Research Article

Linear Multistep Method for Advanced IDEs with Piecewise Constant Arguments

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In this article, based on the linear multistep method, we combined the simplified reproducing kernel method (SRKM) with the optimization method to solve advanced IDEs with piecewise constant arguments. This article also discussed the convergence order and the time complexity of the method. It is proved that the approximate solutions and their derivatives obtained by this algorithm are uniformly convergent. Through two numerical examples, it is proved that the proposed algorithm is obviously better than other methods.

1. Introduction

For almost half a century, impulsive differential equations (IDEs) have been regarded as a summary of many natural phenomena and also the mathematical structure of many practical problems. They play a pivotal role in different fields of biomathematics and applied physics [1–3]. They have been applied to too many practical problems such as biomechanics, population dynamics, and optimal control. Generally speaking, the analytical solutions of IDEs are difficult to obtain, especially for complex cases such as nonlinearity, fractional order, and piecewise constants. In fact, in most practical problems, we can get the approximate solution or numerical solution of IDEs. Therefore, the existence of IDEs and their numerical solutions have attracted more and more attention from scholars [4–7].

Generally speaking, an IDE system with piecewise constant arguments constitutes a very interesting type of problems, which is essentially an important mathematical model. However, the numerical solution problem for IDEs with piecewise constant arguments is rarely noticed. Wiener [8] analyzed many important properties of solutions of IDEs with piecewise constant arguments. Bereketoglu et al. [9] proved the existence of solutions of first-order nonhomogeneous advanced IDEs. Zhang [10–13] studied the oscillation and asymptotic stability of the Runge–Kutta method for IDEs. We know that the Runge–Kutta method and Euler method are mainly used to solve IDEs. Only a few authors adopt the reproducing kernel method to solve IDEs.

In this article, we study the linear multistep method in reproducing kernel space to solve the following advanced IDEs:

\[
\begin{align*}
\dot{u}(t) + au(t) + bu([t]) + cu([t + 1]) &= 0, \quad t \geq 0, t \neq k, k = 1, 2, \ldots, \\
\Delta u(k) &= du(k), \quad k = 1, 2, \ldots, \\
u(0) &= u_0,
\end{align*}
\]
where \([\cdot]\) denotes the greatest integer function, \(a, b, c, d\), and \(u_0\) are real constants, and \(\Delta u(k) = u(k) - u(k^-)\). In addition, we assume that (1) has a unique solution.

Since the reproducing kernel method (RKM) was proposed at the beginning of last century, more and more scholars have used it to solve initial boundary value problems [14–16]. Geng [17, 18] solved the singularly perturbed problem and the nonlocal boundary value problem by RKM. Li [19, 20] applied RKM to solve a variety of fractional models. SRKM can be used to obtain highly smooth analytical solutions easily. In recent years, many scholars have studied SRKM [21–23]. Zhao [24] proposed the convergence analysis and time complexity analysis of the algorithm.

In this section, we need to establish the linear multistep method and SRKM and give the convergence and time complexity analysis of the algorithm.

2. Preliminaries

In order to expand the description of the algorithm, we mainly introduce some definitions of reproducing kernel space and simplification methods in this section. In the full text, \(L^2[a, b] = \left\{ u \int_a^b u^2(t) dt < \infty \right\}\). From [9], the solution \(u(t)\) of (1) is unique.

Definition 1 (see [16]). The simplified reproducing kernel space \(W^2_2\) is defined as follows:

\[
W^2_2[a, b] = \{ u(t)u' \in C[a, b], \quad u'' \in L^2[a, b], \quad \langle u(t), v(t) \rangle = u(a)v(a) + u'(a)v'(a) + \int_a^b u''(t) v(t) dt, \quad u, v \in W^2_2[a, b]. \]

Its reproducing kernel is \(R_t(s)\), and the space \(W^2_2\) can be similarly defined.

Because there are many impulsive points in the solution of (1), this paper presents a piecewise algorithm, that is, we first solve (1) in \([0, 1]\).

If \(t \in [0, 1)\), we would have
\(u([t + 1]) = u(t), u([t]) = u_0\).

On the other hand, \(\Delta u(1) = u(1) - u(1^-) = du(1); \text{ therefore, } u(1) = 1/(1 - d)u(1^-)\).

So, in the interval \([0, 1]\), we can simplify equation (1) into

\[
\begin{cases}
  u'(t) + au(t) = -bu_0 - c/(1 - d)u(1^-), & t \in [0, 1), \\
  u(0) = u_0.
\end{cases}
\]

Let \(-bu_0 - c/(1 - d)u(1^-) = f(u(1^-)); \text{ obviously, } f(\cdot)\) is a continuous function.

In other words, solving (3) is equivalent to finding a function \(u(t)\) that satisfies

\[
\begin{cases}
  u'(t) + au(t) = f(u(1^-)), & t \in [0, 1), \\
  u(0) = u_0,
\end{cases}
\]

where \(a\) and \(u_0\) are real constants and \(u(1^-)\) is an unknown constant.

3. The Linear Multistep Method

In this section, for solving (4), we need to establish the linear multistep method and SRKM and give the convergence analysis and time complexity analysis of the algorithm.

\[
\psi_i(t) = \psi_i(t) = \mathcal{L} R_i(t_i), \quad i = 1, 2, \ldots, \quad \text{where } \psi_i(t) = \mathcal{L} R_i(t_i), \quad \text{and } \mathcal{L}^* \text{ is the adjoint operator of } \mathcal{L}.
\]

\[
\mathcal{L} u = u'(t) + au(t), \quad u \in W^2_2[0, 1].
\]

Wu and Lin [16] proved that \(\mathcal{L}\) is a bounded operator.

So, (4) is equivalent to the following form:

\[
\mathcal{L} u = f(u(1^-)), \quad u(0) = u_0, \quad t \in [0, 1).
\]

Put \(\phi(t) = R_0(t) \in W^2_2\). Take \(\{t_i\}_{i=1}^{\infty} \subset [0, 1]\), which is dense on \([0, 1]\).

Theorem 1. \(\psi_i(t) = \mathcal{L} R_i(t_i), \quad i = 1, 2, \ldots, \quad \text{where } \psi_i(t) = \mathcal{L} R_i(t_i), \quad \text{and } \mathcal{L}^* \text{ is the adjoint operator of } \mathcal{L}.
\]

Proof. \(\psi_i(t) = \langle \mathcal{L}^* R_i, \psi_i \rangle_{W^2_2} = \langle R_i, \mathcal{L} R_i \rangle_{W^2_2} = \mathcal{L} R_i(t_i), \quad i = 1, 2, \ldots\).

From [14], for each fixed \(n\), it follows that the function system \(\{\psi_i(t)\}_{i=1}^{n} \cup \{\phi(t)\}\) is linearly independent on \(W^2_2\). Moreover, \(\{\psi_i(t)\}_{i=1}^{\infty} \cup \{\phi(t)\}\) is completed in \(W^2_2[0, 1]\).

Let

\[
S_n = \text{span}\{\psi_i(t)\}_{i=1}^{n} \cup \{\phi(t)\}.
\]

Then, we can obtain that \(S_n \subset W^2_2[0, 1]\).

Let \(\rho_n : W^2_2[0, 1] \rightarrow S_n\) which is an orthogonal projection operator.

Theorem 2. If \(u \in W^2_2[0, 1]\) satisfies equation 5, then \(\rho_n u\) satisfies
Theorem 3. $u_n$ uniformly converges to $u$, in which $u \in W^1_2[0,1]$ satisfies equation 5.

\[ f(u(1^−)) = f\left(\lim_{n \to \infty} u_n(1^−)\right) = \lim_{n \to \infty} f(u_n(1^−)) = f(u(1^−)) + \varepsilon. \]  \hspace{1cm} (10)

Therefore, $u_n$ is the solution of (11), where $\varepsilon \to 0$ if $n \to \infty$:

\[ \begin{cases} 
\langle u_n, \psi_i \rangle = f(u_n(1^-)) + \varepsilon, & i = 1, 2, \ldots, n, \\
\langle u_n, \phi \rangle = u_0.
\end{cases} \]  \hspace{1cm} (11)

In other words, $u_n$ is the approximate solution of (2) on $[0,1]$.

As $u_n \in S_n$,

\[ \sum_{j=1}^{n} \lambda_j \langle \psi_j, \psi_i \rangle + k \langle \psi_j, \phi \rangle = f(u_n(1^-)) + \varepsilon, \]  \hspace{1cm} (13)

Let

\[ G = \begin{bmatrix} 
\langle \psi_1, \psi_1 \rangle & \langle \psi_1, \psi_2 \rangle & \cdots & \langle \psi_1, \psi_n \rangle \\
\langle \psi_2, \psi_1 \rangle & \langle \psi_2, \psi_2 \rangle & \cdots & \langle \psi_2, \psi_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle \psi_n, \psi_1 \rangle & \langle \psi_n, \psi_2 \rangle & \cdots & \langle \psi_n, \psi_n \rangle \\
\langle \phi, \psi_1 \rangle & \langle \phi, \psi_2 \rangle & \cdots & \langle \phi, \psi_n \rangle \\
\end{bmatrix}_{n \times n}. \]

\[ f(u_n(1^-)) + \varepsilon = \eta, \]

\[ F = (\eta, \eta, \ldots, \eta, u_0)^T. \]  \hspace{1cm} (14)

Considering that $\{\psi_i(t)\}_{i=1}^{n} \cup \{\phi(t)\}$ is linearly independent in $W^1_2[0,1]$, which means $G^{-1}$ exists,

\[ (\lambda_1, \lambda_2, \ldots, \lambda_n, k)^T = G^{-1}F. \]  \hspace{1cm} (15)

So, $\lambda_1, \lambda_2, \ldots, \lambda_n, k$ are expressed by $\eta$. Substituting (15) into (12) yields

\[ u_n(t) = \sum_{i=1}^{n} \lambda_i (\eta) \psi_i(t) + k(\eta) \phi(t). \]  \hspace{1cm} (16)

According to the previous analysis, we have

\[ \eta = f(u(1^-)) = f(u_n(1^-)) + \varepsilon. \]  \hspace{1cm} (17)

Proof. $|u - u_n| = |u - u_n, R_i| \leq \|R_i\|_{W^2_1} \|u - u_n\|_{W^2_1} \leq M \|u - u_n\|_{W^2_1} \to 0.$

Therefore, while $u$ satisfies (5), $u_n \neq \rho_n u$,

\[ \begin{cases} 
\langle u_n, \psi_i \rangle = f(u(1^-)), & i = 1, 2, \ldots, n, \\
\langle u_n, \phi \rangle = u_0.
\end{cases} \]  \hspace{1cm} (9)

Since $f$ is a continuous function and $u_n \to u$ uniformly,

\[ u_n(t) = \sum_{i=1}^{n} \lambda_i \psi_i(t) + k \phi(t). \]  \hspace{1cm} (12)

In order to find approximate solution $u_n$, we must find $\lambda_i, k, i = 1, 2, \ldots, n$. Considering that $\psi_i(t)$ and $\phi(t)$ are known functions, we use $\psi_i(t)$ and $\phi(t)$ to do inner product operation on both sides of (12). The following equations can be obtained:

\[ \sum_{j=1}^{n} \lambda_j \langle \psi_j, \psi_i \rangle + k \langle \psi_j, \phi \rangle = f(u_n(1^-)) + \varepsilon, \]  \hspace{1cm} (13)

\[ \min (\eta - f(u_n(1^-)))^2. \]  \hspace{1cm} (18)

For (18), we use software to solve $\eta$ and substitute $\eta$ into equation (16) to get $u_n$. 

Lemma 1 (see [24]). In $W^1_2$, if $u_n \neq \rho_n u$ is the approximate solution of $\mathcal{L}u = f$ by SRKM, then $\|u(t) - u_n(t)\|_{W^2_1} \leq M h^2$, where $M$ is a constant.

Theorem 4. $u_n$ converges to $u$, and the convergence order is not less than the second order.

Proof. Through the previous analysis and proof, we know that $u_n \neq \rho_n u$ is also the solution of $\mathcal{L}u = f(u(1^-))$ in $W^1_2[0,1]$. According to Lemma 1, $u_n$ converges to $u$, and convergence order is at least second order. Therefore,

\[ \|u(t) - u_n(t)\|_{W^2_1} \leq M_1 \|u(t) - u_n(t)\|_{W^2_1} \leq M_1 (M h^2) = M_2 h^2, \]  \hspace{1cm} (19)

where $h = 1/n, M, M_1, M_2 \in R$. \hfill \Box
Furthermore, the formula for calculating the convergence order of the algorithm is as follows:

\[ \text{C.R.} = \log_2 \left( \frac{|u(t) - u_n(t)|}{|u(t) - u_{2n}(t)|} \right) \]  \hspace{1cm} (20)

**Theorem 5.** \(O(n^3)\) is the time complexity of the algorithm.

**Proof.** According to the previous statement, we can roughly divide it into the following parts to calculate \(u_n(t)\) of equation 12.

1. Solving the coefficient matrix of equation (14): assume that the amount of calculation required to compute each inner product \(\langle \psi_i, \psi_j \rangle\) is \(C\), so the time complexity of calculating all inner products is \(n(n+1)C/2\).

2. Solving \(G^{-1}\) (\(G\) is the coefficient matrix of equation (12)): use the LU decomposition method to solve equation (14); \(LU\) decomposition is a well-known method, the complexity of \(LU\) decomposition is \(O(n^3)\), and the complexity of solving a system of trigonometric equations (such as \(Lu = b, b\) is a vector of \(n \times 1\)) is \(O(n^2)\).

Here we need to solve \(2n\) trigonometric equations, so the total complexity is \(O(n^2)\).

3. The time complexity of calculating \(x_n(t)\) in equation (12) is \(n\).

Therefore, the total time complexity is:

\[ \frac{n(n+1)C}{2} + O(n^3) + n = O(n^3). \]  \hspace{1cm} (21)

Let \(u_{n,0}(t) = u_n(t)\); therefore, \(u_{n,0}\) is the approximate solution of (1) in \([0,1]\).

If \(t \in [1,2]\), then \(u([t+1]) = u(2) = 1/(1-d)u(2^-)\), and we can get the following equation by the same simplification method:

\[ u([t]) = u(1) = \frac{1}{1-d}u_{n,0}(1^-). \]  \hspace{1cm} (22)

We can further solve (23) on the interval \([1,2]\).

\[
\begin{align*}
\psi(t) + au(t) &= -bu(1) - \frac{c}{1-d}u(2^-), & t \in [1,2], \\
\psi(1) &= \frac{1}{1-d}u_{n,0}(1^-),
\end{align*}
\]

where \(u_{n,0}\) is a known function and \(u(2^-)\) is an unknown constant.

(13) are in the same form; therefore, we can use the method proposed in this section to solve (23), that is, the approximate solution \(u_{n,1}\) of (1) in \([1,2]\). Similarly, approximate solutions of (1) in \([k,k+1]\) can be obtained by the linear multistep method, \(k = 1,2,\ldots\).

4. **Numerical Experiments**

**Example 1.** Let us consider the following advanced IDEs [12].

\[
\begin{align*}
\begin{cases}
\psi(t) + u(t) + \psi([t]) - u([t+1]) = 0, & t \geq 0, t \neq k, k = 1,2,\ldots, \\
\Delta u(k) &= \frac{2}{3}u(k), & k = 1,2,\ldots, \\
u(0) &= 1.
\end{cases}
\end{align*}
\]  \hspace{1cm} (24)

The exact solution is

\[
u(t) = \left(2e^{-[t]} - 1\right)\left(\frac{6 - 3e}{3 - 2e}\right)^{[t]} - \left(1 - e^{-[t]}\right)\left(\frac{6 - 3e}{3 - 2e}\right)^{[t+1]}.
\]  \hspace{1cm} (25)

**Example 2.** Consider the following advanced IDEs [13].

\[
\begin{align*}
\begin{cases}
\psi(t) + u(t) + \frac{2}{e-1}u([t]) + \psi([t+1]) + u([t+1]) = 0, & t \geq 0, t \neq k, k = 1,2,\ldots, \\
\Delta u(k) &= \frac{1}{2}u(k), & k = 1,2,\ldots, \\
u(0) &= 1.
\end{cases}
\end{align*}
\]  \hspace{1cm} (26)
The exact solution is

$$u(t) = \left( \frac{e + 1 - e^{-t}}{e - 1} - \frac{2}{e - 1} \right) \left( \frac{2}{2 - 3e} \right)^{[t]} + \left( e^{-[t]} - 1 \right) \left( \frac{2}{2 - 3e} \right)^{[t+1]}.$$  \hspace{1cm} (27)

In Figures 1 and 2, the variation law of error is reflected. As seen in Tables 1 and 2, the more the nodes, the smaller the error of the numerical results. In other words, we can use enough nodes to get a more accurate approximation, which is consistent with the theory presented earlier in this paper. Each curve in the two figures is the result of each step of our algorithm, which shows that the step-by-step solution algorithm in this paper is very suitable for the solution of (1).
Table 2: Comparisons of the results of Example 2.

| n   | \(|u(t) - u_n(t)|\) [13] | Present method \(|u(t) - u_n(t)|\) | C.R. | \(|u(t) - u_n(t)|\) |
|-----|-----------------|-----------------|-----|-----------------|
| 100 | 0.00020         | 1.7828–4        | –   | 2.4088–4        |
| 400 | 4.9657E–4       | 1.1404–5        | 1.9889 | 1.5427E–5 |
| 800 | 2.4833E–4       | 2.8622–6        | 1.9945 | 3.8726E–6 |
| 1600| 1.2418E–4       | 7.1695–7        | 1.9972 | 9.7014E–7 |

5. Conclusion

Based on SRKM and optimization model, this paper first proposes a numerical algorithm for solving IDEs with piecewise constant arguments. Since the SRKM proposed does not consider complex boundary and initial conditions, SRKM is quite simple. Numerical results show the superiority of the method. From the tables and figures of the examples, it can be seen that with the increase of \(n\), the error becomes smaller and smaller. The solution of impulsive differential equations has always been a difficulty in academic circles, which puzzles scholars’ research because of its piecewise smoothness. The idea of step-by-step solution proposed in this paper can solve this problem well, and the algorithm can be applied to other impulsive differential equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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