Hindawi
Journal of Mathematics
Volume 2022, Article ID 5946616, 9 pages
https://doi.org/10.1155/2022/5946616

*Research Article*

# Energy of Certain Classes of Graphs Determined by Their Laplacian Degree Product Adjacency Spectrum

**Asim Khurshid,[1] Muhammad Salman ,[1] Masood Ur Rehman ,[2] and Mohammad Tariq Rahim [3]**

[1]*Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan*
[2]*Department of Mathematics, University College of Zhob, BUITEMS, Zhob 85200, Pakistan*
[3]*Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad 22010, Pakistan*

Correspondence should be addressed to Mohammad Tariq Rahim; tariq.rahim@nu.edu.pk

Received 9 October 2021; Accepted 11 December 2021; Published 10 January 2022

Academic Editor: Feng Feng



In this study, we investigate the Laplacian degree product spectrum and corresponding energy of four families of graphs, namely, complete graphs, complete bipartite graphs, friendship graphs, and corona products of 3 and 4 cycles with a null graph.

## 1. Introduction

The graph energy was firstly introduced by Ivan Gutman in 1978 [1]. His idea was motivated by the well-known Hückel molecular orbital theory by Erich Hückel in 1930s, which permits pharmacologists to imprecise energies associated $\pi$-electron orbital of molecules called conjugated hydrocarbons [2]. The spectrum and the energy of a graph have significant applications and connections in the branches of Mathematics, such as linear algebra and combinatorial optimization fields which have lot to do with graph spectrum and energy. The combinatorial and graph theoretical approaches have strong bonding to solve real-life problems. Many results and methods from the spectral graph theory can be applied for the practicalities and evolution of matrix theory [3]. An ordered pair $\Gamma = (V, E)$, called a graph with vertex set of $\Gamma$, is denoted by $V$ and its edge set by $E$. Two vertices $u, v$ are adjacent if they make an edge in $\Gamma$, and we denote it by $u \sim v$. The number of edges incident to a vertex $v$ of $\Gamma$ is the degree of $v$, and it is denoted by $d(v)$ [4, 5]. The adjacency matrix of $\Gamma$, of order $n$ denoted by $A(\Gamma)$, is a square symmetric matrix of order $n \times n$ whose $ij$th element can be found as [3]

$$a_{ij} = \begin{cases} 0, & \text{if } u \nsim v, \\ \text{number of edges between } u \text{ and } v, & \text{if } u \sim v. \end{cases} \quad (1)$$

For energy and spectrum of graph $\Gamma$, let $A(\Gamma)$ be the adjacency matrix, the summation of absolute values of its eigenvalues compose energy of graph and these eigenvalues related with their multiplicities forms the spectrum of graph [4], i.e.,

$$Sp(\Gamma) = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ n(\lambda_1) & n(\lambda_2) & \ldots & n(\lambda_n) \end{pmatrix}, \quad (2)$$

and

$$E(\Gamma) = \sum_{i=1}^{n} |\lambda_i|, \quad (3)$$

where $n(\lambda_1), n(\lambda_2), \ldots, n(\lambda_n)$ are the multiplicities of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A(\Gamma)$. In [6], the degree product adjacency matrix, for a simple connected graph $\Gamma$ having $n$ vertices say $v_1, v_2, \ldots, v_n$, is a real symmetric matrix, denoted by $DP\ A(\Gamma) = [d_{ij}]$,1 with

$$d_{ij} = \begin{cases} d(v_i)d(v_j), & \text{if } v_i \sim v_j, \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

The Laplacian degree product adjacency matrix of $\Gamma$ is defined as

$$L_{DP\ A}(\Gamma) = DP\ A(\Gamma) - D(\Gamma), \quad (5)$$

where $D(\Gamma)$ is the degree matrix of $\Gamma$ having diagonal entries as the degree of each vertex and all other entries are zero. The spectrum (1) and energy (2) obtained correspond to the eigenvalues of $L_{DP\ A}(\Gamma)$ and are called the Laplacian degree product adjacency spectrum and energy, $LSp_{DP\ A}(\Gamma)$ and $LE_{DP\ A}(\Gamma)$, respectively [7], as the degree sum concept was conceived earlier in [8].

## 2. Main Results

In this module, we study the Laplacian degree product adjacency spectrum and energy of some well-known families of graphs, such as complete graphs, complete bipartite graphs, friendship graphs, and corona products of 3 and 4 cycles with null graph. We also evaluate the correct spectrum and the energy of degree product adjacency matrix of the corona product of 4 cycle with null graphs (thorny 4-cycle ring), which was found incorrect in [6].

*2.1. Complete Graphs $K_x$.* Let $\{v_1, v_2 \ldots, v_x\}$ be the vertex set of $K_x$; then, the following result provides the Laplacian degree product adjacency spectrum and energy of $K_x$.

**Theorem 1.** *For $x \geq 2$, let $K_x$ be a complete graph. Then,*

$$LSp_{DP\ A}(K_x) = \begin{pmatrix} x\left(x^2 - 3x + 2\right) & x(1-x) \\ 1 & x-1 \end{pmatrix}, \quad (6)$$

*and Laplacian degree product adjacency energy of $K_x$ is $2(2x-3)$-times the size of $K_x$.*

*Proof.* First of all note that $d(v_i) = x - 1$, for each $1 \leq i \leq x$. Accordingly, we have the following Laplacian degree product adjacency matrix:

$$L_{DP\ A}(K_x) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \cdots \\ v_x \end{array} \begin{pmatrix} v_1 & v_2 & v_3 & \cdots & v_x \\ (1-x) & (1-x)^2 & (1-x)^2 & \cdots & (1-x)^2 \\ (1-x)^2 & (1-x) & (1-x)^2 & \cdots & (1-x)^2 \\ (1-x)^2 & (1-x)^2 & (1-x) & \cdots & (1-x)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1-x)^2 & (1-x)^2 & (1-x)^2 & \cdots & (1-x) \end{pmatrix}. \quad (7)$$

Eigenvalues of $L_{DP\ A}(K_x)$ are

$$\begin{aligned} & x(1-x), (x-1) - \text{times}, \\ & x\left(x^2 - 3x + 2\right), 1 - \text{time}. \end{aligned} \quad (8)$$

These eigenvalues provide the required spectrum. Moreover, by (3), we have

$$\begin{aligned} LE_{DP\ A}(K_x) &= (x-1)|x(1-x)| + \left|x\left(x^2 - 3x + 2\right)\right| \\ &= 2(2x-3)\begin{pmatrix} x \\ 2 \end{pmatrix}. \end{aligned} \quad (9)$$

Since the size of $K_x$ is $\begin{pmatrix} x \\ 2 \end{pmatrix}$, so the result is proved. □

*2.2. Complete Bipartite Graphs $K_{x,y}$.* Let a complete bipartite graph $K_{x,y}$ with vertex sets $V_x(\Gamma) = \{v_1, v_2, \ldots, v_x\}$ and $V_y(\Gamma) = \{v_{x+1}, v_{x+2}, \ldots, v_y\}$ be as partitions. The order and the size of $K_{x,y}$ graph are $x + y$ and $xy$, respectively. Then, the Laplacian degree product adjacency spectrum and energy of $K_{x,y}$ can be obtained from the following result.

**Theorem 2.** *For $x, y \geq 1$, a complete bipartite graph $K_{x,y}$, then*

$$LSp_{DP\ A}(K_{x,y}) = \begin{cases} \begin{pmatrix} \dfrac{-(x+y)}{2} \pm \dfrac{1}{2}\sqrt{4y^3x^3+1} & -x & -y \\ 1 & y-1 & x-1 \end{pmatrix}, & y = x+1 \text{ with } x \geq 1, \\ \begin{pmatrix} \dfrac{-(x+y)}{2} \pm \dfrac{1}{2}\sqrt{4y^3x^3+(y-x)^2} & -x & -y \\ 1 & y-1 & x-1 \end{pmatrix}, & y > x \geq 1 \text{ with } y \neq x+1, \\ \begin{pmatrix} x(x^2-1) & -x(x^2+1) & -x \\ 1 & 1 & 2(x-1) \end{pmatrix}, & x = y \geq 1, \\ \begin{pmatrix} \dfrac{-(1+y)}{2} \pm \dfrac{1}{2}\sqrt{4y^3+(1-y)^2} & -1 \\ 1 & y-1 \end{pmatrix} & x = 1 \text{ and } y \geq 1. \end{cases} \tag{10}$$

*Moreover,*

$$LE_{DP\ A}(K_{x,y}) = \begin{cases} 2(x^3+x^2-x), & \text{whenever } x = y \geq 1, \\ 2xy, & \text{otherwise.} \end{cases} \tag{11}$$

*Proof.* Note that $d(v_i) = y$, for each $1 \leq i \leq x$ and $d(v_j) = x$, for each $x+1 \leq j \leq y$. Accordingly, the Laplacian degree product adjacency matrix of $K_{x,y}$ is

$$L_{DP\ A}(K_{x,y}) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_x \\ v_{x+1} \\ v_{x+2} \\ \vdots \\ v_{x+y} \end{array} \begin{pmatrix} v_1 & v_2 & v_3 & \vdots & v_x & v_{x+1} & v_{x+2} & \cdots & v_{x+y} \\ -y & 0 & 0 & \cdots & 0 & xy & xy & \cdots & xy \\ 0 & -y & 0 & \cdots & 0 & xy & xy & \cdots & xy \\ 0 & 0 & -y & \cdots & 0 & xy & xy & \cdots & xy \\ \vdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -y & xy & xy & \cdots & xy \\ xy & xy & xy & \cdots & xy & -x & 0 & \cdots & 0 \\ xy & xy & xy & \cdots & xy & 0 & -x & \cdots & 0 \\ \vdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \ddots & \vdots \\ xy & xy & xy & \cdots & xy & 0 & 0 & \cdots & -x \end{pmatrix}. \tag{12}$$

Next, we have four cases to discuss.

Case I ($y = x+1$ with $x \geq 1$): eigenvalues of $L_{DP\ A}(K_{x,y})$ are

$$\begin{aligned} &\frac{-(y+x)}{2} \pm \frac{1}{2}\sqrt{4y^3x^3+1}1 - \text{time}, \\ &-x(y-1) - \text{times}, \\ &-y(x-1) - \text{times}. \end{aligned} \tag{13}$$

The required spectrum can be obtained by these eigenvalues. Furthermore, by (3), we have

$$\begin{aligned} LE_{DP\ A}(K_{x,y}) &= \left|\frac{-(y+x)}{2} + \frac{1}{2}\sqrt{4y^3x^3+1}\right| + \left|\frac{-(y+x)}{2} - \frac{1}{2}\sqrt{4y^3x^3+1}\right| + (y-1).|-x| + (x-1).|-y| \\ &= 2xy. \end{aligned} \tag{14}$$

Case II ($y \neq x+1$ with $y > x \geq 1$): we get the following eigenvalues of $L_{DP\,A}(K_{x,y})$:

$$\begin{aligned} &\frac{-(y+x)}{2} \pm \frac{1}{2}\sqrt{4x^3y^3+(y-x)^2}1-\text{time},\\ &-x(y-1)-\text{times},\\ &-y(x-1)-\text{times}. \end{aligned} \tag{15}$$

The required spectrum can be obtained by these eigenvalues. Moreover, by (3), we have

$$\begin{aligned} LE_{DP\,A}(K_{x,y}) &= \left|\frac{-(y+x)}{2} + \frac{1}{2}\sqrt{4x^3y^3+(y-x)^2}\right| + \left|\frac{-(y+x)}{2} - \frac{1}{2}\sqrt{4x^3y^3+(y-x)^2}\right| + (y-1).|-x| + (x-1).|-y|\\ &= 2xy. \end{aligned} \tag{16}$$

Case III ($x = y \geq 1$): eigenvalues of $L_{DP\,A}(K_{x,x})$ are as follows:

$$\begin{aligned} &x(x^2-1), 1-\text{time},\\ &-x(x^2+1), 1-\text{time},\\ &-x, 2(x-1)-\text{times}. \end{aligned} \tag{17}$$

These eigenvalues provide the required spectrum. Furthermore, by (3), we have

$$\begin{aligned} LE_{DP\,A}(K_{x,x}) &= |x(x^2-1)| + |-x(x^2+1)| + 2(x-1).|-x|\\ &= 2(x^3+x^2-x). \end{aligned} \tag{18}$$

Case IV ($x = 1$ and $y \geq 1$): we get eigenvalues of $L_{DP\,A}(K_{1,y})$ as follows:

$$\begin{aligned} &\frac{-(1+y)}{2} \pm \frac{1}{2}\sqrt{4y^3+(1-y)^2}, 1-\text{time},\\ &-1, (y-1)-\text{times}. \end{aligned} \tag{19}$$

These eigenvalues provide the required spectrum. Using (3), we have the following energy of $K_{1,y}$:

$$\begin{aligned} LE_{DP\,A}(K_{1,y}) &= \left|\frac{-(1+y)}{2} + \frac{1}{2}\sqrt{4y^3+(1-y)^2}\right| + \left|\frac{-(1+y)}{2} - \frac{1}{2}\sqrt{4y^3+(1-y)^2}\right| + (y-1).|-1|\\ &= 2y. \end{aligned} \tag{20}$$

It completes the proof. □

*2.3. Friendship Graphs $F_x$.* A friendship graph $F_x$ has $2x+1$ vertices, and it can be assembled by connecting $x$ clones of the cycle $C_3$ with a common vertex. Let the vertex set of $i$th copy of $C_3$ be $\{v_1^i, v_2^i, v_3^i\}$, where $1 \leq i \leq x$. Let the common vertex be $v = v_1^1 = v_1^2 = \ldots = v_1^x$. Then, the vertex set of $F_x$ is

$$\{v\} \bigcup_{i=1}^{x} \{v_2^i, v_3^i\}. \tag{21}$$

**Theorem 3.** *For $x \geq 2$, let a friendship graph $F_x$; then,*

$$\begin{aligned} LSp_{DP\,A}(F_x) &= \begin{pmatrix} 2 & -6 & (1-x) \pm \sqrt{32x^3+(1+x)^2} \\ x-1 & x & 1 \end{pmatrix},\\ LE_{DP\,A}(F_x) &= 10x-4. \end{aligned} \tag{22}$$

*Proof.* In $F_x$, $d(v) = 2x$ and $d(v_2^i) = d(v_3^i) = 2$ for $1 \leq i \leq x$. Then, the Laplacian degree product adjacency matrix is as follows:

$$L_{DP\,A}(F_x) = \begin{array}{c} \\ v \\ v_2^1 \\ v_3^1 \\ v_2^2 \\ v_3^2 \\ \vdots \\ v_2^x \\ v_3^x \end{array} \begin{array}{c} \begin{array}{cccccccc} v & v_2^1 & v_3^1 & v_2^2 & v_3^2 & \dots & v_2^x & v_3^x \end{array} \\ \begin{pmatrix} -2x & 4x & 4x & 4x & 4x & \dots & 4x & 4x \\ 4x & -2 & 4 & 0 & 0 & \dots & 0 & 0 \\ 4x & 4 & -2 & 0 & 0 & \dots & 0 & 0 \\ 4x & 0 & 0 & -2 & 4 & \dots & 0 & 0 \\ 4x & 0 & 0 & 4 & -2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 4x & 0 & 0 & 0 & 0 & \dots & -2 & 4 \\ 4x & 0 & 0 & 0 & 0 & \dots & 4 & -2 \end{pmatrix} \end{array}. \tag{23}$$

The eigenvalues of Laplacian degree product adjacency matrix of $F_x$ are

$$\begin{gathered} 2,\ (x-1) - \text{times}, \\ -6,\ x - \text{times}, \\ (1-x) \pm \sqrt{32x^3 + (1+x)^2},\ 1 - \text{time}. \end{gathered} \tag{24}$$

The required spectrum can be obtained by these eigenvalues. These eigenvalues provide the following energy:

$$\begin{aligned} LE_{DP\,A}(F_x) &= (x-1).|2| + x.|-6| + \left|(1-x) + \sqrt{32x^3 + (1+x)^2}\right| + \left|(1-x) - \sqrt{32x^3 + (1+x)^2}\right| \\ &= 10x - 4. \end{aligned} \tag{25}$$

$\square$

*2.4. Corona Products of 3 and 4 Cycles with Null Graphs.* The corona product of graphs $\Gamma$ and $\Omega$ is expressed as $\Gamma^{\circ}\Omega$. It can be made by drawing one copy of $\Gamma$ and $|V(\Gamma)|$ copies of $\Omega$ and connecting the $i$th vertex of $\Gamma$ with each vertex of $i$th copy of $\Omega$ [9–11]. Let $\Gamma$ be an $x$-cycle $C_x$ with vertices $v_1, v_2, \dots, v_x$ and $\Omega$ be a null graph $N_k$. Then, the vertex set of $C_x{}^{\circ}N_k$ is

$$V(C_x) \bigcup_{j=1}^{x} \left\{v_i^j; 1 \le i \le k\right\}, \tag{26}$$

where the set $\left\{v_i^j; 1 \le i \le k\right\}$ is the vertex set of $j$th copy of $N_k$ in $C_x{}^{\circ}N_k$. In this portion, we evaluate the Laplacian degree product spectrum and energy of $C_x{}^{\circ}N_k$ for $x = 3$ and 4.

**Theorem 4.** *For $k \ge 1$, let the corona product be $C_x{}^{\circ}N_k$; then,*

$$\begin{aligned} LSp_{DP\,A}(C_3{}^{\circ}N_k) &= \begin{pmatrix} \frac{1}{2}\left(\left(2k^2 + 7k + 5\right) \pm \sqrt{4k^4 + 32k^3 + 93k^2 + 114k + 49}\right) & -\frac{1}{2}\left(\left(k^2 + 5k + 7\right) \mp \sqrt{k^4 + 14k^3 + 51k^2 + 66k + 25}\right) & -1 \\ 1 & 2 & 3(k-1)) \end{pmatrix}, \\ LE_{DP\,A}(C_3{}^{\circ}N_k) &= 4\left(k^2 + 5k + 4\right). \end{aligned} \tag{27}$$

*Proof.* Note that $d(v_j) = k + 2$, for each $j = 1, 2, 3$, and $d(v_i^j) = 1$, for each $1 \le j \le 3$ and $1 \le i \le k$. For the convenience, we let $k + 2 = \alpha$. Then, the Laplacian degree product adjacency matrix of $C_x{}^{\circ}N_k$ is

$$\begin{array}{cc}
 & \begin{array}{cccccccccccc} v_1 & v_2 & v_3 & v_1^1 & \cdots & v_k^1 & v_1^2 & \cdots & v_k^2 & v_1^3 & \cdots & v_k^3 \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_1^1 \\ \vdots \\ v_k^1 \\ v_1^2 \\ \vdots \\ v_k^2 \\ v_1^3 \\ \vdots \\ v_k^3 \end{array} &
\begin{pmatrix}
-\alpha & \alpha^2 & \alpha^2 & \alpha & \cdots & \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\alpha^2 & -\alpha & \alpha^2 & 0 & \cdots & 0 & \alpha & \cdots & \alpha & 0 & \cdots & 0 \\
\alpha^2 & \alpha^2 & -\alpha & 0 & \cdots & 0 & 0 & \cdots & 0 & \alpha & \cdots & \alpha \\
\alpha & 0 & 0 & -1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
\alpha & 0 & 0 & 0 & \cdots & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \alpha & 0 & 0 & \cdots & 0 & -1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
0 & \alpha & 0 & 0 & \cdots & 0 & 0 & \cdots & -1 & 0 & \cdots & 0 \\
0 & 0 & \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & -1
\end{pmatrix}
\end{array} \tag{28}$$

The eigenvalues obtained from the above matrix of $C_x{^\circ}N_k$ are

$$\begin{aligned}
&\frac{2k^2+7k+5}{2} \pm \sqrt{\frac{4k^4+32k^3+93k^2+114k+49}{4}}, 1-\text{times}, \\
&-\frac{k^2+5k+7}{2} \pm \frac{1}{2}\sqrt{k^4+14k^3+51k^2+66k+25}, 2-\text{times}, \\
&\quad -1, 3(k-1)-\text{times}.
\end{aligned} \tag{29}$$

Then, the required spectrum can be obtained by these eigenvalues. Also, by (3), we have

$$\begin{aligned}
LE_{DP\ A}\left(C_3{^\circ}N_k\right) &= \left|\frac{2k^2+7k+5}{2} + \sqrt{\frac{4k^4+32k^3+93k^2+114k+49}{4}}\right| + \left|\frac{2k^2+7k+5}{2} - \sqrt{\frac{4k^4+32k^3+93k^2+114k+49}{4}}\right| + \\
&\quad \left|-\left(\frac{k^2+5k+7}{2}\right) + \frac{1}{2}\sqrt{k^4+14k^3+51k^2+66k+25}\right| + \left|-\left(\frac{k^2+5k+7}{2}\right) - \frac{1}{2}\sqrt{k^4+14k^3+51k^2+66k+25}\right| \\
&\quad + 3(k-1)|-1| \\
&= 4\left(k^2+5k+4\right).
\end{aligned} \tag{30}$$

□

**Theorem 5.** *For $k \geq 1$, let the corona product be $C_x{^\circ}N_k$; then, $LSp_{DP\ A}(C_x{^\circ}N_k)$ is*

$$\begin{aligned}
&\begin{pmatrix}
-\frac{1}{2}\left(\left(2k^2+9k+11\right) \mp \sqrt{4k^4+40k^3+133k^2+178k+81}\right) & -\frac{1}{2}\left((k+3) \mp \sqrt{4k^3+17k^2+18k+1}\right. & \frac{1}{2}\left(\left(2k^2+7k+5\right) \pm \sqrt{4k^4+32k^3+93k^2+114k+49}\right) & -1 \\
1 & 2 & 1 & 4(k-1))
\end{pmatrix} \\
&LE_{DP\ A}\left(C_4{^\circ}N_k\right) = 4k^2+22k+18.
\end{aligned} \tag{31}$$

*Proof.* Note that $d(v_j) = k+2$, for each $j = 1, 2, 3, 4$, and $d(v_i^j) = 1$, for each $1 \leq j \leq 4$ and $1 \leq i \leq k$. For the convenience, we let $k+2 = \alpha$. Then, the Laplacian degree product adjacency matrix of $C_4{^\circ}N_k$ is

$$\begin{array}{cc}
 & \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_1^1 & \cdots & v_k^1 & v_1^4 & \cdots & v_k^4 \end{matrix} \\
\begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_1^1 \\ \vdots \\ v_k^1 \\ v_1^4 \\ \vdots \\ v_k^4 \end{matrix} &
\begin{pmatrix}
-\alpha & \alpha^2 & 0 & \alpha^2 & \alpha & \cdots & \alpha & 0 & \cdots & 0 \\
\alpha^2 & -\alpha & \alpha^2 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \alpha^2 & -\alpha & \alpha^2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\alpha^2 & 0 & \alpha^2 & -\alpha & 0 & \cdots & 0 & \alpha & \cdots & \alpha \\
\alpha & 0 & 0 & 0 & -1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
\alpha & 0 & 0 & 0 & 0 & \cdots & -1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \alpha & 0 & \cdots & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \alpha & 0 & \cdots & 0 & 0 & \cdots & -1
\end{pmatrix}
\end{array}. \tag{32}$$

The eigenvalues of Laplacian degree product adjacency matrix of $C_4{}^{\circ}N_k$ are

$$\begin{aligned}
&-\frac{2k^2+9k+11}{2} \pm \sqrt{\frac{4k^4+40k^3+133k^2+178k+81}{4}}, 1-\text{time},\\
&-\frac{k+3}{2} \pm \sqrt{\frac{4k^3+17k^2+18k+1}{4}}, 2-\text{times},\\
&\frac{2k^2+7k+5}{2} \pm \sqrt{\frac{4k^4+32k^3+93k^2+114k+49}{4}}, 1-\text{time},\\
&-1 4(k-1)-\text{times}.
\end{aligned} \tag{33}$$

Then, the required spectrum can be obtained by these eigenvalues. Furthermore, by (3), we have

$$\begin{aligned}
LE_{DP\ A}\left(C_4{}^{\circ}N_k\right) &= \left|-\frac{2k^2+9k+11}{2} + \sqrt{\frac{4k^4+40k^3+133k^2+178k+81}{4}}\right|\\
&+\left|-\frac{2k^2+9k+11}{2} - \sqrt{\frac{4k^4+40k^3+133k^2+178k+81}{4}}\right|\\
&+2.\left|-\frac{k+3}{2} + \sqrt{\frac{4k^3+17k^2+18k+1}{4}}\right|\\
&+2.\left|-\frac{k+3}{2} - \sqrt{\frac{4k^3+17k^2+18k+1}{4}}\right|\\
&+\left|\frac{2k^2+7k+5}{2} + \sqrt{\frac{4k^4+32k^3+93k^2+114k+49}{4}}\right|\\
&+\left|\frac{2k^2+7k+5}{2} - \sqrt{\frac{4k^4+32k^3+93k^2+114k+49}{4}}\right|\\
&+4(k-1).|-1|\\
&= 2\left(2k^2+11k+9\right).
\end{aligned} \tag{34}$$

□

## 3. Appendix

In [6], Mirajkar and Doddamani considered the corona product $C_4{}^{\circ}N_{k-2}$ for $k \geq 3$ (also called thorny cycle rings $C_{4,k}$) and investigated its energy and spectrum on the base of degree product adjacency matrix. During computations of our results on $C_4{}^{\circ}N_{k-2}$, the eigenvalues investigated in [6] were found incorrect. In this section, we provide the correct energy and spectrum of $C_4{}^{\circ}N_{k-2}$. First of all, note that the degree product adjacency matrix of $C_4{}^{\circ}N_{k-2}$ is

$$DP\ A(C_4{}^{\circ}N_{k-2}) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_1^1 \\ v_2^1 \\ \vdots \\ v_1^4 \\ v_2^4 \end{array} \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_1^1 & v_2^1 & \dots & v_1^4 & v_2^4 \\ 0 & k^2 & 0 & k^2 & k & k & \dots & 0 & 0 \\ k^2 & 0 & k^2 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & k^2 & 0 & k^2 & 0 & 0 & \dots & 0 & 0 \\ k^2 & 0 & k^2 & 0 & 0 & 0 & \dots & k & k \\ k & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & k & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{35}$$

Mirajkar and Doddamani's investigated eigenvalues are

$$\begin{aligned} & k\left(\sqrt{k^2+2} \pm k\right), 1 - \text{time}, \\ & -k\left(\sqrt{k^2+2} \pm k\right), 1 - \text{time}, \\ & \pm k\sqrt{2}, 2 - \text{times}, \\ & 0, 4(k-3) - \text{times}. \end{aligned} \tag{36}$$

Whereas, the correct eigenvalues of $DP\ A(C_4{}^{\circ}N_{k-2})$ are

$$\begin{aligned} & \pm k^2 + k\sqrt{k^2+k-2}, 1 - \text{time}, \\ & \mp k^2 - k\sqrt{k^2+k-2}, 1 - \text{time}, \\ & \pm k\sqrt{k-2}, 2 - \text{times}, \\ & 0, 4(k-3) - \text{times}. \end{aligned} \tag{37}$$

Accordingly, we get the following energy and spectrum of $C_4{}^{\circ}N_{k-2}$ in the corrected form:

$$\begin{aligned} E_{DP\ A}(C_4{}^{\circ}N_{k-2}) &= \left|k^2 + k\sqrt{k^2+k-2}\right| + \left|-k^2 + k\sqrt{k^2+k-2}\right| + \left|-k^2 - k\sqrt{k^2+k-2}\right| \\ &\quad + \left|k^2 - k\sqrt{k^2+k-2}\right| + 2.|-k\sqrt{k-2}| + 2.|k\sqrt{k-2}|, \\ Sp_{DP\ A}(C_4{}^{\circ}N_{k-2}) &= \begin{pmatrix} \pm k^2 + k\sqrt{k^2+k-2} & \mp k^2 - k\sqrt{k^2+k-2} & \pm k\sqrt{k-2} & 0 \\ 1 & 1 & 2 & 4(k-3) \end{pmatrix}. \end{aligned} \tag{38}$$

## 4. Accomplishment Remarks

In this study, we construct the general formulas for spectrums and energies of four different families of graphs, by using Laplacian degree product adjacency matrix. We also obtained faultless and correct eigenvalues of $C_4{}^{\circ}N_{k-2}$, which were defined in [6].

## Data Availability

All kinds of data and materials, used to compute the results, are provided in Section 1.

## Conflicts of Interest

The authors declare no conflicts of interest.

## Authors' Contributions

Asim Khurshid carried out computation and wrote the initial draft; Muhammad Salman supervised the study, administrated the project, analysed data, and developed the methodology; Masood ur Rehman: wrote and reviewed the manuscript, carried out formal analysis, and edited the study; Mohammad Tariq Rahim conceptualized the study and visualized the study.

## References

[1] I. Gutman, "The energy of a graph," *Ber.Math.Stat.Sket. Forschungsz.Graz*, vol. 103, pp. 1–22, 1978.

[2] C. Woods, "My favorite application using graph eigenvalues: graph energy," 2013, http://www.math.ucsd.edu/fan/teach/262/13/262notes/Woods Midterm.pdf.

[3] D. M. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs: Theory and Applications 3rd Revised and Enlarged Edition*, Wiley, New York, USA, 1998.

[4] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Cambridge, UK, 1993.

[5] F. Harary, *Graph Theory*, Addison - Wesely, Boston, MA, USA, 1969.

[6] K. G. Mirajkar and B. R. Doddamani, "On energy and spectrum of degree product adjacency matrix for some class of graphs," *International Journal of Applied Engineering Research*, vol. 14, no. 7, pp. 1546–1554, 2019.

[7] K. G. Mirajkar and B. R. Doddamani, "On the bounds of Laplacian energy for degree product adjacency matrix of regular graph," *Research Guru:Online Journal of Multidisciplinary Subjects*, vol. 1, no. 13, pp. 10–17, 2019.

[8] H. S. Ramane, D. S. Revankar, and J. B. Patil, "Bounds for the degree sum eigenvalues and degree sum energy of a graph," *International Journal of Pure and Applied Mathematical Sciences*, vol. 6, pp. 161–167, 2013.

[9] D. Bonchev and D. J. Klein, "On the wiener number of thorn trees, stars, rings and rods," *Croatica Chemica Acta*, vol. 75, no. 2, pp. 613–620, 2002.

[10] R. Frucht and F. Harary, "On the corona of two graphs," *Aequationes Mathematicae*, vol. 4, no. 1, p. 264, 1970.

[11] H. B. Walikar, H. S. Ramane, L. Sindagi, S. S. Shirakol, and I. Gutman, "Hosoya polynomial of thorn trees, rods, rings and trees," *Kragujevac Journal of Science*, vol. 28, pp. 47–56, 2006.