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Research Article

Exact Formulae for Degree Distance Indices of Sum Graphs

Muhammad Javaid,[1] **Faiz Farid**,[1] **Jia-Bao Liu**,[2] **and Abdulaziz Mohammed Alanazi**[3]

[1]*Department of Mathematics, School of Science, University of Management and Technology, Lahore 54770, Pakistan*
[2]*School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China*
[3]*Department of Mathematics, University of Tabuk, Tabuk, Saudi Arabia*

Correspondence should be addressed to Jia-Bao Liu; liujiabaoad@163.com

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The degree distance index (DDI) is a vertex-degree weighted version of a well-known index that is called by Wiener index (WI). In extremal theory of graphs, improving the bounds with best possible values is a worth investigating problem. In this note, the exact formulae of the DDI for the four different types of the sum graphs in the terms of various indices of their factor graphs are computed. Moreover, a comparison is also presented between the obtained exact and already existing bounded values for the particular sum graphs.

1. Introduction

Molecular structures can be treated as a graph in which atoms are symbolized by nodes and bounds are presented by edges. For these molecular graphs, the graph-related combinatorial methodologies are widely applied to predict the various structural and physicochemical properties of their organic compounds such as molecular weight, density, volume, vaporization refraction, pressure, freezing, and boiling point. Topological indices (TIs) are frequently used in the last two decades for the studies of the molecular graphs. The very first distance-based topological index called by path number or Wiener index (WI) that came into existence when Wiener was working on boiling point of Paraffin [1]. After that, degree-related indices (Zagreb indices) are defined to find total π-electron energy of the molecules [2]. Moreover, TIs play an important role in analysing and modeling of the quantitative structures property and activity relationships (QSPR and QSAR) [3–5]. These all phenomena are studied with the combination (chemoinformatics) of major subjects mathematics, information sciences, and chemistry. There are many TIs; among them, important distance-based TIs are degree distance index (DDI) [6, 7], Gutman index [8], average distance index [9], and Harary index [10, 11]. More importantly, two graphs have equal TIs if they are isomorphic to each other. For further studies of the distance-based TIs, we refer to [12–15] and for degree-related TIs, see [16–18].

In computing graph theory, new graphs are obtained by using the various operations of graphs. For a connected molecular graph H, the graphs $F_1(H)$ (subdivided graph), $F_2(H)$ (triangle parallel graph), $F_3(H)$ (line superposition graph), and $F_4(H)$ (total graph) are constructed with the help of the operations F_1, F_2, F_3, and F_4, respectively, see [19]. Later on, Eliasi and Taeri [20] defined F-sum graphs using product (Cartesian) on the graph $F(H_1)$ and H_2, where $F(H_1)$ is obtained by applying the operation $F \in \{F_1, F_2, F_3, F_4\}$. These F-sum graphs create hexagonal chains, which are isomorphic to various chemical compounds. They also computed the Wiener indices (WIs) of these F-sum graphs $H_1 +_{F_1} H_2$, $H_1 +_{F_2} H_2$, $H_1 +_{F_3} H_2$, and $H_1 +_{F_4} H_2$. First, general Zagreb indices [21] and the general sum-connectivity index [22, 23] are also computed for these F-sum graphs. For Gutman index of these graphs, see [24, 25].

Recently, the upper bounds of the DDIs for all the F-sum graphs are computed [26]. In this note, DDIs are computed for the family of F-sum ($H_1 +_{F_1} H_2$, $H_1 +_{F_2} H_2$, $H_1 +_{F_3} H_2$, and

$H_1+_{F_4}H_2$) graphs in the form of their exact formulae. Moreover, a comparison is also included to show the sharpness of the obtained results with the already existing bounds. The remaining paper is written as follows: Section 2 covers some basic terminologies and related lemmas (statements), Section 3 describes the main results, and Section 4 includes the DDIs for the particular families of the F-sum graphs as the consequence of the main results obtained in Section 3.

2. Preliminaries

Thoroughly, we consider two simple and connected graphs $H_1 = H$ and H_2. The degree of vertex X is the number of edges incident with it and is denoted by $\delta(x)$. The minimum length of the path between both the vertices $x, y \in V(H)$ is called the distance between two vertices and is dented as $d(x, y)$. Two vertices x and y are adjacent ($x \sim y$) if they form an edge xy. Further details can be studied in [27–29]. The more important TIs are defined as follows.

Definition 1 (see [1]). Let H be a connected graph, then its Wiener index (WI) is

$$W(H) = \frac{1}{2}\sum_{y,z\in V(H)} d(y,z). \tag{1}$$

Definition 2 (see [8]). Let H be a connected graph, then the DDI of H is

$$DD(H) = \frac{1}{2}\sum_{y,z\in V(H)} \{d(y,z)(\delta(y)+\delta(z))\}. \tag{2}$$

Yan et al. [19] defined $F_1(H)$, $F_2(H)$, $F_3(H)$ and $F_4(H)$ on the graph H as follows:

(i) $F_1(H)$ is obtained from H if every edge of graph H is replaced by a path of length 2 where old and new vertices of graph H are presented by black vertices and white vertices, respectively.

(ii) $F_2(H)$ is formed by assigning a new vertex to each edge of graph H, and this new vertex is connected with the end vertices of the respective edge.

(iii) $F_3(H)$ is formed from graph $F_1(H)$ if two white vertices of corresponding edges with one common end vertex are further joined together.

(iv) $F_4(H)$ is formed from $F_2(H)$ if two white vertices of corresponding edges with one common end vertex are further joined together.

Figure 1 explains the graphs $F_1(P_5)$, $F_2(P_5)$, $F_3(P_5)$, and $F_4(P_5)$.

Definition 3. Let $H_1+_FH_2$ be a F-sum graph with $V((H_1+_FH_2) = (V(H_1)\cup E(H_1))\times V(H_2)$. For (w,x), $(y,z) \in V(H_1+_FH_2)$, if either $w = y$ and $xz \in E(H_2)$ or $x = z$ and $wy \in E(F(H_1))$, then $wy \sim xz$.

Figure 2 explains $P_3+_{F_1}P_2$, $P_3+_{F_2}P_2$, $P_1+_{F_3}P_2$, and T-sum $P_3+_{F_4}P_2$ graphs. Now, some lemmas of distances between vertices are given as follows.

Lemma 1 (see [20]). *In F-sum graph $H_1+_FH_2$ of two connected and simple graphs H_1 and H_2 where $F \in \{F_1, F_2, F_3, F_4\}$, if either both vertices (w,x) and (y,z) are black or one vertex (w,x) is black and second (y,z) is white, then $d((w,x),(y,z)|H_1+_FH_2) = d(w,y|F(H_1)) + d(x,z|H_2)$.*

Lemma 2 (see [20]). *In F-sum graph $H_1+_FH_2$ of two connected and simple graphs H_1 and H_2 where $F = F_1$ or F_2 and both (w,x) and (y,z) are white vertices, then*

$$d\left((w,x),(y,z)|H_1+_FH_2\right) = \begin{cases} 2 + d\left(x,z|H_2\right)) & \text{If w = y} \\ d\left(w,y|F\left(H_1\right) + d\left(x,z|H_2\right) & \text{If } w \neq y. \end{cases} \tag{3}$$

Lemma 3 (see [20]). *In F-sum graph $H_1+_FH_2$ of two connected and simple graphs H_1 and H_2 graphs where $F = F_3$ or F_4, and both (w,x) and (y,z) are white vertices, then*

$$d\left((w,x),(y,z)|H_1+_FH_2\right) = \begin{cases} 2 + d\left(x,z|H_2\right)) & \text{If w = y} \\ 1 + d\left(w,y|F\left(H_1\right) + d\left(x,z|H_2\right) & \text{If } w \neq y, x \neq z. \end{cases} \tag{4}$$

3. Main Results

This section presents the degree distance indices of F-sum graphs $H_1+_FH_2$ for $F \in \{F_1, F_2, F_3, F_4\}$ with $V(H_1+_FH_2) = \{(x_i, y_j)\colon 1 \le i \le n, 1 \le j \le m\} \cup \{(z_i, y_j)\colon 1 \le i \le k, 1 \le j \le m\}$, where $V(H_1) = \{x_1, x_2, x_3, \ldots, x_n\}$ are black vertices, $Z = \{z_1, z_2, z_3, \ldots, z_k\}$ are white vertices, $V(H_2) = \{y_1, y_2, y_3, \ldots, y_m\}$, $|E(H_1)| = k$, and $|E(H_2)| = l$.

Theorem 1. *If $H_1+_{F_1}H_2$ is the F_1-sum graph of two graphs H_1 and H_2, then*

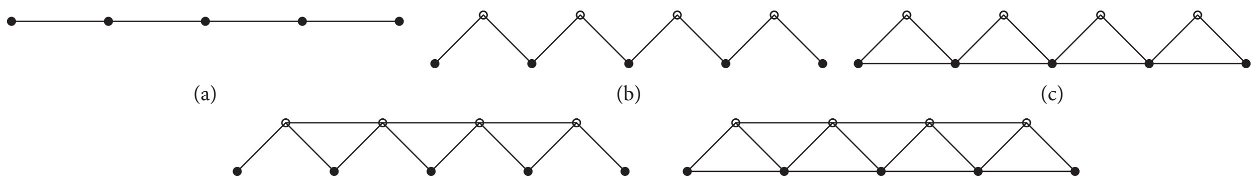

Figure 1: (a) P_5, (b) $F_1(P_5)$, (c) $F_2(P_5)$, (d) $F_3(P_5)$, and (e) $F_4(P_5)$.

$$\begin{aligned} DD\left(H_1 +_{F_1} H_2\right) &= m^2 DD\left(F_1\left(H_1\right)\right) + n(n+k) DD\left(H_2\right) + 8k(n+k) W\left(H_2\right) + 4k\left(m^2 - m\right) \\ &\quad + 2lm \sum_{i,j=1}^{n} \left\{d\left(x_i, y_j\right) | F_1\left(H_1\right)\right\} + 2lm \sum_{r=1}^{k} \sum_{i=1}^{n} \left\{d\left(z_r, x_i\right) | F_1\left(H_1\right)\right\}. \end{aligned} \tag{5}$$

Proof. □

Case 1. Between two old vertices,

$$A = \frac{1}{2} \sum_{\left(x_i, y_p\right), \left(x_j, y_q\right) \in H_1 +_{F_1} H_2} \left\{d\left(\left(x_i, y_p\right), \left(x_j, y_q\right)\right) \left(\delta\left(x_i, y_p\right) + \delta\left(x_j, y_q\right) | H_1 +_{F_1} H_2\right\}\right], \tag{6}$$

where $p, q = 1$ to m and $i, j = 1$ to n, for F_1-sum $\delta\left(x_i, y_p\right) = \delta\left(x_i\right) + \delta\left(y_p\right)$

$$\begin{aligned} &= \frac{1}{2} \sum_{i,j=1}^{n} \sum_{p,q=1}^{m} \left\{d\left(x_i, y_j\right) | F_1\left(H_1\right) + d\left(y_p, y_q | \left(H_2\right)\right\}\right\} \left\{\delta\left(x_i\right) + \delta\left(y_p\right) + \delta\left(x_j\right) + \delta\left(y_q\right)\right\} \\ &= \frac{1}{2} \sum_{i,j=1}^{n} \sum_{p,q=1}^{m} \left\{d\left(x_i, y_j\right) | F_1\left(H_1\right)\right\} \left\{\delta\left(x_i\right) + \delta\left(x_j\right)\right\} + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{p,q=1}^{m} \left\{d\left(y_p, y_q\right) | H_2\right\} \left\{\delta\left(y_p\right) + \delta\left(y_q\right)\right\} \\ &= \frac{1}{2} \sum_{i,j=1}^{n} \sum_{p,q=1}^{m} \left\{d\left(x_i, y_j\right) | F_1\left(H_1\right)\right\} \left\{\delta\left(y_p\right) + \delta\left(y_q\right)\right\} + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{p,q=1}^{m} \left\{d\left(y_p, y_q\right) | H_2\right\} \left\{\delta\left(x_i\right) + \delta\left(x_j\right)\right\} \\ &= \frac{m^2}{2} \sum_{i,j=1}^{n} \left\{d\left(x_i, x_j | F_1\left(H_1\right)\right)\right\} \left\{\delta\left(x_i\right) + \delta\left(x_j\right)\right\} + n^2 DD\left(H_2\right) + \frac{1}{2} \sum_{i,j=1}^{n} \left\{d\left(x_i, x_j | F_1\left(H_1\right)\right)\right\} \sum_{p,q=1}^{m} \left\{\delta\left(y_p\right) + \delta\left(y_q\right)\right\} \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{n} \left\{\delta\left(x_i\right) + \delta\left(x_j\right)\right\} \sum_{p,q=1}^{m} \left\{d\left(y_p, y_q\right) | H_2\right\} \\ A &= \frac{m^2}{2} \sum_{i,j=1}^{n} \left\{d\left(x_i, x_j | F_1\left(H_1\right)\right)\right\} \left\{\delta\left(x_i\right) + \delta\left(x_j\right)\right\} + n^2 DD\left(H_2\right) + 2lm \sum_{i,j=1}^{n} \left\{d\left(x_i, x_j | F_1\left(H_1\right)\right)\right\} + 4kn W\left(H_2\right). \end{aligned} \tag{7}$$

Case 2. Between old and new vertices,

$$B_1 = \frac{1}{2} \sum_{\left(z_r, y_p\right), \left(x_i, y_q\right) \in H_1 +_{F_1} H_2} \left\{d\left(\left(z_r, y_p\right), \left(z_i, y_q\right)\right) \left(\delta\left(z_r, y_p\right) + \delta\left(x_i, y_q\right) | H_1 +_{F_1} H_2\right\}\right, \tag{8}$$

where $p, q = 1$ to m, $i = 1$ to $n\, r = 1$ to k and $\delta\left(z_r, y_p\right) = 2$

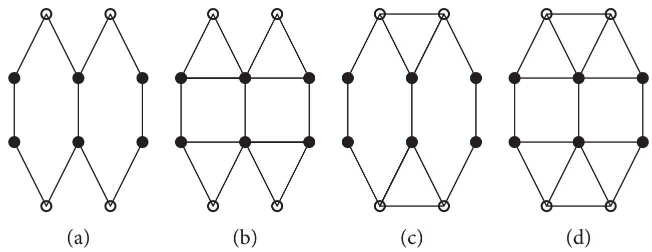

Figure 2: (a) $P_3 +_{F_1} P_2$. (b) $P_3 +_{F_2} P_2$. (c) $P_3 +_{F_3} P_2$. (d) $P_3 +_{F_4} P_2$.

$$
\begin{aligned}
&= \frac{1}{2} \sum_{r=1}^{k} \sum_{i=1}^{n} \sum_{p,q=1}^{m} \{d(z_r, x_i | F_1(H_1)) + d(y_p, y_q | H\}_2 \{2 + \delta(x_i) + \delta(y_q)\} \\
&= \sum_{r=1}^{k} \sum_{i=1}^{n} \sum_{p,q=1}^{m} d(z_r, x_i) | F_1(H_1) + \sum_{r=1}^{k} \sum_{i=1}^{n} \sum_{p,q=1}^{m} \{d(y_p, y_q) | H_2\} + \frac{1}{2} \sum_{r=1}^{k} \sum_{i=1}^{n} \sum_{p,q=1}^{m} d(z_r, x_i | F_1(H_1)) \delta(x_i) \\
&\quad + \frac{1}{2} \sum_{r=1}^{k} \sum_{i=1}^{n} \sum_{p,q=1}^{m} d(z_r, x_i | F_1(H_1)) \delta(y_q) + \frac{1}{2} \sum_{r=1}^{k} \sum_{i=1}^{n} \sum_{p,q=1}^{m} d(y_p, y_q | (H_2)) \delta(x_i) + \frac{1}{2} \sum_{r=1}^{k} \sum_{i=1}^{n} \sum_{p,q=1}^{m} d(y_p, y_q | (H_2)) \delta(y_p), \qquad (9) \\
&= m^2 \sum_{r=1}^{k} \sum_{i=1}^{n} d(z_r, x_i | F_1(H_1)) + 2knW(H_2) + \frac{m^2}{2} \sum_{r=1}^{k} \sum_{i=1}^{n} d(z_r, x_i | F_1(H_1)) \delta(x_i) + lm \sum_{r=1}^{k} \sum_{i=1}^{n} \{d(z_r, x_i | F_1(H_1))\} \\
&\quad + 2k^2 W(H_2) + \frac{kn}{2} DD(H_2).
\end{aligned}
$$

The distance between old and new vertices is twice the B_1, i.e.,

$$
\begin{aligned}
B = 2m^2 \sum_{r=1}^{k} \sum_{i=1}^{n} d(z_r, x_i | F_1(H_1)) + 4knW(H_2) + m^2 \sum_{r=1}^{k} \sum_{i=1}^{n} \{d(z_r, x_i) | F_1(H_1)\} \delta(x_i), \\
+ 2lm \sum_{r=1}^{k} \sum_{i=1}^{n} \{d(z_r, x_i | F_1(H_1)) + 4k^2 W(H_2) + kn\, DD(H_2)
\end{aligned}
\qquad (10)
$$

Case 3. Between two new vertices;

$$
C = \frac{1}{2} \sum_{(z_r, y_p), (z_s, y_q) \in H_1 +_{F_1} H_2} \{d((z_r, y_p), (z_s, y_q)) (\delta(z_r, y_p) + \delta(z_s, y_q) | H_1 +_{F_1} H_2\}. \qquad (11)
$$

Also, $C = C_1 + C_2$ where

$$
\begin{aligned}
C_1 &= \frac{1}{2} \sum_{(z_r, y_p), (z_s, y_q) \in H_1 +_{F_1} H_2} \{d((z_r, y_p), (z_s, y_q)) (\delta(z_r, y_p) + \delta(z_s, y_q) | H_1 +_{F_1} H_2 \colon r = s, p \neq q\}, \\
C_2 &= \frac{1}{2} \sum_{(z_r, y_p), (z_s, y_q) \in H_1 +_{F_1} H_2} \{d((z_r, y_p), (z_s, y_q)) (\delta(z_r, y_p) + \delta(z_s, y_q) | H_1 +_{F_1} H_2 \colon r \neq s\},
\end{aligned}
\qquad (12)
$$

where $\delta(z_r, y_p) = 2$, $r, s = 1$ to k and $p, q = 1$ to m.

$$
\begin{aligned}
C_1 &= \frac{1}{2}\sum_{p,q=1,p\neq q}^{m}\sum_{r,s=1,r=s}^{k}\left(2 + d\left(y_p, y_q\right)(2+2)|H_1 +_{F_1} H_2\right),\\
C_1 &= 4\sum_{p,q=1,p\neq q}^{m}\sum_{r,s=1,r=s}^{k} + 2\sum_{p,q=1,p\neq q}^{m}\sum_{r,s=1,r=s}^{k}\left(d\left(y_p, y_q\right)|H_2\right)\\
&= \left(m^2 - m\right)(4k) + 4kW(H_2),\\
C_2 &= \frac{1}{2}\sum_{r,s=1,r\neq s}^{k}\sum_{p,q=1}^{m}\left\{d(z_r, z_s)|F_1(H_1) + d\left(y_p, y_q\right)\right\}|H_2\}(4),\\
C_2 &= 2\sum_{r,s=1,r\neq s}^{k}\sum_{p,q=1}^{m}\{d(z_r, z_s)|F_1(H_1)\} + 2\sum_{r,s=1,r\neq s}^{k}\sum_{p,q=1}^{m}\left\{d\left(y_p, y_q\right)|H_2\right\},\\
C_2 &= 2m^2\sum_{r,s=1,r\neq s}^{k}\{d(z_r, z_s|F_1(H_1)) + 4\left(k^2 - k\right)W(H_2),\\
C &= \left(m^2 - m\right)(4k) + 4kW(H_2) + 2m^2\sum_{r,s=1,r\neq s}^{k}\{d(z_r, z_s)|F_1(H_1)\} + 4\left(k^2 - k\right)W(H_2),\\
C &= \left(m^2 - m\right)(4k) + 2m^2\sum_{r,s=1,r\neq s}^{k}\{d(z_r, z_s)|F_1(H_1)\} + 4k^2W(H_2)
\end{aligned}
$$

$$
\begin{aligned}
DD\left(H_1 +_{F_1} H_2\right) &= A + B + C,\\
DD\left(H_1 +_{F_1} H_2\right) &= \frac{m^2}{2}\sum_{i,j=1}^{n}\left\{d\left(x_i, x_j\right)|F_1(H_1)\right\}\left\{\delta(x_i) + \delta\left(x_j\right)\right\} + n^2DD(H_2) + 2lm\sum_{i,j=1}^{n}\left\{d\left(x_i, x_j\right)|F_1(H_1)\right\}\\
&\quad + 4knW(H_2) + 2m^2\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r, x_i)|F_1(H_1)\} + 4knW(H_2)\\
&\quad + m^2\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r, x_i)|F_1(H_1)\}\delta(x_i) + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r, x_i)|F_1(H_1)\} + 4k^2W(H_2)\\
&\quad + kn\,DD(H_2) + 4k\left(m^2 - m\right) + 2m^2\sum_{r,s=1,r\neq s}^{k}\{d(z_r, x_i)|F_1(H_1)\} + 4k^2W(H_2),\\
DD\left(H_1 +_{F_1} H_2\right) &= m^2\Big[\frac{1}{2}\sum_{i,j=1}^{n}\left\{d\left(x_i, x_j\right)|F_1(H_1)\right\}\left\{\delta(x_i) + \delta\left(x_j\right)\right\}\\
&\quad + 2\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r, x_i)|F_1(H_1)\} + \sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r, x_i)|F_1(H_1)\}\{\delta(x_i)\} + 2\sum_{r,s=1,r\neq s}^{k}\{d(z_r, z_s)|F_1(H_1)\}\Big]\\
&\quad + n(n+k)DD(H_2) + 8k(n+k)W(H_2) + 4k\left(m^2 - m\right) + 2lm\sum_{i,j=1}^{n}\left\{d\left(x_i, x_j\right)|F_1(H_1)\right\}\\
&\quad + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r, x_i)|F_1(H_1)\}\\
DD\left(H_1 +_{F_1} H_2\right) &= m^2DD(F_1(H_1)) + n(n+k)DD(H_2) + 8k(n+k)W(H_2) + 4k\left(m^2 - m\right)\\
&\quad + 2lm\sum_{i,j=1}^{n}\left\{d\left(x_i, x_j\right)|F_1(H_1)\right\} + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r, x_i)|F_1(H_1)\}.
\end{aligned} \tag{13}
$$

Theorem 2. *If $H_1 +_{F_2} H_2$ is the F_2-sum graph of two graphs H_1 and H_2, then*

$$DD\left(H_1 +_{F_2} H_2\right) = m^2 DD\left(F_2\left(H_1\right)\right) + n(n+k)DD\left(H_2\right) + 12k(n+k)W\left(H_2\right) + 4k\left(m^2 - m\right) + 2lm\sum_{i,j=1}^{n}\left\{d\left(x_i, x_j\right)|F_2\left(H_1\right)\right\} + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\left\{d\left(z_r, x_i\right)|F_2\left(H_1\right)\right\}. \tag{14}$$

Proof. □

Case 4. Between two old vertices,

$$A = \frac{1}{2}\sum_{\left(x_i, y_p\right),\left(x_j, y_q\right)\in H_1 +_{F_2} H_2}\left\{d\left(\left(x_i, y_p\right),\left(x_j, y_q\right)\right)\left(\delta\left(x_i, y_p\right) + \delta\left(x_j, y_q\right)|H_1 +_{F_2} H_2\right\}\right], \tag{15}$$

where $p, q = 1$ to m and $i, j = 1$ to n, for R sum $\delta\left(x_i, y_p\right) = 2\delta\left(x_i\right) + \delta\left(y_p\right)$

$$\begin{aligned}
&= \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{d\left(x_i, x_j|F_2\left(H_1\right)\right) + d\left(y_p, y_q|\left(H_2\right)\right\}\left\{2\delta\left(x_i\right) + \delta\left(y_p\right) + 2\delta\left(x_j\right) + \delta\left(y_q\right)\right\}\right.\\
&= \sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{d\left(x_i, x_j|F_2\left(H_1\right)\right)\right\}\left\{\delta\left(x_i\right) + \delta\left(x_j\right)\right\} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{d\left(y_p, y_q\right)|\left(H_2\right)\right\}\left\{\delta\left(y_p\right) + \delta\left(y_q\right)\right\}\\
&\quad + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{d\left(x_i, x_j|F_2\left(H_1\right)\right)\right\}\left\{\delta\left(y_p\right) + \delta\left(y_q\right)\right\} + \sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{d\left(y_p, y_q\right)|\left(H_2\right)\right\}\left\{\delta\left(x_i\right) + \delta\left(x_j\right)\right\}\\
&= m^2\sum_{i,j=1}^{n}\left\{d\left(x_i, x_j|F_2\left(H_1\right)\right)\right\}\left\{\delta\left(x_i\right) + \delta\left(x_j\right)\right\} + n^2 DD\left(H_2\right)\\
&\quad + \sum_{p,q=1}^{m}\left\{\delta\left(y_p\right) + \delta\left(y_q\right)\right\}\frac{1}{2}\sum_{i,j=1}^{n}\left\{d\left(x_i, x_j|F_2\left(H_1\right)\right)\right\} + \sum_{,q=1}^{m}\left\{d\left(y_p, y_q\right)|\left(H_2\right)\right\}\sum_{i,j=1}^{n}\left\{\delta\left(x_i\right) + \delta\left(x_j\right)\right\}\\
A &= m^2\sum_{i,j=1}^{n}\left\{d\left(x_i, x_j|F_2\left(H_1\right)\right)\right\}\left\{\delta\left(x_i\right) + \delta\left(x_j\right)\right\} + n^2 DD\left(H_2\right) + 2lm\sum_{i,j=1}^{n}\left\{d\left(x_i, x_j|F_2\left(H_1\right)\right)\right\} + 8knW\left(H_2\right).
\end{aligned} \tag{16}$$

Case 5. Between old and new vertices,

$$B_1 = \frac{1}{2}\sum_{\left(z_r, y_p\right),\left(x_i, y_q\right)\in H_1 +_{F_2} H_2}\left\{d\left(\left(z_r, y_p\right),\left(x_i, y_q\right)\right)\left(\delta\left(z_r, y_p\right) + \delta\left(x_i, y_q\right)|H_1 +_{F_2} H_2\right\}\right., \tag{17}$$

where $p, q = 1$ to m, $i = 1$ to $n\,r = 1$ to k and $\delta\left(z_r, y_p\right) = 2$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\left\{d(z_r, x_i)F_2(H_1) + d(y_p, y_q|(H_2)|\right\}\left\{2 + 2\delta(x_i) + \delta(y_q)\right\}\\
&= \sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\{d(z_r, x_i)F_2(H_1)\} + \sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\{d(y_p, y_q|(H_2)\},\\
&\quad + \sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\{d(z_r, x_i)F_2(H_1)\}\{\delta(x_i)\} + \frac{1}{2}\sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\{d(z_r, x_i)F_2(H_1)\}\delta(y_q)\\
&\quad + \sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\left\{d(y_p, y_q|(H_2)\right\}\delta(x_i) + \frac{1}{2}\sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\left\{d(y_p, y_q|(H_2)\right\}\delta(y_q)\\
&= m^2\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r, x_i)F_2(H_1)\} + 2knW(H_2) + m^2\sum_{i=1}^{n}\sum_{r=1}^{k}\{d(z_r, x_i)F_2(H_1)\}\delta(x_i)\\
&\quad + lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r, x_i)F_2(H_1)\} + 4k^2W(H_2) + \frac{kn}{2}DD(H_2).
\end{aligned}
\tag{18}
$$

The distance between old and new vertices is twice the B_1, i.e.,

$$
\begin{aligned}
B &= 2m^2\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r, x_i)F_2(H_1)\} + 4knW(H_2) + 2m^2\sum_{i=1}^{n}\sum_{r=1}^{k}\{d(z_r, x_i)F_2(H_1)\}\{\delta(x_i)\}\\
&\quad + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r, x_i)F_2(H_1)\} + 8k^2W(H_2) + knDD(H_2).
\end{aligned}
\tag{19}
$$

Case 6. Between two new vertices, C can be calculated similarly as in Theorem 1 Case 3:

$$C = 4k\left(m^2 - m\right) + 2m^2 \sum_{r,s=1,r\neq s}^{k} \{d(z_r, z_s)|F_2(H_1)\} + 4k^2 W(H_2),$$

$$DD\left(H_1 +_{F_2} H_2\right) = A + B + C,$$

$$\begin{aligned}
DD\left(H_1 +_{F_2} H_2\right) = {} & m^2 \sum_{i,j=1}^{n} \left\{d\left(x_i, x_j\right)|F_2(H_1)\right\}\left\{\delta(x_i) + \delta\left(x_j\right)\right\} + n^2 DD(H_2) + 2lm \sum_{i,j=1}^{n} \left\{d\left(x_i, x_j\right)|F_2(H_1)\right\} + 8knW(H_2) \\
& + 2m^2 \sum_{r=1}^{k}\sum_{i=1}^{n} \{d(z_r, x_i)|F_2(H_1)\} + 4knW(H_2) \\
& + 2m^2 \sum_{r=1}^{k}\sum_{i=1}^{n} \{d(z_r, x_i)|F_2(H_1)\}\delta(x_i) + 2lm \sum_{r=1}^{k}\sum_{i=1}^{n} \{d(z_r, x_i)|F_2(H_1)\} + 8k^2 W(H_2) + kn\, DD(H_2) \\
& + 4k\left(m^2 - m\right) + 2m^2 \sum_{r,s=1,r\neq s}^{k} \{d(z_r, z_s)|F_2(H_1)\} + 4k^2 W(H_2).
\end{aligned}$$

$$\begin{aligned}
DD\left(H_1 +_{F_2} H_2\right) = {} & m^2 \left[\begin{aligned} & \sum_{i,j=1}^{n} \left\{d\left(x_i, x_j\right)|F_2(H_1)\right\}\left\{\delta(x_i) + \delta\left(x_j\right)\right\} + 2\sum_{r=1}^{k}\sum_{i=1}^{n} \{d(z_r, x_i)|F_2(H_1)\} \\ & + \sum_{r=1}^{k}\sum_{i=1}^{n} \{d(z_r, x_i)|F_2(H_1)\}\{2\delta(x_i)\} + 2 \sum_{r,s=1,r\neq s}^{k} \{d(z_r, z_s)|F_2(H_1)\} \end{aligned}\right] \\
& + n(n+k)DD(H_2) + 12k(n+k)W(H_2) + 4k\left(m^2 - m\right) + 2lm \sum_{i,j=1}^{n} \left\{d\left(x_i, x_j\right)|F_2(H_1)\right\} \\
& + 2lm \sum_{r=1}^{k}\sum_{i=1}^{n} \{d(z_r, x_i)|F_2\}(H_1)
\end{aligned}$$

$$\begin{aligned}
DD\left(H_1 +_{F_2} H_2\right) = {} & m^2 DD(F_2(H_1)) + n(n+k)DD(H_2) + 12k(n+k)W(H_2) + 4k\left(m^2 - m\right) + 2lm \sum_{i,j=1}^{n} \left\{d\left(x_i, x_j\right)|F_2(H_1)\right\} \\
& + 2lm \sum_{r=1}^{k}\sum_{i=1}^{n} \{d(z_r, x_i)|F_2(H_1)\}.
\end{aligned} \tag{20}$$

Theorem 3. *If $H_1 +_{F_3} H_2$ is the F_3-sum graph of two graphs H_1 and H_2, then*

$$\begin{aligned}
DD\left(H_1 +_{F_3} H_2\right) = {} & m^2 DD(F_3(H_1)) + n(n+k)DD(H_2) + 2(6k - t)(n+k)W(H_2) + (k+1)(4k - t)\left(m^2 - m\right) \\
& + 2lm \sum_{i,j=1}^{n} \left\{d\left(x_i, x_j\right)|F_3(H_1)\right\} + 2lm \sum_{r=1}^{k}\sum_{i=1}^{n} \{d(z_r, x_i)|F_3(H_1)\},
\end{aligned} \tag{21}$$

where t is the number of pendant vertices in graph H_1

Proof. □

Case 7. Between two old vertices, A can be calculated similarly as in Theorem 1 Case 1:

$$A = \frac{m^2}{2} \sum_{i,j=1}^{n} \left\{d\left(x_i, x_j\right)|F_3(H_1)\right\}\left\{\delta(x_i) + \delta\left(x_j\right)\right\} + n^2 DD(H_2) + 2lm \sum_{i,j=1}^{n} \left\{d\left(x_i, x_j\right)|F_3(H_1)\right\} + 4knW(H_2). \tag{22}$$

Case 8. Between old and new vertices,

$$B_1 = \frac{1}{2} \sum_{(z_r, y_p), (x_i, y_q) \in H_1 +_{F_3} H_2} \left\{ d\left(\left(z_r, y_p\right), \left(x_i, y_q\right)\right) \left(\delta\left(w_r, y_p\right) + \delta\left(x_i, y_q\right) | H_1 +_{F_3} H_2 \right\}, \tag{23}$$

where $p, q = 1$ to m, $i = 1$ to $n r = 1$ to k and $\delta(x_i, y_q) = \delta(x_i) + \delta(y_j)$

$$\begin{aligned}
&= \frac{1}{2} \sum_{r=1}^{k} \sum_{i=1}^{n} \sum_{p,q=1}^{m} \left\{ d(z_r, x_i) | F_3(H_1) + d(y_p, y_q | (H_2) \right\} \left\{ \delta(z_r, y_p) + d(x_i) + d\left(y_q\right) \right\} \\
&= \frac{1}{2} \sum_{r=1}^{k} \sum_{i=1}^{n} \sum_{p,q=1}^{m} \left\{ d(z_r, x_i) | F_3(H_1) \right\} \delta\left(z_r, y_p\right) + \frac{1}{2} \sum_{r=1}^{k} \sum_{i=1}^{n} \sum_{p,q=1}^{m} \left\{ d\left(y_p, y_q\right) | H_2 \right\} \delta\left(z_r, y_p\right) \\
&\quad + \frac{1}{2} \sum_{r=1}^{k} \sum_{i=1}^{n} \sum_{p,q=1}^{m} \left\{ d(z_r, x_i) | F_3(H_1) \right\} \delta(x_i) + \frac{1}{2} \sum_{r=1}^{k} \sum_{i=1}^{n} \sum_{p,q=1}^{m} \left\{ d(z_r, x_i) | F_3(H_1) \right\} \delta\left(y_q\right) \\
&\quad + \frac{1}{2} \sum_{r=1}^{k} \sum_{i=1}^{n} \sum_{p,q=1}^{m} \left\{ d\left(y_p, y_q\right) | (H_2) \right\} \delta(x_i) + \frac{1}{2} \sum_{r=1}^{k} \sum_{i=1}^{n} \sum_{p,q=1}^{m} \left\{ d\left(y_p, y_q\right) | (H_2) \right\} \delta\left(y_q\right) \\
&= \frac{m^2}{2} \sum_{r=1}^{k} \sum_{i=1}^{n} \left\{ d(z_r, x_i) | F_3(H_1) \right\} \delta\left(z_r, y_p\right) + n(4k - t) W(H_2) + \frac{m^2}{2} \sum_{r=1}^{k} \sum_{i=1}^{n} \left\{ d(z_r, x_i) | F_3(H_1) \right\} \delta(x_i) \\
&\quad + lm \sum_{r=1}^{k} \sum_{i=1}^{n} \left\{ d(z_r, x_i) | F_3(H_1) \right\} + 2k^2 W(H_2) + \frac{kn}{2} DD(H_2).
\end{aligned} \tag{24}$$

The distance between old and new vertices is twice the B_1, i.e.,

$$\begin{aligned}
B &= m^2 \sum_{r=1}^{k} \sum_{i=1}^{n} \left\{ d(z_r, x_i) | F_3(H_1) \right\} \delta\left(z_r, y_p\right) + 2n(4k - t) W(H_2) + m^2 \sum_{i=1}^{n} \sum_{r=1}^{k} \left\{ d(z_r, x_i) | F_3(H_1) \right\} \delta(x_i) \\
&\quad + 2lm \sum_{r=1}^{k} \sum_{i=1}^{n} \left\{ d(z_r, x_i) | F_3(H_1) \right\} + 4k^2 W(H_2) + kn\, DD(H_2).
\end{aligned} \tag{25}$$

Case 9. Between two new vertices:

$$C = \frac{1}{2} \sum_{(z_r, y_p), (z_s, y_q) \in H_1 +_{F_3} H_2} \left\{ d\left(\left(z_r, y_p\right), \left(z_s, y_q\right)\right) \left(\delta\left((z_r, y_p\right) + \delta\left(z_s, y_q\right) | H_1 +_{F_3} H_2 \right\}. \tag{26}$$

Also, $C = C_1 + C_2 + C_3$, where

$$C_1 = \frac{1}{2}\sum_{(z_r,y_p),(z_s,y_q)\in H_1+_{F_3}H_2}\left\{d\left(\left(z_r,y_p\right),\left(z_s,y_q\right)\right)\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)|H_1+_{F_3}H_2\colon\ p\neq q, r=s\right\},$$
$$C_2 = \frac{1}{2}\sum_{(z_r,y_p),(z_s,y_q)\in H_1+_{F_3}H_2}\left\{d\left(\left(z_r,y_p\right),\left(z_s,y_q\right)\right)\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)|H_1+_{F_3}H_2\colon\ p=q, r\neq s\right\}, \quad (27)$$
$$C_3 = \frac{1}{2}\sum_{(z_r,y_p),(z_s,y_q)\in H_1+_{F_3}H_2}\left\{d\left(\left(z_r,y_p\right),\left(z_s,y_q\right)\right)\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)|H_1+_{F_3}H_2\colon\ p\neq q, r\neq s\right\},$$

where $p,q=1$ to m and $r,s=1$ to k.

$$C_1 = \frac{1}{2}\sum_{p,q=1,p\neq q}^{m}\sum_{r,s=1,r=s}^{k}\left\{\left(2+d\left(y_p,y_q\right)|H_2\right)\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)|H_1+_{F_3}H_2\right\},$$

$$\begin{aligned}
C_1 &= \sum_{p,q=1,p\neq q}^{m}\sum_{r,s=1,r=s}^{k}\left(\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)|H_1+_{F_3}H_2\right)+\frac{1}{2}\sum_{p,q=1,p\neq q}^{m}\sum_{r,s=1,r=s}^{k} d\left(y_p,y_q\right)\big)|H_2\}\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)\\
&= 2(4k-t)\left(m^2-m\right)+2(4k-t)W\left(H_2\right),
\end{aligned}$$

$$\begin{aligned}
C_2 &= \frac{1}{2}\sum_{p,q=1,p=q}^{m}\sum_{r,s=1,r\neq s}^{k}\left\{d\left(z_r,z_s\right)|F_3\left(H_1\right)\right\}\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)\\
&= \frac{m}{2}\sum_{r,s=1,r\neq s}^{k}\left\{d\left(z_r,z_s\right)|F_3\left(H_1\right)\right\}\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right),\right.
\end{aligned}$$

$$\begin{aligned}
C_3 &= \frac{1}{2}\sum_{p,q=1,p\neq q}^{m}\sum_{r,s=1,r\neq s}^{k}\left\{1+d\left(z_r,z_s\right)|F_3\left(H_1\right)+d\left(y_p,y_q\right)|H_2\right\}\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)\\
&= \frac{1}{2}\sum_{p,q=1,p\neq q}^{m}\sum_{r,s=1,r\neq s}^{k}\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)+\frac{1}{2}\sum_{r,s=1,r\neq s}^{k}\sum_{p,q=1,p\neq q}^{m}\left\{d\left(z_r,z_s\right)|F_3\left(H_1\right)\right\}\\
&\quad\left(\delta\left(z_r,y_p\right)+\frac{1}{2}\sum_{p,q=1,p\neq q}^{m}\sum_{r,s=1,r\neq s}^{k}\left\{d\left(z_r,z_s\right)|F_3\left(H_1\right)\right\}\delta\left(z_s,y_q\right)+\frac{1}{2}\sum_{p,q=1,p\neq q}^{m}\sum_{r,s=1,r\neq s}^{k}\left\{d\left(y_p,y_q\right)|H_2\right\}\delta\left(z_r,y_p\right)\right)\\
&\quad+\frac{1}{2}\sum_{p,q=1,p\neq q}^{m}\sum_{r,s=1,r\neq s}^{k}\left\{d\left(y_p,y_q\right)|H_2\right\}\delta\left(z_s,y_q\right))\\
&= (k-1)(4k-t)\left(m^2-m\right)+\frac{m(m-1)}{2}\sum_{r,s=1,r\neq s}^{k}\left\{d\left(z_r,z_s\right)|F_3\left(H_1\right)\right\}\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)\\
&\quad+2(k-1)(4k-t)W\left(H_2\right),
\end{aligned}$$

$$\begin{aligned}
C &= 2(4k-t)\left(m^2-m\right)+2(4k-t)W\left(H_2\right)+\frac{m}{2}\sum_{r,s=1,r\neq s}^{k}\left\{d\left(z_r,z_s\right)|F_3\left(H_1\right)\right\}\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)+\left(m^2-m\right)(4k-t)(k-1)\\
&\quad+\frac{m(m-1)}{2}\sum_{r,s=1,r\neq s}^{k}\left\{d\left(z_r,z_s\right)|F_3\left(H_1\right)\right\}\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)+2(4k-t)(k-1)W\left(H_2\right),
\end{aligned}$$

$$
\begin{aligned}
C &= (k+1)(4k-t)\left(m^2-m\right) + 2k(4k-t)W(H_2) + \frac{m^2}{2}\sum_{r,s=1,r\neq s}^{k}\{d(z_r,z_s)|F_3(H_1)\}\left(\delta\left(z_r,v_p\right)+\delta\left(z_s,y_q\right)\right),\\
DD\left(H_1+_{F_3}H_2\right) &= A+B+C,\\
DD\left(H_1+_{F_3}H_2\right) &= \frac{m^2}{2}\sum_{i,j=1}^{n}\left\{d\left(x_i,x_j\right)|F_3(H_1)\right\}\left\{\delta(x_i)+d\left(\delta_j\right)\right\} + n^2DD(H_2) + 4knW(H_2)\\
&\quad + 2lm\sum_{i,j=1}^{n}\left\{d\left(x_i,x_j\right)|F_3(H_1)\right\} + m^2\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r,x_i)|F_3(H_1)\}\delta\left(z_r,y_p\right) + 2n(4k-t)W(H_2)\\
&\quad + m^2\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r,x_i)|F_3(H_1)\}\{\delta(x_i)\}\\
&\quad + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r,x_i)|F_3(H_1)\} + 4k^2W(H_2) + kn\,DD(H_2) + (4k-t)(k+1)\left(m^2-m\right)\\
&\quad + 2k(4k-t)W(H_2) + \frac{m^2}{2}\sum_{r,s=1,r\neq s}^{k}\{d(z_r,z_s)|F_3(H_1)\}\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)\\
DD\left(H_1+_{F_3}H_2\right) &= m^2\begin{bmatrix} \frac{1}{2}\sum_{i,j=1}^{n}\left\{d\left(x_i,x_j\right)|F_3(H_1)\right\}\left\{\delta(x_i)+\delta\left(x_j\right)\right\} + \sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r,x_i)|F_3(H_1)\}\delta\left(z_r,y_p\right)\\ +\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r,x_i)|F_3(H_1)\}\{\delta(x_i)\} + \frac{1}{2}\sum_{r,s=1,r\neq s}^{k}\{\delta(z_r,z_s)|F_3(H_1)\}\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)\end{bmatrix}\\
&\quad + n^2DD(H_2) + kn\,DD(H_2) + W(H_2)\left[4kn + 2n(4k-t) + 2k(4k-t) + 4k^2\right] + (k+1)\left(m^2-m\right)(4k-t)\\
&\quad + 2lm\sum_{i,j=1}^{n}\left\{d\left(x_i,x_j\right)|F_3(H_1)\right\} + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r,x_i)|F_3(H_1)\},\\
DD\left(H_1+_{F_3}H_2\right) &= m^2DD(F_3(H_1)) + n(n+k)DD(H_2) + 2(6k-t)(k+n)W(H_2) + (k+1)\left(m^2-m\right)(4k-t)\\
&\quad + 2lm\sum_{i,j=1}^{n}\left\{d\left(x_i,x_j\right)|F_3(H_1)\right\} + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r,x_i)|F_3(H_1)\}.
\end{aligned}
\tag{28}
$$

Theorem 4. *If $H_1+_{F_4}H_2$ is the F_4-sum graph of two connected and simple graphs H_1 and H_2, then*

$$
\begin{aligned}
DD\left(H_1+_{F_4}H_2\right) &= m^2DD(F_4(H_1)) + n(n+k)DD(H_2) + 2(8k-t)(n+k)W(H_2) + (k+1)(4k-t)\left(m^2-m\right)\\
&\quad + 2lm\sum_{i,j=1}^{n}\left\{d\left(x_i,x_j\right)|F_4(H_1)\right\} + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r,x_i)|F_4(H_1)\},
\end{aligned}
\tag{29}
$$

where t is the number of pendant vertices in graph H_1.

Proof. □

Case 10. Between two old vertices, A can be calculated similarly as in Theorem 2 Case 4:

$$
A = m^2\sum_{i,j=1}^{n}\left\{d\left(x_i,x_j\right)|F_4(H_1)\right\}\left\{d(x_i)+d\left(x_j\right)\right\} + n^2D\,D(H_2) + 2lm\sum_{i,j=1}^{n}\left\{d\left(x_i,x_j\right)|F_4(H_1)\right\} + 8knW(H_2).
\tag{30}
$$

Case 11. Between old and new vertices,

$$B_1 = \frac{1}{2} \sum_{(z_r,y_p),(x_i,y_q)\in H_1+_{F_4}H_2} \left\{d\left(\left(z_r, x_p\right), \left(x_i, y_q\right)\right)\left(\delta\left(z_r, y_p\right) + \delta\left(x_i, y_q\right)|H_1+_{F_4}H_2\right\}, \tag{31}$$

where $p, q = 1$ to m, $i = 1$ to $n r = 1$ to k and $\delta(x_i, y_q) = 2\delta(x_i) + \delta(y_j)$

$$\begin{aligned}
&= \frac{1}{2}\sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\left\{d(z_r, x_i)|F_4(H_1) + d(y_p, y_q|(H_2)\right\}\left\{\delta\left(z_r, y_p\right) + 2\delta(x_i) + \delta\left(y_q\right)\right\}\\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\sum_{r=1}^{k}\{d(z_r, x_i)|F_4(H_1)\}\delta\left(z_r, y_p\right) + \frac{1}{2}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\sum_{r=1}^{k}\{d(y_p, y_q|(H_2)\}\delta\left(z_r, y_p\right)\\
&\quad + \sum_{i=1}^{n}\sum_{p,q=1}^{m}\sum_{r=1}^{k}\{d(z_r, x_i)|F_4(H_1)\}\delta(x_i) + \frac{1}{2}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\sum_{r=1}^{k}\{d(z_r, x_i)|F_4(H_1)\}\delta\left(y_q\right)\\
&\quad + \sum_{i=1}^{n}\sum_{p,q=1}^{m}\sum_{r=1}^{k}\left\{d\left(y_p, y_q\right)|(H_2)\right\}\delta(x_i) + \frac{1}{2}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\sum_{r=1}^{k}\left\{d\left(y_p, y_q\right)|(H_2)\right\}\delta\left(y_q\right)\\
&= \frac{m^2}{2}\sum_{i=1}^{n}\sum_{r=1}^{k}\{d(z_r, x_i)|F_4(H_1)\}\delta\left(z_r, y_p\right) + n(4k - t)W(H_2) + m^2\sum_{i=1}^{n}\sum_{r=1}^{k}\{d(z_r, x_i)|F_4(H_1)\}\delta(x_i)\\
&\quad + lm\sum_{i=1}^{n}\sum_{r=1}^{k}\{d(z_r, x_i)|F_4(H_1)\} + 4k^2W(H_2) + \frac{kn}{2}DD(H_2).
\end{aligned} \tag{32}$$

The distance between old and new vertices is twice the B_1, i.e.,

$$\begin{aligned}
B &= m^2\sum_{i=1}^{n}\sum_{r=1}^{k}\{d(z_r, x_i)|F_4(H_1)\}\delta\left(z_r, y_p\right) + 2n(4k - t)W(H_2) + 2m^2\sum_{i=1}^{n}\sum_{r=1}^{k}\{d(z_r, x_i)|F_4(H_1)\}\delta(x_i)\\
&\quad + 2lm\sum_{i=1}^{n}\sum_{r=1}^{k}\{d(z_r, x_i)|F_4(H_1)\} + 8k^2W(H_2) + kn\ DD(H_2).
\end{aligned} \tag{33}$$

Case 12. Between two new vertices, C can be calculated similarly as in Theorem 3 Case 9:

$$C = (k+1)(4k-t)\left(m^2-m\right) + 2k(4k-t)W(H_2) + \frac{m^2}{2}\sum_{r,s=1,r\neq s}^{k}\{d(z_r,z_s)|F_4(H_1)\}\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)$$

$$DD\left(H_1 +_{F_4} H_2\right) = A + B + C$$

$$\begin{aligned}
DD\left(H_1 +_{F_4} H_2\right) = {} & m^2\sum_{i,j=1}^{n}\left\{d\left(x_i,x_j\right)|F_4(H_1)\right\}\left\{\delta(x_i)+\delta\left(x_j\right)\right\} + n^2 DD(H_2) + 2lm\sum_{i,j=1}^{n}\left\{d\left(x_i,x_j\right)|F_4(H_1)\right\} \\
& + 8knW(H_2) + m^2\sum_{i=1}^{n}\sum_{r=1}^{k}\{d(z_r,x_i)|F_4(H_1)\}\delta\left(z_r,y_p\right) + 2n(4k-t)W(H_2) \\
& + 2m^2\sum_{i=1}^{n}\sum_{r=1}^{k}\{d(z_r,x_i)|F_4(H_1)\}\{\delta(u_i)\} + 2lm\sum_{i=1}^{n}\sum_{r=1}^{k}\{d(z_r,x_i)|F_4(H_1)\} \\
& + 8k^2W(H_2)) + kn\ DD(H_2) + (4k-t)(k+1)\left(m^2-m\right) + 2k(4k-t)W(H_2) \\
& + \frac{m^2}{2}\sum_{r,s=1,r\neq s}^{k}\{d(z_r,z_s)|F_4(H_1)\}\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)
\end{aligned}$$

$$\begin{aligned}
DD\left(H_1 +_{F_4} H_2\right) = {} & m^2\left[\begin{array}{c}\displaystyle\sum_{i,j=1}^{n}\left\{d\left(x_i,x_j\right)|F_4(H_1)\right\}\left(\delta(x_i)+\delta\left(x_j\right)\right) + \sum_{i=1}^{n}\sum_{r=1}^{k}\delta\left(z_r,y_p\right) \\ \displaystyle +2\sum_{i=1}^{n}\sum_{r=1}^{k}\{d(z_r,x_i)|F_4(H_1)\}\{\delta(x_i)\} + \frac{1}{2}\sum_{r,s=1,r\neq s}^{k}\{d(z_r,z_s)|F_4(H_1)\}\left(\delta\left(z_r,y_p\right)+\delta\left(z_s,y_q\right)\right)\end{array}\right] \\
& + n^2DD(H_2) + kn\ DD(H_2) + W(H_2)\left[8kn + 2n(4k-t) + 8k^2 + 2k(4k-t)\right] + (k+1)\left(m^2-m\right)(4k-t) \\
& + 2lm\sum_{i,j=1}^{n}\left\{d\left(x_i,x_j\right)|F_4(H_1)\right\} + 2lm\sum_{i=1}^{n}\sum_{r=1}^{k}\{d(z_r,x_i)|F_4(H_1)\}
\end{aligned}$$

$$\begin{aligned}
DD\left(H_1 +_{F_4} H_2\right) = {} & m^2DD\ \ (F_4(H_1)) + n(n+k)DD(H_2) + 2(8k-t)(k+n)W(H_2) + (k+1)\left(m^2-m\right)(4k-t) \\
& + 2lm\sum_{i,j=1}^{n}\left\{d\left(x_i,x_j\right)|F_4(H_1)\right\} + 2lm\sum_{i=1}^{n}\sum_{r=1}^{k}\{d(z_r,x_i)|F_4(H_1)\}.
\end{aligned} \tag{34}$$

4. Application

The main results of Section 3 are explained in this section. Consider path P_n of order $n \geq 2$, $W(P_n) = n(n^2-1)/6$ and $DD(P_n) = n(n-1)(2n-1)/3$. Now, before to find $DD(P_n +_{F_i} P_m)$, we construct Tables 1 and 2.

Now, using Definition 2, Tables 1 and 2, we obtain

Table 1: WI and DDI of $F_i(P_n)$.

F	$W(F_i(P_n))$	$DD(F_i(P_n))$
F_1	$2n(n-1)(2n-1)/3$	$2(n-1)(2n-1)(4n-3)/3$
F_2	$(n-1)(2n^2+2n-3)/3$	$4(n+1)(n-1)^2$
F_3	$2n(n-1)(n+1)/3$	$4n^3-6n^2+2n-2$
F_4	$n(n-1)(4n+1)/6$	$2/3(8n^3-15n^2+10n-6)$

Table 2: Some particular distance sums of $F_i(P_n)$.

F	$\sum_{i,j=1}^{n}\{d(x_i,x_j)\vert F_i(P_n)\}$	$\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r,x_i)\vert F_i(P_n)\}$
F_1	$2n(n-1)(n+1)/3$	$n(n-1)(2n-1)/3$
F_2	$n(n^2-1)/3$	$n(n^2-1)/3$
F_3	$n(n-1)(n+4)/3$	$n(n-1)(n+1)/3$
F_4	$n(n-1)(n+1)/3$	$n(n-1)(n+1)/3$

$$
\begin{aligned}
&\text{(i) } DD\left(P_n +_{F_1} P_m\right) = \frac{2m^2(n-1)(2n-1)(4n-3)}{3} + \frac{nm(2n-1)(m-1)(2m-1)}{3} + 4m(n-1)(m-1) \\
&\quad + \frac{4m(n-1)(2n-1)\left(m^2-1\right)}{3} + \frac{2mn(m-1)(n-1)(4n+1)}{3}, \\
&\text{(ii) } DD\left(P_n +_{F_2} P_m\right) = 4m^2(n-1)^2(n+1) + \frac{mn(2n-1)(m-1)(2m-1)}{3} + 2m(n-1)(2n-1)\left(m^2-1\right) \\
&\quad + 4m(n-1)(m-1) + \frac{4mn(m-1)\left(n^2-1\right)}{3}, \\
&\text{(iii) } DD\left(P_n +_{F_3} P_m\right) = m^2\left(4n^3-6n^2+2n-2\right) + \frac{nm(2n-1)(m-1)(2m-1)}{3} + 2mn(m-1)(2n-3) \\
&\quad + \frac{2m(2n-1)(3n-4)\left(m^2-1\right)}{3} + \frac{2mn(m-1)(n-1)(2n+5)}{3}, \\
&\text{(iv) } DD\left(P_n +_{F_4} P_m\right) = \frac{2m^2\left(8n^2-15n^2+10n-6\right)}{3} + \frac{nm(2n-1)(m-1)(2m-1)}{3} + 2mn(m-1)(2n-3) \\
&\quad + \frac{2m(2n-1)(4n-5)\left(m^2-1\right)}{3} + \frac{4mn(m-1)(n-1)(n+1)}{3}.
\end{aligned} \tag{35}
$$

Consider cycle C_n of n, then

$$
\begin{aligned}
W(C_n) &= \begin{cases} \dfrac{n^3}{8} & \text{if n is even} \\[2ex] \dfrac{n\left(n^2-1\right)}{8} & \text{if n is odd,} \end{cases} \\
DD(C_n) &= \begin{cases} \dfrac{n^3}{2} & \text{if n is even} \\[2ex] \dfrac{n\left(n^2-1\right)}{2} & \text{if n is odd.} \end{cases}
\end{aligned} \tag{36}
$$

Table 3: WI and DDIs of $F_i(C_n)$.

F	$W(F(C_n))$	$DD(F(C_n))$
F_1	n^3	$4n^3$
F_2	$n(n^2+2n-1)/2$	$n(n+2)(3n-1)$
F_3	$n(n^2+2n-1)/2$	$n(n+2)(3n-1)$
F_4	$n^2(n+1)/2$	$4n^2(n+1)$

Table 4: Some particular distance sums of $F(C_n)$.

F	$\sum_{i,j=1}^{n}\{d(x_i,x_j)\mid F(C_n)\}$	$\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(z_r,x_i)\mid F(C_n)\}$
F_1	$\begin{cases} n^3/2 & \text{if } n \text{ is even} \\ n(n^2-1)/2 & \text{if } n \text{ is odd} \end{cases}$	$\begin{cases} n^3/2 & \text{if } n \text{ is even} \\ n(n^2+1)/2 & \text{if } n \text{ is odd} \end{cases}$
F_2	$\begin{cases} n^3/4 & \text{if } n \text{ is even} \\ n(n^2+1)/4 & \text{if } n \text{ is odd} \end{cases}$	$\begin{cases} n^2(n+2)/4 & \text{if } n \text{ is even} \\ n(n^2+1)/4 & \text{if } n \text{ is odd} \end{cases}$
F_3	$\begin{cases} n(n^2+4n-4)/4 & \text{if } n \text{ is even} \\ n(n^2+4n-5)/4 & \text{if } n \text{ is odd} \end{cases}$	$\begin{cases} n^2(n+2)/4 & \text{if } n \text{ is even} \\ n(n+1)^2/4 & \text{if } n \text{ is odd} \end{cases}$
F_4	$\begin{cases} n^3/4 & \text{if } n \text{ is even} \\ n(n^2-1)^2/4 & \text{if } n \text{ is odd} \end{cases}$	$\begin{cases} n^2(n+2)/4 & \text{if } n \text{ is even} \\ n(n^2+1)^2/4 & \text{if } n \text{ is odd} \end{cases}$

Table 5: Numerical comparison of DD index.

No.	F-sum graph	Exact value	Computed value	Bounded value
1	$DD(P_3+_{F_1}P_2)$	470	470	562
2	$DD(P_3+_{F_2}P_2)$	486	486	570
3	$DD(P_3+_{F_3}P_2)$	486	486	538
4	$DD(P_3+_{F_4}P_2)$	550	550	594
5	$DD(C_3+_{F_1}P_2)$	744	744	852
6	$DD(C_3+_{F_2}P_2)$	828	828	924
7	$DD(C_3+_{F_3}P_2)$	924	924	996
8	$DD(C_3+_{F_4}P_2)$	1068	1068	1140

Now, before to find $DD(C_n+_{F_i}P_m)$, we construct Tables 3 and 4.

Now, using Definition 2, Tables 2 and 4, we obtain

$$\begin{aligned}
&\text{(i) } DD\left(C_n+_{F_1}P_m\right) = 2mn\left[n^2(3m-1)+(m-1)\{n(2m+1)+2\}\right],\\
&\text{(ii) } DD\left(C_n+_{F_2}P_m\right) = mn\left[m(n+2)(3n-1)+(m-1)\left\{\frac{2n}{3}(2m-1)+n(4m+n+5)+4\right\}\right],\\
&\text{(iii) } DD\left(C_n+_{F_3}P_m\right) = mn\left[m(n+2)(3n-1)+(m-1)\left\{\frac{2n}{3}(2m-1)+n(4m+n+11)+2\right\}\right],\\
&\text{(iv) } DD\left(C_n+_{F_4}P_m\right) = mn\left[4mn(n+1)+(m-1)\left\{\frac{2n}{3}(10m+7)+(n+1)(n+4)\right\}\right].
\end{aligned} \tag{37}$$

Now, we construct Table 5 such that values of Columns 3, 4 and 5 are obtained using Theorems 1–4 of Section 3, and aforesaid results of the same section and the bounds obtained in [26], respectively. We note that the computed and exact values are same, and the values of the upper bounds exceed them.

5. Conclusion

In the present study, it is finally concluded that we have improved the already existing bounds of degree distance indices of the F-sum graphs (F_1-sum, F_2-sum, F_3-sum and F_4-sum) ([26]) in the form of their exact values. In the section of Application, the obtained results are also illustrated with the help of the comparison among bounded, computed, and exact values for the particular F-sum graphs. However, the exact values or the improved bounded values of the other topological indices for the F-sum graphs are still to be determined.

Data Availability

All the data are included within this paper. However, the reader may contact the corresponding author for more details of the data.

Conflicts of Interest

The authors have no conflicts of interest.

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