinct principal curvatures is simple and the sectional curvature of M is nonnegative, then M is isometric to one of the following: $\mathbb{S}^1(\coth^2\theta - 1) \times \mathbb{H}^{n-1}(\tanh^2\theta - 1)$ for $c = -1$, or $\mathbb{S}^1(r) \times \mathbb{R}^{n-1}$ for $c = 0$, or $\mathbb{S}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1-r^2})$ for $c = 1$.

Proof. Refer to Theorem 1.1 in [20].

If x: $M^n \longrightarrow N^{n+1}(c)$ is an isoparametric extremal hypersurface with distinct principal curvatures $\kappa_1 > \kappa_2 > \cdots > \kappa_s$. The multiplicity of κ_i is denoted by m_i, $i = 1, 2, \ldots, s$. Then we have the following propositions. □

Proposition 1. *If $s = 1$, M^n has only one distinct principal curvature, then $\rho^2 = 0$, and M is total umbilical.*

Proof. If $s = 1$, the principal curvatures $\kappa_1 = \cdots = \kappa_n = \lambda$. Choose $\widetilde{\lambda} = \lambda - H$; since

$$n\widetilde{\lambda} = \sum_i \widetilde{\lambda} = \sum_i (\lambda - H) = \sum_i \lambda - nH, \tag{40}$$

by using $\sum_i\lambda = nH$, we have $\widetilde{\lambda} = 0$. This implies that $\rho^2 = n\widetilde{\lambda}^2 = 0$, and M is total umbilical. □

Proposition 2. *If $s = 2$, assume that the two distinct principal curvatures $\kappa_1 = \lambda$ and $\kappa_2 = \mu$ with multiplicity k and $n-k$, respectively, $1 \le k \le n-1$, and then either $\rho^2 = 0$, or $n = 2k$, or*

$$\rho^2 = \frac{nk(n-k)(n-4)^2}{4(n-2k)^2}H^2. \tag{41}$$

(a) *If $n = 2k = 4$, then M is the hypersurface $\mathbb{S}^2(\coth^2\theta - 1) \times \mathbb{H}^2(\tanh^2\theta - 1)$ for $c = -1$, or $\mathbb{S}^2(r) \times \mathbb{R}^2$ for $c = 0$, or $\mathbb{S}^2(r) \times \mathbb{S}^2(\sqrt{1 - r^2})$ for $c = 1$, where $\tilde{\lambda} = \lambda - H$, $\tilde{\mu} = \mu - H$, and $\theta > 0$.*

(b) *If $n = 2k$ and $n \neq 4$, then $N^{n+1}(c) = \mathbb{S}^{n+1}$, $M = \mathbb{S}^k(\sqrt{1/2}) \times \mathbb{S}^k(\sqrt{1/2})$, and $\rho^2 = n$.*

Proof. If $s = 2$, the principal curvatures λ and μ have multiplicities k and $n - k$, respectively, let $\tilde{\lambda} = \lambda - H$, and $\tilde{\mu} = \mu - H$. Then, we have

$$H = \frac{k\lambda + (n-k)\mu}{n},$$

$$\tilde{\lambda} = \frac{n-k}{n}(\lambda - \mu), \tag{42}$$

$$\tilde{\mu} = \frac{-k}{n}(\lambda - \mu),$$

$$\begin{aligned}\rho^2 &= k\tilde{\lambda}^2 + (n-k)\tilde{\mu}^2 \\ &= \frac{k(n-k)}{n}(\lambda - \mu)^2.\end{aligned} \tag{43}$$

From equation (38), we have

$$k\tilde{\lambda}^3 + (n-k)\tilde{\mu}^3 = \frac{k(n-k)(n-2k)}{n^2}(\lambda - \mu)^3 = \frac{(n-4)}{2}\rho^2 H. \tag{44}$$

If $n = 4$ or $H = 0$ then $n = 2k$. If $n \neq 4$ and $n \neq 2k$, from (42) and (43), we can get

$$\begin{aligned}k\tilde{\lambda}^3 + (n-k)\tilde{\mu}^3 &= k\frac{(n-k)^3}{n^3}(\lambda - \mu)^3 - (n-k) - \left(\frac{k^3}{n^3}\right)(\lambda - \mu)^3 \\ &= \frac{k(n-k)}{n^2}(\lambda - \mu)^3(n - 2k), \\ \frac{(n-4)}{2}\rho^2 H &= \frac{(n-4)}{2}\frac{k(n-k)}{n}(\lambda - \mu)^2\frac{k\lambda + (n-k)\mu}{n} \\ &= \frac{k(n-k)}{n^2}(\lambda - \mu)^2\frac{(n-4)}{2}[k\lambda + (n-k)\mu].\end{aligned} \tag{45}$$

From (44), we have

$$\begin{aligned}&\frac{k(n-k)}{n^2}(\lambda - \mu)^3(n - 2k) \\ &= \frac{k(n-k)}{n^2}(\lambda - \mu)^2\frac{(n-4)}{2}[k\lambda + (n-k)\mu].\end{aligned} \tag{46}$$

This implies that

$$(\lambda - \mu)(n - 2k) = \frac{(n-4)}{2}[k\lambda + (n-k)\mu]. \tag{47}$$

Hence, we can obtain

$$(k-2)\lambda = -(n-k-2)\mu. \tag{48}$$

Since $n \neq 4$ ($k \neq 2$), we have

$$\lambda = -\frac{(n-k-2)}{k-2}\mu. \tag{49}$$

Substitute into (42); we have

$$H = \frac{k\lambda + (n-k)\mu}{n} = -\frac{2(n-2k)\mu}{n(k-2)} = \frac{2(n-2k)\lambda}{n(n-k-2)}. \tag{50}$$

Substitute (43); we have

$$\begin{aligned}\rho^2 &= \frac{k(n-k)}{n}\frac{(n-4)^2}{(k-2)^2}\mu^2 = \frac{k(n-k)}{n}\frac{(n-4)^2}{(n-k-2)^2}\lambda^2 \\ &= \frac{nk(n-k)(n-4)^2}{4(n-2k)^2}H^2.\end{aligned} \tag{51}$$

(a) If $n = 4$, then $n = 2k$. By using Lemma 6, $M = \mathbb{S}^2(\cot h^2\theta - 1) \times \mathbb{H}^2(\tan h^2\theta - 1)$ for $c = -1$, or $\mathbb{S}^2(r) \times \mathbb{R}^2$ for $c = 0$, or $\mathbb{S}^2(r) \times \mathbb{S}^2(\sqrt{1 - r^2})$ for $c = 1$, where $\theta > 0$.

(b) If $n = 2k$ and $k \neq 2$, from (44), $H = 0$. By using Lemma 6, $M = \mathbb{S}^k(\cot h^2\theta - 1) \times \mathbb{H}^k(\tan h^2\theta - 1)$ for $c = -1$, or $\mathbb{S}^k(r) \times \mathbb{R}^k$ for $c = 0$, or $\mathbb{S}^k(r) \times \mathbb{S}^k(\sqrt{1 - r^2})$ for $c = 1$, where $\theta > 0$. Since $H = 0$, if $c = -1$, we have $\lambda = \cot h\theta$, $\mu = \tan h\theta$, $H = ((\cot h\theta + \tan h\theta)/2) = 0$. Then $\cot h\theta = -\tan h\theta$. This equation has no solution, the contradiction shows that $M \neq \mathbb{S}^k(\cot h^2\theta - 1) \times \mathbb{H}^{n-k}(\tan h^2\theta - 1)$. If $c = 0$, from $\lambda = -\mu$ and $H = 0$, then these two principal curvatures are the same. This contradiction shows that $M \neq \mathbb{S}^k(r) \times \mathbb{R}^k$. If $c = 1$, we have

$$\begin{aligned}\lambda_1 &= \cdots = \lambda_k = \frac{\sqrt{1 - r^2}}{r}, \\ \lambda_{k+1} &= \cdots = \lambda_n = -\frac{r}{\sqrt{1 - r^2}}.\end{aligned} \tag{52}$$

Since $H = 0$, we can obtain $r = \sqrt{1/2}$. Hence, $M = \mathbb{S}^k(\sqrt{1/2}) \times \mathbb{S}^k(\sqrt{1/2})$ and $\rho^2 = n$. □

Proposition 3. *If $s > 2$, then the space form $N^{n+1}(c)$ is a sphere $\mathbb{S}^{n+1}(c)$ and $s = 3, 4$ or 6.*

(a) *If $s = 3$, then either $n = 6$ and M has distinct principal curvatures*

$$\begin{aligned}\kappa_1 &= \lambda, \\ \kappa_2 &= \frac{\lambda - \sqrt{3}}{1 + \sqrt{3}\lambda}, \\ \kappa_3 &= \frac{\lambda + \sqrt{3}}{1 - \sqrt{3}\lambda},\end{aligned} \tag{53}$$

with multiplicities $m_1 = m_2 = m_3 = 2$, *or* M *is minimal and has distinct principal curvatures*

$$\begin{aligned}\kappa_1 &= \sqrt{3},\\ \kappa_2 &= 0,\\ \kappa_3 &= -\sqrt{3},\end{aligned} \tag{54}$$

with multiplicities $m_1 = m_2 = m_3 = 1, 4$ *or* 8 $(n = 3, 12,$ *or* $24)$.

(b) *If* $s = 4$, *then* $m_1 = m_3$, $m_2 = m_4$,

$$\begin{aligned}\kappa_1 &= \lambda,\\ \kappa_2 &= \frac{\lambda - 1}{\lambda + 1},\\ \kappa_3 &= -\frac{1}{\lambda},\\ \kappa_4 &= -\frac{\lambda + 1}{\lambda - 1},\end{aligned} \tag{55}$$

and $A = \lambda - (1/\lambda)$ *is the positive solution of the equation*

$$\begin{aligned}&m_1\left[-5m_1^3 - m_1^2(7m_2 - 10) - 2m_1m_2(m_2 - 3) + 4m_2^2\right]A^6\\ &+ 4m_1\left[-m_1^3 + m_2^2(m_2 + 4) + 2m_1m_2(5m_2 - 9) + 2m_1^2(4m_2 + 1)\right]A^4\\ &- 16m_2\left[m_1^3 - (m_2 - 2)m_2^2 + 2m_1m_2(4m_2 - 9) + 2m_1^2(5m_2 + 2)\right]A^2\\ &+ 64m_2\left[2m_1^2(m_2 - 2) + 5(m_2 - 2)m_2^2 + m_1m_2(7m_2 - 6)\right] = 0,\end{aligned} \tag{56}$$

with multiplicities $(m_1, m_2) = (2, 2)$, *or* $(m_1, m_2) = (4, 5)$, *or* $2^{\varphi(m_1 - 1)}k = m_1 + m_2 + 1$ *for some integer* k. *Here* $\varphi(x)$ *is a number of integers* y *with* $1 \le y \le x$ *and* $y \equiv 0, 1, 2, 4$ *mod* 8.

(c) *If* $s = 6$, *then either* $n = 12$ *and* M *has distinct principal curvatures*

$$\begin{aligned}\kappa_1 &= \lambda,\\ \kappa_2 &= \frac{\sqrt{3}\lambda - 1}{\lambda + \sqrt{3}},\\ \kappa_3 &= \frac{\lambda - \sqrt{3}}{1 + \sqrt{3}\lambda},\\ \kappa_4 &= -\frac{1}{\lambda},\\ \kappa_5 &= \frac{\lambda + \sqrt{3}}{1 - \sqrt{3}\lambda},\\ \kappa_6 &= \frac{1 + \sqrt{3}\lambda}{\sqrt{3} - \lambda},\end{aligned} \tag{57}$$

with multiplicities $m_1 = \cdots = m_6 = 2$, *or* $n = 6$, *and* M *is minimal and has distinct principal curvatures*

$$\begin{aligned}\kappa_1 &= 2 + \sqrt{3},\\ \kappa_2 &= 1,\\ \kappa_3 &= 2 - \sqrt{3},\\ \kappa_4 &= -2 + \sqrt{3},\\ \kappa_5 &= -1,\\ \kappa_6 &= -(2 + \sqrt{3}),\end{aligned} \tag{58}$$

with multiplicities $m_1 = \cdots = m_6 = 1$.

Proof. If $s > 2$, from Lemma 4, we have that the space form $N^{n+1}(c)$ is a sphere $\mathbb{S}^{n+1}(c)$ and $s = 3, 4$ or 6.

(a) If s = 3, from Lemma 4, $m_1 = m_2 = m_3 = m, n = 3m$, $m = 1, 2, 4, 8$, and

$$\begin{aligned}\kappa_1 &= \cot\theta = \lambda,\\ \kappa_2 &= \cot\left(\theta + \frac{\pi}{3}\right) = \frac{\lambda - \sqrt{3}}{1 + \sqrt{3}\lambda},\\ \kappa_3 &= \cot\left(\theta + \frac{2\pi}{3}\right) = \frac{\lambda + \sqrt{3}}{1 - \sqrt{3}\lambda}.\end{aligned} \tag{59}$$

Then, we have

$$H = \frac{\kappa_1 + \kappa_2 + \kappa_3}{3} = \frac{\lambda(\lambda^2 - 3)}{3\lambda^2 - 1}, \tag{60}$$

$$\begin{aligned}\widetilde{\kappa}_1 &= \frac{2\lambda(1 + \lambda^2)}{3\lambda^2 - 1},\\ \widetilde{\kappa}_2 &= \frac{(\sqrt{3} - \lambda)(1 + \lambda^2)}{3\lambda^2 - 1},\\ \widetilde{\kappa}_3 &= -\frac{(\sqrt{3} + \lambda)(1 + \lambda^2)}{3\lambda^2 - 1},\end{aligned} \tag{61}$$

where $\widetilde{\kappa}_i = \kappa_i - H$. By a direct calculation, we can obtain

$$\rho^2 = \frac{6m(1 + \lambda^2)^3}{(3\lambda^2 - 1)^2}, \tag{62}$$

$$\sum_i \widetilde{\kappa}_i^3 = \frac{6m\lambda(\lambda^2 - 3)(1 + \lambda^2)^3}{(3\lambda^2 - 1)^3}. \tag{63}$$

From equation (38) in Lemma 5,

$$0=\sum_i \tilde{\kappa}_i^3-\frac{(n-4)}{2}\rho^2 H=\frac{(6-3m)}{2}\frac{6m\lambda\left(\lambda^2-3\right)\left(1+\lambda^2\right)^3}{\left(3\lambda^2-1\right)^3}. \quad (64)$$

Then, either $m=2$ or $\lambda=\sqrt{3}$. If $m=2$, then $n=6$ and M has distinct principal curvatures:

$$\begin{aligned}\kappa_1&=\lambda,\\ \kappa_2&=\frac{\lambda-\sqrt{3}}{1+\sqrt{3}\lambda},\\ \kappa_3&=\frac{\lambda+\sqrt{3}}{1-\sqrt{3}\lambda},\end{aligned} \quad (65)$$

with multiplicities $m_1=m_2=m_3=2$. If $\lambda=\sqrt{3}$, then $n=3,12$, or 24 and M has distinct principal curvatures $\kappa_1=\sqrt{3}$, $\kappa_2=0$, and $\kappa_3=-\sqrt{3}$.

(b) If $s=4$, from Lemma 4, $m_1=m_3$, $m_2=m_4$, and

$$\begin{aligned}\kappa_1&=\cot\theta=\lambda,\\ \kappa_2&=\cot\left(\theta+\frac{\pi}{4}\right)=\frac{\lambda-1}{\lambda+1},\\ \kappa_3&=\cot\left(\theta+\frac{2\pi}{4}\right)=-\frac{1}{\lambda},\\ \kappa_4&=\cot\left(\theta+\frac{3\pi}{4}\right)=-\frac{\lambda+1}{\lambda-1}.\end{aligned} \quad (66)$$

Let

$$\begin{aligned}A&=\lambda-\frac{1}{\lambda},\\ B&=\frac{\lambda-1}{\lambda+1}-\frac{\lambda+1}{\lambda-1}.\end{aligned} \quad (67)$$

Since $n=2(m_1+m_2)$ and $AB=-4$, we have

$$H=\frac{m_1A+m_2B}{2(m_1+m_2)}=\frac{m_1A^2-4m_2}{2(m_1+m_2)A}, \quad (68)$$

$$\rho^2=\frac{m_1(5m_1+2m_2)A^4+4\left(m_1^2-4m_1m_2+m_2^2\right)A^2+16m_2(2m_1+5m_2)}{2(m_1+m_2)A^2}, \quad (69)$$

$$\sum_i\tilde{\kappa}_i^3=\frac{\left(A^2+4\right)^2m_1m_2\left[(m_1+2m_2)A^2-4(2m_1+m_2)\right]}{2(m_1+m_2)^2A^3}. \quad (70)$$

From Lemma 5, $\sum_i\tilde{\kappa}_i^3-((n-4)/2)\rho^2H=0$; we can obtain

$$\begin{aligned}&m_1\left[-5m_1^3-m_1^2(7m_2-10)-2m_1m_2(m_2-3)+4m_2^2\right]A^6\\ &+4m_1\left[-m_1^3+m_2^2(m_2+4)+2m_1m_2(5m_2-9)+2m_1^2(4m_2+1)\right]A^4\\ &-16m_2\left[m_1^3-(m_2-2)m_2^2+2m_1m_2(4m_2-9)+2m_1^2(5m_2+2)\right]A^2\\ &+64m_2\left[2m_1^2(m_2-2)+5(m_2-2)m_2^2+m_1m_2(7m_2-6)\right]=0.\end{aligned} \quad (71)$$

If $(m_1,m_2)=(2,2)$, then $A=\lambda-(1/\lambda)$ is positive solution of $A^6-12A^4+48A^2-64=0$. This implies that $A=2$,

$$\begin{aligned}\kappa_1&=1+\sqrt{2},\\ \kappa_2&=\sqrt{2}-1,\\ \kappa_3&=-(\sqrt{2}-1),\\ \kappa_4&=-(1+\sqrt{2}),\end{aligned} \quad (72)$$

$H=0$, and $\rho^2=24=3n$. If $(m_1,m_2)=(4,5)$, the $A=\lambda-(1/\lambda)$ is positive solution of $175A^6-1473A^4+6465A^2-21020=0$.

(c) If $s=6$, from Lemma 4, $m_1=m_2=\cdots=m_6=m=1$ or 2, and

$$\begin{aligned}\kappa_1&=\cot\theta=\lambda,\\ \kappa_2&=\cot\left(\theta+\frac{\pi}{6}\right)=\frac{\sqrt{3}\lambda-1}{\lambda+\sqrt{3}},\\ \kappa_3&=\cot\left(\theta+\frac{2\pi}{6}\right)=\frac{\lambda-\sqrt{3}}{1+\sqrt{3}\lambda},\\ \kappa_4&=\cot\left(\theta+\frac{3\pi}{6}\right)=-\frac{1}{\lambda},\\ \kappa_5&=\cot\left(\theta+\frac{4\pi}{6}\right)=\frac{\lambda+\sqrt{3}}{1-\sqrt{3}\lambda},\\ \kappa_6&=\cot\left(\theta+\frac{5\pi}{6}\right)=\frac{1+\sqrt{3}\lambda}{\sqrt{3}-\lambda}.\end{aligned} \quad (73)$$

Then, we can obtain

$$H=\frac{(\lambda-1)(\lambda+1)\left(\lambda^2-4\lambda+1\right)\left(\lambda^2+4\lambda+1\right)}{2\lambda\left(\lambda^2-3\right)\left(3\lambda^2-1\right)}, \quad (74)$$

$$\rho^2=\frac{15m\left(1+\lambda^2\right)^6}{2\lambda^2\left(\lambda^2-3\right)^2\left(3\lambda^2-1\right)^2}, \quad (75)$$

where $m = 1$ or 2. From Lemma 5, $\sum_i \tilde{\kappa}_i^3 - ((n-4)/2)\rho^2 H = 0$; we can obtain

$$\frac{15(12-n)m\left(1+\lambda^2\right)^6(\lambda-1)(\lambda+1)\left(\lambda^2-4\lambda+1\right)\left(\lambda^2+4\lambda+1\right)}{8\lambda^3\left(\lambda^2-3\right)^3\left(3\lambda^2-1\right)^3} = 0. \tag{76}$$

Then either $m = 2$, $n = 12$, and M has distinct principal curvatures

$$\begin{aligned} \kappa_1 &= \lambda, \\ \kappa_2 &= \frac{\sqrt{3}\lambda - 1}{\lambda + \sqrt{3}}, \\ \kappa_3 &= \frac{\lambda - \sqrt{3}}{1 + \sqrt{3}\lambda}, \\ \kappa_4 &= -\frac{1}{\lambda}, \\ \kappa_5 &= \frac{\lambda + \sqrt{3}}{1 - \sqrt{3}\lambda}, \\ \kappa_6 &= \frac{1 + \sqrt{3}\lambda}{\sqrt{3} - \lambda}, \end{aligned} \tag{77}$$

or $m = 1$, $n = 6$, and M is minimal with distinct principal curvatures

$$\begin{aligned} \kappa_1 &= 2 + \sqrt{3}, \\ \kappa_2 &= 1, \\ \kappa_3 &= 2 - \sqrt{3}, \\ \kappa_4 &= -2 + \sqrt{3}, \\ \kappa_5 &= -1, \\ \kappa_6 &= -(2 + \sqrt{3}). \end{aligned} \tag{78}$$

This completes the proof of Proposition 3.

Now, combining Propositions 1–3, the following classification theorem of isoparametric extrema hypersurfaces in a space form can be obtained. □

Theorem 5. *Suppose that* $x: M^n \longrightarrow N^{n+1}(c)$ *is an isoparametric extremal hypersurface with distinct principal curvatures* $\kappa_1 > \kappa_2 > \cdots > \kappa_s$. *The multiplicity of* κ_i *is denoted by* m_i, $i = 1, 2, \ldots, s$. *Then the following properties hold.*

(1) If $s = 1$, *then* $\rho^2 = 0$. *In other words,* M *is total umbilical.*

(2) If $s = 2$, *assume that the two distinct principal curvatures* $\kappa_1 = \lambda$ *and* $\kappa_2 = \mu$ *with multiplicity* k *and* $n - k$ *respectively,* $1 \le k \le n-1$, *and then either* $\rho^2 = 0$, *or* $n = 2k$, *or*

$$\rho^2 = \frac{nk(n-k)(n-4)^2}{4(n-2k)^2} H^2. \tag{79}$$

(a) If $n = 2k = 4$, *then* M *is the hypersurface* $\mathbb{S}^2(\coth^2\theta - 1) \times \mathbb{H}^2(\tan\mathrm{h}^2\theta - 1)$ *for* $c = -1$, *or* $\mathbb{S}^2(r) \times \mathbb{R}^2$ *for* $c = 0$, *or* $\mathbb{S}^2(r) \times \mathbb{S}^2(\sqrt{1-r^2})$ *for* $c = 1$, *where* $\tilde{\lambda} = \lambda - H$, $\tilde{\mu} = \mu - H$, *and* $\theta > 0$.

(b) If $n = 2k$ *and* $n \ne 4$, *then* $N^{n+1}(c) = \mathbb{S}^{n+1}$, $M = \mathbb{S}^k(\sqrt{1/2}) \times \mathbb{S}^k(\sqrt{1/2})$, *and* $\rho^2 = n$.

(c) If $n \ne 4$, *then* $H = 0$ *if and only if* $n = 2k$.

(3) If $s > 2$, *then the space form* $N^{n+1}(c)$ *is a sphere* $\mathbb{S}^{n+1}(c)$ *and* $s = 3, 4$, *or* 6.

(a) If $s = 3$, *then either* $n = 6$ *and* M *has distinct principal curvatures*

$$\begin{aligned} \kappa_1 &= \lambda, \\ \kappa_2 &= \frac{\lambda - \sqrt{3}}{1 + \sqrt{3}\lambda}, \\ \kappa_3 &= \frac{\lambda + \sqrt{3}}{1 - \sqrt{3}\lambda}, \end{aligned} \tag{80}$$

with multiplicities $m_1 = m_2 = m_3 = 2$, *or* M *is minimal and has distinct principal curvatures*

$$\begin{aligned} \kappa_1 &= \sqrt{3}, \\ \kappa_2 &= 0, \\ \kappa_3 &= -\sqrt{3}, \end{aligned} \tag{81}$$

with multiplicities $m_1 = m_2 = m_3 = 1, 4$, *or* 8 *(*$n = 3, 12$, *or* 24*).*

(b) If $s = 4$, *then* $m_1 = m_3$, $m_2 = m_4$,

$$\begin{aligned} \kappa_1 &= \lambda, \\ \kappa_2 &= \frac{\lambda - 1}{\lambda + 1}, \\ \kappa_3 &= -\frac{1}{\lambda}, \\ \kappa_4 &= -\frac{\lambda + 1}{\lambda - 1}, \end{aligned} \tag{82}$$

and $A = \lambda - (1/\lambda)$ *is the positive solution of the equation*

$$\begin{aligned} & m_1\left[-5m_1^3 - m_1^2(7m_2 - 10) - 2m_1m_2(m_2 - 3) + 4m_2^2\right]A^6 \\ & + 4m_1\left[-m_1^3 + m_2^2(m_2 + 4) + 2m_1m_2(5m_2 - 9) + 2m_1^2(4m_2 + 1)\right]A^4 \\ & - 16m_2\left[m_1^3 - (m_2 - 2)m_2^2 + 2m_1m_2(4m_2 - 9) + 2m_1^2(5m_2 + 2)\right]A^2 \\ & + 64m_2\left[2m_1^2(m_2 - 2) + 5(m_2 - 2)m_2^2 + m_1m_2(7m_2 - 6)\right] = 0, \end{aligned} \tag{83}$$

with multiplicities $(m_1, m_2) = (2, 2)$, *or* $(m_1, m_2) = (4, 5)$, *or* $2^{\varphi(m_1 - 1)}k = m_1 + m_2 + 1$ *for some integer* k. *Here* $\varphi(x)$ *is a number of integers* y *with* $1 \le y \le x$ *and* $y \equiv 0, 1, 2, 4$, *mod* 8.

(c) If $s = 6$, *then either* $n = 12$ *and* M *has distinct principal curvatures*

$$\kappa_1 = \lambda,$$
$$\kappa_2 = \frac{\sqrt{3}\lambda - 1}{\lambda + \sqrt{3}},$$
$$\kappa_3 = \frac{\lambda - \sqrt{3}}{1 + \sqrt{3}\lambda},$$
$$\kappa_4 = -\frac{1}{\lambda}, \tag{84}$$
$$\kappa_5 = \frac{\lambda + \sqrt{3}}{1 - \sqrt{3}\lambda},$$
$$\kappa_6 = \frac{1 + \sqrt{3}\lambda}{\sqrt{3} - \lambda},$$

with multiplicities $m_1 = \cdots = m_6 = 2$, or $n = 6$, and M is minimal and has distinct principal curvatures

$$\kappa_1 = 2 + \sqrt{3},$$
$$\kappa_2 = 1,$$
$$\kappa_3 = 2 - \sqrt{3},$$
$$\kappa_4 = -2 + \sqrt{3}, \tag{85}$$
$$\kappa_5 = -1,$$
$$\kappa_6 = -(2 + \sqrt{3}),$$

with multiplicities $m_1 = \cdots = m_6 = 1$.

Proof. It is a combining consequence of Propositions 1–3.

The classification theorem of isoparametric extremal hypersurfaces is too complicated. For the subsequent discussion, we have the following corollary. □

Corollary 1. *Suppose that $x\colon M^n \longrightarrow N^{n+1}(c)$ is an isoparametric extremal hypersurface with distinct principal curvatures $\kappa_1 > \kappa_2 > \cdots > \kappa_s$. The multiplicity of κ_i is denoted by m_i, $i = 1, 2, \ldots, s$. Assume that*

$$\rho^2 = n\left(c + \frac{n-2}{2}H^2\right). \tag{86}$$

Then, M has at most two distinct principal curvatures, i.e., $s = 1$ or 2.

(1) If $s = 1$, then $\rho^2 = 0$.

(2) If $s = 2$, assume that the two distinct principal curvatures $\kappa_1 = \lambda$ and $\kappa_2 = \mu$ with multiplicity k and $n - k$ respectively, $1 \le k \le n - 1$, and then either $\rho^2 = 0$ or $n = 2k$, or

$$k = \frac{n}{2} \pm \frac{1}{2}\sqrt{\frac{n^2H^2(n-4)^2}{n^2H^2 + 16c}}. \tag{87}$$

(a) If $n = 2k = 4$ and $H \neq 0$, then M is the hypersurface $\mathbb{S}^2(\coth^2\theta - 1) \times \mathbb{H}^2(\tanh^2\theta - 1)$ for $c = -1$, or $\mathbb{S}^2(r) \times \mathbb{R}^2$ for $c = 0$, or $\mathbb{S}^2(r) \times \mathbb{S}^2(\sqrt{1 - r^2})$ for $c = 1$.

(b) Either $n = 2k$ and $n \neq 4$, or $n = 2k = 4$ and $H = 0$, we have that $N^{n+1}(c)$ is not a hyperbolic space, $\rho^2 = 0$ for $N^{n+1}(c) = \mathbb{R}^{n+1}$, and $\rho^2 = n$ for $N^{n+1}(c) = \mathbb{S}^{n+1}$. If $N^{n+1}(c) = \mathbb{S}^{n+1}$, M is the hypersurface $\mathbb{S}^k(\sqrt{1/2}) \times \mathbb{S}^k(\sqrt{1/2})$.

Proof. By using Theorem 5, we can easily check this corollary.

(1) If $s = 1$, then $\rho^2 = 0$.

(2) If $s = 2$, the principal curvatures λ and μ have multiplicities k and $n - k$, respectively; let $\tilde{\lambda} = \lambda - H$ and $\tilde{\mu} = \mu - H$. Then either $\rho^2 = 0$, or $n = 2k$, or

$$\rho^2 = \frac{nk(n-k)(n-4)^2}{4(n-2k)^2}H^2. \tag{88}$$

(a) If $n = 2k = 4$ and $H \neq 0$, from Theorem 5, M is the hypersurface $\mathbb{S}^2(\cot h^2\theta - 1) \times \mathbb{H}^2(\tan h^2\theta - 1)$ for $c = -1$, or $\mathbb{S}^2(r) \times \mathbb{R}^2$ for $c = 0$, or $\mathbb{S}^2(r) \times \mathbb{S}^2(\sqrt{1 - r^2})$ for $c = 1$. If $c = -1$, then we have $\lambda = \cot h\theta$, $\mu = \tan hh\theta$, $H = ((\cot h\theta + \tan h\theta)/2)$, and $\rho^2 = (\cot h\theta - \tan h\theta)^2$. If $c = 0$, then $\lambda = \sqrt{r}$, $H = (\sqrt{r}/2)$, and $\rho^2 = r$. If $c = 1$, then

$$\lambda = \frac{\sqrt{1 - r^2}}{r},$$
$$\mu = -\frac{r}{\sqrt{1 - r^2}},$$
$$H = \frac{\left(1 - 2r^2\right)}{2r\sqrt{1 - r^2}}, \tag{89}$$
$$\rho^2 = \frac{1}{r^2\left(1 - r^2\right)}.$$

(b) Either $n = 2k$ and $n \neq 4$, or $n = 2k = 4$ and $H = 0$; we have that $\rho^2 = 2k\tilde{\lambda}^2 = 2k\tilde{\mu}^2 = 2k\lambda^2 = 2k\mu^2$. From (86), if $N^{n+1}(c) = \mathbb{H}^{n+1}$, then $c = -1$ and $\rho^2 = -n < 0$. This contradiction shows that $N^{n+1}(c)$ is not a hyperbolic space. If $N^{n+1}(c) = \mathbb{R}^{n+1}$, then $c = 0$ and $\rho^2 = 0$. If $N^{n+1}(c) = \mathbb{S}^{n+1}$, then $c = 1$ and $\rho^2 = n$. Then we have $\rho^2 = n = 2k\lambda^2 = n\lambda^2$. This implies that $\lambda = 1$, $\mu = -1$, and M is the hypersurface $\mathbb{S}^k(\sqrt{1/2}) \times \mathbb{S}^k(\sqrt{1/2})$.

If $\rho^2 \neq 0, n \neq 4$, $n \neq 2k$, and $H \neq 0$, by using Theorem 5 and equation (60), we have

$$\rho^2 = \frac{nk(n-k)(n-4)^2}{4(n-2k)^2}H^2 = n\left(c + \frac{n-2}{2}H^2\right). \tag{90}$$

Then, k is the positive integer solution of the equation

$$\begin{aligned}&\left(n^2H^2+16c\right)k^2-n\left(n^2H^2+16c\right)k\\&+2n^2\left[(n-2)H^2+2c\right]=0.\end{aligned}\tag{91}$$

Since $H\neq 0$, if $n^2H^2+16c=0$, we have $c\neq 0$ and $(n-2)H^2+2c=0$. Then

$$H^2=-\frac{16c}{n^2}=-\frac{2c}{n-2}.\tag{92}$$

This implies that $n=4$. The contradiction shows that $n^2H^2+16c\neq 0$. Hence, k is a positive integer and

$$k=\frac{n}{2}\pm\frac{1}{2}\sqrt{\frac{n^2H^2(n-4)^2}{n^2H^2+16c}}.\tag{93}$$

(3) If $s>2$, from Theorem 5, the space form $N^{n+1}(c)$ is a sphere $\mathbb{S}^{n+1}(c)$ and $s=3,4$, or 6. Then we can conclude that $c=1$ and

$$\rho^2=n\left(1+\frac{n-2}{2}H^2\right).\tag{94}$$

Now, we shall show that $s\neq 3$, $s\neq 4$, and $s\neq 6$.

(a) Assume that $s=3$; from Theorem 5, either $m=2$, $n=6$, and M has distinct principal curvatures

$$\begin{aligned}\kappa_1&=\lambda,\\\kappa_2&=\frac{\lambda-\sqrt{3}}{1+\sqrt{3}\lambda},\\\kappa_3&=\frac{\lambda+\sqrt{3}}{1-\sqrt{3}\lambda},\end{aligned}\tag{95}$$

with multiplicities $m_1=m_2=m_3=2$, or M is a Cartan minimal hypersurface with distinct principal curvatures

$$\begin{aligned}\kappa_1&=\sqrt{3},\\\kappa_2&=0,\\\kappa_3&=-\sqrt{3},\end{aligned}\tag{96}$$

and multiplicities $m_1=m_2=m_3=1,4$, or 8 ($n=3,12$, or 24). If M is a Cartan minimal hypersurface, then $\rho^2=6m_1=2n$. It contradicts equation (94); $\rho^2=n$. This contradiction shows that M is not a Cartan minimal hypersurface. If $n=6$, then

$$\begin{aligned}H&=\frac{\lambda\left(\lambda^2-3\right)}{3\lambda^2-1},\\\rho^2&=\frac{12\left(1+\lambda^2\right)^3}{\left(3\lambda^2-1\right)^2}.\end{aligned}\tag{97}$$

By using equation (62),

$$\rho^2=\frac{6\left[2\lambda^6-3\lambda^4+12\lambda^2+1\right]}{\left(3\lambda^2-1\right)^2}=\frac{12\left(1+\lambda^2\right)^3}{\left(3\lambda^2-1\right)^2}.\tag{98}$$

We have $\lambda=\sqrt{3}$ and the principal curvatures are

$$\begin{aligned}\kappa_1&=\lambda=\sqrt{3},\\\kappa_2&=0,\\\kappa_3&=-\sqrt{3}.\end{aligned}\tag{99}$$

Hence, M is a Cartan minimal hypersurface with $\rho^2=2n$. This contradiction shows that $s\neq 3$.

(b) If $s=4$, from Theorem 5, $m_1=m_3$, $m_2=m_4$, and

$$\begin{aligned}\kappa_1&=\lambda,\\\kappa_2&=\frac{\lambda-1}{\lambda+1},\\\kappa_3&=-\frac{1}{\lambda},\\\kappa_4&=-\frac{\lambda+1}{\lambda-1}.\end{aligned}\tag{100}$$

Let

$$\begin{aligned}A&=\lambda-\frac{1}{\lambda},\\B&=\frac{\lambda-1}{\lambda+1}-\frac{\lambda+1}{\lambda-1}.\end{aligned}\tag{101}$$

Since $n=2(m_1+m_2)$ and $AB=-4$, we have

$$\begin{aligned}H&=\frac{m_1A+m_2B}{2(m_1+m_2)}=\frac{m_1A^2-4m_2}{2(m_1+m_2)A},\\\rho^2&=\frac{m_1(5m_1+2m_2)A^4+4\left(m_1^2-4m_1m_2+m_2^2\right)A^2+16m_2(2m_1+5m_2)}{2(m_1+m_2)A^2}.\end{aligned}\tag{102}$$

From equation (94),

$$\rho^2 = 2(m_1+m_2)\left(1+\frac{(m_1A^2-4m_2)^2(m_1+m_2-1)}{4(m_1+m_2)^2A^2}\right). \tag{103}$$

Combine these two equations; we have $A=\lambda-(1/\lambda)$ is positive solution of

$$\begin{aligned} &m_1\left[m_1^2+m_1(m_2-6)-2m_2\right]A^4\\ &-8m_1m_2(m_1+m_2-4)A^2\\ &+16m_2\left[m_1(m_2-2)+m_2(m_2-6)\right]=0. \end{aligned} \tag{104}$$

By a direct calculation, we have

$$\begin{aligned} A^2 = &\frac{4m_1m_2(m_1+m_2-4)}{T_1}\\ &-\frac{4\sqrt{2m_1m_2}\sqrt{T_1+T_2+2m_1m_2(m_1+m_2-4)}}{T_1}, \end{aligned} \tag{105}$$

where $T_1=-6m_1^2+m_1^3-2m_1m_2+m_1^2m_2$ and $T_2=-6m_2^2+m_2^3-2m_1m_2+m_1m_2^2$. From Theorem 5 or Proposition 3, $A=\lambda-(1/\lambda)$ is the positive solution of equation (56). But substituting (105) into equation (56), the equation does not hold. This contradiction shows that $s\neq 4$.

(c) Assume that $s=6$; then $m_1=m_2=\cdots=m_6=m=1$ or 2. Either $n=12$ and M has distinct principal curvatures

$$\begin{aligned} \kappa_1 &= \lambda,\\ \kappa_2 &= \frac{\sqrt{3}\lambda-1}{\lambda+\sqrt{3}},\\ \kappa_3 &= \frac{\lambda-\sqrt{3}}{1+\sqrt{3}\lambda},\\ \kappa_4 &= -\frac{1}{\lambda},\\ \kappa_5 &= \frac{\lambda+\sqrt{3}}{1-\sqrt{3}\lambda},\\ \kappa_6 &= \frac{1+\sqrt{3}\lambda}{\sqrt{3}-\lambda}, \end{aligned} \tag{106}$$

with multiplicities $m_1=\cdots=m_6=2$, or $n=6$, and M is minimal and has distinct principal curvatures

$$\begin{aligned} \kappa_1 &= 2+\sqrt{3},\\ \kappa_2 &= 1,\\ \kappa_3 &= 2-\sqrt{3},\\ \kappa_4 &= -2+\sqrt{3},\\ \kappa_5 &= -1,\\ \kappa_6 &= -(2+\sqrt{3}), \end{aligned} \tag{107}$$

with multiplicities $m_1=\cdots=m_6=1$. If $n=12$, we have

$$\begin{aligned} H &= \frac{(\lambda-1)(\lambda+1)(\lambda^2-4\lambda+1)(\lambda^2+4\lambda+1)}{2\lambda(\lambda^2-3)(3\lambda^2-1)},\\ \rho^2 &= \frac{15(1+\lambda^2)^6}{\lambda^2(\lambda^2-3)^2(3\lambda^2-1)^2}, \end{aligned} \tag{108}$$

where $\lambda(\lambda^2-3)(3\lambda^2-1)\neq 0$. By using equation (94),

$$\rho^2 = 12\left(c+\frac{5(\lambda-1)^2(\lambda+1)^2(1-4\lambda+\lambda^2)^2(1+4\lambda+\lambda^2)^2}{4\lambda^2(\lambda^2-3)^2(3\lambda^2-1)^2}\right). \tag{109}$$

Combine these two equations; we have $\lambda^4(3\lambda^2-1)^4(\lambda^2-3)^4(c-5)=0$. This contradiction shows that $n\neq 12$. If $n=6$, M is a minimal hypersurface and $\rho^2=30m=5n$. From equation (94), we have $\rho^2=n$. This contradiction shows that $s\neq 6$.

If x: $M^n \longrightarrow N^{n+1}(c)$ is a minimal isoparametric extrema hypersurface in the space form $N^{n+1}(c)$, we have Corollary 2. □

Corollary 2. *Suppose that x: $M^n \longrightarrow N^{n+1}(c)$ is a minimal isoparametric extrema hypersurface with distinct principal curvatures $\kappa_1>\kappa_2>\cdots>\kappa_s$. The multiplicity of κ_i is denoted by m_i, $i=1,2,\ldots,s$.*

(1) If $s=1$, then $\rho^2=0$.

(2) If $s=2$, assume that the two distinct principal curvatures $\kappa_1=\lambda$ and $\kappa_2=\mu$ with multiplicity k and $n-k$, respectively, $1\le k\le n-1$, and then either $\rho^2=0$ or $n=2k$. In the latter case, $N^{n+1}(c)=\mathbb{S}^{n+1}$, $\rho^2=n$, and $M=\mathbb{S}^k(\sqrt{1/2})\times\mathbb{S}^k(\sqrt{1/2})$.

(3) If $s>2$, then the space form $N^{n+1}(c)$ is a sphere $\mathbb{S}^{n+1}(c)$ and $s=3,4$, or 6.

(a) If $s=3$, then M is a Cartan minimal hypersurface with $\rho^2=2n$.

(b) If $s=4$, then $m_1=m_3$, $m_2=m_4$. Moreover, $\rho^2=3n$.

(c) If $s=6$, then $m_1=m_2=\cdots=m_6=1$ or 2. Moreover, $\rho^2=5n$.

Proof. Since M is minimal, $H=0$; we prove this corollary by using Theorem 5.

(1) By using Theorem 5, if $s=1$, then $\rho^2=0$ and M is totally umbilical.

(2) If $s=2$, since $H=0$, from Theorem 5, either $\rho^2=0$ or $n=2k$. If $n=2k$, then $\rho^2=n$ and $M=\mathbb{S}^k(\sqrt{1/2})\times\mathbb{S}^k(\sqrt{1/2})$.

(3) If $s > 2$, from Theorem 5, the space form $N^{n+1}(c)$ is a sphere $\mathbb{S}^{n+1}(c)$ and $s = 3, 4$, or 6.

(a) If $s = 3$, from (60)

$$H = \frac{\lambda(\lambda^2 - 3)}{3\lambda^2 - 1} = 0. \quad (110)$$

We have $\lambda = \sqrt{3}$ and the principal curvatures

$$\kappa_1 = \lambda = \sqrt{3},\quad \kappa_2 = 0,\quad \kappa_3 = -\sqrt{3}. \quad (111)$$

From (62), $\rho^2 = 2n$.

(b) If $s = 4$, from Theorem 5, $m_1 = m_3$, $m_2 = m_4$. Since M is minimal, by using equation (68)

$$H = \frac{m_1 A^2 - 4m_2}{2(m_1 + m_2)A} = 0, \quad (112)$$

where $A = \lambda - (1/\lambda)$. Then we have $A^2 = -(4m_2/m_1)$ and $m_1\lambda^4 - 2(m_1 + 2m_2)\lambda^2 + m_1 = 0$. This implies that

$$\lambda^2 = \frac{(m_1 + 2m_2) + 2\sqrt{m_2(m_1 + m_2)}}{m_1} = \left(\sqrt{\frac{m_2}{m_1}} + \sqrt{\frac{m_1 + m_2}{m_1}}\right)^2. \quad (113)$$

By using Lemma 5 and equations (69) and (70), we can conclude that

$$(m_1 - m_2)\left(m_1^2 + m_1 m_2 + \sqrt{m_1 m_2}\sqrt{m_1(m_1 + m_2)}\right)^4\left(m_1^2 + 2m_1 m_2 + 2\sqrt{m_1 m_2}\sqrt{m_1(m_1 + m_2)}\right) = 0. \quad (114)$$

Then we have that either $m_1 = m_2$ or $m_1^2 + 2m_1 m_2 + 2\sqrt{m_1 m_2}\sqrt{m_1(m_1 + m_2)} = 0$. Substituting these solutions into equation (69),

$$\rho^2 = \frac{6m_2\left(m_1^2 + m_1 m_2 + \sqrt{m_1 m_2}\sqrt{m_1(m_1 + m_2)}\right)^2\left(m_1^2 + 2m_1 m_2 + 2\sqrt{m_1 m_2}\sqrt{m_1(m_1 + m_2)}\right)}{m_1^2\left[m_1\sqrt{m_1 m_2} + 2m_2\left(\sqrt{m_1 m_2} + \sqrt{m_1(m_1 + m_2)}\right)\right]^2}, \quad (115)$$

then either $\rho^2 = 0$ or $m_1 = m_2$. If $m_1 = m_2$, by a direct calculation of the above equation, we have that $\rho^2 = 12m_1 = 3n$.

(d) If $s = 6$, then $m_1 = m_2 = \cdots = m_6 = m = 1$ or 2. Since M is minimal, from (74),

$$H = \frac{(\lambda - 1)(\lambda + 1)(\lambda^2 - 4\lambda + 1)(\lambda^2 + 4\lambda + 1)}{2\lambda(\lambda^2 - 3)(3\lambda^2 - 1)} = 0. \quad (116)$$

Then we can conclude that $\lambda = 2 + \sqrt{3}$ and the principal curvatures

$$\kappa_1 = 2 + \sqrt{3},\quad \kappa_2 = 1,\quad \kappa_3 = 2 - \sqrt{3},\quad \kappa_4 = -2 + \sqrt{3},\quad \kappa_5 = -1,\quad \kappa_6 = -(2 + \sqrt{3}). \quad (117)$$

From (75), we have $\rho^2 = 30m = 5n$. □

5. Integral Inequality for Compact Extremal Hypersurfaces in the Space Form

Let $N^{n+1}(c)$ be a space form and $x\colon M \longrightarrow N^{n+1}(c)$ is a compact hypersurface; from the structure equations and Lemmas 1 and 2, we have the following lemmas.

Lemma 7. *If $x\colon M^n \longrightarrow N^{n+1}(c)$ is a compact hypersurface in $N^{n+1}(c)$, then*

$$\frac{1}{2}\Delta\rho^2 = \sum_{i,j,k}\tilde{h}_{ijk}^2 + n\sum_{i,j}\tilde{h}_{ij}H_{ij} + \rho^2\left[n\left(c + H^2\right) - \rho^2\right] + n\sum_{i,j,k}H\tilde{h}_{ij}\tilde{h}_{ik}\tilde{h}_{kj}. \quad (118)$$

Proof. From Lemmas 1 and 2, by a direct calculation, we have

$$\sum_{i,j}h_{ij}\Delta h_{ij} = \sum_{i,j,k}h_{ij}h_{ijkk} = n\sum_{i,j}h_{ij}H_{ij} + \sum_{i,j,k,l}h_{ij}h_{il}R_{lkjk} + \sum_{i,j,k,l}h_{ij}h_{lk}R_{lijk}. \quad (119)$$

Since $\tilde{h}_{ij} = h_{ij} - H\delta_{ij}$, we can obtain

$$\sum_{i,j}\widetilde{h}_{ij}\Delta\widetilde{h}_{ij} = n\sum_{i,j}\widetilde{h}_{ij}H_{ij} + \sum_{i,j,k,l}\widetilde{h}_{ij}\widetilde{h}_{il}R_{lkjk} + \sum_{i,j,k,l}\widetilde{h}_{ij}\widetilde{h}_{lk}R_{lijk}. \quad (120)$$

By using equation (23),

$$\sum_{i,j,k,l}\widetilde{h}_{ij}\widetilde{h}_{il}R_{lkjk} + \sum_{i,j,k,l}\widetilde{h}_{ij}\widetilde{h}_{lk}R_{lijk} = n\left(c + H^2\right)\rho^2 + n\sum_{i,j,k}H\widetilde{h}_{ij}\widetilde{h}_{ik}\widetilde{h}_{kj} - \rho^4. \quad (121)$$

Then,

$$\begin{aligned}\frac{1}{2}\Delta\rho^2 &= \sum_{i,j,k}\widetilde{h}_{ijk}^2 + \sum_{i,j}\widetilde{h}_{ij}\Delta\widetilde{h}_{ij} = \sum_{i,j,k}\widetilde{h}_{ijk}^2 + n\sum_{i,j}\widetilde{h}_{ij}H_{ij}\\ &+ \rho^2\left[n\left(c + H^2\right) - \rho^2\right] + n\sum_{i,j,k}H\widetilde{h}_{ij}\widetilde{h}_{ik}\widetilde{h}_{kj}.\end{aligned} \quad (122)$$

We can conclude that Lemma 7 holds. □

Lemma 8. *If $x: M^n \longrightarrow N^{n+1}(c)$ is a compact extrema hypersurface in $N^{n+1}(c)$, then*

$$\frac{1}{2}\Delta\rho^2 = \sum_{i,j,k}\widetilde{h}_{ijk}^2 + n\sum_{i,j}\widetilde{h}_{ij}H_{ij} + \rho^2\left[n\left(c + \frac{n-2}{2}H^2\right) - \rho^2\right] - n(n-1)H\Delta H. \quad (123)$$

Proof. By using Theorem 4, if M is an extremal hypersurface,

$$nH\sum_{i,j,k}\widetilde{h}_{ij}\widetilde{h}_{ik}\widetilde{h}_{kj} = -n(n-1)H\Delta H + \frac{n(n-4)}{2}\rho^2H^2. \quad (124)$$

From Lemma 7,

$$\begin{aligned}\frac{1}{2}\Delta\rho^2 &= \sum_{i,j,k}\widetilde{h}_{ijk}^2 + n\sum_{i,j}\widetilde{h}_{ij}H_{ij}\\ &+ \rho^2\left[n\left(c + \frac{n-2}{2}H^2\right) - \rho^2\right] - n(n-1)H\Delta H.\end{aligned} \quad (125)$$

Hence, Lemma 8 holds.

For the following estimate, also see [4, 11]. □

Lemma 9. *Let M be a compact hypersurface in the space form $N^{n+1}(c)$; then we have the following estimates:*

$$\begin{aligned}|\nabla h|^2 &\geq \frac{3n^2}{n+2}|\nabla H|^2,\\ |\nabla\widetilde{h}|^2 &\geq \frac{2n(n-1)}{n+2}|\nabla H|^2,\end{aligned} \quad (126)$$

where $|\nabla h|^2 = \sum_{i,j,k}h_{ijk}^2$, $|\nabla\widetilde{h}|^2 = \sum_{i,j,k}\widetilde{h}_{ijk}^2$, $|\nabla H|^2 = \sum_k H_k^2$, and $H_k = \nabla_k H$. Equality holds if and only if all the functions h_{ij} are constant for all i, j.

Proof. Following the method of Lemma 4.1 in [4], construct the tensor as

$$F_{ijk} = h_{ijk} - \frac{n}{n+2}\left(H_i\delta_{jk} + H_j\delta_{ik} + H_k\delta_{ij}\right). \quad (127)$$

Then, we can obtain

$$\begin{aligned}|F|^2 = \sum_{i,j,k}F_{ijk}^2 &= |\nabla h|^2 - \frac{3n^2}{n+2}|\nabla H|^2\\ &= |\nabla\widetilde{h}|^2 - \frac{2n(n-1)}{n+2}|\nabla H|^2 \geq 0.\end{aligned} \quad (128)$$

Equality holds if and only if h_{ij} and H are constant for all i, j.

By using Lemmas 7–9, we have the following integral inequality for compact extremal hypersurfaces in $N^{n+1}(c)$. □

Theorem 6. *If $x: M^n \longrightarrow N^{n+1}(c)$ is a compact extremal hypersurface without boundary in the space form $N^{n+1}(c)$, then*

$$\int_M \rho^2\left[n\left(c + \frac{n-2}{2}H^2\right) - \rho^2\right]\mathrm{d}\mu \leq 0. \quad (129)$$

In particular, if

$$0 \leq \rho^2 \leq n\left(c + \frac{n-2}{2}H^2\right), \quad (130)$$

then either $\rho^2 \equiv 0$ or $\rho^2 \equiv n(c + ((n-2)/2)H^2)$. If $\rho^2 \equiv n(c + ((n-2)/2)H^2)$, then either $n = 2k$ or

$$k = \frac{n}{2} \pm \frac{1}{2}\sqrt{\frac{n^2H^2(n-4)^2}{n^2H^2 + 16c}}. \quad (131)$$

If $n = 2k = 4$ and $H \neq 0$, then M is the hypersurface $\mathbb{S}^2(\coth^2\theta - 1) \times \mathbb{H}^2(\tanh^2\theta - 1)$ for $c = -1$, or $\mathbb{S}^2(r) \times \mathbb{R}^2$ for $c = 0$, or $\mathbb{S}^2(r) \times \mathbb{S}^2(\sqrt{1-r^2})$ for $c = 1$. If $n = 2k$ and $n \neq 4$, or $n = 2k = 4$ and $H = 0$, we have that $N^{n+1}(c) = \mathbb{S}^{n+1}$ and M is the hypersurface $\mathbb{S}^k(\sqrt{1/2}) \times \mathbb{S}^k(\sqrt{1/2})$.

Proof. From Lemma 8, we have

$$\int_M \frac{1}{2}\Delta\rho^2\mathrm{d}\mu = \int_M\left[\sum_{i,j,k}\widetilde{h}_{ijk}^2 + n\sum_{i,j}\widetilde{h}_{ij}H_{ij} + \rho^2\left[n\left(c + \frac{n-2}{2}H^2\right) - \rho^2\right] - n(n-1)H\Delta H\right]\mathrm{d}\mu. \quad (132)$$

Since M is compact without boundary, by Stokes' theorem and $\sum_i \widetilde{h}_{ii}^{\alpha} = 0$,

$$\begin{aligned} 0 &= \int_M \left\{ \sum_{i,j,k} \widetilde{h}_{ijk}^2 + \rho^2\left[n\left(c + \frac{n-2}{2}H^2\right) - \rho^2\right]\right\} d\mu \\ &\geq \int_M \left\{ \frac{2n(n-1)}{n+2}|\nabla H|^2 + \rho^2\left[n\left(c + \frac{n-2}{2}H^2\right) - \rho^2\right]\right\} d\mu \\ &\geq \int_M \rho^2\left[n\left(c + \frac{n-2}{2}H^2\right) - \rho^2\right] d\mu, \end{aligned} \tag{133}$$

where we used Lemma 8, $|\nabla H|^2 \geq 0$, and

$$\begin{aligned} &\int_M \left[n\sum_{i,j} \widetilde{h}_{ij}H_{ij} - n(n-1)H\Delta H \right] d\mu \\ &= \int_M \left[-n\sum_{i,j} \widetilde{h}_{ijj}H_i + n(n-1)\left|\nabla^{\perp}H\right|^2 \right] d\mu \\ &= \int_M \left[-n\sum_{\alpha,i,j} \left(\widetilde{h}_{ijj} + H_i\delta_{jj} - H_j\delta_{ij}\right)H_i + n(n-1)\left|\nabla^{\perp}H\right|^2 \right] d\mu \\ &= \int_M \left[-n(n-1)\left|\nabla^{\perp}H\right|^2 + n(n-1)\left|\nabla^{\perp}H\right|^2 \right] d\mu = 0. \end{aligned} \tag{134}$$

Then, we can obtain the integral inequality

$$\int_M \rho^2\left[n\left(c + \frac{n-2}{2}H^2\right) - \rho^2\right] d\mu \leq 0. \tag{135}$$

In particular, if

$$0 \leq \rho^2 \leq n\left(c + \frac{n-2}{2}H^2\right), \tag{136}$$

then either $\rho^2 = 0$ (i.e., M is totally umbilical) or $\rho^2 = (c + ((n-2)/2)H^2)$. If $\rho^2 \neq 0$, by using Corollary 1, when $n = 4$ and $H \neq 0$, then M is the hypersurface $\mathbb{S}^2(\coth^2\theta - 1) \times \mathbb{H}^2(\tanh^2\theta - 1)$ for $c = -1$, or $\mathbb{S}^2(r) \times \mathbb{R}^2$ for $c = 0$, or $\mathbb{S}^2(r) \times \mathbb{S}^2(\sqrt{1-r^2})$ for $c = 1$. In other cases, $N^{n+1}(c) = \mathbb{S}^{n+1}$ and M is the hypersurface $\mathbb{S}^k(\sqrt{1/2})\times \mathbb{S}^k(\sqrt{1/2})$. □

6. Comparison of Integral Inequalities for Compact Extremal Hypersurfaces

Guo and Li studied a compact extremal submanifold without boundary in the unit sphere $\mathbb{S}^{n+p}$, and they obtained an integral inequality. This integral inequality is described in the compact extremal hypersurface as the following theorem.

Theorem 7 (see [9]). *If x: $M^n \longrightarrow \mathbb{S}^{n+1}$ is a compact extremal hypersurface without boundary in a sphere, then*

$$\int_M \rho^2\left(n - \rho^2\right) d\mu \leq 0. \tag{137}$$

In particular, if

$$0 \leq \rho^2 \leq n, \tag{138}$$

then either $\rho^2 \equiv 0$ or $\rho^2 \equiv n$. If $\rho^2 \equiv n$, then $n = 2k$ and $M = \mathbb{S}^k(\sqrt{1/2}) \times \mathbb{S}^k(\sqrt{1/2})$.

Proof. Refer to Theorem 1.3 in [9].

The integral inequality,

$$\int_M \rho^2\left[n\left(c + \frac{n-2}{2}H^2\right) - \rho^2\right] d\mu \leq 0, \tag{139}$$

is better and more useful than the integral inequality

$$\int_M \rho^2\left(n - \rho^2\right) d\mu \leq 0. \tag{140}$$

When isoparametric extremal hypersurfaces are with

$$n \leq \rho^2 \leq n\left(1 + \frac{n-2}{2}H^2\right), \tag{141}$$

the integral inequality,

$$\int_M \rho^2\left(n - \rho^2\right) d\mu \leq 0, \tag{142}$$

tells us nothing. But the integral inequality,

$$\int_M \rho^2\left[n\left(c + \frac{n-2}{2}H^2\right) - \rho^2\right] d\mu \leq 0, \tag{143}$$

tells us either $\rho^2 = 0$ (i.e., M is totally umbilical) or $\rho^2 = (c + ((n-2)/2)H^2)$. In the latter case, M is the hypersurface $\mathbb{S}^2(\coth^2\theta - 1) \times \mathbb{H}^2(\tanh^2\theta - 1)$, or $\mathbb{S}^2(r)\times \mathbb{R}^2$, or $\mathbb{S}^2(r) \times \mathbb{S}^2(\sqrt{1-r^2})$, or $\mathbb{S}^k(\sqrt{1/2})\times \mathbb{S}^k(\sqrt{1/2})$. So, the sharp estimate in (130) is meaningful, if M is not minimal and

$$n \leq \rho^2 \leq n\left(1 + \frac{n-2}{2}H^2\right). \tag{144}$$

There are many special compact extremal hypersurfaces that Guo and Li have not considered. The following examples state that there exist many isoparametric extremal hypersurfaces which is different form $\mathbb{S}^k(\sqrt{1/2})\times \mathbb{S}^k(\sqrt{1/2})$ and has the properties $\rho^2 = n(c + ((n-2)/2)H^2)$ with $H \neq 0$. Example 1 states that if $n = 4$, even if $n = 2k$, there exist isoparametric extremal hypersurfaces that satisfy the sharp estimate which is different form $\mathbb{S}^k(\sqrt{1/2})\times \mathbb{S}^k(\sqrt{1/2})$. Example 2 states that there exist isoparametric extremal hypersurfaces that satisfy the sharp estimate with $n \neq 2k$. □

Example 1. Suppose that r is a constant and $r \neq \sqrt{1/2}$. The tori

$$M = \mathbb{S}^2(r) \times \mathbb{S}^2\left(\sqrt{1-r^2}\right), \tag{145}$$

has two district principal curvatures and constant mean curvature

$$\kappa_1 = \kappa_2 = \frac{\sqrt{1-r^2}}{r},$$
$$\kappa_3 = \kappa_4 = -\frac{r}{\sqrt{1-r^2}}, \quad (146)$$
$$H = \frac{1-2r^2}{2r\sqrt{1-r^2}} \neq 0.$$

Let

$$\widetilde{\kappa}_1 = \widetilde{\kappa}_2 = \kappa_1 - H = \frac{1}{2r\sqrt{1-r^2}},$$
$$\widetilde{\kappa}_3 = \widetilde{\kappa}_4 = \kappa_3 - H = -\frac{1}{2r\sqrt{1-r^2}}. \quad (147)$$

Then,

$$\rho^2 = \frac{1}{r^2(1-r^2)}, \quad (148)$$

since

$$\sum_i \widetilde{\kappa}_i^3 = 0 = \frac{(n-4)}{2}\rho^2 H. \quad (149)$$

We can obtain that $M = \mathbb{S}^2(r) \times \mathbb{S}^2(\sqrt{1-r^2})$ is an extrema hypersurface in $\mathbb{S}^5$. Now,

$$n\left(c + \frac{n-2}{2}H^2\right) = 4\left(1+H^2\right) = 4\left(1 + \frac{1-4r^2+4r^4}{4r^2(1-r^2)}\right)$$
$$= \frac{1}{r^2(1-r^2)} = \rho^2. \quad (150)$$

Hence, $M = \mathbb{S}^2(r) \times \mathbb{S}^2(\sqrt{1-r^2})$ is an extrema hypersurface in $\mathbb{S}^5$ with $\rho^2 = n(c + ((n-2)/2)H^2)$ and $H \neq 0$.

Since $M = \mathbb{S}^2(r) \times \mathbb{S}^2(\sqrt{1-r^2})$ is an extrema hypersurface, we have the integral inequality by using Theorem 7,

$$\int_M \rho^2(n-\rho^2)\mathrm{d}\mu \leq 0. \quad (151)$$

From Theorem 6, we have the integral inequality

$$\int_M \rho^2\left[n\left(c + \frac{n-2}{2}H^2\right) - \rho^2\right]\mathrm{d}\mu \leq 0. \quad (152)$$

In fact,

$$\int_M \rho^2(n-\rho^2)\mathrm{d}\mu < \int_M \rho^2\left[n\left(c + \frac{n-2}{2}H^2\right) - \rho^2\right]\mathrm{d}\mu \leq 0. \quad (153)$$

This means that the integral inequality of Theorem 6 is better and more useful than the integral inequality of Theorem 7.

Example 2. The tori

$$M = \mathbb{S}^4(r) \times \mathbb{S}^3\left(\sqrt{1-r^2}\right) = \mathbb{S}^4\left(\sqrt{\frac{2}{3}}\right) \times \mathbb{S}^3\left(\frac{\sqrt{3}}{3}\right), \quad (154)$$

has two district principal curvatures and constant mean curvature

$$\kappa_1 = \cdots = \kappa_4 = \frac{\sqrt{2}}{2},$$
$$\kappa_5 = \cdots = \kappa_7 = -\sqrt{2}, \quad (155)$$
$$H = -\frac{\sqrt{2}}{7}.$$

Let

$$\widetilde{\kappa}_1 = \cdots = \widetilde{\kappa}_4 = \kappa_1 - H = \frac{9\sqrt{2}}{14},$$
$$\widetilde{\kappa}_5 = \cdots = \widetilde{\kappa}_7 = \kappa_5 - H = -\frac{6\sqrt{2}}{7}. \quad (156)$$

Then,

$$\rho^2 = \frac{54}{7}, \quad (157)$$

since

$$\sum_i \widetilde{\kappa}_i^3 = -\frac{81\sqrt{2}}{49} = \frac{(n-4)}{2}\rho^2 H. \quad (158)$$

We can obtain that $M = \mathbb{S}^4(\sqrt{2/3}) \times \mathbb{S}^3(\sqrt{3}/3)$ is an extrema hypersurface in $\mathbb{S}^8$. Now,

$$n\left(c + \frac{n-2}{2}H^2\right) = 7\left(1 + \frac{5}{2}H^2\right) = \frac{54}{7} = \rho^2. \quad (159)$$

In particular,

$$k = \frac{n}{2} \pm \frac{1}{2}\sqrt{\frac{n^2H^2(n-4)^2}{n^2H^2 + 16c}} = 4. \quad (160)$$

Hence, $M = \mathbb{S}^4(\sqrt{2/3}) \times \mathbb{S}^3(\sqrt{3}/3)$ is an extrema hypersurface in $\mathbb{S}^8$ with $\rho^2 = n(c + ((n-2)/2)H^2)$ and $H \neq 0$.

Since $M = \mathbb{S}^4(\sqrt{2/3}) \times \mathbb{S}^3(\sqrt{3}/3)$ is an extrema hypersurface, we have the integral inequality by using Theorem 7,

$$\int_M \rho^2(n-\rho^2)\mathrm{d}\mu \leq 0. \quad (161)$$

From Theorem 6, we have the integral inequality

$$\int_M \rho^2\left[n\left(c + \frac{n-2}{2}H^2\right) - \rho^2\right]\mathrm{d}\mu \leq 0. \quad (162)$$

The fact

$$\int_M \rho^2(n-\rho^2)\mathrm{d}\mu < \int_M \rho^2\left[n\left(c + \frac{n-2}{2}H^2\right) - \rho^2\right]\mathrm{d}\mu \leq 0, \quad (163)$$

shows that the integral inequality of Theorem 6 is better and more useful than the integral inequality of Theorem 7.

7. Conclusion

In this paper, we derive the Euler-Lagrange equation for extremal hypersurface M^n in the space form $N^{n+1}(c)$ as Theorem 1 and then classify the isoparametric extremal hypersurfaces in the space form as Theorem 2. Since the classification theorem of isoparametric extremal hypersurfaces is too complicated, we decompose Theorem 5 into Propositions 1–3. We also reduce Theorem 5 to two corollaries when $\rho^2 = n(c + ((n-2)/2)H^2)$ and when M^n is a minimal. Finally, we obtain an integral inequality for the compact extremal hypersurfaces as Theorem 6. Comparing with the integral inequality in Theorem 7 obtained by Guo and Li, we can find that the integral inequality in Theorem 6 is better and more useful than Guo and Li's integral inequality.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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References

[1] A. Gray, *Modern Differential Geometry of Curves and Surfaces with Mathematica*, CRC Press, Boca Raton, FL, USA, 3rd edition, 2005.

[2] Z. Hu and H. Li, "Willmore submanifolds in a Riemannian manifold," *Contemporary Geometry and Related Topics*, pp. 251–275, World Scientific, Singapore, 2004.

[3] G. Wei, "On the minimizers of curvature functionals in asymptotically flat manifolds," *Journal of Geometric Analysis*, vol. 31, no. 6, pp. 5837–5853, 2021.

[4] H. Li, "Willmore submanifolds in a sphere," *Mathematical Research Letters*, vol. 9, no. 6, pp. 771–790, 2002.

[5] G. Zhen, "Willmore submanifolds in the unit sphere," *Collectanea Mathematica*, vol. 55, pp. 279–288, 2004.

[6] H. Li, "Willmore hypersurfaces in a sphere," *Asian Journal of Mathematics*, vol. 5, no. 2, pp. 365–377, 2001.

[7] D. Seminara, J. Sisti, and E. Tonni, "Holographic entanglement entropy in AdS4/BCFT3 and the Willmore functional," *Journal of High Energy Physics*, vol. 2018, no. 164, pp. 1–55, 2018.

[8] Z. Deng and T. Li, "Conformally flat Willmore spacelike hypersurfaces in ${R^{n+1}}_1$," *Turkish Journal of Mathematics*, vol. 44, no. 1, pp. 252–273, 2020.

[9] Z. Guo and H. Li, "A variational problem for submanifolds in a sphere," *Monatshefte für Mathematik*, vol. 152, no. 4, pp. 295–302, 2007.

[10] M. Rigoli and I. M. C. Salavessa, "Willmore submanifolds of the Möbius space and a Bernstein-type theorem," *Manuscripta Mathematica*, vol. 81, no. 1, pp. 203–222, 1993.

[11] J. Liu and H. Jian, "F-Willmore submanifold in space forms," *Frontiers of Mathematics in China*, vol. 6, no. 5, pp. 871–886, 2011.

[12] R. C. Reilly, "Variational properties of functions of the mean curvatures for hypersurfaces in space forms," *Journal of Differential Geometry*, vol. 8, no. 3, pp. 465–477, 1973.

[13] T. Levi-Civita, "Famiglie di superficie isoparametriche nell' ordinario spacio enclideo," *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti*, vol. 26, pp. 255–362, 1937.

[14] B. Segre, "Famiglie di ipersurficie isoparametriche negli spazi euclidei ad un qualunque numero di dimensioni," *Atti della Accademia Nazionale dei Lincei, Classe di Scienze Fisiche, Matematiche e Naturali*, vol. 27, pp. 203–207, 1938.

[15] E. Cartan, "Familles de surfaces isoparamétriques dans les espaces á courbure constante," *Annali di Matematica Pura ed Applicata*, vol. 17, no. 1, pp. 177–191, 1938.

[16] E. Cartan, "Sur des families remarquables d'hypersurfaces isoparametriques dans les espace spheriques," *Mathematische Zeitschrift*, vol. 45, no. 1, pp. 335–367, 1939.

[17] U. Abresch, "Isoparametric hypersurfaces with four or six distinct principal curvatures," *Mathematische Annalen*, vol. 264, no. 3, pp. 283–302, 1983.

[18] H. F. Munzner, "Isoparametrische hyperflachen in sharen I," *Mathematische Annalen*, vol. 251, pp. 57–71, 1980.

[19] S. Stolz, "Multiplicities of dupin hypersurfaces," *Inventiones Mathematicae*, vol. 138, no. 2, pp. 253–279, 1999.

[20] S. Shu and S. Liu, "Hypersurfaces with two distinct principal curvatures in a real space form," *Monatshefte für Mathematik*, vol. 164, no. 2, pp. 225–236, 2011.