Research Article

On Edge H -Irregularity Strength of Hexagonal and Octagonal Grid Graphs

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The edge H -irregularity strength, $ehs(\Gamma, H)$, of a graph Γ is the smallest integer k , such that Γ has an H -irregular edge k -labeling. In this study, we compute the exact value of edge H -irregularity strength of hexagonal and octagonal grid graphs.

1. Introduction

Let Γ be a connected graph. A mapping that assigns numbers to graph elements (vertices and edges) is called labeling. A graph labeling is called vertex labeling or edge labeling if its domain set is the vertex set or the edge set of the graph, respectively. The concept of graph labeling plays an important role to construct models for a wide range of engineering applications such as coding theory, X-rays crystallography, astronomy, radar, circuit design, wired communication, and wireless communications. For further detail related to graph labeling, refer to [1].

For an edge k -labeling $\zeta: E(\Gamma) \rightarrow \{1, 2, \dots, k\}$, $k \in \mathbb{Z}^+$, the corresponding weight of $u \in V(\Gamma)$ is $wt_{\zeta}(u) = \sum_{uv \in E(\Gamma)} \zeta(uv)$. The edge k -labeling ζ is called irregular if $wt_{\zeta}(u) \neq wt_{\zeta}(v)$, $\forall u \neq v \in V(\Gamma)$. The irregularity strength of Γ , $s(\Gamma)$, is the minimum $k \in \mathbb{Z}^+$, such that there exists an edge irregular k -labeling of Γ . The concept of the irregularity strength of a graph was introduced by Chartrand et al. [2]. There has been a flurry of research work on the irregularity strength in the last few years [3–12].

A vertex k -labeling $\zeta^*: V(\Gamma) \rightarrow \{1, 2, \dots, k\}$ of Γ is called an edge irregular k -labeling if for every two distinct edges uv and $u'v'$, $wt_{\zeta^*}(uv) \neq wt_{\zeta^*}(u'v')$, where $wt_{\zeta^*}(uv) = \zeta^*(u) + \zeta^*(v)$.

The edge irregularity strength of Γ , $es(\Gamma)$, is the minimum k , such that the graph Γ has an edge irregular k -labeling. The concept of the edge irregularity strength was given by Ahmad et al. [13]. Ashraf et al. [14] have introduced two new graph parameters, i.e., vertex (edge) H -irregularity strength of a graph. These parameters are considered as extensions of the irregularity strength and the edge irregularity strength of Γ .

A family $\mathcal{H} = \{H_i, 1 \leq i \leq t\}$ of subgraphs of a graph Γ , such that each edge of Γ belonging to at least one member of $\mathcal{H}$ is called an edge-covering of Γ . If every member of $\mathcal{H}$ is isomorphic to a graph H , then Γ admits an H -covering. An edge k -labeling ζ is called an H -irregular edge k -labeling of the graph Γ that admits H -covering if for every two distinct subgraphs H_1 and H_2 , which are isomorphic to H , $wt_{\zeta}(H_1) \neq wt_{\zeta}(H_2)$. The edge H -irregularity strength of a graph Γ , $ehs(\Gamma, H)$, is the smallest integer k , such that Γ has an H -irregular edge k -labeling.

In proving lower bound concerning a graph Γ admitting H -covering, the following result is useful.

Theorem 1 (see [14]). *Let Γ be a graph admitting H -covering given by t subgraphs isomorphic to H . Then, $ehs(\Gamma, H) \geq \lceil 1 + (t - 1/|E(H)|) \rceil$.*

In this study, we compute the exact values of the edge H -irregularity strength of the hexagonal grid graph and octagonal grid graph.

2. Motivation and Applications of H -Covering of Graph

The topic of H -covering of the graph is newly addressed by Ashraf et al. in [14] and found the H -covering for different graphs. Still now, the problem of H -covering was addressed on grids by any author. As the topic is related to the networking and communication system, it is needed to work on some networks. So, we choose these two graphs as these graphs are related to network. The H -covering has a wide range of applications in X-rays, circuit design, and especially in communication and networks. The graphs studied in the present study are hexagonal and octagonal grids. These graphs consist of 6 and 8 vertices in each face of the graph, and each face made a cycle consisting of 6 and 8 vertices and edges as well. These graphs and their faces could be served as models for surveillance or security systems, electrical switchboards, circuit design, and communication networks. These networks can be extended in both horizontal and vertical directions, so that the extension in the graphs and networks can be handled and made easy. Moreover, these networks are used especially in communication networks, and the efficiency of the networks may be improved as the weights of the each face have distinct numbers.

3. Hexagonal Grid Graph

For finite $r, s \geq 3$, the hexagonal grid graph (honeycomb), H_r^s , is a graph with s rows and r columns of hexagons [15]. The vertex and the edge sets of this graph are defined as $V(H_r^s) = \{v_j^i, u_j^i; 1 \leq i \leq s, 1 \leq j \leq r\}$; $E(H_r^s) = \{v_j^i u_j^i, v_j^{i+1} u_j^{i+1}; 1 \leq i \leq s-1, 1 \leq j \leq r\} \cup \{v_j^i u_{j+1}^i, u_j^{i+1} v_{j+1}^{i+1}; 1 \leq i \leq s-1, 1 \leq j \leq r-1\} \cup \{u_j^i v_j^{i+1}; 1 \leq i \leq s-1, 1 \leq j \leq r\}$. The face set of $F(H_r^s)$ consists of $|F(H_r^s)| - 1$ 6-sided faces and one external face which is infinite. Also, $|V(H_r^s)| = 2rs - 2$ and $|E(H_r^s)| = |V(H_r^s)| + s(r-1)$, while $|F(H_r^s)| = 4(r+s) - 9$. For $r, s \geq 3$, the hexagonal grid graph is shown in Figure 1.

In the following theorem, we determine the edge H -irregularity strength of H_r^s , where $r, s \geq 3$ are finite.

Theorem 2. For hexagonal grid graph H_r^s , $r, s \geq 3$ admitting an H -covering given by $(r-1)(s-1)$ subgraphs isomorphic to C_6 , $\text{ehs}(H_r^s, C_6) = \lceil 1 + ((r-1)(s-1) - 1/6) \rceil$.

Proof. The graph H_r^s , $r, s \geq 3$ obviously admits a C_6 -covering with exactly $(r-1)(s-1)$ subcovers of C_6 . Set $k = \lceil 1 + ((r-1)(s-1) - 1/6) \rceil$; then, k is the lower bound of $\text{ehs}(H_r^s, C_6)$ by Theorem 1. Now, to prove the converse inequality, we have to describe a C_6 -irregular edge k -labeling $\zeta: E(C_6) \rightarrow \{1, 2, \dots, k\}$ as follows:

Case 1: when s is odd,

$$\zeta(v_j^i u_j^i) = \zeta(v_j^{i+1} u_j^{i+1}) = \begin{cases} 1, & \text{for } i = 1; 1 \leq j \leq r, \\ \left\lceil \frac{j}{2} \right\rceil, & \text{for } i = 2; 1 \leq j \leq 2 \left\lceil \frac{i(r-1)+5}{6} \right\rceil, \\ \left\lceil \frac{i(r-1)+5}{6} \right\rceil, & \text{for } i = 2; 2 \left\lceil \frac{i(r-1)+5}{6} \right\rceil + 1 \leq j \leq r, \\ \left\lceil \frac{(i-1)(r-1)+5}{6} \right\rceil, & \text{for } 3 \leq i \leq s-1; (i \text{ is odd}), 1 \leq j \leq r, \\ \left\lceil \frac{i(r-1)+5}{6} \right\rceil, & \text{for } 4 \leq i \leq s-1; (i \text{ is even}), 1 \leq j \leq r, \end{cases} \quad (1)$$

FIGURE 1: Hexagonal grid graph H_r^s .

$$\zeta(v_j^i u_{j+1}^i) = \zeta(u_j^{i+1} v_{j+1}^{i+1}) = \begin{cases} 1, & \text{for } i = 1; 1 \leq j \leq r-1, (j \text{ is odd}), \\ 2, & \text{for } i = 1; 2 \leq j \leq r-1, (j \text{ is even}), \\ \left\lfloor \frac{j}{2} \right\rfloor, & \text{for } i = 1; 1 \leq j \leq 2 \left\lceil \frac{i(r-1)+5}{6} \right\rceil, \\ \left\lfloor \frac{(i+1)(r-1)+5}{6} \right\rfloor, & \text{for } i = 2; 2 \left\lceil \frac{i(r-1)+5}{6} \right\rceil + 1 \leq j \leq r-1, \\ \left\lfloor \frac{(i-1)(r-1)+5}{6} \right\rfloor, & \text{for } 3 \leq i \leq s-1; (i \text{ is odd}), 1 \leq j \leq r-1, \\ j, & \text{for } i = 4; 1 \leq j \leq \left\lceil \frac{i(r-1)+5}{6} \right\rceil, \\ \left\lfloor \frac{i(r-1)+5}{6} \right\rfloor, & \text{for } i = 4; \left\lceil \frac{i(r-1)+5}{6} \right\rceil + 1 \leq j \leq r-1, \\ \left\lfloor \frac{i(r-1)+5}{6} \right\rfloor, & \text{for } 6 \leq i \leq s-1; (i \text{ is even}), 1 \leq j \leq r-1, \end{cases} \quad (2)$$

$$\zeta(u_j^i v_j^{i+1}) = \begin{cases} 1, & \text{for } i = 1; 1 \leq j \leq r, \\ \left\lfloor \frac{r+3-2\lceil i(r-1)+5/6 \rceil}{2} \right\rfloor, & \text{for } i = 2; \text{for } 1 \leq j \leq 2\lceil \frac{i(r-1)+5}{6} \rceil + (j \text{ is odd}), \\ \left\lceil \frac{r+3-2\lceil i(r-1)+5/6 \rceil}{2} \right\rceil, & \text{for } i = 2; \text{for } 2 \leq j \leq 2\lceil \frac{i(r-1)+5}{6} \rceil + (j \text{ is even}), \\ \left\lceil \frac{j-2\lceil i(r-1)+5/6 \rceil}{2} \right\rceil + \left\lceil \frac{r+3-2\lceil i(r-1)+5/6 \rceil}{2} \right\rceil, & \text{for } i = 2; \text{for } 2\lceil \frac{i(r-1)+5}{6} \rceil + 1 \leq j \leq r, (j \text{ is odd}), \\ \left\lceil r+3-2\lceil \frac{i(r-1)+5}{2} \rceil \right\rceil - \left\lceil \frac{j-2\lceil i(r-1)+5/6 \rceil}{2} \right\rceil, & \text{for } i = 2; \text{for } 2\lceil \frac{i(r-1)+5}{6} \rceil + 1 \leq j \leq r, (j \text{ is even}), \\ \left\lfloor \frac{(i-1)(r-1)+5-2\lceil (i-1)(r-1)+5/6 \rceil - \lceil (i+1)(r-1)+5/6 \rceil}{2} \right\rfloor, & \text{for } i = 3; \text{for } 1 \leq j \leq \left\lceil \frac{(i+1)(r-1)+5}{6} \right\rceil + 1, (j \text{ is odd}), \\ \left\lceil \frac{j-1\lceil (i+1)(r-1)+5/6 \rceil}{2} \right\rceil + \left\lfloor \frac{2(r-1)+6-2\lceil (i-1)(r-1)+5/6 \rceil - \lceil (i+1)(r-1)+5/6 \rceil}{2} \right\rfloor, & \text{for } i = 3; \left\lceil \frac{(i+1)(r-1)+5}{6} \right\rceil + 2 \leq j \leq r, \\ \left\lceil \frac{2(r-1)+5-2\lceil (i-1)(r-1)+5/6 \rceil - \lceil (i+1)(r-1)+5/6 \rceil}{2} \right\rceil, & \text{for } i = 3; \text{for } 2 \leq j \leq \left\lceil \frac{(i+1)(r-1)+5}{6} \right\rceil + 1, (j \text{ is even}), \end{cases} \quad (3)$$

$$\zeta(u_j^i v_j^{i+1}) = \begin{cases} \lceil \frac{j-1 - \lceil (i+1)(r-1) + 5/6 \rceil}{2} \rceil + \lceil \frac{2(r-1) + 5 - 2 \lceil (i-1)(r-1) + 5/6 \rceil - \lceil (i+1)(r-1) + 5/6 \rceil}{2} \rceil, & \text{for } i = 3; \lceil \frac{(i+1)(r-1) + 5}{6} \rceil + 2 \leq j \leq r, \\ \lfloor \frac{3(r-1) + 5 - 3 \lceil i(r-1) + 5/6 \rceil}{2} \rfloor, & \text{for } i = 4; 1 \leq j \leq \lceil \frac{i(r-1) + 5}{6} \rceil + 1, (j \text{ is odd}), \\ \lfloor \frac{3(r-1) + 5 - 3 \lceil i(r-1) + 5/6 \rceil}{2} \rfloor + \lceil \frac{j-1 - \lceil i(r-1) + 5/6 \rceil}{2} \rceil, & \text{for } i = 4; \lceil \frac{i(r-1) + 5}{6} \rceil + 2 \leq j \leq r, \\ \lfloor \frac{3(r-1) + 5 - 3 \lceil i(r-1) + 5/6 \rceil}{2} \rfloor, & \text{for } i = 4; 2 \leq j \leq \lceil \frac{i(r-1) + 5}{6} \rceil + 1, (j \text{ is even}), \\ \lceil \frac{3(r-1) + 5 - 3 \lceil i(r-1) + 5/6 \rceil}{2} \rceil + \lceil \frac{j-1 - \lceil i(r-1) + 5/6 \rceil}{2} \rceil, & \text{for } i = 4; \lceil \frac{i(r-1) + 5}{6} \rceil + 2 \leq j \leq r, \\ \lfloor \frac{(i-1)(r-1) + 6 - 2 \lceil (i-1)(r-1) + 5/6 \rceil - 2 \lceil (i+1)(r-1) + 5/6 \rceil}{2} \rfloor + \lceil \frac{j}{2} \rceil - 1, & \text{for } 5 \leq i \leq s-1; (i \text{ is odd}), 1 \leq j \leq r, (j \text{ is odd}), \\ \lfloor \frac{(i-1)(r-1) + 6 - 2 \lceil (i-1)(r-1) + 5/6 \rceil - 2 \lceil (i+1)(r-1) + 5/6 \rceil}{2} \rfloor + \lceil \frac{j}{2} \rceil - 1, & \text{for } 5 \leq i \leq s-1; (i \text{ is odd}), 2 \leq j \leq r, (j \text{ is even}), \\ \lfloor \frac{(i-1)(r-1) + 6 - 4 \lceil i(r-1) + 5/6 \rceil}{2} \rfloor + \lceil \frac{j}{2} \rceil - 1, & \text{for } 6 \leq i \leq s-1; (i \text{ is even}), 1 \leq j \leq r, (j \text{ is odd}), \\ \lceil \frac{(i-1)(r-1) + 6 - 4 \lceil i(r-1) + 5/6 \rceil}{2} \rceil + \lceil \frac{j}{2} \rceil - 1, & \text{for } 6 \leq i \leq s-1; (i \text{ is even}), 1 \leq j \leq r, (j \text{ is even}). \end{cases} \quad (4)$$

Case 2: when s is even,

$$\zeta(v_j^i u_j^i) = \zeta(v_j^{i+1} u_j^{i+1}) = \begin{cases} \lceil \frac{j}{3} \rceil, & \text{for } i = 1; 1 \leq j \leq 3, \\ \lceil \frac{j+3}{6} \rceil, & \text{for } i = 1; 4 \leq j \leq r, \\ \lceil \frac{j}{6} \rceil, & \text{for } i = 2; 1 \leq j \leq r, \\ \lceil \frac{i(r-1) + 5}{6} \rceil - 1, & \text{for } r = 3, i = 3; j = 1, \\ \lceil \frac{i(r-1) + 5}{6} \rceil - 2, & \text{for } r > 3, i = 3; j = 1, \\ \lceil \frac{i(r-1) + 5}{6} \rceil - 1, & \text{for } i = 3; j = 2, \\ \lceil \frac{i(r-1) + 5}{6} \rceil, & \text{for } i = 3; 3 \leq j \leq r, \\ \lceil \frac{(i-1)(r-1) + 5}{6} \rceil, & \text{for } 4 \leq i \leq s-1, (i \text{ is even}); 1 \leq j \leq r, \\ \lceil \frac{i(r-1) + 5}{6} \rceil - 2, & \text{for } i = 5; j = 1, \\ \lceil \frac{i(r-1) + 5}{6} \rceil - 1, & \text{for } i = 5; j = 2, \\ \lceil \frac{i(r-1) + 5}{6} \rceil, & \text{for } i = 5; 3 \leq j \leq r, \\ \lceil \frac{i(r-1) + 5}{6} \rceil - 1, & \text{for } 7 \leq i \leq s-1, (i \text{ is odd}); 1 \leq j \leq r, \end{cases}$$

$$\zeta(v_j^i u_{j+1}^i) = \zeta(u_j^{i+1} v_{j+1}^{i+1}) = \begin{cases} \lceil \frac{j}{5} \rceil, & \text{for } i = 1; 1 \leq j \leq 5, \\ \lceil \frac{j+1}{6} \rceil, & \text{for } i = 1; 6 \leq j \leq r-1, \\ \lceil \frac{j}{4} \rceil, & \text{for } i = 2; 1 \leq j \leq 4, \\ \lceil \frac{j+2}{6} \rceil, & \text{for } i = 2; 5 \leq j \leq r-1, \\ \lceil \frac{(i)(r-1)+5}{6} \rceil, & \text{for } 3 \leq i \leq s-1; (i \text{ is odd}), 1 \leq j \leq r-1, \\ \lceil \frac{(i-1)(r-1)+5}{6} \rceil, & \text{for } 4 \leq i \leq s-1; (i \text{ is even}), 1 \leq j \leq r-1, \end{cases} \quad (5)$$

$$\zeta(u_j^i v_j^{i+1}) = \begin{cases} \lceil \frac{j}{2} \rceil, & \text{for } i = 1; 1 \leq j \leq 2, \\ \lceil \frac{j+4}{6} \rceil, & \text{for } i = 1; 3 \leq j \leq r, \\ \lceil \frac{(i-1)(r-1)+4-2\lceil (i+1)(r-1)+5/6 \rceil + \lceil r/13 \rceil}{2} \rceil, & \text{for } i = 2; j = 1, 3, \\ \lfloor \frac{(i-1)(r-1)+4-2\lceil (i+1)(r-1)+5/6 \rceil + \lceil r/14 \rceil}{2} \rfloor, & \text{for } i = 2; j = 2, \\ \lfloor (i-1)(r-1)+4-2\lceil \frac{(i+1)(r-1)+5}{6} \rceil \rfloor + \lfloor \frac{j-1}{3} \rfloor, & \text{for } (r \text{ is even}), i = 2; \text{for } 4 \leq j \leq r, (j \text{ is even}), \\ \lfloor (i-1)(r-1)+4-2\lceil \frac{(i+1)(r-1)+5}{6} \rceil + \lceil \frac{r}{14} \rceil \rfloor + \lfloor \frac{j-1}{3} \rfloor, & \text{for } (r \text{ is even}), i = 2; \text{for } 5 \leq j \leq r, (j \text{ is odd}), \\ \lfloor (i-1)(r-1)+2-2\lceil \frac{(i+1)(r-1)+5}{6} \rceil + \lceil \frac{r}{14} \rceil \rfloor + \lfloor \frac{j-1}{3} \rfloor, & \text{for } (r \text{ is odd}), i = 2; \text{for } 4 \leq j \leq r, (j \text{ is even}), \\ \lceil \frac{(i-1)(r-1)+4-2\lceil (i+1)(r-1)+5/6 \rceil + \lceil r/13 \rceil}{2} \rceil + \lfloor \frac{j-2}{3} \rfloor, & \text{for } (r \text{ is odd}), i = 2; \text{for } 5 \leq j \leq r, (j \text{ is odd}), \\ \frac{(i-1)(r-1)+6-4\lceil i(r-1)+5/6 \rceil + 2\lceil r/13 \rceil}{2}, & \text{for } i = 3; 1 \leq j \leq 4, \\ \frac{(i-1)(r-1)+6-4\lceil i(r-1)+5/6 \rceil + 2\lceil r/13 \rceil}{2} + \lceil \frac{j-4}{2} \rceil, & \text{for } i = 3; 5 \leq j \leq r, \\ \lceil \frac{i(r-1)+6-3\lceil (i+1)(r-1)+5/6 \rceil - 2\lceil (i-1)(r-1)+5/6 \rceil + \lceil r/14 \rceil}{2} \rceil, & \text{for } (r \text{ is odd}), i = 2; j = 1, 3, \\ \lceil \frac{i(r-1)+4-3\lceil (i+1)(r-1)+5/6 \rceil - 2\lceil (i-1)(r-1)+5/6 \rceil - \lfloor r/12 \rfloor + 3\lceil r/13 \rceil}{2} \rceil, & \text{for } (r \text{ is even}), i = 4; j = 1, 3, \\ \lfloor \frac{i(r-1)+9-3\lceil (i+1)(r-1)+5/6 \rceil - 2\lceil (i-1)(r-1)+5/6 \rceil}{2} \rfloor, & \text{for } (r = 3, 5), i = 4; j = 2, \\ \lfloor \frac{i(r-1)+4-3\lceil (i+1)(r-1)+5/6 \rceil - 2\lceil (i-1)(r-1)+5/6 \rceil + 3\lceil r/14 \rceil}{2} \rfloor, & \text{for } (r \text{ is odd}); r > 5, i = 4; j = 2, \\ \lceil \frac{i(r-1)+5-3\lceil (i+1)(r-1)+5/6 \rceil - 2\lceil (i-1)(r-1)+5/6 \rceil + 2\lceil r/13 \rceil - \lfloor r/12 \rfloor}{2} \rceil, & \text{for } (r \text{ is even}), i = 4; j = 2, \\ \lfloor \frac{i(r-1)+9-3\lceil (i+1)(r-1)+5/6 \rceil - 2\lceil (i-1)(r-1)+5/6 \rceil}{2} \rfloor + \lceil \frac{j-3}{2} \rceil, & \text{for } (r = 5), i = 4; 4 \leq j \leq r, \\ \lfloor \frac{i(r-1)+4-3\lceil (i+1)(r-1)+5/6 \rceil - 2\lceil (i-1)(r-1)+5/6 \rceil + 3\lceil r/14 \rceil}{2} \rfloor + \lceil \frac{j-3}{2} \rceil, & \text{for } (r \text{ is odd}); r \neq 5, i = 4; \text{for } 4 \leq j \leq r, \\ \lceil \frac{i(r-1)+6-3\lceil (i+1)(r-1)+5/6 \rceil - 2\lceil (i-1)(r-1)+5/6 \rceil + \lceil r/13 \rceil - \lfloor r/12 \rfloor}{2} \rceil + \lceil \frac{j-3}{2} \rceil, & \text{for } (r \text{ is even}), i = 4; \text{for } 4 \leq j \leq r, (j \text{ is even}), \\ \lceil \frac{i(r-1)+4-3\lceil (i+1)(r-1)+5/6 \rceil - 2\lceil (i-1)(r-1)+5/6 \rceil + 3\lceil r/13 \rceil}{2} \rceil + \lceil \frac{j-3}{2} \rceil, & \text{for } (r \text{ is odd}), i = 4; \text{for } 5 \leq j \leq r, (j \text{ is odd}), \\ \frac{(i-1)(r-1)+4-4\lceil i(r-1)+5/6 \rceil + 4\lceil r/13 \rceil}{2}, & \text{for } i = 5; 1 \leq j \leq 4, \\ \frac{(i-1)(r-1)+4-4\lceil i(r-1)+5/6 \rceil + 4\lceil r/13 \rceil + \lceil j-4/2 \rceil}{2}, & \text{for } i = 5; 5 \leq j \leq r, \end{cases} \quad (6)$$

$$\zeta(u_j^i v_j^{i+1}) = \begin{cases} \lfloor \frac{(i-1)(r-1) + 6 - 2\lceil (i-1)(r-1) + 5/6 \rceil - 2\lceil (i+1)(r-1) + 5/6 \rceil + 5\lceil r/13 \rceil}{2} \rfloor + \lceil \frac{j}{2} \rceil - 1, & \text{for } 6 \leq i \leq s-1; (i \text{ is even}); \text{for } 1 \leq j \leq r, (j \text{ is odd}), \\ \lfloor \frac{(i-1)(r-1) + 6 - 2\lceil (i-1)(r-1) + 5/6 \rceil - 2\lceil (i+1)(r-1) + 5/6 \rceil + 5\lceil r/13 \rceil}{2} \rfloor + \lceil \frac{j}{2} \rceil - 1, & \text{for } 6 \leq i \leq s-1; (i \text{ is even}); \text{for } 1 \leq j \leq r, (j \text{ is even}), \\ \frac{(i-1)(r-1) - 2\lceil (i-1)(r-1) + 5/6 \rceil - 2\lceil (i+1)(r-1) + 5/6 \rceil + 4\lceil r/13 \rceil}{2} + \lceil \frac{j}{2} \rceil - 1, & \text{for } r=3; 1 \leq j \leq r; \text{for } 7 \leq i \leq s-1, (i \text{ is odd}), \\ \frac{(i-1)(r-1) + 6 - 2\lceil (i-1)(r-1) + 5/6 \rceil - 2\lceil (i+1)(r-1) + 5/6 \rceil}{2} + \lceil \frac{j}{2} \rceil - 1, & \text{for } (r \text{ is odd}), 3 < r \leq 13; 1 \leq j \leq r; \text{for } 7 \leq i \leq s-1, (i \text{ is odd}), \\ \frac{(i-1)(r-1) + 6 - 2\lceil (i-1)(r-1) + 5/6 \rceil - 2\lceil (i+1)(r-1) + 5/6 \rceil + 4\lceil r/14 \rceil}{2} + \lceil \frac{j}{2} \rceil - 1, & \text{for } r=15+6k, k \geq 0; 1 \leq j \leq r; \text{for } 7 \leq i \leq s-1, (i \text{ is odd}), \\ \frac{(i-1)(r-1) + 8 - 2\lceil (i-1)(r-1) + 5/6 \rceil - 2\lceil (i+1)(r-1) + 5/6 \rceil + 4\lceil r/14 \rceil}{2} + \lceil \frac{j}{2} \rceil - 1, & \text{for } (r \text{ is odd}), r \geq 17; 1 \leq j \leq r; \text{for } 7 \leq i \leq s-1, (i \text{ is odd}), \\ \frac{(i-1)(r-1) + 2 - 2\lceil (i-1)(r-1) + 5/6 \rceil - 2\lceil (i+1)(r-1) + 5/6 \rceil + 4\lceil r/14 \rceil}{2} + \lceil \frac{j}{2} \rceil - 1, & \text{for } r = \text{even} (0 \bmod 6); 1 \leq j \leq r; \text{for } 7 \leq i \leq s-1, (i \text{ is odd}), \\ \frac{(i-1)(r-1) + 4 - 2\lceil (i-1)(r-1) + 5/6 \rceil - 2\lceil (i+1)(r-1) + 5/6 \rceil + 4\lceil r/13 \rceil}{2} + \lceil \frac{j}{2} \rceil - 1, & \text{for } r = \text{even} (2 \bmod 6); 1 \leq j \leq r; \text{for } 7 \leq i \leq s-1, (i \text{ is odd}), \\ \frac{(i-1)(r-1) - 2\lceil (i-1)(r-1) + 5/6 \rceil - 2\lceil (i+1)(r-1) + 5/6 \rceil + 4\lceil r/15 \rceil}{2} + \lceil \frac{j}{2} \rceil - 1, & \text{for } r = \text{even} (4 \bmod 6); 1 \leq j \leq r; \text{for } 7 \leq i \leq s-1, (i \text{ is odd}). \end{cases} \quad (7)$$

Figure 2 shows the weights of the edge C_6 -covering of the hexagonal grid graph, and the formula for the weights of the edge C_6 -covering is given as follows: $wt_{\zeta}(C_6^{(v_j^i u_j^i)(v_{j+1}^{i+1} u_{j+1}^{i+1})(v_j^i u_{j+1}^i)(u_j^{i+1} v_{j+1}^{i+1})(u_j^i v_j^{i+1})}) = 5 + (i-1)(r-1) + j$, $1 \leq i \leq s$; $1 \leq j \leq r$. Observe that all the weights of subcovers of C_6 are distinct. Hence, $\text{ehs}(H_r^s, C_6) = \lceil 1 + ((r-1)(s-1) - 1/6) \rceil$, which completes the proof of this theorem. $\square$

4. Octagonal Grid Graph

For finite $r, s \geq 3$, the octagonal grid graph, $\mathcal{O}_r^s$, is a graph with s rows and r columns of octagons [16]. The vertex and the edge sets of this graph are defined as $V(\mathcal{O}_r^s) = \{v_j^i, x_j^i; \text{for } 1 \leq i \leq s, 1 \leq j \leq r\} \cup \{u_j^i, x_j^i; 1 \leq i \leq s-1, 1 \leq j \leq r\}$; $E(\mathcal{O}_r^s) = \{v_j^i u_j^i, x_j^i u_{j+1}^{i+1}, w_{j+1}^{i+1} v_{j+1}^{i+1}, x_{j+1}^{i+1} w_{j+1}^{i+1}; 1 \leq i \leq s-1, 1 \leq j \leq r-1\} \cup \{u_j^i w_j^i; 1 \leq i \leq s-1, 1 \leq j \leq r\} \cup \{v_j^i x_j^i; \text{for } 1 \leq i \leq s, 1 \leq j \leq r-1\}$. The face set of $F(\mathcal{O}_r^s)$ consists of $|F(\mathcal{O}_r^s)| - 1$ faces as 8-sided faces and one external face which is infinite.

Also, $|V(\mathcal{O}_r^s)| = 4rs - 2(r+s)$ and $|E(\mathcal{O}_r^s)| = |V(\mathcal{O}_r^s)| + r(2s-1) - 3s + 2$, while $|F(\mathcal{O}_r^s)| = (2r-1)(2s-1)$.

For $r, s \geq 3$, the octagonal grid graph is shown in Figure 3.

The following result establishes the edge H -irregularity strength of $\mathcal{O}_r^s$ for $3 \leq r \neq 12, 14 \leq 15$ and $s \geq 3$.

Theorem 3. Let $\mathcal{O}_r^s$ be an octagonal grid graph admitting C_8 -covering given by $(r-1)(s-1)$ subgraphs isomorphic to H ; then, $\text{ehs}(\mathcal{O}_r^s) = \lceil 1 + ((r-1)(s-1) - 1/8) \rceil$.

Proof. It is clear that the graph $\mathcal{O}_r^s$ for $3 \leq r \leq 15$ and $s \geq 3$ and $r \neq 12, 14$ admits C_8 -covering with exactly $(r-1)(s-1)$ subcovers of C_8 . Set $k = \lceil 1 + ((r-1)(s-1) - 1/8) \rceil$; then, k is the lower bound of $\text{ehs}(\mathcal{O}_r^s, C_8)$ by Theorem 1. Now, to prove the converse inequality, we have to describe a C_8 -irregular edge k -labeling $\zeta: E(C_8) \rightarrow \{1, 2, \dots, k\}$ given as follows:

Case 1: when $3 \leq r \leq 7$ and $s \geq 3$, $1 \leq i \leq s-1$; $1 \leq j \leq r-1$,

$$\begin{aligned} \zeta(v_j^i u_j^i) &= \lceil \frac{(r-1)(i-1) + j}{8} \rceil, \\ \zeta(x_j^i u_{j+1}^{i+1}) &= \lceil \frac{(r-1)i - \lceil r/2 \rceil + j}{8} \rceil, \\ \zeta(x_{j+1}^{i+1} w_{j+1}^{i+1}) &= \lceil \frac{(r-1)i \lfloor 5 - r/2 \rfloor + j}{8} \rceil, \\ \zeta(w_j^i v_j^{i+1}) &= \begin{cases} \lceil \frac{(r-1)(i-1) + 1 + j}{8} \rceil, & \text{for } (r=4, 5, 6), 1 \leq i \leq s-1; 1 \leq j \leq r-1, \\ \lceil \frac{(r-1)(i-1) + 2 + j}{8} \rceil, & \text{for } (r=3, 7), 1 \leq i \leq s-1; 1 \leq j \leq r-1, \end{cases} \end{aligned} \quad (8)$$

$$\zeta(w_j^i v_j^{i+1}) = \begin{cases} \lceil \frac{(r-1)(i-1) + 1 + j}{8} \rceil, & \text{for } (r=4, 5, 6), 1 \leq i \leq s-1; 1 \leq j \leq r-1, \\ \lceil \frac{(r-1)(i-1) + 2 + j}{8} \rceil, & \text{for } (r=3, 7), 1 \leq i \leq s-1; 1 \leq j \leq r-1, \end{cases} \quad (9)$$

FIGURE 2: Labeling and weights of hexagonal grid graph H_6^6 .

FIGURE 3: Octagonal grid graph $\mathcal{O}_r^s$.

$$\zeta(u_j^i w_j^i) = \begin{cases} \lceil \frac{(r-1)(i-1) + 5 + j}{8} \rceil, & \text{for } (r = 4, 5, 6, 7), 1 \leq i \leq s-1; 1 \leq j \leq r, \\ \lceil \frac{(r-1)i + 4}{8} \rceil, & \text{for } (r = 3), 1 \leq i \leq s-1; 1 \leq j \leq r, \end{cases} \quad (10)$$

$$\zeta(v_j^i x_j^i) = \left\lceil \frac{(r-1)(i-2) + 7 + j}{8} \right\rceil, \quad \text{for } 1 \leq i \leq s; 1 \leq j \leq r-1. \quad (11)$$

Case 2: when $8 \leq r \leq 13$, $r \neq 12, 14$ and $s \geq 3$, $1 \leq i \leq s-1$;
 $1 \leq j \leq r-1$,

$$\begin{aligned} \zeta(x_j^i u_{j+1}^i) &= \left\lceil \frac{(r-1)(i-1) + 3 + j}{8} \right\rceil, \\ \zeta(w_j^i v_j^{i+1}) &= \left\lceil \frac{(r-1)(i-1) + 2 + j}{8} \right\rceil, \end{aligned} \quad (12)$$

$$\begin{aligned} \zeta(x_j^{i+1} w_{j+1}^i) &= \left\lceil \frac{(r-1)(i-1) + 4 + j}{8} \right\rceil, \\ \zeta(v_j^i u_j^i) &= \begin{cases} \left\lceil \frac{(r-1)(i-1) + 1 + j}{8} \right\rceil, & \text{for } (r = 8, 9, 10, 11), 1 \leq i \leq s-1; 1 \leq j \leq r-1, \\ \left\lceil \frac{(r-1)(i-1) + 2 + j}{8} \right\rceil, & \text{for } (r = 13), 1 \leq i \leq s-1; 1 \leq j \leq r-1, \end{cases} \end{aligned} \quad (13)$$

$$\zeta(u_j^i w_j^{i+1}) = \left\lceil \frac{(r-1)(i-1) + 5 + j}{8} \right\rceil, \quad \text{for } 1 \leq i \leq s-1; 1 \leq j \leq r, \quad (14)$$

$$\zeta(v_j^i x_j^{i+1}) = \begin{cases} \left\lceil \frac{(r-1)(i-1) + j}{8} \right\rceil, & \text{for } (r = 8), 1 \leq i \leq s; 1 \leq j \leq r-1, \\ \left\lceil \frac{(r-1)i - r + j}{8} \right\rceil + 1, & \text{for } (r = 9), 1 \leq i \leq s, (i \text{ is odd}); j = 1, \\ \left\lceil \frac{(r-1)i - r + j}{8} \right\rceil, & \text{for } (r = 9), 1 \leq i \leq s, (i \text{ is odd}); 2 \leq j \leq r-1, \\ \left\lceil \frac{(r-1)i - r + j}{8} \right\rceil, & \text{for } (r = 9), 1 \leq i \leq s, (i \text{ is even}); 1 \leq j \leq r-1, \\ \left\lceil \frac{(r-1)i - r + j}{8} \right\rceil - (-1)^i, & \text{for } (r = 10, 11), 1 \leq i \leq s; j = 1, \\ \left\lceil \frac{(r-1)i - r + j}{8} \right\rceil, & \text{for } (r = 10, 11), 1 \leq i \leq s; 1 \leq j \leq r-1, \\ \left\lceil \frac{(r-1)i - r + j}{8} \right\rceil - (-1)^i, & \text{for } (r = 13), 1 \leq i \leq s; 1 \leq j \leq 4, \\ \left\lceil \frac{(r-1)i - r + j}{8} \right\rceil, & \text{for } (r = 13), 1 \leq i \leq s; 5 \leq j \leq r-1. \end{cases} \quad (15)$$

FIGURE 4: Labeling and weights of octagonal grid graph $\mathcal{O}_5^5$.

Case 3: when $r = 15$ and $s \geq 3$, $1 \leq i \leq s - 1$; $1 \leq j \leq r - 1$,

$$\zeta(v_j^i u_j^i) = \lceil \frac{(r-1)(i-1) + 1 + j}{8} \rceil,$$

$$\zeta(x_j^i u_{j+1}^i) = \lceil \frac{(r-1)(i-1) + 3 + j}{8} \rceil, \quad (16)$$

$$\zeta(w_j^i v_j^{i+1}) = \lceil \frac{(r-1)(i-1) + 4 + j}{8} \rceil,$$

$$\zeta(x_j^{i+1} w_{j+1}^i) = \lceil \frac{(r-1)(i-1) + 5 + j}{8} \rceil, \quad (17)$$

$$\zeta(u_j^i w_j^i) = \left\lceil \frac{(r-1)(i-1) + 6 + j}{8} \right\rceil, \quad \text{for } 1 \leq i \leq s; 1 \leq j \leq r, \quad (18)$$

$$\zeta(v_j^i x_j^i) = \begin{cases} \left\lceil \frac{(r-1)(i-1) - 4 + j}{8} \right\rceil - (-1)^i, & \text{for } 1 \leq i \leq s-1; 1 \leq j \leq 6, \\ \left\lceil \frac{(r-1)(i-1) - 4 + j}{8} \right\rceil, & \text{for } 1 \leq i \leq s; 1 \leq j \leq r-1. \end{cases} \quad (19)$$

Figure 4 shows the weights of the edge C_8 -covering of the octagonal grid graph, and the formula for the weights of the edge C_8 -covering is given as follows:

$$wt_{\zeta} \left(c_8 \left(v_j^i u_j^i \right) \left(x_j^i u_{j+1}^i \right) \left(w_j^i v_j^{i+1} \right) \left(x_j^{i+1} w_{j+1}^i \right) \left(u_j^i w_j^i \right) \left(v_j^i x_j^i \right) \right) \quad (20)$$

$$= 7 + (i-1)(r-1) + j, 1 \leq i \leq s; 1 \leq j \leq r-1.$$

Observe that all the weights of subcovers of C_8 are distinct. Hence, $\text{ehs}(\mathcal{O}_r^s, C_8) = \lceil 1 + ((r-1)(s-1) - 1/8) \rceil$, which completes the proof of this theorem. $\square$

5. Conclusion

In this study, we have determined the edge H -irregularity strength for the hexagonal grid graphs H_r^s for $r, s \geq 3$ and the octagonal grid graphs $\mathcal{O}_r^s$ for $r, s \geq 3$. We have tried to find the edge H -irregularity strength of octagonal grid graph $\mathcal{O}_r^s$ for $r \geq 12$ (r is even) and $s \geq 17$ (s is odd), but so far without success. Hence, we conclude the study with the following open problems.

Open Problem 1. Let $\mathcal{O}_r^s$ be an octagonal grid graph admitting C_8 -covering. Then, $\text{ehs}(\mathcal{O}_r^s, C_8) = \lceil 1 + ((r-1)(s-1) - 1/8) \rceil$ for $r > 12$ (r is even) and for $r \geq 17$ (r is odd).

Open Problem 2. Let $\mathcal{O}_r^s$ be an octagonal grid graph admitting C_8 -covering. Then, $\text{ehs}(\mathcal{O}_r^s, C_8) = \lceil 1 + ((r-1)(s-1) - 1/8) \rceil$ for $r \geq 3, s \geq 3$ for any choice of r and s .

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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