

Research Article

Double Metric Resolvability in Convex Polytopes

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Nowadays, the study of source localization in complex networks is a critical issue. Localization of the source has been investigated using a variety of feasible models. To identify the source of a network's diffusion, it is necessary to find a vertex from which the observed diffusion spreads. Detecting the source of a virus in a network is equivalent to finding the minimal doubly resolving set (MDRS) in a network. This paper calculates the doubly resolving sets (DRSs) for certain convex polytope structures to calculate their double metric dimension (DMD). It is concluded that the cardinality of MDRSs for these convex polytopes is finite and constant.

1. Introduction and Preliminaries

Consider a connected and undirected graph $\Gamma = (V_\Gamma, E_\Gamma)$, where V_Γ denotes the set of vertices, while E_Γ represents the collection of edges. The distance $d(b, c)$ between two vertices $b, c \in V_\Gamma$ is calculated by counting the length of the shortest path between the two vertices. $d(c, f_1) \neq d(c, f_2) \neq \dots \neq d(c, f_n)$ implies that the vertex set $\{f_1, f_2, \dots, f_n\}$ is resolved by the vertex $c \in V_\Gamma$. Let $M_\Gamma \subseteq V_\Gamma$ be a set of order l . The vector of metric coordinates of a vertex u with respect to M_Γ , represented by $r(u|M_\Gamma)$, is the l -vector $r(u|M_\Gamma) = (d(u, m) : m \in M_\Gamma)$. If distinct vertices of Γ have unique metric coordinate vectors with respect to M_Γ , the set M_Γ is said to be resolving. A minimum-order resolving set is referred to as the basis of graph Γ and its order is called the metric dimension (MD) of Γ , denoted as $\dim(\Gamma)$.

Blumenthal [1] first proposed the notion of the MD of general metric space in 1953. Slater [2] introduced MD in 1975, and Harary and Melter [3] developed it as a resolving set of graphs in 1976. It was originally intended that the resolving sets have been used to locate an intruder in a network, but Chartrand and Zang [4] have since presented

numerous uses of resolving sets in robotics, chemistry, and biology. The concept of trilateration can be generalized from a two-dimensional real plane in terms of the MD of graphs. The Global Positioning System (GPS) uses distance measurements to detect the exact location of various objects on our planet. For studying the MD theoretically, there are several applications in navigation of robots [5] and molecular chemistry [6]. The MD of Hamming graphs has been investigated in a variety of applications, ranging from coinweighing difficulties [7, 8] to efficient Mastermind game strategies [9]. Studies on combinatorial optimization [10] and the difficulty of locating facilities, as well as the Coast Guard's loran and sonar systems [2], have all made use of the MD approach. Moreover, different fields including robotics [5], discovery and verification in networks [11], and routing protocols geographically [12] have examined this notion extensively.

Determining the exact MD value of any given graph is a computationally challenging task [5, 13]. Several helpful bounds for different types of graphs were investigated; for instance, see [14]. Both Buczkowski et al. [15] and Ali et al. [16] explored the minimal order resolving sets of wheel

graphs and Mobius ladders, respectively. Chartrand and others performed a classification of all graphs using MD 1, $n - 2$, and $n - 1$ (see [6]). In [17, 18], Imran et al. find the MD for different convex polytope structures. Tomescu et al. in [19] and Ahmad et al. [20] calculated the MD for Jahangir graphs, chorded cycles and kayak paddle graphs, respectively. Baca et al. provides the results for the MD of regular bipartite graphs [21]. The MD of necklace graphs was studied in [22], and certain plane graphs for MD were considered in [23].

As a research tool for studying minimal resolving sets of cartesian product graphs, Caceres et al. [24] established the concept of DRSs. Consider a graph Γ , where $|\Gamma| \geq 2$. Two vertices h' and $k' \in V_\Gamma$ doubly resolve a pair of vertices h, k of Γ if $d(h, h') - d(h, k') \neq d(k, h') - d(k, k')$. A subset D_Γ is a DRS of Γ if some two vertices of D_Γ doubly resolve any pair of vertices in Γ . The MDRS is a DRS that is the smallest possible. The DMD of Γ is the cardinality of an MDRS and is represented by $\psi(\Gamma)$. For all graphs Γ , any DRS is obviously a resolving set, with $\dim(\Gamma) \leq \psi(\Gamma)$.

Kratika et al. [25] studied the computational complexity of the DRSs of a graph Γ . An investigation of the Harary graph family (resp. circulant graph family) for MD and MDRSs was conducted by Ahmad et al. [26, 27]. Chen et al. [28] provided the first explicitly approximated lower and upper limits for the MDRSs problem. Ahmad et al. studied the line graphs of n -Sunlet, prism [29], and kayak paddle graphs [30] for the MD and MDRSs, respectively. To find the smallest possible DRS, a variety of graph families have been examined, such as those involving prisms [31], convex polytopes [32], and Hamming graphs [33]. The MDRSs of different convex polytope structures were examined by Pan et al. [34] and Ahmad et al. [35]. Minimal order resolving sets and MDRSs of cocktail and jellyfish graphs were calculated by Liu and others [36]. For the layer-sun graphs and their related line graphs, the MDRS problem was also studied in [37]. In [38], the minimal resolving sets and DMD of the line graph of chorded cycles were examined. Authors demonstrated that the DMD of $L(C'_n)$ is exactly one greater than its MD.

Thus, DRSs are essential for examining the MD of cartesian products. We were inspired by the idea of achieving upper bounds in the cartesian product of graphs to examine the DRSs of other graph classes. In addition, this parameter can be helpful in a wide range of domains, including rumor spreading on online social networks and the origin of a disease outbreak.

It is feasible to determine the diffusion source in complex networks using MDRSs. Finding the source of wide-area network propagation might be challenging. As an example, in the case of an unknown virus source, all that is required to locate it is the contamination time of a subset of the sensor nodes. These sensor nodes may record their infection time. How many sensors are required to locate the viral source? A property called DMD provides the answer to this problem (for details see [39, 40]). It would be simple to determine the

origins of the pandemic if we were able to see it from beginning to end. Data for source localization is sometimes scarce due to the cost and complexity of collecting and storing information. Even if the initial virus propagation period is unknown, a doubly resolving sensor set may detect infection sources reliably.

In a star-like network, identifying the viral origin is much more complicated than in a path-like network. The DMD is $n - 1$ for a star network of n nodes, but the DMD is 2 for a path network. As a result, this demonstrates that the DMD is always dependent on the network's topology [39].

It has been challenging to solve convex polytopes for the DMD in the previous few years. Here, Imran et al. [18, 41] calculate DMD for T_n and S_n convex polytopes. In the following theorems, the MD of the convex polytopes T_n and S_n is shown.

Theorem 1. *If T_n is the graph of convex polytope, $\dim(T_n)$ is 3, and $\forall n \geq 6$.*

Theorem 2. *If S_n is the graph of convex polytope, $\dim(S_n)$ is 3, and $\forall n \geq 6$.*

The following sections elaborate on the rest of the article:

- (i) In Section 2, we define the graph of T_n and computed the MDRSs of the convex polytope T_n for $n \geq 6$.
- (ii) In Section 3, we define the graph of S_n and calculated the DMD for the convex polytope structure S_n , for $n \geq 6$.
- (iii) Section 4 concludes that the DMD for the convex polytopes T_n and S_n is finite and constant.

2. Double Metric Dimension for the Convex Polytope T_n

The results of the DMD computation for the convex polytope T_n are presented here in this section.

There are 3-sided, 5-sided, and n -sided faces on the convex polytope T_n as shown in Figure 1.

There are three types of vertex labels used here: $\{r_\omega: \forall 0 \leq \omega \leq n - 1\}$ represents the inner cycle vertex, $\{s_\omega: 0 \leq \omega \leq n - 1\}$ represents the central cycle vertex, and $\{t_\omega: \forall 0 \leq \omega \leq n - 1\}$ represents the outer cycle vertex as shown in Figure 1.

As a result of applying Theorem 1, $\psi(T_n) \geq 3$ for $n \geq 6$ is obtained. Also, $\psi(T_n) = 3$ for $n \geq 6$ will be proved in this section. Vertex distances of the graph T_n can be computed using the following procedure:

Assume that the set $S_\omega(s_0) = \{s \in V_{T_n}: d(s_0, s) = \omega\}$ is a vertex set in V_{T_n} at a distance of ω from s_0 . The Table 1 is simply constructed for $S_\omega(s_0)$, and it will be used to figure out how far two vertices in V_{T_n} are from each other.

Because of the symmetry of T_n for $n \geq 6$, it can be demonstrated that

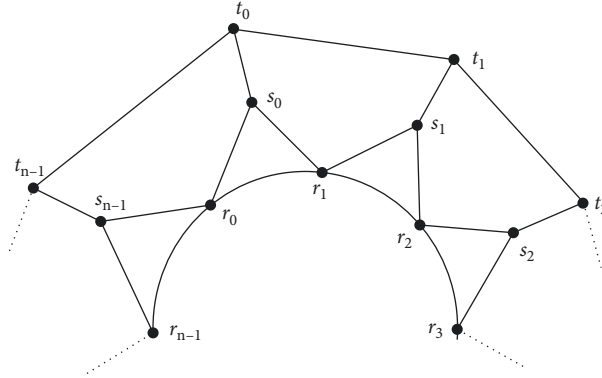


FIGURE 1: Convex polytope T_n .

TABLE 1: $S_\omega(s_0)$ for T_n .

n	ω	$S_\omega(s_0)$
	1	$\{r_0, r_1, t_0\}$
even	$2 \leq \omega \leq \lfloor (n/2) \rfloor$ $(n + 2/2)$	$\{r_\omega, r_{n-\omega+1}, s_{\omega-1}, s_{n-\omega+1}, t_{\omega-1}, t_{n-\omega+1}\}$ $\{s_{(n/2)}, t_{(n/2)}\}$
odd	$(n + 1/2)$	$\{r_{(n+1/2)}, s_{(n-1/2)}, s_{(n+1/2)}, t_{(n-1/2)}, t_{(n+1/2)}\}$

$$d(r_\omega, r_\nu) = \begin{cases} d(s_0, r_{|\omega-\nu|}) - 1, & \text{if } |\omega - \nu| = 0, \\ d(s_0, r_{|\omega-\nu|}), & \text{if } 1 \leq |\omega - \nu| \leq \frac{n}{2}, \\ d(s_0, r_{|\omega-\nu|}) - 1, & \text{if } \frac{n+2}{2} \leq |\omega - \nu| \leq n-1, \end{cases} \quad (1)$$

$$d(s_\omega, t_\nu) = d(s_0, t_{|\omega-\nu|}), \quad \text{if } 0 \leq |\omega - \nu| \leq n-1,$$

$$d(t_\omega, t_\nu) = d(s_0, t_{|\omega-\nu|}) - 1, \quad \text{if } 0 \leq |\omega - \nu| \leq n-1,$$

$$d(s_\omega, s_\nu) = d(s_0, s_{|\omega-\nu|}), \quad \text{if } 0 \leq |\omega - \nu| \leq n-1.$$

If n is odd

$$d(r_\omega, t_\nu) = \begin{cases} d(s_0, t_{|\omega-\nu|}), & \text{if } 1 \leq |\omega - \nu| \leq \frac{n}{2} \text{ for } \omega > \nu, \\ d(s_0, t_{|\omega-\nu|}) + 1, & \text{if } \frac{n+2}{2} \leq |\omega - \nu| \leq n-1 \text{ for } \omega > \nu, \\ d(s_0, t_{|\omega-\nu|}) + 1, & \text{if } 0 \leq |\omega - \nu| \leq \frac{n}{2} \text{ for } \nu \geq \omega, \\ d(s_0, t_{|\omega-\nu|}), & \text{if } \frac{n+2}{2} \leq |\omega - \nu| \leq n-1 \text{ for } \nu > \omega, \end{cases} \quad (2)$$

$$d(r_\omega, s_\nu) = \begin{cases} d(s_0, r_{|\omega-\nu|}) + 1, & \text{if } 1 \leq |\omega - \nu| \leq \frac{n}{2} \text{ for } \nu > \omega, \\ d(s_0, r_{|\omega-\nu|}) - 1, & \text{if } \frac{n+2}{2} \leq |\omega - \nu| \leq n-1 \text{ for } \nu > \omega, \\ d(s_0, r_{|\omega-\nu|}), & \text{if } 0 \leq |\omega - \nu| \leq n-1 \text{ for } \omega \geq \nu. \end{cases}$$

If n is even

$$\begin{aligned}
 d(r_\omega, t_\nu) &= \begin{cases} d(s_0, t_{|\omega-\nu|}), & \text{if } 1 \leq |\omega - \nu| \leq \frac{n}{2} \text{ for } \omega > \nu, \\ d(s_0, t_{|\omega-\nu|}) + 1, & \text{if } \frac{n+2}{2} \leq |\omega - \nu| \leq n-1 \text{ for } \omega > \nu, \\ d(s_0, t_{|\omega-\nu|}) + 1, & \text{if } 0 \leq |\omega - \nu| \leq \frac{n-2}{2} \text{ for } \nu \geq \omega, \\ d(s_0, t_{|\omega-\nu|}), & \text{if } \frac{n}{2} \leq |\omega - \nu| \leq n-1 \text{ for } \nu > \omega, \end{cases} \\
 d(r_\omega, s_\nu) &= \begin{cases} d(s_0, r_{|\omega-\nu|}) + 1, & \text{if } 1 \leq |\omega - \nu| \leq \frac{n-2}{2} \text{ for } \nu > \omega, \\ d(s_0, r_{|\omega-\nu|}), & \text{if } |\omega - \nu| = \frac{n}{2} \text{ for } \nu > \omega, \\ d(s_0, r_{|\omega-\nu|}) - 1, & \text{if } \frac{n+2}{2} \leq |\omega - \nu| \leq n-1 \text{ for } \nu > \omega, \\ d(s_0, r_{|\omega-\nu|}), & \text{if } 0 \leq |\omega - \nu| \leq n-1 \text{ for } \omega \geq \nu. \end{cases}
 \end{aligned} \tag{3}$$

Due to the fact that, in order to calculate the distance between any pair of vertices in V_{T_n} , we must know the distance $d(s_0, s)$ for each $s \in V_{T_n}$.

Lemma 1. For the odd positive integer $n \geq 6$, $\psi(T_n) = 3$.

Proof. To demonstrate that $\psi(T_n) = 3$ for the odd positive integer, finding a DRS with cardinality 3 is enough. Using the sets $S_\omega(s_0)$ listed in Table 1, the Table 2 shows the metric coordinate vectors for all vertices of T_n with respect to $D_{T_n} = \{r_0, s_{(n-1/2)}, t_0\}$.

Table 2 shows that the representation $r(g_1, D_{T_n}) - r(g_2, D_{T_n}) \neq 0$ for any two vertices $g_1, g_2 \in S_\omega(s_0)$, where $\omega \in \{1, 2, \dots, (n+1/2)\}$. Consequently, for any $\omega, \nu \in \{1, 2, \dots, (n+1/2)\}$ such as $\omega \neq \nu$ if the vertices $g_1 \in S_\omega(s_0)$ and $g_2 \in S_\nu(s_0)$, then the representation $r(g_1, D_{T_n}) - r(g_2, D_{T_n}) \neq \omega - \nu$. So the collection $D_{T_n} = \{r_0, s_{(n-1/2)}, t_0\}$ represents the MDRS of T_n . As a result, Lemma 1 holds. \square

Lemma 2. For the even positive integer $n \geq 6$, $\psi(T_n) = 3$.

Proof. To demonstrate that $\psi(T_n) = 3$ for the even positive integer $n \geq 6$, finding a DRS with cardinality 3 is enough. Using the sets $S_\omega(s_0)$ listed in Table 1, the Table 3 shows the metric coordinate vectors for all vertices of T_n with respect to $D_{T_n} = \{r_0, r_{(n/2)}, t_0\}$.

Table 3 shows that the representation $r(g_1, D_{T_n}) - r(g_2, D_{T_n}) \neq 0$ for any two vertices $g_1, g_2 \in S_\omega(s_0)$, where $\omega \in \{1, 2, \dots, (n+2/2)\}$. Consequently, for any $\omega, \nu \in \{1, 2, \dots, (n+2/2)\}$ such as $\omega \neq \nu$ if the vertices $g_1 \in S_\omega(s_0)$ and $g_2 \in S_\nu(s_0)$, then the representation $r(g_1, D_{T_n}) - r(g_2, D_{T_n}) \neq \omega - \nu$. So the collection

$D_{T_n} = \{r_0, r_{(n/2)}, t_0\}$ represents the MDRS of T_n . As a result, Lemma 2 holds.

The entire technique clearly demonstrates that $\psi(T_n) = 3$, for $n \geq 6$. The main theorem is stated below using Lemmas 1 and 2: \square

Theorem 3. Let T_n be a convex polytope, then $\psi(T_n) = 3$ for $n \geq 6$.

3. Double Metric Dimension for the Convex Polytope S_n

This section contains the results of the DMD computation for the convex polytope S_n .

There are 3-sided, 4-sided, and n -sided faces on the convex polytope S_n as shown in Figure 2.

There are three types of vertex labels used here: $\{p_\omega: \forall 0 \leq \omega \leq n-1\}$ represents the inner cycle vertex, $\{q_\omega: \forall 0 \leq \omega \leq n-1\}$ represents the interior cycle vertex, $\{r_\omega: \forall 0 \leq \omega \leq n-1\}$ represents the exterior cycle vertex, and $\{t_\omega: \forall 0 \leq \omega \leq n-1\}$ represents the outer cycle vertex as shown in Figure 2.

As a result of applying Theorem 2, $\psi(S_n) \geq 3$ for $n \geq 6$ is obtained. Also, $\psi(S_n) = 3$ for $n \geq 6$ will be proved in this section. Vertex distances of the graph S_n can be computed using the following procedure:

Assume that the set $S_\omega(q_0) = \{q \in V_{S_n}: d(q_0, q) = \omega\}$ is a vertex set in V_{S_n} at a distance of ω from q_0 . Table 4 is simply constructed for $S_\omega(q_0)$, and it will be used to figure out how far two vertices in V_{S_n} are from each other.

Because of the symmetry of S_n for $n \geq 6$, it can be demonstrated that

TABLE 2: Metric coordinate vectors of T_n for odd $n \geq 6$.

ω	$S_\omega(f_0)$	$D_{T_n} = \{r_0, s_{(n-1/2)}, t_0\}$
0	s_0 r_0	$(1, (n+1/2), 1)$ $(0, (n+1/2), 2)$
1	r_1 t_0	$(1, (n-1/2), 2)$ $(2, (n+1/2), 0)$
$2 \leq \omega \leq (n-1/2)$	r_ω	$(\omega, (n-2\omega+1/2), \omega+1)$
	$r_{n-\omega+1}$	$(\omega-1, (n-2\omega+3/2), \omega+1)$
	$s_{\omega-1}$	$(\omega, (n-2\omega+3/2), \omega)$
	$s_{n-\omega+1}$	$(\omega-1, (n-2\omega+5/2), \omega)$
	$t_{\omega-1}$ $t_{n-\omega+1}$	$(\omega+1, (n-2\omega+3/2), \omega-1)$ $(\omega, (n-2\omega+5/2), \omega-1)$
$(n+1/2)$	$r_{(n+1/2)}$	$(n-1/2, 1, (n+3/2))$
	$s_{(n-1/2)}$	$(n+1/2, 0, (n+1/2))$
	$s_{(n+1/2)}$	$(n-1/2, 2, (n+1/2))$
	$t_{(n-1/2)}$	$(n+3/2, 1, (n-1/2))$
	$t_{(n+1/2)}$	$(n+1/2, 2, (n-1/2))$

TABLE 3: Metric coordinate vectors of T_n for even $n \geq 6$.

ω	$S_\omega(s_0)$	$D_{T_n} = \{r_0, r_{(n/2)}, t_0\}$
0	s_0	$(1, (n/2), 1)$ $(0, (n/2), 2)$
1	s_1 t_0	$(1, (n-2/2), 2)$ $(2, (n+2/2), 0)$
$2 \leq \omega \leq (n/2)$	r_ω	$(\omega, (n-2\omega/2), \omega+1)$
	$r_{n-\omega+1}$	$(\omega-1, (n-2\omega+2/2), \omega+1)$
	$s_{\omega-1}$	$(\omega, (n-2\omega+2/2), \omega)$
	$s_{n-\omega+1}$	$(\omega-1, (n-2\omega+4/2), \omega)$
	$t_{\omega-1}$ $t_{n-\omega+1}$	$(\omega+1, (n-2\omega+4/2), \omega-1)$ $(\omega, (n-2\omega+6/2), \omega-1)$
$(n+2/2)$	$s_{(n/2)}$	$(n/2, 1, (n+2/2))$
	$t_{(n/2)}$	$(n+2/2, 2, (n/2))$

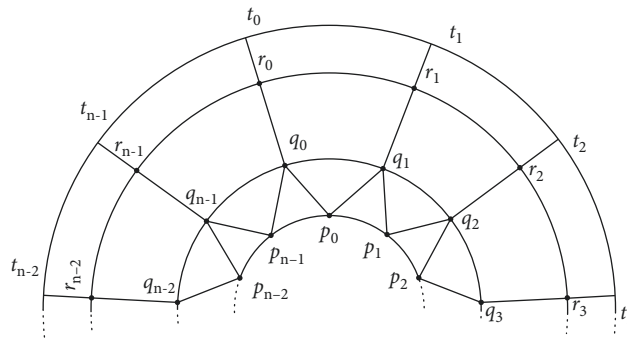


FIGURE 2: Convex polytope S_n .

TABLE 4: $S_\omega(q_0)$ for S_n .

n	ω	$S_\omega(q_0)$
	1	$\{p_0, p_{n-1}, q_1, q_{n-1}, r_0\}$
	2	$\{p_1, p_{n-2}, q_2, q_{n-2}, r_1, r_{n-1}, t_0\}$
	$3 \leq \omega \leq \lfloor n/2 \rfloor - 1$	$\{p_{\omega-1}, p_{n-\omega}, q_\omega, q_{n-\omega}, r_{\omega-1}, r_{n-\omega+1}, t_{\omega-2}, t_{n-\omega+2}\}$
even	$n/2$	$\{p_{(n-2)/2}, p_{n/2}, q_{n/2}, r_{(n-2)/2}, r_{(n+2)/2}, t_{(n-4)/2}, t_{(n+4)/2}\}$
	$(n+2)/2$	$\{r_{(n/2)}, t_{(n-2)/2}, t_{(n+2)/2}\}$
	$(n+4)/2$	$\{t_{(n/2)}\}$
odd	$(n-1)/2$	$\{p_{(n-3)/2}, p_{(n+1)/2}, q_{(n-1)/2}, q_{(n+1)/2}, r_{(n-3)/2}, r_{(n+3)/2}, t_{(n-5)/2}, t_{(n+5)/2}\}$
	$(n+1)/2$	$\{p_{(n-1)/2}, r_{(n-1)/2}, r_{(n+1)/2}, t_{(n-3)/2}, t_{(n+3)/2}\}$
	$(n+3)/2$	$\{t_{(n-1)/2}, t_{(n+1)/2}\}$

$$\begin{aligned}
 d(q_\omega, t_\nu) &= d(q_0, t_{|\omega-\nu|}), \quad \text{if } 0 \leq |\omega - \nu| \leq n - 1, \\
 d(q_\omega, r_\nu) &= d(q_0, r_{|\omega-\nu|}), \quad \text{if } 0 \leq |\omega - \nu| \leq n - 1, \\
 d(r_\omega, t_\nu) &= d(q_0, t_{|\omega-\nu|}) - 1, \quad \text{if } 0 \leq |\omega - \nu| \leq n - 1, \\
 d(r_\omega, r_\nu) &= d(t_\omega, t_\nu) = d(q_0, r_{|\omega-\nu|}) - 1, \quad \text{if } 0 \leq |\omega - \nu| \leq n - 1, \\
 d(q_\omega, q_\nu) &= d(q_0, q_{|\omega-\nu|}), \quad \text{if } 0 \leq |\omega - \nu| \leq n - 1.
 \end{aligned} \tag{4}$$

If n is odd

$$\begin{aligned}
 d(p_\omega, p_\nu) &= \begin{cases} d(q_0, p_{|\omega-\nu|}) - 1, & \text{if } 0 \leq |\omega - \nu| \leq \frac{n-1}{2}, \\ d(q_0, p_{|\omega-\nu|}), & \text{if } \frac{n+1}{2} \leq |\omega - \nu| \leq n - 1, \end{cases} \\
 d(q_\omega, p_\nu) &= \begin{cases} d(q_0, p_{|\omega-\nu|}) - 1, & \text{if } 1 \leq |\omega - \nu| \leq \frac{n-1}{2} \text{ for } \omega > \nu, \\ d(q_0, p_{|\omega-\nu|}) + 1, & \text{if } \frac{n+1}{2} \leq |\omega - \nu| \leq n - 1 \text{ for } \omega > \nu, \\ d(q_0, p_{|\omega-\nu|}), & \text{if } 0 \leq |\omega - \nu| \leq n - 1 \text{ for } \nu \geq \omega, \end{cases} \\
 d(p_\omega, r_\nu) &= \begin{cases} d(q_0, r_{|\omega-\nu|}) + 1, & \text{if } 0 \leq |\omega - \nu| \leq \frac{n-1}{2} \text{ for } \omega \geq \nu, \\ d(q_0, r_{|\omega-\nu|}), & \text{if } \frac{n+1}{2} \leq |\omega - \nu| \leq n - 1 \text{ for } \omega > \nu, \\ d(q_0, r_{|\omega-\nu|}), & \text{if } 1 \leq |\omega - \nu| \leq \frac{n-1}{2} \text{ for } \nu > \omega, \\ d(q_0, r_{|\omega-\nu|}) + 1, & \text{if } \frac{n+1}{2} \leq |\omega - \nu| \leq n - 1 \text{ for } \nu > \omega, \end{cases} \\
 d(p_\omega, t_\nu) &= \begin{cases} d(q_0, h_{|\omega-\nu|}) + 1, & \text{if } 0 \leq |\omega - \nu| \leq \frac{n-1}{2} \text{ for } \omega \geq \nu, \\ d(q_0, h_{|\omega-\nu|}), & \text{if } \frac{n+1}{2} \leq |\omega - \nu| \leq n - 1 \text{ for } \omega > \nu, \\ d(q_0, h_{|\omega-\nu|}), & \text{if } 1 \leq |\omega - \nu| \leq \frac{n-1}{2} \text{ for } \nu > \omega, \\ d(q_0, h_{|\omega-\nu|}) + 1, & \text{if } \frac{n+1}{2} \leq |\omega - \nu| \leq n - 1 \text{ for } \nu > \omega. \end{cases}
 \end{aligned} \tag{5}$$

If n is even

$$\begin{aligned}
 d(p_\omega, p_\nu) &= \begin{cases} d(q_0, p_{|\omega-\nu|}) - 1, & \text{if } 0 \leq |\omega - \nu| \leq \frac{n-2}{2}, \\ d(q_0, p_{q\omega-\nu}), & \text{if } \frac{n}{2} \leq |\omega - \nu| \leq n-1, \end{cases} \\
 d(p_\omega, r_\nu) &= \begin{cases} d(q_0, r_{|\omega-\nu|}) + 1, & \text{if } 0 \leq |\omega - \nu| \leq \frac{n-2}{2} \text{ for } \omega \geq \nu, \\ d(q_0, r_{|\omega-\nu|}), & \text{if } \frac{n}{2} \leq |\omega - \nu| \leq n-1 \text{ for } \omega > \nu, \\ d(q_0, r_{|\omega-\nu|}), & \text{if } 1 \leq |\omega - \nu| \leq \frac{n}{2} \text{ for } \nu > \omega, \\ d(q_0, r_{|\omega-\nu|}) + 1, & \text{if } \frac{n+2}{2} \leq |\omega - \nu| \leq n-1 \text{ for } \nu > \omega, \end{cases} \\
 d(q_\omega, p_\nu) &= \begin{cases} d(q_0, p_{|\omega-\nu|}) - 1, & \text{if } 1 \leq |\omega - \nu| \leq \frac{n-2}{2} \text{ for } \omega > \nu, \\ d(q_0, p_{|\omega-\nu|}), & \text{if } |\omega - \nu| = \frac{n}{2} \text{ for } \omega > \nu, \\ d(q_0, p_{|\omega-\nu|}) + 1, & \text{if } \frac{n+2}{2} \leq |\omega - \nu| \leq n-1 \text{ for } \omega > \nu, \\ d(q_0, p_{|\omega-\nu|}), & \text{if } 0 \leq |\omega - \nu| \leq n-1 \text{ for } \nu \geq \omega, \end{cases} \tag{6} \\
 d(p_\omega, t_\nu) &= \begin{cases} d(q_0, t_{|\omega-\nu|}) + 1, & \text{if } 0 \leq |\omega - \nu| \leq \frac{n-2}{2} \text{ for } \omega \geq \nu, \\ d(q_0, t_{|\omega-\nu|}), & \text{if } \frac{n}{2} \leq |\omega - \nu| \leq n-1 \text{ for } \omega > \nu, \\ d(q_0, t_{|\omega-\nu|}), & \text{if } 1 \leq |\omega - \nu| \leq \frac{n}{2} \text{ for } \nu > \omega, \\ d(q_0, t_{|\omega-\nu|}) + 1, & \text{if } \frac{n+2}{2} \leq |\omega - \nu| \leq n-1 \text{ for } \nu > \omega. \end{cases}
 \end{aligned}$$

Due to the fact that, in order to calculate the distance between any pair of vertices in V_{S_n} , we need to know $d(q_0, q)$ for each $q \in V_{S_n}$.

Lemma 3. For any odd positive integer $n \geq 6$, $\psi(S_n) = 3$.

Proof. The MDRSs for S_7, S_9 , and S_{11} are

$$D_{S_n} = \begin{cases} \{p_0, p_3, t_2\}, & \text{if } n = 7, \\ \{p_0, p_4, t_7\}, & \text{if } n = 9, \\ \{p_0, p_5, t_3\}, & \text{if } n = 11, \end{cases} \tag{7}$$

to demonstrate that $\psi(S_n) = 3$ for any odd positive integer $n \geq 13$, finding a DRS with cardinality 3 is enough. Using the sets $S_\omega(q_0)$ listed in Table 4, the Table 5 shows the metric coordinate vectors for all vertices of S_n in relation to the set $D_{S_n} = \{p_0, p_{(n-1/2)}, t_{(n+7/2)}\}$.

Table 5 shows that the representation $r(h_1, D_{S_n}) - r(h_2, D_{S_n}) \neq 0$ for any two vertices $h_1, h_2 \in S_\omega(q_0)$, where $\omega \in \{1, 2, \dots, (n+3/2)\}$. Consequently, for any $\omega, \nu \in \{1, 2, \dots, (n+3/2)\}$ such as $\omega \neq \nu$ if the vertices $h_1 \in S_\omega(q_0)$ and $h_2 \in S_\nu(q_0)$, then the

representation $r(h_1, D_{S_n}) - r(h_2, D_{S_n}) \neq \omega - \nu$. So the collection $D_{S_n} = \{p_0, p_{(n-1/2)}, t_{(n+7/2)}\}$ represents the MDRS of T_n . As a result, Lemma 3 holds. \square

Lemma 4. For any even positive integer $n \geq 6$, $\psi(S_n) = 3$.

Proof. To demonstrate that $\psi(S_n) = 3$ for any even positive integer $n \geq 6$, finding a DRS with cardinality 3 is enough. Using the sets $S_\omega(q_0)$ listed in Table 4, the Table 6 shows the metric coordinate vectors for all vertices of S_n in relation to the set $D_{S_n} = \{p_0, p_{(n-2/2)}, t_1\}$.

Table 6 shows that the representation $r(h_1, D_{S_n}) - r(h_2, D_{S_n}) \neq 0$ for any two vertices $h_1, h_2 \in S_\omega(q_0)$, where $\omega \in \{1, 2, \dots, (n+4/2)\}$. Consequently, for any $\omega, \nu \in \{1, 2, \dots, (n+4/2)\}$ such as $\omega \neq \nu$ if the vertices $h_1 \in S_\omega(q_0)$ and $h_2 \in S_\nu(q_0)$, then the representation $r(h_1, D_{S_n}) - r(h_2, D_{S_n}) \neq \omega - \nu$. So the collection $D_{S_n} = \{p_0, p_{(n-2/2)}, t_1\}$ represents the MDRS of T_n . As a result, Lemma 4 holds.

The entire technique clearly demonstrates that $\psi(S_n) = 3$, for $n \geq 6$. The main theorem is stated below using Lemmas 3 and 4: \square

TABLE 5: Metric coordinate vectors of S_n for odd $n \geq 13$.

ω	$S_\omega(f_0)$	$D_{S_n} = \{p_0, p_{(n-1/2)}, t_{(n+7/2)}\}$
0	q_0	$(1, (n+1/2), (n-3/2))$
	p_0	$(0, (n-1/2), (n-1/2))$
	p_{n-1}	$(1, (n-1/2), (n-3/2))$
1	q_1	$(1, (n-1/2), (n-1/2))$
	q_{n-1}	$(2, (n-1/2), (n-5/2))$
	r_0	$(2, (n+3/2), (n-5/2))$
2	p_1	$(1, (n-3/2), (n+1/2))$
	p_{n-2}	$(2, (n-3/2), (n-5/2))$
	q_2	$(2, (n-3/2), (n+1/2))$
	q_{n-2}	$(3, (n-3/2), (n-7/2))$
	r_1	$(2, (n+1/2), (n-3/2))$
	r_{n-1}	$(3, (n+1/2), (n-7/2))$
	t_0	$(3, (n+5/2), (n-7/2))$
$3 \leq \omega \leq (n-3/2)$	$p_{\omega-1}$	$= \begin{cases} (2, (n-5/2), (n+3/2)), & \text{if } \omega = 2, \\ (\omega-1, (n-2\omega+1/2), (n-2\omega+13/2)), & \text{if } \omega \neq 2. \end{cases}$
	$p_{n-\omega}$	$= \begin{cases} (\omega, (n-2\omega+1/2), (n-2\omega-1/2)), & \text{if } \omega \leq (n-7/2) \\ (\omega, (n-2\omega+1/2), (2\omega-n+11/2)), & \text{if } \omega > (n-7/2) \end{cases}$
	q_ω	$= \begin{cases} (3, (n-5/2), (n+3/2)), & \text{if } \omega = 3 \\ (\omega, (n-2\omega+1/2), (n-2\omega+11/2)), & \text{if } \omega \neq 3 \end{cases}$
	$q_{n-\omega}$	$= \begin{cases} (\omega+1, (n-2\omega+1/2), (n-2\omega-3/2)) & \text{if } \omega \leq (n-7/2) \\ (\omega+1, (n-2\omega+1/2), (2\omega-n+11/2)), & \text{if } \omega > (n-7/2) \end{cases}$
	$r_{\omega-1}$	$= \begin{cases} (\omega, (n-2\omega+5/2), (n+2\omega-7/2)), & \text{if } \omega \leq 4 \\ (\omega, (n-2\omega+5/2), (n-2\omega+11/2)) & \text{if } \omega > 4 \end{cases}$
	$r_{n-\omega+1}$	$= \begin{cases} (\omega+1, (n-2\omega+5/2), (n-2\omega-3/2)), & \text{if } \omega \neq (n-3/2) \\ ((n-1/2), 4, 2), & \text{if } \omega = (n-3/2) \end{cases}$
	$t_{\omega-2}$	$= \begin{cases} (\omega, (n-2\omega+9/2), (n+2\omega-13/2)), & \text{if } \omega \leq 5 \\ (\omega, (n-2\omega+9/2), (n-2\omega+11/2)), & \text{if } \omega > 5 \end{cases}$
$t_{n-\omega+2}$	$(\omega+1, (n-2\omega+9/2), (n-2\omega-3/2))$	
$(n-1/2)$	$p_{(n-3/2)}$	$(n-3/2, 1, 7)$
	$p_{(n+1/2)}$	$(n-1/2, 1, 5)$
	$q_{(n-1/2)}$	$(n-1/2, 1, 6)$
	$q_{(n+1/2)}$	$(n+1/2, 1, 5)$
	$r_{(n-3/2)}$	$(n-1/2, 3, 6)$
	$r_{(n+3/2)}$	$(n+1/2, 3, 3)$
	$t_{(n-5/2)}$	$(n-1/2, 5, 6)$
$t_{(n+5/2)}$	$(n+1/2, 5, 1)$	
$(n+1/2)$	$p_{(n-1/2)}$	$(n-1/2), 0, 6)$
	$r_{(n-1/2)}$	$(n+1/2), 2, 5)$
	$r_{(n+1/2)}$	$(n+3/2), 2, 4)$
	$t_{(n-3/2)}$	$(n+1/2), 4, 5)$
	$t_{(n+3/2)}$	$(n+3/2), 4, 2)$
$(n+3/2)$	$t_{(n-1/2)}$	$(n+3/2), 3, 4)$
	$t_{(n+1/2)}$	$(n+5/2), 3, 3)$

TABLE 6: Metric coordinate vectors of S_n for even $n \geq 6$.

ω	$S_\omega(q_0)$	$D_{S_n} = \{p_0, p_{(n-2/2)}, t_1\}$
0	q_0	$(1, (n/2), 2)$
	p_0	$(0, (n - 2/2), 2)$
	p_{n-1}	$(1, (n/2), 3)$
1	q_1	$(1, (n - 2/2), 1)$
	q_{n-1}	$(2, (n/2), 3)$
	r_0	$(2, (n + 2/2), 1)$
2	p_1	$(1, (n - 4/2), 3)$
	p_{n-2}	$(2, (n - 2/2), 5)$
	q_2	$(2, (n - 4/2), 3)$
	q_{n-2}	$(3, (n - 2/2), 5)$
	r_1	$(2, (n/2), 1)$
	r_{n-1}	$(3, (n + 2/2), 3)$
$3 \leq \omega \leq (n - 2/2)$	t_0	$(3, (n + 4/2), 1)$
	$p_{\omega-1}$	$(\omega - 1, (n - 2\omega/2), \omega + 1)$
	$p_{n-\omega}$	$(\omega, (n - 2\omega + 2/2), \omega + 3)$
	q_ω	$(\omega, (n - 2\omega/2), \omega + 1)$
	$q_{n-\omega}$	$(\omega + 1, (n - 2\omega + 2/2), \omega + 3)$
	$r_{\omega-1}$	$(\omega, (n - 2\omega + 4/2), \omega - 1)$
	$r_{n-\omega+1}$	$(\omega + 1, (n - 2\omega + 6/2), \omega + 1)$
$(n/2)$	$t_{\omega-2}$	$(\omega, (n - 2\omega + 8/2), \omega - 3)$
	$t_{n-\omega+2}$	$(\omega + 1, (n - 2\omega + 10/2), \omega - 1)$
	$p_{(n-2/2)}$	$(n - 2/2), 0, (n + 2/2)$
	$p_{n/2}$	$(n/2), 1, (n + 4/2)$
	$q_{n/2}$	$(n/2), 1, (n + 4/2)$
	$r_{(n-2/2)}$	$(n/2), 2, (n - 2/2)$
	$r_{(n+2/2)}$	$(n + 2/2), 3, (n + 2/2)$
$(n + 2/2)$	$t_{(n-4/2)}$	$(n/2), 4, (n - 6/2)$
	$t_{(n+4/2)}$	$(n + 2/2), 5, (n - 2/2)$
	$r_{(n/2)}$	$(n + 2/2), 2, (n/2)$
$(n + 4/2)$	$t_{(n-2/2)}$	$(n + 2/2), 3, (n - 4/2)$
	$t_{(n+2/2)}$	$(n + 4/2), 4, (n/2)$
$(n + 4/2)$	$t_{n/2}$	$(n + 6/2), 3, (n - 2/2)$

Theorem 4. Let S_n be the convex polytope for $n \geq 6$. Then $\psi(S_n) = 3$.

4. Conclusion

In this paper, we investigate the idea of calculating MDRSs of graphs using an integer linear programming formulation previously presented in the literature. We calculate the DMD for the convex polytopes T_n and S_n , which is the minimum cardinality over all the DRs of T_n and S_n . It is interesting to consider these families of the convex polytopes because their DMD is finite and independent of the parity of n . Finally, we get $\psi(T_n) = \psi(S_n) = 3$ for $n \geq 6$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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