Research Article

κ_G -Contractive Mappings in Fuzzy Metric Spaces

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This article is concerned with a class of contractions in the framework of a fuzzy metric space endowed with a graph. The main results obtained not only broaden but also generalize a number of relevant results in the literature. Particularly, we get some results for cyclic contractions in fuzzy metric spaces. We also consider an integral equation.

1. Introduction

In recent years, fixed-point theory has become a tantalizing field of research in nonlinear analysis due to its wide range of applications in computer sciences, game theory, physics, economics, and various other fields. It has become a tool to find solutions for nonlinear equations such as differential equations, integrodifferential equations, and matrix equations [1, 2]. After the fundamental Banach's theorem [3] in 1922, a huge amount of literature appeared for the investigation of fixed points of self-operators. Among those results, two aspects must be considered: underlying geometric structure on which self-operator is defined and an appropriate contraction condition on operator.

The investigation of existence of a fixed point of self-mappings on a metric space $(\mathcal{N}, d)$ equipped with a partial ordering was originated by the works of Ran and Reurings [4]. By replacing a partial order with a graph, Jachymski [5] studied some prolific results about the existence of a fixed point involving the graph concept. Later on, several researchers conducted further studies in this area by relaxing the contractive condition and analyzing the

connectivity condition of a graph. For more details, see [6–8].

On the other hand, Kramosil and Michalek [9] gave the concept of a fuzzy metric space. Afterward Grabiec [10] extended the Banach contraction principle (BCP) to fuzzy metric spaces in the sense of Kramosil and Michalek. Many authors undertook further advancements in this direction by generalizing the contraction inequalities and by providing the underlying space with rich geometric structure to make the results more viable. In this context, recently a number of authors contributed new and exciting developments in fixed-point theory using fuzzy metric spaces; see [9–18] and the references included therein. For some very recent studies on the topic, we refer the reader to [19–21].

In particular, Altun [11] proved some useful results about the existence of fixed points of fuzzy order κ -contractive mappings with the setting of non-Archimedean fuzzy metric spaces. The work of Jachymski motivated us to introduce the notion of fuzzy order κ_G -contractive mappings and study related fixed-point results in the context of a fuzzy metric space equipped with a graph. Our introduced notion unifies the concept of Altun and Mihet, and Jachymski. By a

successful application of our main result, we also ensure the existence of a solution for an integral equation.

2. Preliminaries

In this section, we recall some basic notions and results which are used in the sequel. Throughout this article, we use $J = [0, 1]$.

Definition 1 (see [9]). A binary operation $\theta: J \times J \longrightarrow J$ is a continuous t -norm if, for all $u, v, w, z \in J$, it satisfies the following conditions:

- (1) θ is continuous,
- (2) $(u\theta v)\theta w = u\theta(v\theta w)$ and $u\theta v = v\theta u$,
- (3) $u\theta 1 = u$,
- (4) $w \geq u$ and $z \geq v$ imply that $w\theta z \geq u\theta v$.

If there exists a strictly increasing sequence $\{v_n\} \subset (0, 1)$ such that $v_n\theta v_n = v_n$ for all $n \in \mathbb{N}$, then the continuous t -norm θ is known as Hadzic type.

Definition 2 (see [9]). Let $\mathcal{N}$ be an arbitrary set, let $\Gamma: \mathcal{N} \times \mathcal{N} \times [0, \infty) \longrightarrow J$ be a fuzzy set, and let θ be a continuous t -norm. $(\mathcal{N}, \Gamma, \theta)$ is said to be a fuzzy metric space if, for all $\alpha, \beta, \gamma \in \mathcal{N}$ and $\iota, \lambda > 0$, the following conditions hold:

- (1) $[(FM_1)] \Gamma(\alpha, \beta, 0) = 0$,
- (2) $[(FM_2)] \Gamma(\alpha, \beta, \iota) = 1$ if and only if $\alpha = \beta$,
- (3) $[(FM_3)] \Gamma(\alpha, \beta, \iota) = \Gamma(\beta, \alpha, \iota)$,
- (4) $[(FM_4)] \Gamma(\alpha, \beta, \iota)\theta\Gamma(\alpha, \gamma, \lambda) \leq \Gamma(\alpha, \gamma, \iota + \lambda)$,
- (5) $[(FM_5)] \Gamma(\alpha, \beta, \cdot): [0, \infty) \longrightarrow J$ is left continuous.

Remark 1 (see [11]). If we take the inequality (NA) rather than (FM_4) of Definition 2,

$$\Gamma(\alpha, \beta, \iota)\theta\Gamma(\alpha, \gamma, \lambda) \leq \Gamma(\alpha, \gamma, \max\{\iota, \lambda\}), \quad (1)$$

for all $\alpha, \beta, \gamma \in \mathcal{N}$ and $\iota, \lambda > 0$, (NA),

then we call $(\mathcal{N}, \Gamma, \theta)$ a non-Archimedean fuzzy metric space. It is worthwhile to note that every non-Archimedean fuzzy metric space is a fuzzy metric space. From now on, we denote a fuzzy metric space simply by $\mathcal{N}$.

Definition 3 (see [9, 11]). A sequence $\{\alpha_n\}$ in $\mathcal{N}$ is called

- (i) Cauchy if $\lim_{n \rightarrow \infty} \Gamma(\alpha_n, \alpha_{n+p}, \iota) = 1$ for each $\iota > 0$ and $p > 0$;
- (ii) convergent to $\alpha \in \mathcal{N}$ if $\lim_{n \rightarrow \infty} \Gamma(\alpha_n, \alpha, \iota) = 1$ for all $\iota > 0$;
- (iii) M -Cauchy if for each $\varepsilon \in (0, 1)$ and $\iota > 0$ there is a natural number n_0 such that $\Gamma(\alpha_n, \alpha_m, \iota) > 1 - \varepsilon$ for all $m, n > n_0$;
- (iv) G -Cauchy if $\lim_{n \rightarrow \infty} \Gamma(\alpha_n, \alpha_{n+1}, \iota) = 1$, for all $\iota > 0$.

$\mathcal{N}$ is called complete if every Cauchy sequence is convergent. Additionally, it is called M -complete if every M -Cauchy

sequence is convergent and G -complete if every G -Cauchy sequence is convergent.

In the following, we take into account the main results of [11]. A self-map $\mathcal{T}$ on an ordered set $(\mathcal{N}, <)$ is named nondecreasing (nonincreasing) w.r.t $<$ if $\alpha < \beta$ implies $\mathcal{T}\alpha < \mathcal{T}\beta$ ($\mathcal{T}\alpha \geq \mathcal{T}\beta$).

Definition 4 (see [11]). Let $(\mathcal{N}, <)$ be a partially order set and let $(\mathcal{N}, \Gamma, \theta)$ be a fuzzy metric space. Let $\kappa: J \longrightarrow J$ be a function. $\mathcal{T}: \mathcal{N} \longrightarrow \mathcal{N}$ is called a fuzzy order κ -contractive mapping if the following holds:

$$\alpha, \beta \in \mathcal{N}, \alpha < \beta \Rightarrow \Gamma(\mathcal{T}\alpha, \mathcal{T}\beta, \iota) \geq \kappa(\Gamma(\alpha, \beta, \iota)), \quad \text{for all } \iota > 0. \quad (2)$$

Theorem 1 (see [11], Theorem 2.3). Let $(\mathcal{N}, \Gamma, \theta)$ be an M -complete non-Archimedean fuzzy metric space with the t -norm θ of Hadzic type, in which $\mathcal{N}$ is a set with the partial order $<$. Let $\mathcal{T}: \mathcal{N} \longrightarrow \mathcal{N}$ be a fuzzy order κ -contractive mapping, where $\kappa: J \longrightarrow J$ is a continuous, nondecreasing function. Assume that

- (i) $\mathcal{T}$ is nondecreasing;
- (ii) either for every nondecreasing sequence $\{\alpha_n\}$ in $\mathcal{N}$ with $\alpha_n \longrightarrow \alpha \in \mathcal{N}$ implies $\alpha_n < \alpha$ for all n or $\mathcal{T}$ is continuous;
- (iii) there exists $\alpha_0 \in \mathcal{N}$ such that $\alpha_0 < \mathcal{T}\alpha_0$ and $\lim_{n \rightarrow \infty} \kappa^n(\Gamma(\alpha_0, \mathcal{T}\alpha_0, \iota)) = 1$ for each $\iota > 0$.

Then there is $\omega \in \mathcal{N}$ such that $\mathcal{T}\omega = \omega$.

Theorem 2 (see [11], Theorem 2.4). Let $(\mathcal{N}, \Gamma, \theta)$ be an M -complete non-Archimedean fuzzy metric space with the t -norm θ of Hadzic type in which $\mathcal{N}$ is a set with the partial order $<$. Let $\mathcal{T}: \mathcal{N} \longrightarrow \mathcal{N}$ be a fuzzy order κ -contractive mapping, where $\kappa: J \longrightarrow J$ is a continuous mapping such that $\kappa(\iota) > \iota$ for all $\iota \in (0, 1)$. Assume that

- (i) $\mathcal{T}$ is nondecreasing;
- (ii) either for every nondecreasing sequence $\{\alpha_n\}$ in $\mathcal{N}$ with $\alpha_n \longrightarrow \alpha \in \mathcal{N}$ implies $\alpha_n < \alpha$ for all n or $\mathcal{T}$ is continuous;
- (iii) there is $\alpha_0 \in \mathcal{N}$ such that $\alpha_0 < \mathcal{T}\alpha_0$, $\Gamma(\alpha_0, \mathcal{T}\alpha_0, \iota) > 0$ for all $\iota > 0$.

Then there exists $\omega \in \mathcal{N}$ such that $\mathcal{T}\omega = \omega$.

Following [5], we include basic notions about graph which are used throughout this paper. Let $(\mathcal{N}, \Gamma, \theta)$ be a fuzzy metric space. The diagonal of $\mathcal{N} \times \mathcal{N}$ is represented by Δ . Consider a directed graph G such that its diagonal is a subset of $E(G)$ and $V(G)$ is equal to the whole set $\mathcal{N}$. We assume that G does not have any edges which are parallel and describe G with the pair $(V(G), E(G))$. By G^{-1} , we mean the converse of graph G and it is defined as

$$E(G^{-1}) = \{(\alpha, \beta) \in \mathcal{N} \times \mathcal{N}: (\beta, \alpha) \in E(G)\}. \quad (3)$$

The undirected graph is denoted by $\tilde{G}$ and it is obtained by neglecting the direction of edges. $E(\tilde{G})$ is given as

$$E(\tilde{G}) = E(G^{-1}) \cup E(G). \quad (4)$$

If V' is a subset of $V(G)$, E' is a subset of $E(G)$ and, for any edge $(\alpha, \beta) \in E'$, $\alpha, \beta \in V'$; then (V', E') is called a subgraph of G . The sequence $\{\alpha_j\}_{j=0}^p$ of $(p+1)$ vertices, such that

$$\alpha = \alpha_0, \alpha_p = \beta \text{ and } (\alpha_{j-1}, \alpha_j) \in E(G), \quad (5)$$

where $j \in \{1, 2, \dots, p\}$,

is called the path of length p between two vertices α and β in G , where p is a nonnegative integer. If between any two vertices there is always a path, then we call G a connected graph. G is said to be weakly connected if $\tilde{G}$ is connected. Let α be a vertex in G and let $E(G)$ be symmetric; then the set of vertices and edges that are comprised in certain path starting at α is known as component of G which contains α ; it is a subgraph of G and is denoted by G_α . The set of all edges in G_α is denoted by $E(G_\alpha)$ and $V(G_\alpha) = [\alpha]_G$. It is worthwhile to indicate that G_α is connected. Here, $[\alpha]_G$ is the equivalence class of R and the relation R on vertices of V is defined as

$$\beta R \gamma, \text{ if there is a path amidst } \beta \text{ and } \gamma \text{ in } G. \quad (6)$$

Let $\{\alpha_n\}$ be any sequence in $\mathcal{N}$. If for $\{\alpha_n\}$ with $\alpha_n \rightarrow \alpha$ and $(\alpha_n, \alpha_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ implies that there exists a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ such that $(\alpha_{n_k}, \alpha) \in E(G)$ for $k \in \mathbb{N}$, then G is called (C)-graph.

3. Main Results

From now on, we denote a fuzzy metric space endowed with a graph G simply by $\mathcal{N}$.

Definition 5. A mapping $\mathcal{T}: \mathcal{N} \rightarrow \mathcal{N}$ is called fuzzy order κ_G -contractive if, for all $\alpha, \beta \in \mathcal{N}$, we have

$$(\alpha, \beta) \in E(G) \text{ implies } (\mathcal{T}\alpha, \mathcal{T}\beta) \in E(G), \quad (7)$$

$$\Gamma(\mathcal{T}\alpha, \mathcal{T}\beta, \iota) \geq \kappa(\Gamma(\alpha, \beta, \iota)), \quad (8)$$

for all $\iota > 0$ whenever $(\alpha, \beta) \in E(G)$,

where $\kappa: J \rightarrow J$ is a continuous and nondecreasing function.

Example 1. Suppose that G is any graph such that $V(G) = \mathcal{N}$. Define $\mathcal{T}: \mathcal{N} \rightarrow \mathcal{N}$ by $\mathcal{T}\alpha = u$ for all $\alpha \in \mathcal{N}$, where u is any constant; then $\mathcal{T}$ is a fuzzy order κ_G -contractive mapping.

Example 2. Let G_0 be a graph such that $E(G_0) = \mathcal{N} \times \mathcal{N}$. Then any fuzzy order κ_G -contractive mapping is a fuzzy order κ_{G_0} -contractive mapping.

Definition 6. Two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $\mathcal{N}$ are said to be equivalent if $\lim_{n \rightarrow \infty} \Gamma(\alpha_n, \beta_n, \iota) = 1$ for all $\iota > 0$. In addition, $\{\alpha_n\}$ and $\{\beta_n\}$ are called Cauchy equivalent if each of them is Cauchy.

Lemma 1. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be equivalent sequences in $\mathcal{N}$. Then the following hold:

- (i) if $\alpha_n \rightarrow \alpha$, then $\beta_n \rightarrow \alpha$ and its converse holds;
- (ii) if $\{\alpha_n\}$ is a Cauchy sequence, then $\{\beta_n\}$ is also a Cauchy sequence and vice versa.

Proof. By using condition (FM_4) of Definitions 2 and 3, one can easily invoke the conclusions. $\square$

Proposition 1. Let $\mathcal{T}: \mathcal{N} \rightarrow \mathcal{N}$ be a fuzzy order κ_G -contractive mapping, where $\kappa: J \rightarrow J$ is a continuous, nondecreasing function. Then the following hold:

- (i) $\mathcal{T}$ is both a fuzzy order κ_G -contractive mapping and a fuzzy order $\kappa_{G^{-1}}$ -contractive mapping,
- (ii) $[\alpha_0]_{\tilde{G}}$ is $\mathcal{T}$ -invariant, and $\mathcal{T}|_{[\alpha_0]_{\tilde{G}}}$ is a fuzzy order $\kappa_{\tilde{G}}$ -contractive mapping provided that $\alpha_0 \in \mathcal{N}$ is such that $\mathcal{T}\alpha_0 \in [\alpha_0]_{\tilde{G}}$.

Proof. (i) It could be proved easily by taking into account (FM_3) of Definition 3. (ii) Let $\alpha \in [\alpha_0]_{\tilde{G}}$. Then there exists a path between α and α_0 ; that is, there exists a sequence $\{z_j\}_{j=0}^p$ of $(p+1)$ vertices such that $\alpha = z_0, z_p = \alpha_0$ and $(z_{j-1}, z_j) \in E(G)$, for $j = 1, 2, \dots, p$. Since $\mathcal{T}$ is fuzzy order κ_G -contractive, one writes $(\mathcal{T}z_{j-1}, \mathcal{T}z_j) \in E(G)$ for $j \in \{1, 2, \dots, p\}$. Thus $\mathcal{T}\alpha \in [\mathcal{T}\alpha_0]_{\tilde{G}} = [\alpha_0]_{\tilde{G}}$. Now, suppose that $(\alpha, \beta) \in E(G)$. Since $\mathcal{T}$ is fuzzy order κ_G -contractive, we have $(\mathcal{T}\alpha, \mathcal{T}\beta) \in E(G)$. But $[\alpha_0]_{\tilde{G}}$ is $\mathcal{T}$ -invariant, which implies that $(\mathcal{T}\alpha, \mathcal{T}\beta) \in E(\tilde{G}_{\alpha_0})$. As $\tilde{G}_{\alpha_0}$ is a subgraph of G , we have

$$\Gamma(\mathcal{T}\alpha, \mathcal{T}\beta, \iota) \geq \kappa(\Gamma(\alpha, \beta, \iota)), \quad (9)$$

for all $\iota > 0$ whenever $(\alpha, \beta) \in E(\tilde{G}_{\alpha_0})$. $\square$

Lemma 2. Let $\mathcal{T}: \mathcal{N} \rightarrow \mathcal{N}$ be a fuzzy order κ_G -contractive mapping, where $\kappa: J \rightarrow J$ is a continuous, nondecreasing function. Assume that, for $\alpha \in \mathcal{N}$ and $\beta \in [\alpha]_{\tilde{G}}$, we have

$$\lim_{n \rightarrow \infty} \kappa^n(\Gamma(z_{j-1}, z_j, \iota)) = 1, \text{ for all } \iota > 0, \quad (10)$$

where $\{z_j\}_{j=0}^p$ is any path between α and β . Then $\{\mathcal{T}^n \alpha\}$ and $\{\mathcal{T}^n \beta\}$ are equivalent.

Proof. Let $\alpha \in \mathcal{N}$ and $\beta \in [\alpha]_{\tilde{G}}$; then a path $\alpha = \alpha_0, \alpha_1, \dots, \alpha_p = \beta$ exists between α and β with $(\alpha_{j-1}, \alpha_j) \in E(\tilde{G})$ for $j \in \{1, 2, \dots, p\}$. From Proposition 1, $\mathcal{T}$ is a fuzzy order κ_G -contractive mapping. So, for all $n \in \mathbb{N}$ and $j \in \{1, 2, \dots, p\}$, we obtain

$$\Gamma(\mathcal{T}^n \alpha_{j-1}, \mathcal{T}^n \alpha_j, \iota) \geq \kappa(\Gamma(\mathcal{T}^{n-1} \alpha_{j-1}, \mathcal{T}^{n-1} \alpha_j, \iota)), \quad (11)$$

for all $\iota > 0$,

whenever $(\mathcal{T}^n \alpha_{j-1}, \mathcal{T}^n \alpha_j) \in E(\tilde{G})$. Now, for all $n \in \mathbb{N}$ and $j \in \{1, 2, \dots, p\}$, we obtain

$$\begin{aligned} \Gamma(\mathcal{T}^n \alpha_{j-1}, \mathcal{T}^n \alpha_j, \iota) &\geq \kappa(\Gamma(\mathcal{T}^{n-1} \alpha_{j-1}, \mathcal{T}^{n-1} \alpha_j, \iota)) \\ &\geq \dots \geq \kappa^n(\Gamma(\alpha_{j-1}, \alpha_j, \iota)). \end{aligned} \quad (12)$$

It yields that $\{\mathcal{T}^n \alpha_j\}_{j=0}^p$ is a path in $\tilde{G}$ from $\mathcal{T}^n \alpha$ to $\mathcal{T}^n \beta$. Thus,

$$\begin{aligned} \Gamma(\mathcal{T}^n \alpha, \mathcal{T}^n \beta, \iota) &\geq \Gamma\left(\mathcal{T}^n \alpha_0, \mathcal{T}^n \alpha_1, \frac{\iota}{p}\right) \theta^{(p \text{ times})} \Gamma\left(\mathcal{T}^n \alpha_{p-1}, \mathcal{T}^n \alpha_p, \frac{\iota}{p}\right) \\ &\geq \kappa^n\left(\Gamma\left(\alpha_0, \alpha_1, \frac{\iota}{p}\right)\right) \theta^{(p \text{ times})} \kappa^n\left(\Gamma\left(\alpha_{p-1}, \alpha_p, \frac{\iota}{p}\right)\right). \end{aligned} \quad (13)$$

By letting $n \rightarrow \infty$, we conclude $\lim_{n \rightarrow \infty} \Gamma(\mathcal{T}^n \alpha, \mathcal{T}^n \beta, \iota) = 1$. $\square$

Proposition 2. Let $\mathcal{T}: \mathcal{N} \rightarrow \mathcal{N}$ be a fuzzy order κ_G -contractive mapping, where $\kappa: J \rightarrow J$ is a continuous, nondecreasing function. Assume that, for $\alpha_0 \in \mathcal{N}$, we have $\mathcal{T} \alpha_0 \in [\alpha_0]_{\tilde{G}}$ and

$$\lim_{n \rightarrow \infty} \kappa^n(\Gamma(z_{j-1}, z_j, \iota)) = 1, \quad \text{for all } \iota > 0, \quad (14)$$

where $\{z_j\}_{j=0}^p$ is any path between α_0 and $\mathcal{T} \alpha_0$. Suppose that, for any sequence $\{\mathcal{T}^n \alpha_0\}$ in $\mathcal{N}$ with $\mathcal{T}^n \alpha_0 \in [\alpha_0]_{\tilde{G}}$, $n \geq 1$ and $\Gamma(\mathcal{T}^n \alpha_0, \mathcal{T}^{n+1} \alpha_0, \iota) \rightarrow 1$, there is $n_0 \in \mathbb{N}$ satisfying $(\mathcal{T}^m \alpha_0, \mathcal{T}^n \alpha_0) \in E(G)$ for each $m, n \geq n_0$.

Then $\{\mathcal{T}^n \alpha_0\}$ is an M -Cauchy sequence in $\mathcal{N}$.

Proof. Since $\mathcal{T} \alpha_0 \in [\alpha_0]_{\tilde{G}}$, from Proposition 1, it follows that $\mathcal{T}^n \alpha_0 \in [\alpha_0]_{\tilde{G}}$ for all $n \geq 1$. Now, we suppose that $\{\beta_j\}_{j=0}^p$ is a

path connecting α_0 and $\mathcal{T} \alpha_0$. Using the similar reasoning like in the proof of Lemma 2, we have

$$\lim_{n \rightarrow \infty} \Gamma(\mathcal{T}^n \alpha_0, \mathcal{T}^{n+1} \alpha_0, \iota) = 1, \quad \text{for all } n \in \mathbb{N}. \quad (15)$$

Now, we show that $\{\mathcal{T}^n \alpha_0\}$ is an M -Cauchy sequence. Suppose on the contrary that $\{\mathcal{T}^n \alpha_0\}$ is not an M -Cauchy sequence; then there exist $\varepsilon \in (0, 1)$ and two subsequences $\{\mathcal{T}^{m(k)} \alpha_0\}$ and $\{\mathcal{T}^{n(k)} \alpha_0\}$ of $\{\mathcal{T}^n \alpha_0\}$ with $n(k) > m(k) > k$ such that

$$\Gamma(\mathcal{T}^{m(k)} \alpha_0, \mathcal{T}^{n(k)} \alpha_0, \iota) \leq 1 - \varepsilon. \quad (16)$$

Let, for each k , $m(k)$ be the least integer exceeding $n(k)$ satisfying (16), but

$$\Gamma(\mathcal{T}^{m(k)-1} \alpha_0, \mathcal{T}^{n(k)} \alpha_0, \iota) > 1 - \varepsilon. \quad (17)$$

Now, for each $k \in \mathbb{N}$, we obtain

$$\begin{aligned} 1 - \varepsilon &\geq \Gamma(\mathcal{T}^{m(k)} \alpha_0, \mathcal{T}^{n(k)} \alpha_0, \iota) \geq \Gamma\left(\mathcal{T}^{m(k)-1} \alpha_0, \mathcal{T}^{n(k)} \alpha_0, \frac{\iota}{2}\right) \theta \Gamma\left(\mathcal{T}^{m(k)-1} \alpha_0, \mathcal{T}^{m(k)} \alpha_0, \frac{\iota}{2}\right) \\ &> (1 - \varepsilon) \theta \Gamma\left(\mathcal{T}^{m(k)-1} \alpha_0, \mathcal{T}^{m(k)} \alpha_0, \frac{\iota}{2}\right). \end{aligned} \quad (18)$$

By taking $k \rightarrow \infty$ and using (15), we obtain

$$\lim_{k \rightarrow \infty} \Gamma(\mathcal{T}^{m(k)} \alpha_0, \mathcal{T}^{n(k)} \alpha_0, \iota) = 1 - \varepsilon. \quad (19)$$

Since $\lim_{n \rightarrow \infty} \Gamma(\mathcal{T}^n \alpha_0, \mathcal{T}^{n+1} \alpha_0, \iota) = 1$, for all $n \in \mathbb{N}$ and $\iota \geq 0$, by condition (i), there exists a positive integer n_0 such that $(\mathcal{T}^m \alpha_0, \mathcal{T}^n \alpha_0) \in E(G)$ for each $m, n \in \mathbb{N}$ with $m, n > n_0$. Now, by using triangular inequality for all $k \geq n_0$, we obtain

$$\begin{aligned} \Gamma(\mathcal{T}^{m(k)} \alpha_0, \mathcal{T}^{n(k)} \alpha_0, \iota) &\geq \Gamma\left(\mathcal{T}^{m(k)} \alpha_0, \mathcal{T}^{m(k)+1} \alpha_0, \frac{\iota}{3}\right) \theta \Gamma\left(\mathcal{T}^{m(k)+1} \alpha_0, \mathcal{T}^{n(k)+1} \alpha_0, \frac{\iota}{3}\right) \\ &\quad \theta \Gamma\left(\mathcal{T}^{n(k)+1} \alpha_0, \mathcal{T}^{n(k)} \alpha_0, \frac{\iota}{3}\right) \\ &\geq \Gamma\left(\mathcal{T}^{m(k)} \alpha_0, \mathcal{T}^{m(k)+1} \alpha_0, \frac{\iota}{3}\right) \theta \kappa\left(\Gamma\left(\mathcal{T}^{m(k)} \alpha_0, \mathcal{T}^{n(k)} \alpha_0, \frac{\iota}{3}\right)\right) \\ &\quad \theta \Gamma\left(\mathcal{T}^{n(k)+1} \alpha_0, \mathcal{T}^{n(k)} \alpha_0, \frac{\iota}{3}\right). \end{aligned} \quad (20)$$

On letting $k \rightarrow \infty$ and by using (15) and (19), we acquire

$$1 - \varepsilon \geq 1\theta\kappa(1 - \varepsilon)\theta 1 = \kappa(1 - \varepsilon) > 1 - \varepsilon. \quad (21)$$

Hence, $\{\mathcal{T}^n\alpha_0\}$ is an M -Cauchy sequence in $\mathcal{N}$. $\square$

Remark 2. It is important to note that assertion of Proposition 2 still holds if we replace the condition $\mathcal{T}\alpha_0 \in [\alpha_0]_{\tilde{G}}$ with $(\alpha_0, \mathcal{T}\alpha_0) \in E(G)$. In that case, condition (i) is also replaced by the following: “for any sequence $\{\mathcal{T}^n\alpha_0\}$ in $\mathcal{N}$ with $(\mathcal{T}^n\alpha_0, \mathcal{T}^{n+1}\alpha_0) \in E(G)$ and $\Gamma(\mathcal{T}^n\alpha_0, \mathcal{T}^{n+1}\alpha_0, \iota) \rightarrow 1$, there exists $n_0 \in \mathbb{N}$ such that $(\mathcal{T}^m\alpha_0, \mathcal{T}^n\alpha_0) \in E(G)$ for all $m, n \geq n_0$.”

Below we extend the definitions of (H_f) - and (C_f) -graphs to the setting of fuzzy metric spaces.

Definition 7. Suppose that $\mathcal{T}: \mathcal{N} \rightarrow \mathcal{N}$, $\beta \in \mathcal{N}$, and the sequence $\{\mathcal{T}^n\beta\}$ in $\mathcal{N}$ is such that $\mathcal{T}^n\beta \rightarrow \alpha \in \mathcal{N}$.

- (i) If there exists a subsequence $\{\mathcal{T}^{n_k}\beta\}$ of $\{\mathcal{T}^n\beta\}$ such that $(\mathcal{T}^{n_k}\beta, \alpha) \in E(G)$ for all $k \geq p \in \mathbb{N}$, then G is called a (C_f) -graph.
- (ii) A graph G is called an (H_f) -graph if $\mathcal{T}^n\beta \in [\alpha]_{\tilde{G}}$ for $n \geq 1$; then $r(\mathcal{T}^n\beta, \alpha, \iota) \rightarrow 1$ (as $n \rightarrow \infty$), where $r(\mathcal{T}^n\beta, \alpha, \iota) = \Gamma(z_0, z_1, \iota/p)\theta^{(p-\text{times})}\theta\Gamma(z_{p-1}, z_p, \iota/p)$ and $\{z_j\}_{j=0}^p$ is a path from $\mathcal{T}^n\beta$ to α .

It is evident that every (C) -graph is a (C_f) -graph for the identity mapping I , but the other side is not always true. Similarly, every (H) -graph is an (H_f) -graph for the identity mapping I , but the other side is not always true (see [8], Examples 2.4 and 2.5).

Lemma 3. Let $\mathcal{T}: \mathcal{N} \rightarrow \mathcal{N}$ be fuzzy order κ_G -contractive, where $\kappa: J \rightarrow J$ is a continuous, nondecreasing function. Then, for any $\alpha \in \mathcal{N}$ and $\beta \in [\alpha]_{\tilde{G}}$ and for each $n \in \mathbb{N}$, there is $r_n(\alpha, \beta, \iota) \in J$ such that

$$\Gamma(\mathcal{T}^n\alpha, \mathcal{T}^n\beta, \iota) \geq r_n(\alpha, \beta, \iota). \quad (22)$$

Furthermore, if $\lim_{n \rightarrow \infty} \kappa^n(\Gamma(z_{j-1}, z_j, \iota)) = 1$ for all $\iota > 0$, where $\{z_j\}_{j=1}^p$ is any path between α and β , then

$$\lim_{n \rightarrow \infty} \Gamma(\mathcal{T}^n\alpha, \mathcal{T}^n\beta, \iota) = \lim_{n \rightarrow \infty} r_n(\alpha, \beta, \iota) = 1. \quad (23)$$

Proof. Let $\alpha \in \mathcal{N}$ and $\beta \in [\alpha]_{\tilde{G}}$; then there exists a path $\alpha = \alpha_0, \alpha_1, \dots, \alpha_p = \beta$ between α and β with $(\alpha_{j-1}, \alpha_j) \in E(\tilde{G})$ for $j \in \{1, 2, \dots, p\}$. From Proposition 1 for all $\iota > 0$, we have

$$\Gamma(\mathcal{T}^n\alpha_{j-1}, \mathcal{T}^n\alpha_j, \iota) \geq \kappa(\Gamma(\mathcal{T}^{n-1}\alpha_{j-1}, \mathcal{T}^{n-1}\alpha_j, \iota)). \quad (24)$$

Since $\mathcal{T}$ is a fuzzy order κ_G contraction such that $(\mathcal{T}^n\alpha_{j-1}, \mathcal{T}^n\alpha_j) \in E(\tilde{G})$ for each natural number n and j varies from 1 to p , for a positive integer n and $j \in \{1, 2, \dots, p\}$, we have

$$\begin{aligned} \Gamma(\mathcal{T}^n\alpha_{j-1}, \mathcal{T}^n\alpha_j, \iota) &\geq \kappa(\Gamma(\mathcal{T}^{n-1}\alpha_{j-1}, \mathcal{T}^{n-1}\alpha_j, \iota)) \\ &\geq \dots \geq \kappa^n(\Gamma(\alpha_{j-1}, \alpha_j, \iota)). \end{aligned} \quad (25)$$

Note that $\{\mathcal{T}^n\alpha_j\}_{j=0}^p$ is a path in $\tilde{G}$ from $\mathcal{T}^n\alpha$ to $\mathcal{T}^n\beta$. Thus,

$$\begin{aligned} \Gamma(\mathcal{T}^n\alpha, \mathcal{T}^n\beta, \iota) &\geq \Gamma\left(\mathcal{T}^n\alpha_0, \mathcal{T}^n\alpha_1, \frac{\iota}{p}\right)\theta^{(p-\text{times})}\theta\Gamma\left(\mathcal{T}^n\alpha_{p-1}, \mathcal{T}^n\alpha_p, \frac{\iota}{p}\right) \\ &\geq \kappa^n\left(\Gamma\left(\alpha_0, \alpha_1, \frac{\iota}{p}\right)\right)\theta^{(p-\text{times})}\theta\kappa^n\left(\Gamma\left(\alpha_{p-1}, \alpha_p, \frac{\iota}{p}\right)\right). \end{aligned} \quad (26)$$

It suffices to set

$$r_n(\alpha, \beta, \iota) = \kappa^n\left(\Gamma\left(\alpha_0, \alpha_1, \frac{\iota}{p}\right)\right)\theta\kappa^n\left(\Gamma\left(\alpha_1, \alpha_2, \frac{\iota}{p}\right)\right)\theta^{(p-\text{times})}\theta\kappa^n\left(\Gamma\left(\alpha_{p-1}, \alpha_p, \frac{\iota}{p}\right)\right). \quad (27)$$

On letting $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} \Gamma(\mathcal{T}^n\alpha, \mathcal{T}^n\beta, \iota) = \lim_{n \rightarrow \infty} r_n(\alpha, \beta, \iota) = 1. \quad (28)$$

Now, we proceed to establish our first main result. From now on, $\mathcal{N}$ represents an M -complete fuzzy metric space endowed with a graph G . $\square$

Theorem 3. Let $\mathcal{T}: \mathcal{N} \rightarrow \mathcal{N}$ be a fuzzy order κ_G -contractive mapping, where $\kappa: J \rightarrow J$ is a continuous, nondecreasing function. Assume that, for a given $\alpha_0 \in \mathcal{N}$, $(\alpha_0, \mathcal{T}\alpha_0) \in E(G)$ and

$$\lim_{n \rightarrow \infty} \kappa^n(\Gamma(\alpha_0, \mathcal{T}\alpha_0, \iota)) = 1, \quad \text{for all } \iota > 0. \quad (29)$$

Moreover, if

- (i) G is a (C_f) -graph,
- (ii) for any sequence $\{\mathcal{T}^n \alpha_0\}$ in $\mathcal{N}$ with $(\mathcal{T}^n \alpha_0, \mathcal{T}^{n+1} \alpha_0) \in E(G), n \geq 1$ and $\Gamma(\mathcal{T}^n \alpha_0, \mathcal{T}^{n+1} \alpha_0, \iota) \longrightarrow 1$, then there is a natural number n_0 so that $(\mathcal{T}^m \alpha_0, \mathcal{T}^n \alpha_0) \in E(G)$ for every $m, n \geq n_0$.

Then there is $\omega \in [\alpha_0]_{\tilde{G}}$ in such a way that $\mathcal{T}\omega = \omega$. In addition, if, for every $\beta \in [\alpha_0]_{\tilde{G}}$, we have

$$\lim_{n \rightarrow \infty} \kappa^n(\Gamma(z_{j-1}, z_j, \iota)) = 1, \quad \text{for all } \iota > 0, \quad (30)$$

where $\{z_j\}_{j=0}^p$ is any path between α and β , then $\mathcal{T}|_{[\alpha_0]_{\tilde{G}}}$ is a Picard operator.

Proof. Let $(\alpha_0, \mathcal{T}\alpha_0) \in E(G)$; then $\mathcal{T}\alpha_0 \in [\alpha_0]_{\tilde{G}}$. From Proposition 2, we infer that $\{\mathcal{T}^n \alpha_0\}$ is an M -Cauchy sequence. Since $\mathcal{N}$ is an M -complete space, there exists $\omega \in \mathcal{N}$ such that $\mathcal{T}^n \alpha_0 \longrightarrow \omega$. Since G is a (C_f) -graph, there are $\{\mathcal{T}^{n_k} \alpha_0\}$, a subsequence of $\{\mathcal{T}^n \alpha_0\}$, and a positive integer q in such a way that $(\mathcal{T}^{n_k} \alpha_0, \omega) \in E(G)$ for every positive integer k and $q \leq k$. Thus, for all $p \leq k$, we obtain

$$\begin{aligned} \Gamma(\mathcal{T}\omega, \omega, \iota) &\geq \Gamma\left(\mathcal{T}\omega, \mathcal{T}^{n_k+1} \alpha_0, \frac{\iota}{2}\right) \theta \Gamma\left(\mathcal{T}^{n_k} \alpha_0, \omega, \frac{\iota}{2}\right) \\ &\geq \kappa\left(\Gamma\left(\omega, \mathcal{T}^{n_k} \alpha_0, \frac{\iota}{2}\right)\right) \theta \Gamma\left(\mathcal{T}^{n_k} \alpha_0, \omega, \frac{\iota}{2}\right). \end{aligned} \quad (31)$$

Using the continuity of κ and taking $n \longrightarrow \infty$, one gets

$$\Gamma(\mathcal{T}\omega, \omega, \iota) = 1. \quad (32)$$

This infers that $\mathcal{T}\omega = \omega$. We observe that $\{\alpha_0, \mathcal{T}\alpha_0, \dots, \mathcal{T}^{n_1} \alpha_0, \dots, \mathcal{T}^p \alpha_0, \omega\}$ is a path between α_0 and ω in $\tilde{G}$. Thus, $\omega \in [\alpha_0]_{\tilde{G}}$. Now, take a random element $\beta \in [\alpha_0]_{\tilde{G}}$ and, using Lemma 2, we obtain

$$\lim_{n \rightarrow \infty} \Gamma(\mathcal{T}^n \beta, \mathcal{T}^n \alpha_0, \iota) = 1. \quad (33)$$

Thus, $\lim_{n \rightarrow \infty} \mathcal{T}^n \beta = \omega$. To ensure the uniqueness, we argue by contradiction. Suppose that ω and π are two distinct fixed points of $\mathcal{T}$ in $[\alpha_0]_{\tilde{G}}$, so

$$\Gamma(\omega, \pi, \iota) = \Gamma(\mathcal{T}^n \omega, \mathcal{T}^n \pi, \iota) \geq \Gamma\left(\mathcal{T}^n \omega, \mathcal{T}^n \alpha_0, \frac{\iota}{2}\right) \theta \Gamma\left(\mathcal{T}^n \alpha_0, \mathcal{T}^n \pi, \frac{\iota}{2}\right). \quad (34)$$

On letting $n \longrightarrow \infty$, we obtain $\omega = \pi$. $\square$

Remark 3. Observe that, in Theorem 3, $\mathcal{T}$ is a Picard operator on $\mathcal{N}$ if the underlying graph G is weakly connected such that $[\alpha_0]_{\tilde{G}} = \mathcal{N}$.

By analyzing condition on graph G and imposing a strict condition on κ , we have the following variant.

Theorem 4. Let $\mathcal{T}: \mathcal{N} \longrightarrow \mathcal{N}$ be a fuzzy order κ_G -contractive mapping, where $\kappa: J \longrightarrow J$ is a continuous, nondecreasing function such that $\kappa(\iota) > \iota$ for all $\iota \in (0, 1)$. Assume that for $\alpha_0 \in \mathcal{N}$ and $\beta \in [\alpha_0]_{\tilde{G}}$ we get

$$\lim_{n \rightarrow \infty} \kappa^n(\Gamma(z_{j-1}, z_j, \iota)) = 1, \quad \text{for all } \iota > 0, \quad (35)$$

where $\{z_j\}_{j=0}^p$ is any path between α_0 and β . Moreover, if

- (i) G is an (H_f) -graph,
- (ii) for any sequence $\{\mathcal{T}^n \alpha_0\}$ in $\mathcal{N}$ with $(\mathcal{T}^n \alpha_0, \mathcal{T}^{n+1} \alpha_0) \in E(G), n \geq 1$ and $\Gamma(\mathcal{T}^n \alpha_0, \mathcal{T}^{n+1} \alpha_0, \iota) \longrightarrow 1$, then there is $n_0 \in \mathcal{N}$ in such a way that $(\mathcal{T}^m \alpha_0, \mathcal{T}^n \alpha_0) \in E(G), \forall m, n \geq n_0$;
- (iii) G is weakly connected.

Then $\mathcal{T}$ is a Picard operator.

Proof. Assume that $\alpha_0 \in \mathcal{N}$ and condition (iii) holds. Then $\mathcal{T}\alpha_0 \in [\alpha_0]_{\tilde{G}}$; that is, there is a path between α_0 and $\mathcal{T}\alpha_0$ in $\tilde{G}$. Take into account Proposition 2; we infer that the sequence $\{\mathcal{T}^n \alpha_0\}$ is an M -Cauchy sequence. Since $\mathcal{N}$ is an M -complete fuzzy metric space, there exists $\omega \in \mathcal{N}$ such that $\mathcal{T}^n \alpha_0 \longrightarrow \omega$. Since condition (iii) holds, we have, for each natural number n , $\mathcal{T}^n \alpha_0 \in [\omega]_{\tilde{G}}$. Then there exists a path $\{z_j^n\}_{j=0}^{p_n}$ of finite length p_n between $\mathcal{T}^n \alpha_0$ and ω with $z_0^n = \mathcal{T}^n \alpha_0, z_{p_n}^n = \omega$. Using the conditions that $\mathcal{T}$ is a κ_G -contractive mapping and $\kappa(\iota) > \iota$, we obtain

$$\begin{aligned} \Gamma(\omega, \mathcal{T}\omega, \iota) &\geq \Gamma\left(\omega, \mathcal{T}^{n+1} \alpha_0, \frac{\iota}{2}\right) \theta \Gamma\left(\mathcal{T}^{n+1} \alpha_0, \mathcal{T}\omega, \frac{\iota}{2}\right) \\ &\geq \Gamma\left(\omega, \mathcal{T}^{n+1} \alpha_0, \frac{\iota}{2}\right) \theta \left[\Gamma\left(\mathcal{T} z_0^n, \mathcal{T} z_1^n, \frac{\iota}{2l_n}\right) \theta^{(p_n \text{ times})} \theta \Gamma\left(\mathcal{T} z_{j-1}^n, \mathcal{T} z_j^n, \frac{\iota}{2l_n}\right) \right] \\ &\geq \Gamma\left(\omega, \mathcal{T}^{n+1} \alpha_0, \frac{\iota}{2}\right) \theta \left[\kappa\left(\Gamma\left(z_0^n, z_1^n, \frac{\iota}{2l_n}\right)\right) \theta^{(p_n \text{ times})} \theta \kappa\left(\Gamma\left(z_{j-1}^n, z_j^n, \frac{\iota}{2l_n}\right)\right) \right] \\ &= \Gamma\left(\omega, \mathcal{T}^{n+1} \alpha_0, \frac{\iota}{2}\right) \theta r(\mathcal{T}^n \alpha_0, \omega, \iota). \end{aligned} \quad (36)$$

By letting $n \longrightarrow \infty$ and using condition (i), we get $\Gamma(\omega, \mathcal{T}^{n+1} \alpha_0, \iota/2) \theta r(\mathcal{T}^n \alpha_0, \omega, \iota) \longrightarrow 1$. Hence, we have

$\Gamma(\omega, \mathcal{T}\omega, \iota) = 1$. Thus, $\omega = \mathcal{T}\omega$. Now, let β be a random element of $[\alpha_0]_{\tilde{G}}$; then Lemma 2 implies

$$\lim_{n \rightarrow \infty} \Gamma(\mathcal{T}^n \beta, \mathcal{T}^n \alpha_0, \iota) = 1. \quad (37)$$

Thus, $\lim_{n \rightarrow \infty} \mathcal{T}^n \beta = \omega$. Now, for uniqueness, suppose that $\omega \neq \pi$ are two fixed points of $\mathcal{T}$ in $[\alpha_0]_{\widetilde{G}}$; we have

$$\Gamma(\omega, \pi, \iota) = \Gamma(\mathcal{T}^n \omega, \mathcal{T}^n \pi, \iota) \geq \Gamma\left(\mathcal{T}^n \omega, \mathcal{T}^n \alpha_0, \frac{\iota}{2}\right) \theta \Gamma\left(\mathcal{T}^n \alpha_0, \mathcal{T}^n \pi, \frac{\iota}{2}\right). \quad (38)$$

On letting $n \rightarrow \infty$, we obtain $\omega = \pi$. $\square$

Definition 8. Let $\mathcal{T}: \mathcal{N} \rightarrow \mathcal{N}$. If, for all $\alpha, \beta \in \mathcal{N}$ and some sequence $\{p_j\}_{j \in \mathbb{N}}$ of a positive integer, $\mathcal{T}^{p_j} \alpha \rightarrow \beta$ implies $\mathcal{T}(\mathcal{T}^{p_j} \alpha) \rightarrow \mathcal{T} \beta$ as $j \rightarrow \infty$, then $\mathcal{T}$ is called orbitally continuous. If, for each $\alpha, \beta \in \mathcal{N}$ and some sequence $\{p_j\}_{j \in \mathbb{N}}$ of a positive integer $\mathcal{T}^{p_j} \alpha \rightarrow \beta$ and $(\mathcal{T}^{p_j} \alpha, \mathcal{T}^{p_j+1} \alpha) \in E(G)$ for each positive integer n implies $\mathcal{T}(\mathcal{T}^{p_j} \alpha) \rightarrow \mathcal{T} \beta$, then $\mathcal{T}$ is known as orbitally G -continuous.

Theorem 5. Let $\mathcal{T}: \mathcal{N} \rightarrow \mathcal{N}$ be a fuzzy order κ_G -contractive mapping, where $\kappa: J \rightarrow J$ is a continuous and nondecreasing function. Assume that, for given $\alpha_0 \in \mathcal{N}$,

$$(\alpha_0, \mathcal{T} \alpha_0) \in E(G) \text{ and } \lim_{n \rightarrow \infty} \kappa^n(\Gamma(\alpha_0, \mathcal{T} \alpha_0, \iota)) = 1, \quad (39)$$

for all $\iota > 0$.

Moreover, if

- (i) $\mathcal{T}$ is orbitally G -continuous or $\mathcal{T}$ is orbitally continuous,
- (ii) for any sequence $\{\mathcal{T}^n \alpha_0\}$ in $\mathcal{N}$ with $(\mathcal{T}^n \alpha_0, \mathcal{T}^{n+1} \alpha_0) \in E(G), n \geq 1$ and $\Gamma(\mathcal{T}^n \alpha_0, \mathcal{T}^{n+1} \alpha_0, \iota) \rightarrow 1$, there is $n_0 \in \mathbb{N}$ in such a way that $(\mathcal{T}^m \alpha_0, \mathcal{T}^n \alpha_0) \in E(G), \forall m, n \geq n_0$.

Then there is an element ω in $[\alpha_0]_{\widetilde{G}}$ such that $\mathcal{T} \omega = \omega$. In addition, if, for every $\beta \in [\alpha_0]_{\widetilde{G}}$, we have

$$\lim_{n \rightarrow \infty} \kappa^n(\Gamma(z_{j-1}, z_j, \iota)) = 1, \quad \text{for all } \iota > 0, \quad (40)$$

where $\{z_j\}_{j=0}^p$ is any path between α and β , then $\mathcal{T}|_{[\alpha_0]_{\widetilde{G}}}$ is a Picard operator.

Proof. Since $(\alpha_0, \mathcal{T} \alpha_0) \in E(G)$, we have $\mathcal{T} \alpha_0 \in [\alpha_0]_{\widetilde{G}}$. From Proposition 2, $\{\mathcal{T}^n \alpha_0\}$ is an M -Cauchy sequence. Since $\mathcal{N}$ is an M -complete fuzzy metric space, there exists $\omega \in \mathcal{N}$ such that $\mathcal{T}^n \alpha_0 \rightarrow \omega$. As $(\mathcal{T}^n \alpha_0, \mathcal{T}^{n+1} \alpha_0) \in E(G)$, for all $n \geq 1$, and $\mathcal{T}$ is orbitally G -continuous, we obtain $\mathcal{T} \mathcal{T}^n \alpha_0 \rightarrow \mathcal{T} \omega$. Hence, $\mathcal{T} \omega = \omega$. Let $\beta \in [\alpha_0]_{\widetilde{G}}$ be arbitrary, so Lemma 3 yields $\mathcal{T}^n \beta \rightarrow \omega$. Uniqueness can be easily followed.

Now, we include another result when the underlying fuzzy metric space is endowed with a partial order. $\square$

Corollary 1. Let $(\mathcal{N}, <)$ be a partially ordered set and let $(\mathcal{N}, \Gamma, \theta)$ be an M -complete fuzzy metric space. Let $\mathcal{T}: \mathcal{N} \rightarrow \mathcal{N}$ be a fuzzy order κ -contractive mapping, where

$\kappa: J \rightarrow J$ is a continuous and nondecreasing function. Assume that

- (i) $\mathcal{T}$ is nondecreasing (or nonincreasing),
- (ii) either for every nondecreasing (nonincreasing) sequence $\{\alpha_n\}$ in $\mathcal{N}$ with $\alpha_n \rightarrow \alpha \in \mathcal{N}$ implies $\alpha_n < \alpha$ (or $\alpha_n \geq \alpha$) for all n or $\mathcal{T}$ is continuous, and
- (iii) there exists $\alpha_0 \in \mathcal{N}$ such that $\alpha_0 < \mathcal{T} \alpha_0$ ($\alpha_0 \geq \mathcal{T} \alpha_0$) and $\lim_{n \rightarrow \infty} \kappa^n(\Gamma(\alpha_0, \mathcal{T} \alpha_0, \iota)) = 1$ for each $\iota > 0$.

Then $\mathcal{T}$ has a fixed point.

Proof. Define G by $V(G) = \mathcal{N}$ and $E(G) = \{(\alpha, \beta) \in \mathcal{N} \times \mathcal{N} : \beta < \alpha\}$.

Since $\mathcal{T}$ is nondecreasing, we have that (7) is satisfied and (8) holds in the light of condition (2). Thus, $\mathcal{T}$ is a κ_G -contractive mapping. Also, note that condition (ii) implies that G is a (C_f) -graph. Thus, taking into account Theorem 3, we invoke the result. $\square$

Remark 4. Note that Corollary 1 extends/generalizes ([22], Theorem 2.3) to non-Archimedean M -complete fuzzy metric spaces.

Example 3. Let $\mathcal{N} = (0, \infty)$ be equipped with a fuzzy metric $\Gamma(\alpha, \beta, \iota) = (\min\{\alpha, \beta\} / \max\{\alpha, \beta\})$ so that $\alpha \theta \beta = \alpha \beta$. Then $(\mathcal{N}, \Gamma, \theta)$ is an M -complete fuzzy metric space (see [23]). Define $\mathcal{T}: \mathcal{N} \rightarrow \mathcal{N}$ by

$$\mathcal{T} \alpha = \begin{cases} 2\alpha, & 0 < \alpha < 1, \\ \frac{\alpha + 5}{3}, & 1 \leq \alpha \leq 4, \\ 2\alpha - 5, & \alpha > 4. \end{cases} \quad (41)$$

Let G be a graph in such a way that $V(G) = \mathcal{N}$ and $E(G) = \Delta \cup \{(\alpha, \beta) : \alpha, \beta \in [1, 4], \alpha \leq \beta\}$. Condition (7) is trivial. We show that $\mathcal{T}$ is fuzzy order κ_G -contractive with $\kappa(u) = \sqrt{u}$. Indeed, let $\alpha, \beta \in \mathcal{N}$ with $(\alpha, \beta) \in E(G)$. Now, if $\alpha \leq \beta$, then

$$\Gamma(\mathcal{T} \alpha, \mathcal{T} \beta, \iota) = 1 \geq \kappa(1) = \kappa(\Gamma(\alpha, \beta, \iota)). \quad (42)$$

If $\alpha, \beta \in [1, 4]$ with $\alpha \leq \beta$, then

$$\begin{aligned} \Gamma(\mathcal{T} \alpha, \mathcal{T} \beta, \iota) &= \frac{\min\{(\alpha + 5)/3, (\beta + 5)/3\}}{\max\{(\alpha + 5)/3, (\beta + 5)/3\}} \\ &= \frac{\alpha + 5}{\beta + 5} \geq \sqrt{\frac{\alpha}{\beta}} = \kappa(\Gamma(\alpha, \beta, \iota)). \end{aligned} \quad (43)$$

Therefore, $\mathcal{T}$ is a fuzzy order κ_G -contractive mapping with $\kappa(u) = \sqrt{u}$. Moreover, for $\alpha_0 = 1$, $(\alpha_0, \mathcal{T} \alpha_0) \in E(G)$. Hence, all the conditions of Theorem 3 are satisfied and $\mathcal{T}$ has a fixed point.

4. Application

In [24], the following definition was initiated: let $\mathcal{N}$ be a nonempty set and let $\{G_j\}_{j=1}^k$ be nonempty closed subsets of

$\mathcal{N}$, where n is any positive integer. Then, $D = \cup_{j=1}^k G_j$ is called a cyclic representation of $\mathcal{N}$ with respect to the mapping $\mathcal{T}: D \longrightarrow D$ if

$$\mathcal{T}(G_1) \subset G_2, \dots, \mathcal{T}(G_{m-1}) \subset G_m, \mathcal{T}(G_m) \subset G_1. \quad (44)$$

Then $\mathcal{T}$ is called a cyclic operator.

Theorem 6. Let $(\mathcal{N}, \Gamma, \theta)$ be an M -complete fuzzy metric space. Let $\{G_j\}_{j=1}^k$ be nonempty closed subsets of $\mathcal{N}$, $D = \cup_{j=1}^k G_j$, and $\mathcal{T}: D \longrightarrow D$. Further, suppose the following:

- (i) $\cup_{j=1}^k G_j$ is a cyclic representation of D with respect to $\mathcal{T}$.
- (ii) There exists $\kappa: J \longrightarrow J$, where κ is a continuous and nondecreasing function such that, for $\alpha \in G_j, \beta \in G_{j+1}$, we have

$$\Gamma(\mathcal{T}\alpha, \mathcal{T}\beta, t) \geq \kappa(\Gamma(\alpha, \beta, t)), \quad \forall t > 0. \quad (45)$$

- (iii) Let $\alpha_0 \in D$; for a sequence $\{\mathcal{T}^n \alpha_0\}$ in D with $\Gamma(\mathcal{T}^n \alpha_0, \mathcal{T}^{n+1} \alpha_0, t) \longrightarrow 1$, there are a natural number n_0 and some $j \in \{1, 2, \dots, k\}$ such that $\mathcal{T}^m \alpha_0 \in G_j, \mathcal{T}^n \alpha_0 \in G_{j+1}$, for all $m, n \geq n_0$ and

$$\lim_{n \rightarrow \infty} \kappa^n(\Gamma(\alpha_0, \mathcal{T}\alpha_0, t)) = 1, \quad \forall t > 0. \quad (46)$$

Then $\mathcal{T}$ has a fixed point $w \in \cap_{j=1}^k G_j$ and $\mathcal{T}^n \alpha_0 \longrightarrow w$.

Proof. Since $\{G_j\}_{j=1}^k$ are closed subsets, one can assert that (D, Γ, θ) is an M -complete fuzzy metric space. Let G be a graph such that

$$V(G) = D \text{ and } E(G) = \Delta \cup \{(\alpha, \beta) \in D \times D: \alpha \in G_j, \beta \in G_{j+1}; j = 1, 2, \dots, k\}. \quad (47)$$

As $D = \cup_{j=1}^k G_j$ is a cyclic representation of D with respect to $\mathcal{T}$, it follows that $\mathcal{T}$ preserves edges. Also, observe that, in light of inequality (45), $\mathcal{T}$ is a κ_G -contractive mapping. Now, for a sequence $\{\mathcal{T}^n \alpha_0\}$ in D , we have $(\mathcal{T}^n \alpha_0, \mathcal{T}^{n+1} \alpha_0) \in E(G), \forall n \geq 1$ and if $\Gamma(\mathcal{T}^n \alpha_0, \mathcal{T}^{n+1} \alpha_0, t) \longrightarrow 1$, then, from condition (iii), there is $n_0 \in \mathbb{N}$ such that $(\mathcal{T}^m \alpha_0, \mathcal{T}^n \alpha_0) \in E(G)$ for all $m, n \geq n_0$. Let $\mathcal{T}^n \alpha \longrightarrow w$ in D be such that $(\mathcal{T}^n \alpha, \mathcal{T}^{n+1} \alpha) \in E(G)$ for all $n \geq 1$; then, from (44) in every G_j , the sequence $\{\mathcal{T}^n \alpha\}$ has infinitely many terms; thus we have a subsequence of $\{\mathcal{T}^n \alpha\}$ which converges to w in each G_j . As G_j 's are all closed subsets, we get $w \in \cap_{j=1}^k G_j$. Thus, there exists a subsequence $\{\mathcal{T}^{n_r} \alpha\}$ in some $G_j, j \in \{1, 2, \dots, k\}$, such that $(\mathcal{T}^{n_r} \alpha, w) \in E(G)$ for $r \geq 1$, which shows that G is a weakly connected (C_f) -graph. Taking into account Theorem 3, we invoke the conclusion.

Now, we present an application of our result to find a solution of an integral equation. Consider the below integral equation:

$$\alpha(r) = \int_0^r K(r, s, \alpha(s)) ds + g(r), \quad \forall r \in [0, I] \text{ where } I > 0. \quad (48)$$

Let $\mathcal{N} = C([0, I])$ be the set of all real valued continuous functions defined on $[0, I]$. Then $(\mathcal{N}, d)$ is a complete metric space endowed with the metric

$$d(\alpha, \beta) = \sup_{r \in [0, I]} |\alpha(r) - \beta(r)| \quad \alpha, \beta \in \mathcal{N}. \quad (49)$$

For $a \theta b = ab$ for all $a, b \in J$, define a fuzzy metric by

$$\Gamma(\alpha, \beta, t) = \frac{t}{t + d(\alpha, \beta)}, \quad \text{for all } \alpha, \beta \in \mathcal{N} \text{ and } t > 0. \quad (50)$$

Then $(\mathcal{N}, \Gamma, \theta)$ is an M -complete fuzzy metric space.

In the following, we discuss the existence of the solution for the integral equation (48). $\square$

Theorem 7. Let $\mathcal{N}$ be the set of all real valued continuous functions defined on $[0, I]$. Let $\mathcal{T}: \mathcal{N} \longrightarrow \mathcal{N}$ be the integral operator given by

$$\mathcal{T}\alpha(r) = \int_0^r K(r, s, \alpha(s)) ds + g(r), \quad \forall r \in [0, I] \text{ and } g \in \mathcal{N}, \quad (51)$$

where $K: [0, I] \times [0, I] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. Assume that

- (i) there exist a nondecreasing, continuous function $\kappa: J \longrightarrow J$ and a continuous function $\mathcal{T}: [0, I] \times [0, I] \longrightarrow \mathbb{R}^+$ such that

$$|K(r, s, \alpha(s)) - K(r, s, \beta(s))| \leq f(r, s) \kappa(|\alpha(s) - \beta(s)|), \quad (52)$$

for each $r, s \in [0, I]$ and $\alpha(r) \leq \beta(r)$ for all $r \in [0, I]$;

- (ii) $K(r, s, \cdot): \mathbb{R} \longrightarrow \mathbb{R}$ is nondecreasing for each $r, s \in [0, I]$;

- (iii) $\sup_{r \in [0, I]} \int_0^r f(r, s) ds \leq 1$;

- (iv) there exists $\alpha_0 \in \mathcal{N}$ such that $\alpha_0(r) \leq \int_0^r K(r, s, \alpha_0(s)) ds + g(r)$ for all $r \in [0, I]$;

- (v) for any sequence $\{\alpha_n\}$ in $\mathcal{N}$ with $\Gamma(\alpha_n, \alpha_{n+1}, t) \longrightarrow 1$ there exists $n_0 \in \mathbb{N}$ such that $\alpha_m(i) \leq \alpha_n(i)$ for all $m, n \in \mathbb{N}, m, n \geq n_0$.

Then the integral equation (48) has a unique solution in the set $\{\alpha \in \mathcal{N}: \alpha(r) \leq \alpha_0(r), \text{ or } \alpha(r) \geq \alpha_0(r) \text{ for all } r \in [0, I]\}$.

Proof. Let G be a graph such that $V(G) = \mathcal{N}$ and $E(G) = \{(\alpha, \beta) \in \mathcal{N} \times \mathcal{N}: \alpha(r) \leq \beta(r) \text{ for all } r \in [0, I]\}$. It follows from (ii) that T is nondecreasing and thus preserves edges. Also, for every nondecreasing sequence $\{\alpha_n\} \subset \mathcal{N}$ which converges to $\omega \in \mathcal{N}$, it implies $\alpha_n(r) \leq \omega(r)$ for every

$r \in [0, I]$, which shows that G is a (C_f) -graph. Now, for all $\alpha, \beta \in \mathcal{N}$, we obtain

$$\begin{aligned} |\mathcal{T}\alpha(r) - \mathcal{T}\beta(r)| &\leq \int_0^r |K(r, s, \alpha(s)) - K(r, s, \beta(s))| ds \\ &\leq \int_0^r f(r, s) |\alpha(s) - \beta(s)| ds \\ &\leq \kappa(|\alpha(s) - \beta(s)|) \int_0^r f(r, s) ds. \end{aligned} \quad (53)$$

Hence, $d(\mathcal{T}\alpha, \mathcal{T}\beta) \leq \kappa(d(\alpha, \beta))$. Thus,

$$\frac{t}{t + d(\mathcal{T}\alpha, \mathcal{T}\beta)} \geq \kappa(d(\alpha, \beta)) \geq \kappa\left(\frac{t}{t + d(\alpha, \beta)}\right). \quad (54)$$

It follows that

$$\Gamma(\mathcal{T}\alpha, \mathcal{T}\beta, t) \geq \kappa(\Gamma(\alpha, \beta, t)). \quad (55)$$

From (iv), we have $(\alpha_0, \mathcal{T}\alpha_0) \in E(G)$ and the conclusion follows from Theorem 3. Moreover, the solution lies in $[\alpha_0]_{\bar{G}} = \{\alpha \in \mathcal{N} : \alpha(r) \leq \alpha_0, \text{ or } \alpha(r) \geq \alpha_0 \text{ for all } r \in [0, I]\}$. $\square$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

M. S. contributed to methodology, supervision, and data analysis; S. H. contributed to data analysis, study conception, and investigation; H. A. contributed to investigation, validation, and design; Y. U. G. contributed to investigation, editing the revised version, and visualization. All authors approve and agree on the final version.

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