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Research Article

On Efficient Estimation of the Population Mean under Stratified Ranked Set Sampling

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This paper considers some efficient combined and separate classes of estimators of the population mean in the presence of bivariate auxiliary information under stratified ranked set sampling. The mean square error (MSE) expressions of the proffered combined and separate classes of estimators are derived to the first order of approximation. The theoretical conditions are obtained under which the proffered combined and separate classes of estimators perform better than the existing combined and separate class of estimators. Subsequently, numerical and simulation studies are performed using real and artificially generated populations. The numerical and simulation results are found to be rewarding, showing the superiority of the proffered estimators over the existing estimators.

1. Introduction

It is evident from sample surveys that the convenient utilization of auxiliary information may always boost the efficiency of the estimator. This information may be used either at the design phase (sampling design) or at the estimation phase or at both phases. It is a well-known fact that when this information is employed for the estimation phase, the ratio, exponential, product, and regression-type estimators are often used in distinct aspects. Various esteemed authors have considered this auxiliary information at the estimation stage and envisaged a wide range of modified estimators till date. However, the literature encompasses a variety of estimators in the presence of multiple auxiliary information. Koyuncu and Kadilar [1] introduced a family of estimators of the population mean using stratified simple random sampling (SSRS). Tailor et al. [2] suggested the ratio-cum-product estimator in SSRS for estimating the population mean. Tailor and Chouhan [3] envisaged the ratio-cum-product-type exponential estimator of the finite population mean. Lone et al. [4] employed a generalized ratio-cum-product-type exponential estimator in SSRS, whereas Lone et al. [5] mooted enhanced separate classes of estimators of the population mean. Muneer et al. [6] investigated a class of combined estimators in SSRS. Recently, Muneer et al. [7] investigated the apparent family of chain exponential estimators in SSRS.

The concept of ranked set sampling (RSS) was initiated by Mclntyre [8] but did not furnish any mathematical support. Takahasi and Wakimoto [9] improved the lacuna of the RSS method by furnishing the necessary mathematical theory. Dell and Clutter [10] demonstrated that the mean under RSS is an unbiased estimator of the population mean under the condition of perfect and imperfect ranking.

Muttlak [11] suggested the estimation of parameters in simple linear regression under RSS. Samawi and Muttlak [12] shown that ranking of the denominator variable of the ratio estimator improves the efficiency. The interested readers may refer to some recent works such as Bhushan and Kumar [13–16] and Bhushan et al. [17, 18] for a comprehensive study about RSS. In extensive surveys, when the information on more than one auxiliary variable is available, then, Abu-Dayyeh et al. [19] introduced two estimators under RSS for estimating the population mean. Following Olkin [20], Mehta and Mandowara [21] suggested an improved ratio estimator under RSS. Khan and Shabbir [22] introduced a generalized exponential-type ratio-cum-ratio estimators using RSS and SRSS for estimating the population mean. This study is aimed at proffering some efficient classes of combined and separate estimators in the presence of bivariate auxiliary information using SRSS.

Neutrosophic statistics is an extension of classical statistics and is applied when the data is coming from a complex process or from an uncertain environment. Smarandache [23] considered a neutrosophic set as a generalization of the intuitionistic fuzzy set. Smarandache [24] discussed the neutrosophic measure, neutrosophic integral, and neutrosophic probability. Alhabib et al. [25] discussed some neutrosophic probability distributions. Aslam et al. [26] proposed a new diagnosis test under the neutrosophic statistics with an application to diabetic patients, whereas Aslam et al. [27] devised a vague data analysis using neutrosophic the Jarque-Bera test. Aslam [27] suggested the neutrosophic statistical test for counts in climatology. Tahir et al. [28] investigated neutrosophic ratio-type estimators for estimating the population mean. Recently, Vishwakarma and Singh [29] introduced a generalized estimator for computation of the population mean under neutrosophic RSS with an application to solar energy data. In future, we intend to extend the current study using neutrosophic statistics.

1.1. Sampling Methodology. The procedure of ranked set sampling is based on selecting m independent random samples of size m units with equal probability and with replacement from a population of size N units. The units of each random sample are now ranked with respect to the variable of choice. Let the study variable be denoted by Y and the two auxiliary variables by X and Z. Then, m^2 trivariate samples are selected randomly from the population and distributed into m sets, each of size m units. Let the ranking be performed over each unit of the auxiliary variable X. Now, from the first sample, the unit with the smallest rank of X, together with the variables Y and Z associated with the smallest rank of X, is measured. From the second sample of the same size, the variables Y and Z associated with the second smallest rank of X are measured. This procedure is continued until the variables Y and Z associated with the highest rank of X are measured from the m^{th} sample. This process completes a single cycle. The whole procedure is repeated r times to get a desired sample of size $n = mr$ units.

The procedure of stratified ranked set sampling (SRSS) is consisting of quantifying m_h^2 trivariate random samples from the h^{th} stratum of the population. These quantified random samples are fixed up into m_h sets, each of size m_h units. The procedure of RSS is now employed on each set to get m_h sets of ranked set samples, each of size m_h units. These ranked set samples are jointly formed m_h sets, each of size m_h units. Iterate this procedure r times for each stratum to get the desired stratified ranked set sample of size $n_h = m_h r$ units.

Consider a finite population $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_N)$ based on N identifiable units with a study variable Y and two auxiliary variables X and Z associated with each unit κ_i, $i = 1, 2, \cdots, N$ of the population. Let the population be divided into L disjoint strata with stratum h based on N_h, $h = 1, 2, ,\cdots, L$ units. Let $(Y_{h[1]j}, X_{h(1)j}, Z_{h[1]j})$, $(Y_{h[2]j}, X_{h(2)j}, Z_{h[2]j})$,…,$(Y_{h[m_h]j}, X_{h(m_h)j}, Z_{h[m_h]j})$ be the stratified ranked set sample for j^{th}, $j = 1, 2, \cdots, r$ cycle in the h^{th} stratum. Here, $Y_{h[i]j}$ and $Z_{h[i]j}$ are the i^{th} judgement order for the variables Y and Z and $X_{h(i)j}$ is the i^{th} order statistics for variable X in the i^{th} sample at the j^{th} cycle of the h^{th} stratum. Let $\bar{y}_{[\text{SRSS}]} = \sum_{h=1}^{L} W_h \bar{y}_{h[\text{RSS}]}$, $\bar{x}_{(\text{SRSS})} = \sum_{h=1}^{L} W_h \bar{x}_{h(\text{RSS})}$, and $\bar{z}_{[\text{SRSS}]} = \sum_{h=1}^{L} W_h \bar{z}_{h[\text{RSS}]}$ be the stratified ranked set sample means corresponding to the population means $\bar{Y} = \sum_{h=1}^{L} W_h \bar{Y}_h$, $\bar{X} = \sum_{h=1}^{L} W_h \bar{X}_h$, and $\bar{Z} = \sum_{h=1}^{L} W_h \bar{Z}_h$ of variables Y, X, and Z, where $W_h = N_h/N$ is the weight in stratum h. Let $\bar{y}_{h[\text{RSS}]} = \sum_{i=1}^{m_h} \sum_{j=1}^{r} Y_{h[i]j}/m_h r$, $\bar{x}_{h(\text{RSS})} = \sum_{i=1}^{m_h} \sum_{j=1}^{r} X_{h(i)j}/m_h r$, and $\bar{z}_{h[\text{RSS}]} = \sum_{i=1}^{m_h} \sum_{j=1}^{r} Z_{h[i]j}/m_h r$ be the stratified ranked set sample means corresponding to the population means $\bar{Y}_h = \sum_{j=1}^{N_h} Y_{h[i]j}/N_h$, $\bar{X}_h = \sum_{j=1}^{N_h} X_{h(i)j}/N_h$, and $\bar{Z}_h = \sum_{j=1}^{N_h} Z_{h[i]j}/N_h$ of variables Y, X, and Z in stratum h. Let $s_{y_h}^2 = \sum_{h=1}^{L} (Y_{h[i]} - \bar{y}_{h[\text{RSS}]})^2/(n_h - 1)$, $s_{x_h}^2 = \sum_{h=1}^{L} (X_{h(i)} - \bar{x}_{h(\text{RSS})})^2/(n_h - 1)$, $s_{z_h}^2 = \sum_{h=1}^{L} (Z_{h[i]} - \bar{z}_{h[\text{RSS}]})^2/(n_h - 1)$, $s_{xy_h} = \sum_{h=1}^{L} (X_{h(i)} - \bar{x}_{h(\text{RSS})})(Y_{h[i]} - \bar{y}_{h[\text{RSS}]})/(n_h - 1)$, $s_{yz_h} = \sum_{h=1}^{L} (Y_{h[i]} - \bar{y}_{h[\text{RSS}]})(Z_{h[i]} - \bar{z}_{h[\text{RSS}]})/(n_h - 1)$, and $s_{xz_h} = \sum_{h=1}^{L} (X_{h(i)} - \bar{x}_{h(\text{RSS})})(Z_{h[i]} - \bar{z}_{h[\text{RSS}]})/(n_h - 1)$ be the sample variances and covariances corresponding to the population variances and covariances $S_{y_h}^2 = \sum_{h=1}^{L} (Y_{h[i]} - \bar{Y}_h)^2/(N_h - 1)$, $S_{x_h}^2 = \sum_{h=1}^{L} (X_{h(i)} - \bar{X}_h)^2/(N_h - 1)$, $S_{z_h}^2 = \sum_{h=1}^{L} (Z_{h[i]} - \bar{Z}_h)^2/(N_h - 1)$, $S_{xy_h} = \sum_{h=1}^{L} (X_{h(i)} - \bar{X}_h)(Y_{h[i]} - \bar{Y}_h)/(N_h - 1)$, $S_{yz_h} = \sum_{h=1}^{L} (Y_{h(i)} - \bar{Y}_h)(Z_{h[i]} - \bar{Z}_h)/(N_h - 1)$, and $S_{xz_h} = \sum_{h=1}^{L} (X_{h(i)} - \bar{X}_h)(Z_{h[i]} - \bar{Z}_h)/(N_h - 1)$ in the stratum h. Let C_{y_h}, C_{x_h}, and C_{z_h} be the population coefficient of variation of variables Y, X, and Z, respectively.

To derive the MSE of the proffered combined estimators, the following notations will be utilized throughout the paper.

$$\bar{y}_{[\text{SRSS}]} = \bar{Y}(1 + \varepsilon_0), \bar{x}_{(\text{SRSS})} = \bar{X}(1 + \varepsilon_1), \bar{z}_{[\text{SRSS}]} = \bar{Z}(1 + \varepsilon_2), \quad (1)$$

such that $E(\varepsilon_0) = E(\varepsilon_1) = E(\varepsilon_2) = 0$ and

$$V_{rst} = \sum_{h=1}^{L} W_h^{r+s+t} \frac{E\left[\left(\bar{y}_{[\text{SRSS}]} - \bar{Y}\right)^r \left(\bar{x}_{(\text{SRSS})} - \bar{X}\right)^s \left(\bar{z}_{[\text{SRSS}]} - \bar{Z}\right)^t\right]}{\bar{Y}^r \bar{X}^s \bar{Z}^t}. \quad (2)$$

On the basis of (2), it can be written as follows:

$$
\begin{aligned}
E\left(\varepsilon_0{}^2\right) &= \sum_{h=1}^{L} W_h^2\left(\gamma_h C_{y_h}^2 - D_{y_{h[i]}}^2\right) = V_{200},\\
E\left(\varepsilon_1{}^2\right) &= \sum_{h=1}^{L} W_h^2\left(\gamma_h C_{x_h}^2 - D_{x_{h(i)}}^2\right) = V_{020},\\
E\left(\varepsilon_2{}^2\right) &= \sum_{h=1}^{L} W_h^2\left(\gamma_h C_{z_h}^2 - D_{z_{h[i]}}^2\right) = V_{002},\\
E(\varepsilon_0, \varepsilon_1) &= \sum_{h=1}^{L} W_h^2\left(\gamma_h \rho_{xy_h} C_{x_h} C_{y_h} - D_{xy_{h[i]}}\right) = V_{110},\\
E(\varepsilon_1, \varepsilon_2) &= \sum_{h=1}^{L} W_h^2\left(\gamma_h \rho_{xz_h} C_{x_h} C_{z_h} - D_{xz_{h[i]}}\right) = V_{011},\\
E(\varepsilon_0, \varepsilon_2) &= \sum_{h=1}^{L} W_h^2\left(\gamma_h \rho_{yz_h} C_{y_h} C_{z_h} - D_{yz_{h[i]}}\right) = V_{101},
\end{aligned}
\tag{3}
$$

where $\gamma_h = 1/m_h r$, $C_{x_h} = S_{x_h}/\bar{X}$, $C_{z_h} = S_{z_h}/\bar{Z}$, $C_{y_h} = S_{y_h}/\bar{Y}$, $D_{x_{h(i)}}^2 = \sum_{i=1}^{m_h} \tau_{x_{h(i)}}^2 / m_h^2 r \bar{X}^2$, $D_{z_{h[i]}}^2 = \sum_{i=1}^{m_h} \tau_{z_{h[i]}}^2 / m_h^2 r \bar{Z}^2$, $D_{y_{h[i]}}^2 = \sum_{i=1}^{m_h} \tau_{y_{h[i]}}^2 / m_h^2 r \bar{Y}^2$, $D_{yx_{h[i]}} = \sum_{i=1}^{m_h} \tau_{yx_{h[i]}} / m_h^2 r \bar{Y}\bar{X}$, $D_{yz_{h[i]}} = \sum_{i=1}^{m_h} \tau_{yz_{h[i]}} / m_h^2 r \bar{Y}\bar{Z}$, $D_{xz_{h[i]}} = \sum_{i=1}^{m_h} \tau_{xz_{h[i]}} / m_h^2 r \bar{X}\bar{Z}$, $\tau_{x_{h(i)}} = (X_{h(i)} - \bar{X}_h)$, $\tau_{y_{h[i]}} = (Y_{h[i]} - \bar{Y}_h)$, $\tau_{xy_{h[i]}} = (X_{h(i)} - \bar{X}_h)(Y_{h[i]} - \bar{Y}_h)$, $\tau_{yz_{h[i]}} = (Y_{h[i]} - \bar{Y}_h)(Z_{h(i)} - \bar{Z}_h)$, and $\tau_{xz_{h[i]}} = (X_{h(i)} - \bar{X}_h)(Z_{h[i]} - \bar{Z}_h)$.

Again, to determine the properties of the separate estimators, the following notations will be used throughout the paper.

$$
\text{Let } \bar{y}_{h[\text{RSS}]} = \bar{Y}_h(1 + \varepsilon_{0h}), \bar{x}_{h(\text{RSS})} = \bar{X}_h(1 + \varepsilon_{1h}), \bar{z}_{h[\text{RSS}]} = \bar{Z}_h(1 + \varepsilon_{2h}),
\tag{4}
$$

such that $E(\varepsilon_{0h}) = E(\varepsilon_{1h}) = E(\varepsilon_{2h}) = 0$, $E(\varepsilon_{0h}{}^2) = (\gamma_h C_{y_h}^2 - M_{y_{h[i]}}^2) = U_{200}$, $E(\varepsilon_{1h}{}^2) = (\gamma_h C_{x_h}^2 - M_{x_{h(i)}}^2) = U_{020}$, $E(\varepsilon_{2h}{}^2) = (\gamma_h C_{z_h}^2 - M_{z_{h[i]}}^2) = U_{002}$, $E(\varepsilon_{0h}, \varepsilon_{1h}) = (\gamma_h \rho_{xy_h} C_{x_h} C_{y_h} - M_{xy_{h[i]}}) = U_{110}$, $E(\varepsilon_{1h}, \varepsilon_{2h}) = (\gamma_h \rho_{x_h z_h} C_{x_h} C_{z_h} - M_{xz_{h[i]}}) = U_{011}$, and $E(\varepsilon_{0h}, \varepsilon_{2h}) = (\gamma_h \rho_{y_h z_h} C_{y_h} C_{z_h} - M_{yz_{h[i]}}) = U_{101}$, where $C_{x_h} = S_{x_h}/\bar{X}_h$, $C_{z_h} = S_{z_h}/\bar{Z}_h$, $C_{y_h} = S_{y_h}/\bar{Y}_h$, $M_{x_{h(i)}}^2 = \sum_{i=1}^{m_h} \tau_{x_{h(i)}}^2 / m_h^2 r \bar{X}_h^2$, $M_{z_{h[i]}}^2 = \sum_{i=1}^{m_h} \tau_{z_{h[i]}}^2 / m_h^2 r \bar{Z}_h^2$, $M_{y_{h[i]}}^2 =$, $M_{xy_{h[i]}} =$, $M_{yz_{h[i]}} = \sum_{i=1}^{m_h} \tau_{yz_{h[i]}} / m_h^2 r \bar{Y}_h \bar{Z}_h$, and $M_{xz_{h[i]}} = \sum_{i=1}^{m_h} \tau_{xz_{h[i]}} / m_h^2 r \bar{X}_h \bar{Z}_h$.

The paper is divided into some sections. Section 2 is devoted to the existing combined and separate estimators, whereas Section 3 deals with the proffered combined classes of estimators. The suggested combined and separate estimators are given in Section 3 along with their properties. The theoretical comparisons of the proffered and existing combined and separate estimators are given in Section 4. The theoretical results are illustrated through a simulation study in Section 5. Lastly, the study is concluded in Section 6.

2. Review of Literature

This section considers the review of existing combined and separate classes of estimators.

2.1. Combined Estimators. The classical combined ratio estimator of population mean $\bar{Y}$ using bivariate auxiliary information under SRSS is defined as follows:

$$
\bar{y}_r^c = \bar{y}_{[\text{SRSS}]} \left(\frac{\bar{X}}{\bar{x}_{(\text{SRSS})}}\right)\left(\frac{\bar{Z}}{\bar{z}_{[\text{SRSS}]}}\right).
\tag{5}
$$

The classical combined regression estimator of population mean $\bar{Y}$ based on bivariate auxiliary information under SRSS is defined as follows:

$$
\bar{y}_{lr}^c = \bar{y}_{[\text{SRSS}]} + \beta_1\left(\bar{X} - \bar{x}_{(\text{SRSS})}\right) + \beta_2\left(\bar{Z} - \bar{z}_{[\text{SRSS}]}\right),
\tag{6}
$$

where β_1 and β_2 are the regression coefficients of Y on X and Z, respectively.

Motivated by Khoshnevisan et al. [30] and Koyuncu and Kadilar [1], we introduce a general family of combined estimators for population mean $\bar{Y}$ under SRSS as follows:

$$
\begin{aligned}
\bar{y}_{kk}^c = \bar{y}_{[\text{SRSS}]} &\left[\frac{a\bar{X} + b}{\lambda_1\left(a\bar{x}_{(\text{SRSS})} + b\right) + (1 - \lambda_1)\left(a\bar{X} + b\right)}\right]^{g_1}\\
&\cdot \left[\frac{c\bar{Z} + d}{\lambda_2\left(c\bar{z}_{[\text{SRSS}]} + d\right) + (1 - \lambda_2)\left(c\bar{Z} + d\right)}\right]^{g_2},
\end{aligned}
\tag{7}
$$

where λ_1, λ_2, g_1, and g_2 are suitably chosen scalars, whereas $a(\neq 0)$, b and $c(\neq 0)$, d are either real numbers or the function of known parameters of variables X and Z such as standard deviation, coefficients of variation, coefficient of skewness, coefficient of kurtosis, and correlation coefficient.

On the lines of Tailor and Chouhan [3], one may suggest the following combined class of the estimator of population mean under SRSS as follows:

$$
\bar{y}_{tc}^c = k\bar{y}_{[\text{SRSS}]} \exp\left(\frac{\bar{X} - \bar{x}_{(\text{SRSS})}}{\bar{X} + \bar{x}_{(\text{SRSS})}}\right) \exp\left(\frac{\bar{z}_{[\text{SRSS}]} - \bar{Z}}{\bar{z}_{[\text{SRSS}]} + \bar{Z}}\right),
\tag{8}
$$

where k is a duly opted scalar.

Following Lone et al. [4], we define a generalized combined ratio-cum-product-type exponential estimator in SRSS as follows:

$$
\bar{y}_{l_1}^c = \bar{y}_{[\text{SRSS}]} \exp\left[L_1\left(\frac{\bar{X} - \bar{x}_{(\text{SRSS})}}{\bar{X} + \bar{x}_{(\text{SRSS})}}\right) + L_2\left(\frac{\bar{z}_{[\text{SRSS}]} - \bar{Z}}{\bar{z}_{[\text{SRSS}]} + \bar{Z}}\right)\right],
\tag{9}
$$

where L_1 and L_2 are suitably chosen scalars.

Following Lone et al. [5], one may introduce a combined estimator for estimating population mean $\bar{Y}$ under SRSS as follows:

$$\bar{y}_{l_2} = \bar{y}_{[\mathrm{SRSS}]}\left(\frac{a\bar{X}+b}{a\bar{x}_{(\mathrm{SRSS})}+b}\right)\left(\frac{c\bar{z}_{\mathrm{SRSS}}+d}{c\bar{Z}+d}\right). \quad (10)$$

On the lines of Muneer et al. [6], one may investigate a combined class of estimators under SRSS as follows:

$$\bar{y}^{c}_{mu} = \left[k_3\bar{y}_{[\mathrm{SRSS}]} - k_4\left(\bar{x}_{(\mathrm{SRSS})}-\bar{X}\right)\right]\left[\Theta\left\{2-\exp\left(\frac{\bar{z}_{[\mathrm{SRSS}]}-\bar{Z}}{\bar{z}_{[\mathrm{SRSS}]}+\bar{Z}}\right)\right\} + (1-\Theta)\left\{\exp\left(\frac{\bar{Z}-\bar{z}_{[\mathrm{SRSS}]}}{\bar{Z}-\bar{z}_{[\mathrm{SRSS}]}}\right)\right\}\right], \quad (11)$$

where k_3 and k_4 are duly opted scalars.

Khan and Shabbir [22] suggested a generalized combined exponential-type ratio-cum-ratio estimators under SRSS as follows:

$$\bar{y}^{c}_{ks} = \bar{y}_{[\mathrm{SRSS}]}\exp\left(\frac{\bar{X}-\bar{x}_{(\mathrm{SRSS})}}{\bar{X}+(\eta_1-1)\bar{x}_{(\mathrm{SRSS})}}\right)\exp\left(\frac{\bar{Z}-\bar{z}_{[\mathrm{SRSS}]}}{\bar{Z}+(\eta_2-1)\bar{z}_{[\mathrm{SRSS}]}}\right), \quad (12)$$

where η_1 and η_2 are duly opted scalars.

Following Muneer et al. [7], one can introduce a combined chain ratio exponential family of estimators in SRSS as follows:

$$\bar{y}^{c}_{mu_1} = \left[\bar{y}_{[\mathrm{SRSS}]}\left(\frac{\bar{X}^*}{w_1\bar{x}^*_{(\mathrm{SRSS})}+(1-w_1\bar{X}^*)}\right)^{\alpha_1}\left(\frac{\bar{Z}^*}{w_2\bar{z}^*_{[\mathrm{SRSS}]}+(1-w_2\bar{Z}^*)}\right)^{\alpha_2} \times\exp\left\{\frac{\alpha_3\left(\bar{X}-\bar{x}_{(\mathrm{SRSS})}\right)}{\left(\bar{X}+\bar{x}_{(\mathrm{SRSS})}\right)}\right\}\exp\left\{\frac{\alpha_4\left(\bar{Z}-\bar{z}_{[\mathrm{SRSS}]}\right)}{\left(\bar{Z}+\bar{z}_{[\mathrm{SRSS}]}\right)}\right\}\right]. \quad (13)$$

On the lines of Searls [31], the above-combined chain ratio exponential family of estimators becomes

$$\bar{y}^{c}_{mu_2} = \left[k_1\bar{y}_{[\mathrm{SRSS}]}\left(\frac{\bar{X}^*}{w_1\bar{x}^*_{(\mathrm{SRSS})}+(1-w_1\bar{X}^*)}\right)^{\alpha_1}\left(\frac{\bar{Z}^*}{w_2\bar{z}^*_{[\mathrm{SRSS}]}+(1-w_2\bar{Z}^*)}\right)^{\alpha_2} \times\exp\left\{\frac{\alpha_3\left(\bar{X}-\bar{x}_{(\mathrm{SRSS})}\right)}{\left(\bar{X}+\bar{x}_{(\mathrm{SRSS})}\right)}\right\}\exp\left\{\frac{\alpha_4\left(\bar{Z}-\bar{z}_{[\mathrm{SRSS}]}\right)}{\left(\bar{Z}+\bar{z}_{[\mathrm{SRSS}]}\right)}\right\}\right], \quad (14)$$

where $\bar{X}^* = a\bar{X}+b$, $\bar{x}^*_{(\mathrm{SRSS})} = a\bar{x}_{(\mathrm{SRSS})}+b$, $\bar{Z}^* = c\bar{Z}+d$, $\bar{z}^*_{[\mathrm{SRSS}]} = c\bar{z}_{[\mathrm{SRSS}]}+d$, $w_1, w_2 = (0,1)$, and k_1 are duly opted scalars; $\alpha_j, j = 1,2,3,4$ assumes values -1, 0, and $+1$ in order to form different new and existing estimators.

It is noticed that the minimum MSE of the Koyuncu and Kadilar- [1] type estimator $\bar{y}^{c}_{kk}$, [4, 5] estimator $\bar{y}^{c}_{l_1}$ and $\bar{y}^{c}_{l_2}$ and Khan and Shabbir- [22] estimator $\bar{y}^{c}_{ks}$ are equal to the minimum MSE of classical regression estimator $\bar{y}^{c}_{lr}$.

The MSE of the estimators considered in this section is discussed in Appendix A.

2.2. Separate Estimators. The classical separate ratio estimator of population mean $\bar{Y}$ using bivariate auxiliary information under SRSS can be defined as follows:

$$\bar{y}^{s}_{r} = \sum_{h=1}^{L} W_h\bar{y}_{h[\mathrm{RSS}]}\left(\frac{\bar{X}_h}{\bar{x}_{h(\mathrm{RSS})}}\right)\left(\frac{\bar{Z}_h}{\bar{z}_{h[\mathrm{RSS}]}}\right). \quad (15)$$

The classical separate regression estimator of population mean $\bar{Y}$ under bivariate auxiliary information using SRSS is as follows:

$$\bar{y}^{s}_{lr} = \sum_{h=1}^{L} W_h\left[\bar{y}_{h[\mathrm{RSS}]} + \beta_{1_h}\left(\bar{X}_h-\bar{x}_{h(\mathrm{RSS})}\right) + \beta_{2_h}\left(\bar{Z}_h-\bar{z}_{h[\mathrm{RSS}]}\right)\right], \quad (16)$$

where β_{1_h} and β_{2_h} are the regression coefficients of Y on X and Z in stratum h, respectively.

Following Koyuncu and Kadilar [1], one may consider a general family of separate estimators for population mean $\bar{Y}$ under SRSS as follows:

$$\bar{y}^{s}_{kk} = \sum_{h=1}^{L} W_h\bar{y}_{h[\mathrm{RSS}]}\left[\frac{a_h\bar{X}_h+b_h}{\lambda_{1_h}\left(a_h\bar{x}_{h(\mathrm{RSS})}+b_h\right)+\left(1-\lambda_{1_h}\right)\left(a_h\bar{X}_h+b_h\right)}\right]^{g_1} \cdot\left[\frac{c_h\bar{Z}_h+d_h}{\lambda_{2_h}\left(c_h\bar{z}_{h[\mathrm{RSS}]}+d_h\right)+\left(1-\lambda_{2_h}\right)\left(c_h\bar{Z}_h+d_h\right)}\right]^{g_2}, \quad (17)$$

where λ_{1_h}, λ_{2_h}, g_1, and g_2 are suitably chosen constants whereas $a_h(\neq 0)$, b_h and $c_h(\neq 0)$, d_h are either real numbers or the function of known parameters of variables X and Z, respectively, in stratum h.

Following Tailor and Chouhan [3], one may suggest the following separate class of the estimator of the population mean using bivariate auxiliary information under SRSS as follows:

$$\bar{y}^{s}_{tc} = \sum_{h=1}^{L} W_h k_h\bar{y}_{h[\mathrm{RSS}]}\exp\left(\frac{\bar{X}_h-\bar{x}_{h(\mathrm{RSS})}}{\bar{X}_h+\bar{x}_{h(\mathrm{RSS})}}\right)\exp\left(\frac{\bar{z}_{h[\mathrm{RSS}]}-\bar{Z}_h}{\bar{z}_{h[\mathrm{RSS}]}+\bar{Z}_h}\right), \quad (18)$$

where k_h is a duly opted scalar.

On the lines of Lone et al. [4], we consider a generalized separate ratio-cum-product-type exponential estimator in SRSS as follows:

$$\bar{y}^s_{l_1} = \sum_{h=1}^{L} W_h \bar{y}_{h[\mathrm{RSS}]} \exp\left[L_{1_h}\left(\frac{\bar{X}_h - \bar{x}_{h(\mathrm{RSS})}}{\bar{X}_h + \bar{x}_{h(\mathrm{RSS})}}\right) + L_{2_h}\left(\frac{\bar{z}_{h[\mathrm{RSS}]} - \bar{Z}_h}{\bar{z}_{h[\mathrm{RSS}]} + \bar{Z}_h}\right)\right], \tag{19}$$

where L_{1_h} and L_{2_h} are suitably chosen scalars.

The separate version of the Lone et al. [5] estimator for estimating population mean $\bar{Y}$ under SRSS is defined as follows:

$$\bar{y}^s_{l_2} = \sum_{h=1}^{L} W_h \bar{y}_{h[\mathrm{RSS}]}\left(\frac{a_h \bar{X}_h + b_h}{a_h \bar{x}_{h(\mathrm{RSS})} + b_h}\right)\left(\frac{c_h \bar{z}_{h[\mathrm{RSS}]} + d_h}{c_h \bar{Z}_h + d_h}\right). \tag{20}$$

Following Muneer et al. [6], we introduce a separate class of estimators in SRSS as follows:

$$\begin{aligned}\bar{y}^s_{mu} = \sum_{h=1}^{L} W_h \left[k_{3_h}\bar{y}_{h[\mathrm{RSS}]} - k_{4_h}\left(\bar{x}_{h(\mathrm{RSS})} - \bar{X}_h\right)\right]\left[\Theta_h\left\{2 - \exp\left(\frac{\bar{z}_{h[\mathrm{RSS}]} - \bar{Z}_h}{\bar{z}_{h[\mathrm{RSS}]} + \bar{Z}_h}\right)\right\}\right.\\ \left. + (1-\Theta_h)\left\{\exp\left(\frac{\bar{Z}_h - \bar{z}_{h[\mathrm{RSS}]}}{\bar{Z}_h - \bar{z}_{h[\mathrm{RSS}]}}\right)\right\}\right],\end{aligned} \tag{21}$$

where k_{3_h} and k_{4_h} are duly opted scalars.

Khan and Shabbir [22] suggested a generalized separate exponential-type ratio-cum-ratio estimator under SRSS as follows:

$$\begin{aligned}\bar{y}^s_{ks} = \sum_{h=1}^{L} W_h \bar{y}_{h[\mathrm{RSS}]} \exp\left(\frac{\bar{X}_h - \bar{x}_{h(\mathrm{RSS})}}{\bar{X}_h + \left(\eta_{1_h} - 1\right)\bar{x}_{h(\mathrm{RSS})}}\right) \exp\\ \cdot\left(\frac{\bar{Z}_h - \bar{z}_{h[\mathrm{RSS}]}}{\bar{Z}_h + \left(\eta_{2_h} - 1\right)\bar{z}_{h[\mathrm{RSS}]}}\right),\end{aligned} \tag{22}$$

where η_{1_h} and η_{2_h} are duly opted scalars.

Motivated by Muneer et al. [7], a separate chain ratio exponential family of estimators is defined in SRSS as follows:

$$\begin{aligned}\bar{y}^s_{mu_1} = \sum_{h=1}^{L} W_h \left[\bar{y}_{h[\mathrm{RSS}]}\left(\frac{\bar{X}^*_h}{w_{1_h}\bar{x}^*_{h(\mathrm{RSS})} + \left(1 - w_{1_h}\bar{X}^*_h\right)}\right)^{\alpha_{1_h}}\right.\\ \cdot\left(\frac{\bar{Z}^*_h}{w_{2_h}\bar{z}^*_{h[\mathrm{RSS}]} + \left(1 - w_{2_h}\bar{Z}^*_h\right)}\right)^{\alpha_{2_h}}\\ \left.\times \exp\left\{\frac{\alpha_{3_h}\left(\bar{X}_h - \bar{x}_{h(\mathrm{RSS})}\right)}{\left(\bar{X}_h + \bar{x}_{h(\mathrm{RSS})}\right)}\right\} \exp\left\{\frac{\alpha_{4_h}\left(\bar{Z}_h - \bar{z}_{h[\mathrm{RSS}]}\right)}{\left(\bar{Z}_h + \bar{z}_{h[\mathrm{RSS}]}\right)}\right\}\right].\end{aligned} \tag{23}$$

On the lines of Searls [31], the above separate chain ratio exponential family of estimators becomes

$$\begin{aligned}\bar{y}^s_{mu_2} = \sum_{h=1}^{L} W_h \left[k_{1_h}\bar{y}_{h[\mathrm{RSS}]}\left(\frac{\bar{X}^*_h}{w_{1_h}\bar{x}^*_{h(\mathrm{RSS})} + \left(1 - w_{1_h}\bar{X}^*_h\right)}\right)^{\alpha_{1_h}}\right.\\ \cdot\left(\frac{\bar{Z}^*_h}{w_{1_h}\bar{z}^*_{h[\mathrm{RSS}]} + \left(1 - w_{1_h}\bar{Z}^*_h\right)}\right)^{\alpha_{2_h}}\\ \left.\times \exp\left\{\frac{\alpha_{3_h}\left(\bar{X}_h - \bar{x}_{h(\mathrm{RSS})}\right)}{\left(\bar{X}_h + \bar{x}_{h(\mathrm{RSS})}\right)}\right\} \exp\left\{\frac{\alpha_{4_h}\left(\bar{Z}_h - \bar{z}_{h[\mathrm{RSS}]}\right)}{\left(\bar{Z}_h + \bar{z}_{h[\mathrm{RSS}]}\right)}\right\}\right],\end{aligned} \tag{24}$$

where $\bar{X}^* = a_h\bar{X}_h + b_h$, $\bar{x}^*_{h(\mathrm{RSS})} = a_h\bar{x}_{h(\mathrm{RSS})} + b_h$, $\bar{Z}^* = c_h\bar{Z}_{h[\mathrm{RSS}]} + d_h$, $\bar{z}^*_{h[\mathrm{RSS}]} = c_h\bar{z}_{h[\mathrm{RSS}]} + d_h$, and $w_{1_h}, w_{2_h} = (0,1)$ and k_{1_h} is a duly opted scalar; $\alpha_{j_h}, j = 1,2,3,4$ assumes values −1, 0, and +1 in order to form different new and existing estimators.

It is to be noted that the minimum MSE of Koyuncu and Kadilar- [1] type estimator $\bar{y}^s_{kk}$, Lone et al.- [4, 5] type estimator $\bar{y}^s_{l_1}$, $\bar{y}^s_{l_2}$, and Khan and Shabbir- [22] estimator $\bar{y}^s_{ks}$ are similar to the minimum MSE of classical regression estimator $\bar{y}^s_{lr}$.

The MSE of the estimators discussed in this section is given in Appendix B for ready reference.

3. Proffered Estimators

Motivated by the works of Bhushan et al. [32, 33], we have extended the work of Bhushan et al. [17] using bivariate auxiliary information under SRSS.

3.1. Combined Estimators. We propose some improved combined classes of estimators based on bivariate auxiliary information under SRSS as follows:

$$\begin{aligned}
\bar{y}^c_{s_1} &= \xi_1\bar{y}_{[\mathrm{SRSS}]}\left[1 + \log\left(\frac{\bar{x}_{(\mathrm{SRSS})}}{\bar{X}}\right)\right]^{\theta_1}\left[1 + \log\left(\frac{\bar{z}_{[\mathrm{SRSS}]}}{\bar{Z}}\right)\right]^{\delta_1},\\
\bar{y}^c_{s_2} &= \xi_2\bar{y}_{[\mathrm{SRSS}]}\left[1 + \theta_2\log\left(\frac{\bar{x}_{(\mathrm{SRSS})}}{\bar{X}}\right)\right]\left[1 + \delta_2\log\left(\frac{\bar{z}_{[\mathrm{SRSS}]}}{\bar{Z}}\right)\right],\\
\bar{y}^c_{s_3} &= \xi_3\bar{y}_{[\mathrm{SRSS}]} + \theta_3\left(\bar{x}_{(\mathrm{SRSS})} - \bar{X}\right) + \delta_3\left(\bar{z}_{[\mathrm{SRSS}]} - \bar{Z}\right),\\
\bar{y}^c_{s_4} &= \xi_4\bar{y}_{[\mathrm{SRSS}]}\left(\frac{\bar{X}}{\bar{x}_{(\mathrm{SRSS})}}\right)^{\theta_4}\left(\frac{\bar{Z}}{\bar{z}_{[\mathrm{SRSS}]}}\right)^{\delta_4},\\
\bar{y}^c_{s_5} &= \xi_5\bar{y}_{[\mathrm{SRSS}]}\left[\frac{\bar{X}}{\bar{X} + \theta_5\left(\bar{x}_{(\mathrm{SRSS})} - \bar{X}\right)}\right]\left[\frac{\bar{Z}}{\bar{Z} + \delta_5\left(\bar{z}_{[\mathrm{SRSS}]} - \bar{Z}\right)}\right],
\end{aligned} \tag{25}$$

where ξ_i, θ_i, δ_i, $i = 1, 2, \cdots, 5$ are suitably chosen scalars.

Theorem 1. *The MSE of the proffered combined class estimators to the first order of approximation is given by*

$$\begin{aligned}
\mathrm{MSE}\left(\bar{y}^{c}_{s_1}\right) &= \bar{Y}^2\Bigg[1+\xi_1^2\left\{1+V_{200}+2\theta_1^2V_{020}+2\delta_1^2V_{002}-2\theta_1V_{020}-2\delta_1V_{002}+4\theta_1V_{110}+4\delta_1V_{101}+4\theta_1\delta_1V_{011}\right\}\\
&\quad-2\xi_1\left\{1+\frac{\theta_1^2}{2}V_{020}+\frac{\delta_1^2}{2}V_{002}-\theta_1V_{020}-\delta_1V_{002}+\theta_1V_{110}+\delta_1V_{101}+\theta_1\delta_1V_{011}\right\}\Bigg],\\
\mathrm{MSE}\left(\bar{y}^{c}_{s_2}\right) &= \bar{Y}^2\Bigg[1+\xi_2^2\left\{1+V_{200}+\theta_2^2V_{020}+\delta_2^2V_{002}-\theta_2V_{020}-\delta_2V_{002}+4\theta_2V_{110}+4\delta_2V_{101}+4\theta_2\delta_2V_{011}\right\}\\
&\quad-2\xi_2\left\{1-\frac{\theta_2}{2}V_{020}-\frac{\delta_2}{2}V_{002}+\theta_2V_{110}+\delta_2V_{101}+\theta_2\delta_2V_{011}\right\}\Bigg],\\
\mathrm{MSE}\left(\bar{y}^{c}_{s_3}\right) &= \bar{Y}^2\left[(\xi_3-1)^2\bar{Y}^2+\xi_3^2\bar{Y}^2V_{200}+\theta_3^2\bar{X}^2V_{020}+\delta_3^2\bar{Z}^2V_{002}+2\xi_3\theta_3\bar{X}\bar{Y}V_{110}+2\xi_3\delta_3\bar{Z}\bar{Y}V_{101}+2\theta_3\delta_3\bar{X}\bar{Z}V_{011}\right],\\
\mathrm{MSE}\left(\bar{y}^{c}_{s_4}\right) &= \bar{Y}^2\Bigg[1+\xi_4^2\left\{1+V_{200}+\theta_4V_{020}+\delta_4V_{002}+2\theta_4^2V_{020}+2\delta_4^2V_{002}-4\theta_4V_{110}-4\delta_4V_{101}+4\theta_4\delta_4V_{011}\right\}\\
&\quad-2\xi_4\left\{1+\frac{\theta_4(\theta_4+1)}{2}V_{020}+\frac{\delta_4(\delta_4+1)}{2}V_{002}-\theta_4V_{110}-\delta_4V_{101}+\theta_4\delta_4V_{011}\right\}\Bigg],\\
\mathrm{MSE}\left(\bar{y}^{c}_{s_5}\right) &= \bar{Y}^2\Bigg[1+\xi_5^2\left\{1+V_{200}+3\theta_5^2V_{020}+3\delta_5^2V_{002}-4\theta_5V_{110}-4\delta_5V_{101}+4\theta_5\delta_5V_{011}\right\}\\
&\quad-2\xi_5\left\{1+\theta_5^2V_{020}-\theta_5V_{110}+\delta_5^2V_{002}-\delta_5V_{101}+\theta_5\delta_5V_{011}\right\}\Bigg].
\end{aligned}\tag{26}$$

Proof. The outline of the derivation is given in Appendix C for ready reference. □

Corollary 2. *The minimum MSE at the optimum values of ξ_i, θ_i, and δ_i, $i = 1, 2, \cdots, 5$, is given by*

$$\begin{aligned}
minMSE\left(\bar{y}^{c}_{s_i}\right) &= \bar{Y}^2\left[1-\frac{B_i^2}{A_i}\right],\quad i=1,2,4,5,\\
minMSE\left(\bar{y}^{c}_{s_3}\right) &= \bar{Y}^2\left[1-\xi_{3(opt)}\right]=\bar{Y}^2\left[1-\frac{B_3^2}{A_3}\right].
\end{aligned}\tag{27}$$

Proof. The outline of the derivation is given in Appendix C for ready reference. □

3.2. Separate Estimators. We propose some improved separate classes of estimators based on bivariate auxiliary information under SRSS as follows:

$$\begin{aligned}
\bar{y}^{s}_{s_1} &= \sum_{h=1}^{L}W_h\xi_{1_h}\bar{y}_{h[\mathrm{RSS}]}\left[1+\log\left(\frac{\bar{x}_{h(\mathrm{RSS})}}{\bar{X}_h}\right)\right]^{\theta_{1_h}}\left[1+\log\left(\frac{\bar{z}_{h[\mathrm{RSS}]}}{\bar{Z}_h}\right)\right]^{\delta_{1_h}},\\
\bar{y}^{s}_{s_2} &= \sum_{h=1}^{L}W_h\xi_{2_h}\bar{y}_{h[\mathrm{RSS}]}\left[1+\theta_{2_h}\log\left(\frac{\bar{x}_{h(\mathrm{RSS})}}{\bar{X}_h}\right)\right]\left[1+\delta_{2_h}\log\left(\frac{\bar{z}_{h[\mathrm{RSS}]}}{\bar{Z}_h}\right)\right],\\
\bar{y}^{s}_{s_3} &= \sum_{h=1}^{L}W_h\left[\xi_{3_h}\bar{y}_{h[\mathrm{RSS}]}+\theta_{3_h}\left(\bar{x}_{h(\mathrm{RSS})}-\bar{X}_h\right)+\delta_{3_h}\left(\bar{z}_{h[\mathrm{RSS}]}-\bar{Z}_h\right)\right],\\
\bar{y}^{s}_{s_4} &= \sum_{h=1}^{L}W_h\xi_{4_h}\bar{y}_{h[\mathrm{RSS}]}\left(\frac{\bar{X}_h}{\bar{x}_{h(\mathrm{RSS})}}\right)^{\theta_{4_h}}\left(\frac{\bar{Z}_h}{\bar{z}_{h[\mathrm{RSS}]}}\right)^{\delta_{4_h}},\\
\bar{y}^{s}_{s_5} &= \sum_{h=1}^{L}W_h\xi_{5_h}\bar{y}_{h[\mathrm{RSS}]}\left[\frac{\bar{X}_h}{\bar{X}_h+\theta_{5_h}\left(\bar{x}_{h(\mathrm{RSS})}-\bar{X}_h\right)}\right]\cdot\left[\frac{\bar{Z}_h}{\bar{Z}_h+\delta_{5_h}\left(\bar{z}_{h[\mathrm{RSS}]}-\bar{Z}_h\right)}\right],
\end{aligned}\tag{28}$$

where ξ_{i_h}, θ_{i_h}, and δ_{i_h}, $i = 1, 2, \cdots, 5$, are suitably chosen scalars.

Theorem 3. *The MSE of the proffered separate classes of estimators to the first order of approximation is given by*

$$
\begin{aligned}
\mathrm{MSE}\left(\bar{y}^{s}_{s_1}\right) &= \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Bigg[1 + \xi_{1_h}^2 \left\{1 + U_{200} + 2\theta_{1_h}^2 U_{020} + 2\delta_{1_h}^2 U_{002} - 2\delta_{1_h} U_{020} - 2\delta_{1_h} U_{002} + 4\theta_{1_h} U_{110} + 4\delta_{1_h} U_{101} + 4\theta_{1_h}\delta_{1_h} U_{011}\right\} \\
&\quad - 2\xi_{1_h}\left\{1 + \frac{\theta_{1_h}^2}{2} U_{020} + \frac{\delta_{1_h}^2}{2} U_{002} - \theta_1 U_{020} - \delta_{1_h} U_{002} + \theta_{1_h} U_{110} + \delta_{1_h} U_{101} + \theta_{1_h}\delta_{1_h} U_{011}\right\}\Bigg], \\
\mathrm{MSE}\left(\bar{y}^{s}_{s_2}\right) &= \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Bigg[1 + \xi_{2_h}^2 \left\{1 + U_{200} + \theta_{2_h}^2 U_{020} + \delta_{2_h}^2 U_{002} - \theta_{2_h} U_{020} - \delta_{2_h} U_{002} + 4\theta_{2_h} U_{110} + 4\delta_{2_h} U_{101} + 4\theta_{2_h}\delta_{2_h} U_{011}\right\} \\
&\quad - 2\xi_{2_h}\left\{1 - \frac{\theta_{2_h}}{2} U_{020} - \frac{\delta_{2_h}}{2} U_{002} + \theta_{2_h} U_{110} + \delta_{2_h} U_{101} + \theta_{2_h}\delta_{2_h} U_{011}\right\}\Bigg], \\
\mathrm{MSE}\left(\bar{y}^{s}_{s_3}\right) &= \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[\left(\xi_{3_h} - 1\right)^2 \bar{Y}_h^2 + \xi_{3_h}^2 \bar{Y}_h^2 U_{200} + \theta_{3_h}^2 \bar{X}_h^2 U_{020} + \delta_{3_h}^2 \bar{Z}_h^2 U_{002} + 2\xi_{3_h}\theta_{3_h} \bar{X}_h \bar{Y}_h U_{110} + 2\xi_{3_h}\delta_{3_h} \bar{Z}_h \bar{Y}_h U_{101} + 2\theta_{3_h}\delta_{3_h} \bar{X}_h \bar{Z}_h U_{011}\right], \\
\mathrm{MSE}\left(\bar{y}^{s}_{s_4}\right) &= \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Bigg[1 + \xi_{4_h}^2 \left\{1 + U_{200} + \theta_{4_h} U_{020} + \delta_{4_h} U_{002} + 2\theta_{4_h}^2 U_{020} + 2\delta_{4}^2 U_{002} - 4\theta_{4_h} U_{110} - 4\delta_{4_h} U_{101} + 4\theta_{4_h}\delta_{4_h} U_{011}\right\} \\
&\quad - 2\xi_{4_h}\left\{1 + \frac{\theta_{4_h}\left(\theta_{4_h} + 1\right)}{2} U_{020} + \frac{\delta_{4_h}\left(\delta_{4_h} + 1\right)}{2} U_{002} - \theta_{4_h} U_{110} - \delta_{4_h} U_{101} + \theta_{4_h}\delta_{4_h} U_{011}\right\}\Bigg], \\
\mathrm{MSE}\left(\bar{y}^{s}_{s_5}\right) &= \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Big[1 + \xi_{5_h}^2 \left\{1 + U_{200} + 3\theta_{5_h}^2 U_{020} + 3\delta_{5_h}^2 U_{002} - 4\theta_{5_h} U_{110} - 4\delta_{5_h} U_{101} + 4\theta_{5_h}\delta_{5_h} U_{011}\right\} \\
&\quad - 2\xi_{5_h}\left\{1 + \theta_{5_h}^2 U_{020} - \theta_{5_h} U_{110} + \delta_{5_h}^2 U_{002} - \delta_{5_h} U_{101} + \theta_{5_h}\delta_{5_h} U_{011}\right\}\Big].
\end{aligned} \tag{29}
$$

Proof. The outline of the derivation is given in Appendix C for ready reference. □

Corollary 4. *The minimum MSE at the optimum values of ξ_{i_h}, θ_{i_h}, and δ_{i_h}, $i = 1, 2, \cdots, 5$, is given by*

$$
\begin{aligned}
\mathit{minMSE}\left(\bar{y}^{s}_{s_i}\right) &= \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right), \quad i = 1, 2, 4, 5, \\
\mathit{minMSE}\left(\bar{y}^{s}_{s_3}\right) &= \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left(1 - \xi_{3_h(opt)}\right) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left(1 - \frac{B_{3_h}^2}{A_{3_h}}\right).
\end{aligned} \tag{30}
$$

Proof. The outline of the derivation is given in Appendix D for ready reference. □

4. Theoretical Conditions

4.1. Combined Estimators. On comparing the minimum MSE of the proffered combined classes of estimators $\bar{y}^{c}_{s_i}$, $i = 1, 2, \cdots, 5$ and the existing combined estimators from (27) with (47), (48), (50), (52), (56), (58), (60), (62), (63), and (65), we obtain the theoretical conditions given as follows:

$$
\mathrm{MSE}(\bar{y}^{c}_{m}) > \mathrm{MSE}\left(\bar{y}^{c}_{s_i}\right) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200},
$$

$$
\begin{aligned}
\mathrm{MSE}(\bar{y}^{c}_{r}) > \mathrm{MSE}\left(t^{c}_{s_i}\right) \Rightarrow \frac{B_i^2}{A_i} &> 1 - V_{200} - V_{020} - V_{002} \\
&\quad + 2V_{110} + 2V_{101} - 2V_{011},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{MSE}(\bar{y}^{c}_{lr}) > \mathrm{MSE}\left(\bar{y}^{c}_{s_i}\right) \Rightarrow \frac{B_i^2}{A_i} &> 1 - V_{200} \\
&\quad + \frac{\left(V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{MSE}(\bar{y}^{c}_{kk}) > \mathrm{MSE}\left(\bar{y}^{c}_{s_i}\right) \Rightarrow \frac{B_i^2}{A_i} &> 1 - V_{200} \\
&\quad + \frac{\left(V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)},
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{MSE}(\bar{y}^{c}_{tc}) > \mathrm{MSE}\left(\bar{y}^{c}_{s_i}\right) \\
&\quad \Rightarrow \frac{B_i^2}{A_i} > \Bigg[1 - k^{*^2}\left\{V_{200} + \frac{1}{4}\left(V_{020} + V_{002} - 2V_{002}\right) + V_{101} - V_{110}\right\} \\
&\quad - 2k^*\left(k^* - 1\right)\left\{\frac{1}{8}\left(3V_{020} - V_{002} - 2V_{011}\right) + \frac{1}{2}\left(V_{101} - V_{110}\right)\right\} \\
&\quad - \left(k^* - 1\right)^2\Bigg],
\end{aligned}
$$

$$\mathrm{MSE}\left(\bar{y}^{c}_{l_1}\right) > \mathrm{MSE}\left(\bar{y}^{c}_{s_i}\right) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} + \frac{\left(V_{101}^2 V_{020} + V_{110}^2 V_{002} - 2V_{011}V_{101}V_{110}\right)}{V_{020}V_{002}\left(1-\rho^2_{x_h z_h}\right)},$$

$$\mathrm{MSE}\left(\bar{y}^{c}_{l_2}\right) > \mathrm{MSE}\left(\bar{y}^{c}_{s_i}\right) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} + \frac{\left(V_{002} V_{110}^2 + V_{020} V_{101}^2 - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)},$$

$$\mathrm{MSE}(\bar{y}^{c}_{mu}) > \mathrm{MSE}\left(\bar{y}^{c}_{s_i}\right) \Rightarrow \frac{B_i^2}{A_i} > \frac{V_{011}^2}{4V_{020}} + \frac{A_m^2}{B_m},$$

$$\mathrm{MSE}(\bar{y}^{c}_{ks}) > \mathrm{MSE}\left(\bar{y}^{c}_{s_i}\right) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} + \frac{\left(V_{002} V_{110}^2 + V_{020} V_{101}^2 - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)},$$

$$\mathrm{MSE}\left(\bar{y}^{c}_{mu_1}\right) > \mathrm{MSE}\left(\bar{y}^{c}_{s_i}\right) \Rightarrow \frac{B_i^2}{A_i} > 2O - P,$$

$$\mathrm{MSE}\left(\bar{y}^{c}_{mu_2}\right) > \mathrm{MSE}\left(\bar{y}^{c}_{s_i}\right) \Rightarrow \frac{B_i^2}{A_i} > \frac{O^2}{P}. \quad (31)$$

If condition (31) holds, then, the proffered combined classes of estimators $\bar{y}^{c}_{s_i}, i = 1, 2, \cdots, 5$, perform better than the other existing combined estimators.

4.2. Separate Estimators. On comparing the minimum MSE of the proffered separate classes of estimator $\bar{y}^{s}_{s_i}, i = 1, 2, \cdots, 5$, with the existing separate estimators from (30) with (81), (82), (84), (86), (90), (92), (94), (96), (97), and (99), we get the following theoretical conditions.

$$\mathrm{MSE}(\bar{y}^{s}_{m}) > \mathrm{MSE}\left(\bar{y}^{s}_{s_i}\right), \quad \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 U_{200}, \quad (32)$$

$$\mathrm{MSE}(\bar{y}^{s}_{r}) > \mathrm{MSE}\left(\bar{y}^{s}_{s_i}\right), \quad \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \left[U_{200} + U_{020} + U_{002} - 2U_{110} - 2U_{101} + 2U_{011}\right], \quad (33)$$

$$\mathrm{MSE}(\bar{y}^{s}_{lr}) > \mathrm{MSE}\left(\bar{y}^{s}_{s_i}\right), \quad \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left[U_{200} - \frac{\left(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)}\right], \quad (34)$$

$$\mathrm{MSE}(\bar{y}^{s}_{kk}) > \mathrm{MSE}\left(\bar{y}^{s}_{s_i}\right), \quad \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left[U_{200} - \frac{\left(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)}\right], \quad (35)$$

$$\mathrm{MSE}(\bar{y}^{s}_{tc}) > \mathrm{MSE}\left(\bar{y}^{s}_{s_i}\right), \quad \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left[k_h^{*^2}\left\{U_{200} + \frac{1}{4}(U_{020} + U_{002} - 2U_{002}) + U_{101} - U_{110}\right\} + 2k_h^*(k_h^* - 1)\left\{\frac{1}{8}(3U_{020} - U_{002} - 2U_{011}) + \frac{1}{2}(U_{101} - U_{110})\right\} + (k_h^* - 1)^2\right], \quad (36)$$

$$\mathrm{MSE}\left(\bar{y}^{s}_{l_1}\right) > \mathrm{MSE}\left(\bar{y}^{s}_{s_i}\right), \quad \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left[U_{200} - \frac{\left(U_{101}^2U_{020} + U_{110}^2U_{002} - 2U_{011}U_{101}U_{110}\right)}{U_{020}U_{002}\left(1-\rho^2_{x_h z_h}\right)}\right], \quad (37)$$

$$\mathrm{MSE}\left(\bar{y}^{s}_{l_2}\right) > \mathrm{MSE}\left(\bar{y}^{s}_{s_i}\right), \quad \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left[U_{200} - \frac{\left(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)}\right], \quad (38)$$

$$\mathrm{MSE}(\bar{y}^{s}_{mu}) > \mathrm{MSE}\left(\bar{y}^{s}_{s_i}\right), \quad \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left[1 - \frac{U_{011}^2}{4U_{020}} - \frac{A_{m_h}^2}{B_{m_h}}\right], \quad (39)$$

$$\mathrm{MSE}(\bar{y}^{s}_{ks}) > \mathrm{MSE}\left(\bar{y}^{s}_{s_i}\right), \quad \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left[U_{200} - \frac{\left(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)}\right], \quad (40)$$

Table 1: MSE and PRE of different combined and separate estimators using real populations.

Combined estimators	Population 5 MSE	Population 5 PRE	Population 6 MSE	Population 6 PRE	Separate Estimators	Population 5 MSE	Population 5 PRE	Population 6 MSE	Population 6 PRE
$\bar{y}_m^c$	127.24	100.00	0.1485	100.00	$\bar{y}_m^s$	129.42	100.00	0.1617	100.00
$\bar{y}_r^c$	85.53	148.76	0.0406	365.76	$\bar{y}_r^s$	86.11	150.29	0.0439	368.33
$\bar{y}_{lr}^c$	69.78	182.34	0.0263	563.72	$\bar{y}_{lr}^s$	70.13	184.54	0.0285	567.36
$\bar{y}_{kk}^c$	69.78	182.34	0.0263	563.72	$\bar{y}_{kk}^s$	70.13	184.54	0.0285	567.36
$\bar{y}_{tc}^c$	156.75	81.17	0.0985	150.78	$\bar{y}_{tc}^s$	155.65	83.14	0.0989	163.49
$\bar{y}_{l_1}^c$	69.78	182.34	0.0263	563.72	$\bar{y}_{l_1}^s$	70.13	184.54	0.0285	567.36
$\bar{y}_{l_2}^c$	69.78	182.34	0.0263	563.72	$\bar{y}_{l_2}^s$	70.13	184.54	0.0285	567.36
$\bar{y}_{ks}^c$	69.78	182.34	0.0263	563.72	$\bar{y}_{ks}^s$	70.13	184.54	0.0285	567.36
$\bar{y}_{mu}^c$	88.03	144.52	0.0193	769.43	$\bar{y}_{mu}^s$	88.78	145.77	0.0210	770.00
$\bar{y}_{mu_1}^c$	99.64	127.70	0.0195	758.29	$\bar{y}_{mu_1}^s$	100.33	128.99	0.0212	762.73
$\bar{y}_{mu_2}^c$	98.56	129.09	0.0195	758.60	$\bar{y}_{mu_2}^s$	99.51	130.05	0.0213	759.15
$\bar{y}_{s_1}^c$	61.44	207.07	0.0189	785.71	$\bar{y}_{s_1}^s$	62.18	208.13	0.0201	804.47
$\bar{y}_{s_2}^c$	65.79	193.40	0.0192	773.43	$\bar{y}_{s_2}^s$	66.21	195.46	0.0208	777.40
$\bar{y}_{s_3}^c$	66.01	192.75	0.0192	774.24	$\bar{y}_{s_3}^s$	66.97	193.25	0.0208	778.15
$\bar{y}_{s_4}^c$	64.69	196.66	0.0190	781.57	$\bar{y}_{s_4}^s$	65.29	198.22	0.0206	784.95
$\bar{y}_{s_5}^c$	65.13	195.36	0.0191	777.48	$\bar{y}_{s_5}^s$	65.98	196.15	0.0207	779.27

$$\operatorname{MSE}\left(\bar{y}_{mu_1}^s\right) > \operatorname{MSE}\left(\bar{y}_{s_i}^s\right),$$
$$\sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 (1 + P_h - 2O_h), \tag{41}$$

$$\operatorname{MSE}\left(\bar{y}_{mu_2}^s\right) > \operatorname{MSE}\left(\bar{y}_{s_i}^s\right),$$
$$\sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left(1 - \frac{O_h^2}{P_h}\right). \tag{42}$$

If conditions (32)–(42) hold, then, the proffered separate classes of estimators $\bar{y}_{s_i}^s$, $i = 1, 2, \cdots, 5$, become superior than the other existing separate estimators.

4.3. Comparison of Proffered Combined and Separate Estimators. On comparing the minimum MSE of proffered combined and separate classes of estimators $\bar{y}_{s_i}^c$ and $\bar{y}_{s_i}^s$, $i = 1, 2, \cdots, 5$, we get

$$\operatorname{minMSE}\left(\bar{y}_{s_i}^c\right) - \operatorname{minMSE}\left(\bar{y}_{s_i}^s\right) = \sum_{h=1}^{L}\left[\left(\bar{Y}^2 - W_h^2 \bar{Y}_h^2\right) - \left(\bar{Y}^2 \frac{B_i^2}{A_i} - W_h^2 \bar{Y}_h^2 \frac{B_{i_h}^2}{A_{i_h}}\right)\right]. \tag{43}$$

If the ratio estimate is veritable and the relationship between auxiliary and study variables within each stratum is a straight line passing through the origin, then, the last term of (43) is broadly small and it vanishes.

Moreover, unless R_h is invariant from stratum to stratum, separate estimators probably become more efficient in each stratum if the sample in each stratum is large enough so that the approximate formula for $\operatorname{MSE}(\bar{y}_{s_i}^s)$, $i = 1, 2, \cdots, 5$, is valid and the cumulative bias that can affect the proffered estimators is negligible, whereas the proffered combined estimators are to be preferably recommended with only a small sample in each stratum (see [34]).

5. Numerical Study

To enhance the credibility of the theoretical development of the proposed combined and separate classes of estimators, we have conducted a numerical study using two real populations which are described as follows.

Population 5 (Sarndal et al. [35], p. 529). y is the population in thousands during 1985, x is the population in thousands during 1975, z is the total number of seats in the municipal council, $N = 199$, $n = 47$, $N_1 = 25$, $N_2 = 48$, $N_3 = 32$, $N_4 = 38$, $N_5 = 56$, $n_1 = 6$, $n_2 = 11$, $n_3 = 8$, $n_4 = 9$, $n_5 = 13$, $\bar{Y}_1 = 62.4400$, $\bar{Y}_2 = 29.6042$, $\bar{Y}_3 = 24.0625$, $\bar{Y}_4 = 31.0000$, $\bar{Y}_5 = 29.41071$, $\bar{X}_1 = 59.5200$, $\bar{X}_2 = 29.1667$, $\bar{X}_3 = 23.9375$, $\bar{X}_4 = 30.6316$, $\bar{X}_5 = 28.7143$, $\bar{Z}_1 = 51.1600$, $\bar{Z}_2 = 47.6667$, $\bar{Z}_3 = 50.2500$, $\bar{Z}_4 = 48.4737$, $\bar{Z}_5 = 46.3572$, $S_{y1}^2 = 15521.5900$, $S_{y2}^2 = 1320.2442$, $S_{y3}^2 = 445.3508$, $S_{y4}^2 = 1536.3784$, $S_{y5}^2 = 3219.8464$, $S_{x1}^2 = 16564.7600$, $S_{x2}^2 = 1228.2270$, $S_{x3}^2 = 437.1573$, $S_{x4}^2 = 1721.1579$, $S_{x5}^2 = 3565.1169$, $S_{z1}^2 = 197.9734$, $S_{z2}^2 = 106.3546$, $S_{z3}^2 = 106.7742$, $S_{z4}^2 = 82.0939$, $S_{z5}^2 = 97.9065$, $S_{yx1} =$

Table 2: MSE and PRE of different combined estimators.

ρ_{xy}	0.6		0.7		0.8		0.9		0.9		0.8		0.7		0.6	
ρ_{yz}	0.6		0.7		0.8		0.9		0.8		0.7		0.6		0.5	
ρ_{xz}	0.6		0.7		0.8		0.9		0.7		0.6		0.5		0.4	
Estimators	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE
$\bar{y}_m^c$	0.6839	100.00	0.6801	100.00	0.6753	100.00	0.5721	100.00	0.6639	100.00	0.6737	100.00	0.6860	100.00	0.6950	100.00
$\bar{y}_r^c$	1.5128	45.20	1.4878	45.71	1.4461	46.69	1.4122	40.51	1.4490	45.27	1.4664	45.94	1.4983	45.78	1.5053	46.17
$\bar{y}_{lr}^c$	0.5858	116.74	0.5804	117.17	0.5719	118.08	0.4442	128.79	0.5622	118.08	0.5742	117.32	0.5923	115.81	0.5946	116.88
$\bar{y}_{kk}^c$	0.5858	116.74	0.5804	117.17	0.5719	118.08	0.4442	128.79	0.5622	118.08	0.5742	117.32	0.5923	115.81	0.5946	116.88
$\bar{y}_{tc}^c$	1.1548	59.22	1.1386	59.73	1.1290	59.81	1.0609	53.92	1.1452	57.97	1.1549	58.33	1.1622	59.02	1.1821	58.79
$\bar{y}_{l_1}^c$	0.5858	116.74	0.5804	117.17	0.5719	118.08	0.4442	128.79	0.5622	118.08	0.5742	117.32	0.5923	115.81	0.5946	116.88
$\bar{y}_{l_2}^c$	0.5858	116.74	0.5804	117.17	0.5719	118.08	0.4442	128.79	0.5622	118.08	0.5742	117.32	0.5923	115.81	0.5946	116.88
$\bar{y}_{ks}^c$	0.5858	116.74	0.5804	117.17	0.5719	118.08	0.4442	128.79	0.5622	118.08	0.5742	117.32	0.5923	115.81	0.5946	116.88
$\bar{y}_{mu}^c$	0.5775	118.42	0.5735	118.58	0.5633	119.88	0.4381	130.58	0.5584	118.89	0.5682	118.56	0.5855	117.16	0.5893	117.93
$\bar{y}_{mu_1}^c$	0.7804	87.63	0.7761	87.63	0.7744	87.20	0.6565	87.14	0.7554	87.88	0.7642	88.15	0.7829	87.62	0.7951	87.41
$\bar{y}_{mu_2}^c$	0.7340	93.17	0.7288	93.31	0.7262	92.99	0.6132	93.29	0.7123	93.20	0.7221	93.29	0.7377	92.99	0.7454	93.23
$\bar{y}_{s_1}^c$	0.5667	120.68	0.5616	121.10	0.5532	122.07	0.4184	136.73	0.5443	121.97	0.5545	121.49	0.5726	119.80	0.5858	118.64
$\bar{y}_{s_2}^c$	0.5732	119.31	0.5681	119.71	0.5602	120.54	0.4287	133.44	0.5508	120.53	0.5618	119.91	0.5792	118.43	0.5896	117.87
$\bar{y}_{s_3}^c$	0.5728	119.39	0.5678	119.78	0.5597	120.65	0.4254	134.48	0.5505	120.59	0.5621	119.85	0.5791	118.45	0.5899	117.81
$\bar{y}_{s_4}^c$	0.5710	119.77	0.5660	120.15	0.5578	121.06	0.4210	135.89	0.5487	120.99	0.5596	120.38	0.5761	119.07	0.5888	118.03
$\bar{y}_{s_5}^c$	0.5720	119.56	0.5678	119.77	0.5601	120.56	0.4259	134.32	0.5505	120.59	0.5616	119.96	0.5771	118.87	0.5891	117.97

Table 3: MSE and PRE of different separate estimators.

ρ_{xy}	0.6		0.7		0.8		0.9		0.9		0.8		0.7		0.6	
ρ_{yz}	0.6		0.7		0.8		0.9		0.8		0.7		0.6		0.5	
ρ_{xz}	0.6		0.7		0.8		0.9		0.7		0.6		0.5		0.4	
Estimators	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE
$\bar{y}_m^s$	0.7347	100.00	0.6643	100.00	0.6584	100.00	0.6510	100.00	0.6503	100.00	0.6571	100.00	0.6594	100.00	0.6653	100.00
$\bar{y}_r^s$	1.6797	43.73	1.5155	43.96	1.4746	44.64	1.4213	45.80	1.4505	44.83	1.5082	43.56	1.5483	42.58	1.6388	40.59
$\bar{y}_{lr}^s$	0.6297	116.67	0.5560	117.72	0.5390	122.15	0.5282	123.24	0.5278	123.20	0.5386	122.00	0.5425	121.54	0.5607	118.65
$\bar{y}_{kk}^s$	0.6297	116.67	0.5560	117.72	0.5390	122.15	0.5282	123.24	0.5278	123.20	0.5286	122.00	0.5425	121.54	0.5607	118.65
$\bar{y}_{tc}^s$	1.3078	56.17	1.1427	58.30	1.1233	58.61	1.1107	58.61	1.0990	59.17	1.1285	58.22	1.1432	57.68	1.1134	59.75
$\bar{y}_{l_1}^s$	0.6297	116.67	0.5560	117.72	0.5390	122.15	0.5282	123.24	0.5278	123.20	0.5286	122.00	0.5425	121.54	0.5607	118.65
$\bar{y}_{l_2}^s$	0.6297	116.67	0.5560	117.72	0.5390	122.15	0.5282	123.24	0.5278	123.20	0.5286	122.00	0.5425	121.54	0.5607	118.65
$\bar{y}_{ks}^s$	0.6297	116.67	0.5560	117.72	0.5390	122.15	0.5282	123.24	0.5278	123.20	0.5286	122.00	0.5425	121.54	0.5607	118.65
$\bar{y}_{mu}^s$	0.6127	119.91	0.5470	121.80	0.5304	124.13	0.5186	125.53	0.5176	125.63	0.5289	124.23	0.5328	123.76	0.5536	120.17
$\bar{y}_{mu_1}^s$	0.8901	82.54	0.7929	84.03	0.7539	87.33	0.7366	88.37	0.7404	87.83	0.7567	86.83	0.7856	83.93	0.8156	81.57
$\bar{y}_{mu_2}^s$	0.8258	90.05	0.7316	91.07	0.6995	94.12	0.6866	94.81	0.6886	94.43	0.7022	93.57	0.7257	90.86	0.7484	88.89
$\bar{y}_{s_1}^s$	0.5965	123.16	0.5348	124.58	0.5175	127.22	0.5065	128.52	0.5076	128.11	0.5145	127.71	0.5223	126.24	0.5405	123.09
$\bar{y}_{s_2}^s$	0.6101	120.42	0.5421	122.91	0.5256	125.26	0.5157	126.23	0.5157	126.10	0.5244	125.30	0.5301	124.39	0.5469	121.64
$\bar{y}_{s_3}^s$	0.6115	120.14	0.5398	123.43	0.5238	125.69	0.5137	126.72	0.5133	126.69	0.5247	125.23	0.5274	125.02	0.5447	122.14
$\bar{y}_{s_4}^s$	0.5978	122.90	0.5390	123.61	0.5224	126.03	0.5122	127.09	0.5123	126.93	0.5110	126.12	0.5267	125.19	0.5441	122.27
$\bar{y}_{s_5}^s$	0.6100	120.44	0.5397	123.45	0.5237	125.72	0.5136	126.75	0.5131	126.73	0.5239	125.42	0.5272	125.07	0.5459	121.87

10125.1783, $S_{yx2} = 1270.2376$, $S_{yx3} = 440.4234$, $S_{yx4} = 1622.6216$, $S_{yx5} = 3385.5013$, $S_{xz1} = 383.6633$, $S_{xz2} = 409.8440$, $S_{xz3} = 189.1129$, $S_{xz4} = 232.3954$, $S_{xz5} = 417.5584$, $S_{yz1} = 150.0933$, $S_{yz2} = 422.9929$, $S_{yz3} = 189.1129$, $S_{yz4} = 229.8378$, and $S_{yz5} = 399.9052$.

Population 6 (National Horticulture Baard [36]). y is productivity (MT/hectare), x is production (000 tons), z is the area (000 hectares), $N = 10$, $n = 7$, $N_1 = 5$, $N_2 = 5$, $n_1 = 3$, $n_2 = 4$, $\bar{Y}_1 = 1.70$, $\bar{Y}_2 = 3.67$, $\bar{X}_1 = 10.41$, $\bar{X}_2 = 309.14$, $\bar{Z}_1 = 6.20$, $\bar{Z}_2 = 80.67$, $S_{y1}^2 = 0.2916$, $S_{y2}^2 = 1.9881$, $S_{x1}^2 = 1.4116$, $S_{x2}^2 = 3486.6916$, $S_{z1}^2 = 1.4116$, $S_{z2}^2 = 116.8561$, $S_{yx1} = 1.6000$, $S_{yx2} = 83.47$, $S_{xz1} = 1.7500$, $S_{xz2} = 64.9700$, $S_{yz1} = 0.2000$, and $S_{yz2} = 5.5800$.

We have calculated the MSE and PRE of different estimators (T) by using the above populations. The PRE is calculated with respect to the usual mean estimator using the following expression.

$$\text{PRE} = \frac{\text{MSE}\left(\bar{y}_{[srss]}\right)}{\text{MSE}(T)} \times 100, \tag{44}$$

where $\text{MSE}(T)$ is the MSE of existing and proposed combined and separate estimators. The results of the numerical study for the above populations are reported in Table 1 by MSE and PRE. The numerical results show the dominance of the proposed combined and separate classes of estimators $\bar{y}_{s_i}^c$ and $\bar{y}_{s_i}^s$, $i = 1, 2, \cdots, 5$, respectively, in terms of lesser MSE and greater PRE over the combined and separate usual mean estimator, classical ratio, and regression estimators, Koyuncu and Kadilar- [1] type estimator, Tailor and Chouhan- [3] type estimator, [4, 5] type estimators, Khan and Shabbir- [22] type estimator, and Muneer et al.- [6, 7] type estimators. Also, the proposed combined and separate class of estimators $\bar{y}_{s_1}^c$ and $\bar{y}_{s_1}^s$ attain the lesser MSE and greater PRE among the proposed classes of estimators in both the populations.

6. Simulation Study

To generalize the results of the numerical study, we have conducted a simulation study over a hypothetically generated normal population. The simulation procedure is explained in the following points:

(i) Trivariate random observations of size 600 units are drawn from a trivariate normal distribution with parameters $\bar{Y} = 20$, $\bar{X} = 15$, $\bar{Z} = 10$, $\sigma_y = 15$, $\sigma_x = 10$, and $\sigma_z = 5$ and different amounts of correlation coefficients ρ_{xy}, ρ_{yz} and ρ_{xz}

(ii) The population generated above is divided into 3 equal strata, and a stratified ranked set sample of size 12 units with a number of cycles 4 and set size 3 is drawn from each stratum

(iii) Compute the required statistics

(iv) Iterate the above steps 10000 times to calculate the MSE and PRE of various combined and separate classes of estimators using the following expression:

$$\text{MSE}(T) = \frac{1}{10{,}000} \sum_{i=1}^{10{,}000} \left(T_i - \bar{Y}\right)^2, \tag{45}$$

$$\text{PRE} = \frac{\text{MSE}\left(\bar{y}_{[\text{SRSS}]}\right)}{\text{MSE}(T)} \times 100 \tag{46}$$

The MSE and PRE of the combined and separate classes of estimators are calculated using (45) and (46), respectively, and the results are reported for various values of correlation coefficients in Tables 2 and 3 which exhibit the ascendancy of the proposed combined and separate classes of estimators over the existing combined and separate classes of estimators.

7. Conclusion

In this paper, we proffer some improved classes of estimators along with their properties using bivariate auxiliary information in SRSS. The proffered estimators dominate the other existing estimators under the conditions stated in Section 4. The numerical and simulation studies are performed using real and artificially generated populations with various amounts of correlation coefficients. The results of numerical and simulation studies are reported in terms of MSE and PRE from Tables 1 to 3. From the perusal of the results of Tables 1–3, the following conclusions are drawn:

(i) In Table 1, the proffered combined classes of estimators $\bar{y}_{s_i}^c, i = 1, 2, \cdots, 5$, perform better than the existing combined estimators, namely, conventional mean estimator $\bar{y}_m^c$, classical ratio and regression estimators $\bar{y}_r^c$ & $\bar{y}_{lr}^c$, Koyuncu and Kadilar- [1] type estimator $\bar{y}_{kk}^c$, Tailor and Chouhan- [3] type estimator $\bar{y}_{tc}^c$, Lone et al.- [4, 5] type estimators $\bar{y}_{l_1}^c$ & $\bar{y}_{l_2}^c$, Khan and Shabbir [22] estimator $\bar{y}_{ks}^c$, and Muneer et al.- [6, 7] type estimators $\bar{y}_{mu}^c$, $\bar{y}_{mu_1}^c$, & $\bar{y}_{mu_2}^c$ in both the populations

(ii) In Table 1, the proffered separate classes of estimators $\bar{y}_{s_i}^s, i = 1, 2, \cdots, 5$, dominate the existing separate estimators, namely, conventional mean estimator $\bar{y}_m^s$, classical ratio and regression estimators $\bar{y}_r^s$ & $\bar{y}_{lr}^s$, Koyuncu and Kadilar- [1] type estimator $\bar{y}_{kk}^c$, Tailor and Chouhan- [3] type estimator $\bar{y}_{tc}^s$, Lone et al.- [4, 5] type estimators $\bar{y}_{l_1}^s$ & $\bar{y}_{l_2}^s$, Khan and Shabbir [22] estimator $\bar{y}_{ks}^c$, and Muneer et al.- [6, 7] type estimators $\bar{y}_{mu}^s$, $\bar{y}_{mu_1}^s$, & $\bar{y}_{mu_2}^s$

(iii) In Table 2, the proffered combined classes of estimators $\bar{y}_{s_i}^c, i = 1, 2, \cdots, 5$, perform better than the existing combined estimators. The similar inclination in the results of separate estimators can also be observed in Table 3

(iv) In Tables 1, 2, and 3, the proffered combined and separate classes of estimators $\bar{y}^c_{s_1}$ and $\bar{y}^s_{s_1}$, respectively, become superior among the proffered combined and separate classes of estimators

(v) In Tables 1, 2, and 3, the proffered separate classes of estimators $\bar{y}^s_{s_i}, i = 1, 2, \cdots, 5$, perform better than the proffered combined classes of estimators $\bar{y}^c_{s_i}, i = 1, 2, \cdots, 5$, in terms of lesser MSE and greater PRE

(vi) From the simulation results of Tables 2 and 3, it has been observed that the PRE increases as the correlation coefficients decrease

Thus, the proffered classes of estimators are preferred for the computation of the population mean when bivariate auxiliary information is available.

Appendix

A. MSEs of Existing Combined Estimators

The MSE of the existing combined classes of estimators is tabulated as follows:

$$\mathrm{MSE}(\bar{y}^c_m) = \bar{Y}^2 V_{200}, \tag{47}$$

$$\mathrm{MSE}(\bar{y}^c_r) = \bar{Y}^2 [V_{200} + V_{020} + V_{002} - 2V_{110} - 2V_{101} + 2V_{011}], \tag{48}$$

$$\mathrm{MSE}(\bar{y}^c_{lr}) = \left[\bar{Y}^2 V_{200} + \beta^2 \bar{X}^2 V_{020} + \theta^2 \bar{Z}^2 V_{002} - 2\beta \bar{X}\bar{Y} V_{110} - 2\theta \bar{Z}\bar{Y} V_{101} + 2\beta\theta \bar{X}\bar{Z} V_{011}\right], \tag{49}$$

$$\mathrm{minMSE}(\bar{y}^c_{lr}) = \bar{Y}^2 \left[V_{200} - \frac{\left(V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}\right], \tag{50}$$

$$\mathrm{MSE}(\bar{y}^c_{kk}) = \bar{Y}^2 [V_{200} + l_1^2 V_{020} + l_2^2 V_{002} - 2l_1 V_{110} - 2l_2 V_{101} + 2l_1 l_2 V_{011}], \tag{51}$$

$$\mathrm{minMSE}(\bar{y}^c_{kk}) = \bar{Y}^2 \left[V_{200} - \frac{\left(V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}\right], \tag{52}$$

$$\mathrm{MSE}(\bar{y}^c_{tc}) = \bar{Y}^2 \left[k^2 \left\{V_{200} + \frac{1}{4}(V_{020} + V_{002} - 2V_{002}) + V_{101} - V_{110}\right\} + (k-1)^2 + 2k(k-1)\left\{\frac{1}{8}(3V_{020} - V_{002} - 2V_{011}) + \frac{1}{2}(V_{101} - V_{110})\right\}\right], \tag{53}$$

$$\mathrm{minMSE}(\bar{y}^c_{tc}) = \bar{Y}^2 \left[k^{*^2} \left\{V_{200} + \frac{1}{4}(V_{020} + V_{002} - 2V_{002}) + V_{101} - V_{110}\right\} + (k^* - 1)^2 + 2k^*(k^* - 1)\left\{\frac{1}{8}(3V_{020} - V_{002} - 2V_{011}) + \frac{1}{2}(V_{101} - V_{110})\right\}\right], \tag{54}$$

$$\mathrm{MSE}\left(\bar{y}^c_{l_1}\right) = \bar{Y}^2 \left[V_{200} + \frac{1}{4}\left(L_1^2 V_{020} + L_2^2 V_{002} - 2L_1 L_2 V_{011}\right) + L_2 V_{101} - L_1 V_{110}\right], \tag{55}$$

$$\mathrm{minMSE}\left(\bar{y}^c_{l_1}\right) = \bar{Y}^2 \left[V_{200} - \frac{\left(V_{101}^2 V_{020} + V_{110}^2 V_{002} - 2V_{011}V_{101}V_{110}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}\right], \tag{56}$$

$$\mathrm{MSE}\left(\bar{y}^c_{l_2}\right) = \bar{Y}^2 [V_{200} + \lambda^2 V_{020} + \psi^2 V_{002} - 2\lambda V_{110} + 2\psi V_{101} - 2\lambda\psi V_{011}], \tag{57}$$

$$\mathrm{minMSE}\left(\bar{y}^c_{l_2}\right) = \bar{Y}^2 \left[V_{200} - \frac{\left(V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}\right], \tag{58}$$

$$\mathrm{MSE}(\bar{y}^c_{mu}) = \left[\bar{Y}^2(1-k_3)^2 + k_3^2 \bar{Y}^2 \left\{V_{200} + \left(1 - \frac{\alpha}{2}\right)V_{002} - 2V_{101}\right\} + k_4^2 \bar{X}^2 V_{020} - 2k_3 \bar{Y}^2 \left\{\left(\frac{3}{8} - \frac{\alpha}{4}\right)V_{002} - \frac{V_{101}}{2}\right\} - k_4 \bar{X}\bar{Y} V_{011} + 2k_3 k_4 \bar{X}\bar{Y}(V_{011} - V_{110})\right], \tag{59}$$

$$\mathrm{minMSE}(\bar{y}^c_{mu}) = \bar{Y}^2 \left[1 - \frac{V_{011}^2}{4V_{020}} - \frac{A_m^2}{B_m}\right], \tag{60}$$

$$\mathrm{MSE}(\bar{y}^c_{lk}) = \bar{Y}^2 \left[V_{200} + \frac{V_{020}}{\eta_1^2} + \frac{V_{002}}{\eta_2^2} - 2\frac{V_{110}}{\eta_1} - 2\frac{V_{101}}{\eta_2} + 2\frac{V_{011}}{\eta_1 \eta_2}\right], \tag{61}$$

$$\mathrm{minMSE}(\bar{y}^c_{lk}) = \bar{Y}^2 \left[V_{200} - \frac{\left(V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}\right], \tag{62}$$

$$\mathrm{MSE}\left(\bar{y}^c_{mu_1}\right) = \bar{Y}^2 [1 + P - 2O], \tag{63}$$

$$\mathrm{MSE}\left(\bar{y}^{c}_{mu_2}\right) = \bar{Y}^2\left[1 + k_1^2 P - 2k_1 O\right], \tag{64}$$

$$\mathrm{minMSE}\left(\bar{y}^{c}_{mu_2}\right) = \bar{Y}^2\left[1 - \frac{O^2}{P}\right], \tag{65}$$

where

$$P = 1 + R + 2S, O = 1 + S,$$

$$\begin{aligned} R = \Bigg[& V_{200} + \left(\alpha_1 g_1 + \frac{1}{2}\alpha_3\right)^2 V_{020} + \left(\alpha_2 g_2 + \frac{1}{2}\alpha_4\right)^2 V_{002} \\ & - 2\left(\alpha_1 g_1 + \frac{1}{2}\alpha_3\right) V_{110} - 2\left(\alpha_2 g_2 + \frac{1}{2}\alpha_4\right) V_{101} \\ & + 2\left(\alpha_1 g_1 + \frac{1}{2}\alpha_3\right)\left(\alpha_2 g_2 + \frac{1}{2}\alpha_4\right) V_{011}\Bigg], \end{aligned}$$

$$\begin{aligned} S = \Bigg[& -\left(\alpha_1 g_1 + \frac{1}{2}\alpha_3\right) V_{110} - \left(\alpha_2 g_2 + \frac{1}{2}\alpha_4\right) V_{101} \\ & + \left\{\frac{1}{2}\alpha_1(\alpha_1 + 1) g_1^2 + \frac{3}{8}\alpha_3(\alpha_3 + 1) + \frac{1}{2}\alpha_1\alpha_3 g_1\right\} U_{020} \\ & + \left\{\frac{1}{2}\alpha_2(\alpha_2 + 1) g_2^2 + \frac{3}{8}\alpha_4(\alpha_4 + 1) + \frac{1}{2}\alpha_2\alpha_4 g_2\right\} V_{002} \\ & + \left\{\alpha_1\alpha_2 g_1 g_2 + \frac{1}{2}\alpha_2\alpha_3 g_2 + \frac{1}{2}\alpha_1\alpha_4 g_1 + \frac{1}{4}\alpha_3\alpha_4\right\} V_{011}\Bigg]. \end{aligned} \tag{66}$$

The optimum values of the scalars involved in the estimators are given as follows:

$$\beta_{1(\mathrm{opt})} = \frac{\bar{Y}}{\bar{X}}\left[\frac{V_{002}V_{110} - V_{101}V_{011}}{V_{020}V_{002} - V_{011}^2}\right]. \tag{67}$$

$$\beta_{2(\mathrm{opt})} = \frac{\bar{Y}}{\bar{Z}}\left[\frac{V_{020}V_{101} - V_{110}V_{011}}{V_{020}V_{002} - V_{011}^2}\right], \tag{68}$$

$$l_{1(\mathrm{opt})} = \left[\frac{V_{002}V_{110} - V_{101}V_{011}}{V_{020}V_{002} - V_{011}^2}\right], \tag{69}$$

$$l_{2(\mathrm{opt})} = \left[\frac{V_{020}V_{101} - V_{110}V_{011}}{V_{020}V_{002} - V_{011}^2}\right], \tag{70}$$

$$\begin{aligned} k_{(\mathrm{opt})} &= \frac{\left(1 + \{(1/8)(3V_{020} - V_{002} - 2V_{011}) + (1/2)(V_{101} - V_{110})\}\right)}{\begin{pmatrix} 1 + 2\{(1/8)(3V_{020} - V_{002} - 2V_{011}) + (1/2)(V_{101} - V_{110})\} \\ +k^2\{V_{200} + (1/4)(V_{020} + V_{002} - 2V_{002}) + V_{101} - V_{110}\} \end{pmatrix}} \\ &= k^*(say), \end{aligned} \tag{71}$$

$$L_{1(\mathrm{opt})} = \frac{2(V_{002}V_{110} - V_{101}V_{011})}{(V_{020}V_{002} - V_{011}^2)}, \tag{72}$$

$$L_{2(\mathrm{opt})} = \frac{2(V_{110}V_{011} - V_{020}V_{101})}{(V_{020}V_{002} - V_{011}^2)}, \tag{73}$$

$$\lambda_{(\mathrm{opt})} = \frac{(V_{002}V_{110} - V_{101}V_{011})}{(V_{020}V_{002} - V_{011}^2)}, \tag{74}$$

$$\psi_{(\mathrm{opt})} = \frac{-(V_{020}V_{101} - V_{110}V_{011})}{(V_{020}V_{002} - V_{011}^2)}, \tag{75}$$

$$\begin{aligned} k_{3(\mathrm{opt})} &= \frac{1 + ((3/8) - (\alpha/4))V_{002} - (V_{101}/2) - (V_{011}(V_{011} - V_{110})/2V_{020})}{1 + V_{200} + (1 - (\alpha/2))V_{002} - 2V_{101} - \left((V_{011} - V_{110})^2/V_{020}\right)} \\ &= \frac{A_m}{B_m}(say), \end{aligned} \tag{76}$$

$$k_{4(\mathrm{opt})} = \frac{\bar{Y}}{\bar{X}}\left[\frac{V_{011}}{2V_{020}} - k_3\left(\frac{V_{011} - V_{110}}{V_{020}}\right)\right], \tag{77}$$

$$\eta_{1(\mathrm{opt})} = \frac{(V_{020}V_{002} - V_{011}^2)}{(V_{002}V_{110} - V_{101}V_{011})}, \tag{78}$$

$$\eta_{2(\mathrm{opt})} = \frac{(V_{020}V_{002} - V_{011}^2)}{(V_{020}V_{101} - V_{110}V_{011})}, \tag{79}$$

$$k_{1(\mathrm{opt})} = \frac{O}{P}. \tag{80}$$

B. MSEs of Existing Separate Estimators

This section considers the MSEs of the existing separate estimators.

$$\mathrm{MSE}(\bar{y}^s_m) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 U_{200}, \tag{81}$$

$$\mathrm{MSE}(\bar{y}^s_r) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 [U_{200} + U_{020} + U_{002} - 2U_{110} - 2U_{101} + 2U_{011}], \tag{82}$$

$$\begin{aligned} \mathrm{MSE}(\bar{y}^s_{lr}) = \sum_{h=1}^{L} W_h^2 \Big[& \bar{Y}_h^2 U_{200} + \beta_h^2 \bar{X}^2 U_{020} + \theta_h^2 \bar{Z}^2 U_{002} \\ & - 2\beta_h \bar{X}\bar{Y} U_{110} - 2\theta_h \bar{Z}\bar{Y} U_{101} + 2\beta_h\theta_h \bar{X}\bar{Z} U_{011}\Big], \end{aligned} \tag{83}$$

$$\begin{aligned} & \mathrm{minMSE}(\bar{y}^s_{lr}) \\ & = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[U_{200} - \frac{(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011})}{(U_{020}U_{002} - U_{011}^2)}\right], \end{aligned} \tag{84}$$

$$\mathrm{MSE}(\bar{y}^s_{kk}) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Big[U_{200} + l_{1_h}^2 U_{020} + l_{2_h}^2 U_{002} - 2 l_{1_h} U_{110} - 2 l_{2_h} U_{101} + 2 l_{1_h} l_{2_h} U_{011} \Big], \tag{85}$$

$$\mathrm{minMSE}(\bar{y}^s_{kk}) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[U_{200} - \frac{\left(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)} \right], \tag{86}$$

$$\mathrm{MSE}(\bar{y}^s_{tc}) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Big[k_h^2 \Big\{ U_{200} + \frac{1}{4}(U_{020} + U_{002} - 2U_{002}) + U_{101} - U_{110} \Big\} + 2k_h(k_h - 1)\Big\{ \frac{1}{8}(3U_{020} - U_{002} - 2U_{011}) + \frac{1}{2}(U_{101} - U_{110}) \Big\} + (k_h - 1)^2 \Big], \tag{87}$$

$$\mathrm{minMSE}(\bar{y}^s_{tc}) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Big[k_h^{*^2} \Big\{ U_{200} + \frac{1}{4}(U_{020} + U_{002} - 2U_{002}) + U_{101} - U_{110} \Big\} + 2k_h^*(k_h^* - 1)\Big\{ \frac{1}{8}(3U_{020} - U_{002} - 2U_{011}) + \frac{1}{2}(U_{101} - U_{110}) \Big\} + (k_h^* - 1)^2 \Big], \tag{88}$$

$$\mathrm{MSE}\left(\bar{y}^s_{l_1}\right) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[\begin{array}{c} U_{200} + \frac{1}{4}\left(L_{1_h}^2 U_{020} + L_{2_h}^2 U_{002} - 2L_{1_h} L_{2_h} U_{011} \right) \\ + L_{2_h} U_{101} - L_{1_h} U_{110} \end{array} \right], \tag{89}$$

$$\mathrm{minMSE}\left(\bar{y}^s_{l_1}\right) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[U_{200} - \frac{\left(U_{101}^2 U_{020} + U_{110}^2 U_{002} - 2U_{011}U_{101}U_{110}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)} \right], \tag{90}$$

$$\mathrm{MSE}\left(\bar{y}^s_{l_2}\right) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \big[U_{200} + \lambda_h^2 U_{020} + \psi_h^2 U_{002} - 2\lambda_h U_{110} + 2\psi_h U_{101} - 2\lambda_h \psi_h U_{011} \big], \tag{91}$$

$$\mathrm{minMSE}\left(\bar{y}^s_{l_2}\right) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[U_{200} - \frac{\left(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)} \right], \tag{92}$$

$$\mathrm{MSE}(\bar{y}^s_{mu}) = \sum_{h=1}^{L} W_h^2 \Big[\bar{Y}_h^2 + \bar{Y}_h^2 k_{3_h}^2 - 2k_{3_h} \bar{Y}_h^2 \Big\{ U_{200} + \Big(1 - \frac{\alpha_h}{2U_{002} - 2U_{101}} \Big) \Big\} + k_{4_h}^2 \bar{X}_h^2 U_{020} - 2k_{3_h} \bar{Y}_h^2 \Big\{ \Big(\frac{3}{8} - \frac{\alpha_h}{4} \Big) U_{002} - \frac{U_{101}}{2} \Big\} - k_{4_h} \bar{X}_h \bar{Y}_h U_{011} + 2k_{3_h} k_{4_h} \bar{X}_h \bar{Y}_h (U_{011} - U_{110}) \Big], \tag{93}$$

$$\mathrm{minMSE}(\bar{y}^s_{mu}) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[1 - \frac{U_{011}^2}{4U_{020}} - \frac{A_{m_h}^2}{B_{m_h}} \right], \tag{94}$$

$$\mathrm{MSE}(\bar{y}^s_{lk}) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[U_{200} + \frac{U_{020}}{\eta_{1_h}^2} + \frac{U_{002}}{\eta_{2_h}^2} - 2\frac{U_{110}}{\eta_{1_h}} - 2\frac{U_{101}}{\eta_{2_h}} + 2\frac{U_{011}}{\eta_{1_h}\eta_{2_h}} \right], \tag{95}$$

$$\mathrm{minMSE}(\bar{y}^s_{lk}) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \cdot \left[U_{200} - \frac{\left(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)} \right], \tag{96}$$

$$\mathrm{MSE}\left(\bar{y}^s_{mu_1}\right) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 [1 + P_h - 2O_h], \tag{97}$$

$$\mathrm{MSE}\left(\bar{y}^s_{mu_2}\right) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[1 + k_{1_h}^2 P_h - 2k_{1_h} O_h\right], \tag{98}$$

$$\mathrm{minMSE}\left(\bar{y}^s_{mu_2}\right) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[1 - \frac{O_h^2}{P_h}\right], \tag{99}$$

where

$$P_h = 1 + R_h + 2S_h,$$

$$O_h = 1 + S_h,$$

$$R_h = \Big[U_{200} + \Big(\alpha_{1_h} g_{1_h} + \frac{1}{2}\alpha_{3_h} \Big)^2 U_{020} + \Big(\alpha_{2_h} g_{2_h} + \frac{1}{2}\alpha_{4_h} \Big)^2 U_{002} - 2\Big(\alpha_{1_h} g_{1_h} + \frac{1}{2}\alpha_{3_h} \Big) U_{110} - 2\Big(\alpha_{2_h} g_{2_h} + \frac{1}{2}\alpha_{4_h} \Big) U_{101} + 2\Big(\alpha_{1_h} g_{1_h} + \frac{1}{2}\alpha_{3_h} \Big)\Big(\alpha_{2_h} g_{2_h} + \frac{1}{2}\alpha_{4_h} \Big) U_{011} \Big],$$

$$S_h = \left[\begin{array}{c} -\Big(\alpha_{1_h} g_{1_h} + \frac{1}{2}\alpha_{3_h} \Big) U_{110} - \Big(\alpha_{2_h} g_{2_h} + \frac{1}{2}\alpha_{4_h} \Big) U_{101} \\ + \Big\{ \frac{1}{2}\alpha_{1_h}\left(\alpha_{1_h} + 1\right) g_{1_h}^2 + \frac{3}{8}\alpha_{3_h}\left(\alpha_{3_h} + 1\right) + \frac{1}{2}\alpha_{1_h}\alpha_{3_h} g_{1_h} \Big\} U_{020} \\ + \Big\{ \frac{1}{2}\alpha_{2_h}\left(\alpha_{2_h} + 1\right) g_{2_h}^2 + \frac{3}{8}\alpha_{4_h}\left(\alpha_{4_h} + 1\right) + \frac{1}{2}\alpha_{2_h}\alpha_{4_h} g_{2_h} \Big\} U_{002} \\ + \Big\{ \alpha_{1_h}\alpha_{2_h} g_{1_h} g_{2_h} + \frac{1}{2}\alpha_{2_h}\alpha_{3_h} g_{2_h} + \frac{1}{2}\alpha_{1_h}\alpha_{4_h} g_{1_h} + \frac{1}{4}\alpha_{3_h}\alpha_{4_h} \Big\} U_{011} \end{array} \right]. \tag{100}$$

The optimum values of the constants involved in the estimators are obtained by minimizing MSE expression w.r.t. the constants as follows:

$$\beta_{1_h(\text{opt})} = \frac{\bar{Y}_h}{\bar{X}_h}\left[\frac{U_{002}U_{110} - U_{101}U_{011}}{U_{020}U_{002} - U_{011}^2}\right], \tag{101}$$

$$\beta_{2_h(\text{opt})} = \frac{\bar{Y}_h}{\bar{Z}_h}\left[\frac{U_{020}U_{101} - U_{110}U_{011}}{U_{020}U_{002} - U_{011}^2}\right], \tag{102}$$

$$l_{1_h(\text{opt})} = \left[\frac{U_{002}U_{110} - U_{101}U_{011}}{U_{020}U_{002} - U_{011}^2}\right], \tag{103}$$

$$l_{2_h(\text{opt})} = \left[\frac{U_{020}U_{101} - U_{110}U_{011}}{U_{020}U_{002} - U_{011}^2}\right], \tag{104}$$

$$k_{h(\text{opt})} = \frac{\left(1 + \{(1/8)(3U_{020} - U_{002} - 2U_{011}) + (1/2)(U_{101} - U_{110})\}\right)}{\left(1 + 2\{(1/8)(3U_{020} - U_{002} - 2U_{011}) + (1/2)(U_{101} - U_{110})\} + k^2\{U_{200} + (1/4)(U_{020} + U_{002} - 2U_{002}) + U_{101} - U_{110}\}\right)} = k_h^*(say), \tag{105}$$

$$L_{1_h(\text{opt})} = \frac{2(U_{002}U_{110} - U_{101}U_{011})}{\left(U_{020}U_{002} - U_{011}^2\right)}, \tag{106}$$

$$L_{2_h(\text{opt})} = \frac{2(U_{110}U_{011} - U_{020}U_{101})}{\left(U_{020}U_{002} - U_{011}^2\right)}, \tag{107}$$

$$\lambda_{h(\text{opt})} = \frac{(U_{002}U_{110} - U_{101}U_{011})}{\left(U_{020}U_{002} - U_{011}^2\right)}, \tag{108}$$

$$\psi_{h(\text{opt})} = \frac{-(U_{020}U_{101} - U_{110}U_{011})}{\left(U_{020}U_{002} - U_{011}^2\right)}, \tag{109}$$

$$k_{3_h(\text{opt})} = \frac{1 + ((3/8) - (\alpha_h/4))U_{002} - (U_{101}/2) - (U_{011}(U_{011} - U_{110})/2U_{020})}{1 + U_{200} + (1 - (\alpha_h/2))U_{002} - 2U_{101} - \left((U_{011} - U_{110})^2/U_{020}\right)} = \frac{A_{m_h}}{B_{m_h}}(say), \tag{110}$$

$$k_{4_h(\text{opt})} = \frac{\bar{Y}_h}{\bar{X}_h}\left[\frac{U_{011}}{2U_{020}} - k_{3_h(\text{opt})}\left(\frac{U_{011} - U_{110}}{U_{020}}\right)\right], \tag{111}$$

$$\eta_{1_h(\text{opt})} = \frac{\left(U_{020}U_{002} - U_{011}^2\right)}{(U_{002}U_{110} - U_{101}U_{011})}, \tag{112}$$

$$\eta_{2_h(\text{opt})} = \frac{\left(U_{020}U_{002} - U_{011}^2\right)}{(U_{020}U_{101} - U_{110}U_{011})}, \tag{113}$$

$$k_{1_h(\text{opt})} = \frac{O_h}{P_h}. \tag{114}$$

C. MSEs of Proposed Combined Estimators

This section addresses the outline of the proof of Theorem 1 and Corollary 2 of Section 3.1.

Consider the estimator

$$\bar{y}_{s_3}^c = \xi_3\bar{y}_{[\text{SRSS}]} + \theta_3\left(\bar{x}_{(\text{SRSS})} - \bar{X}\right) + \delta_3\left(\bar{z}_{[\text{SRSS}]} - \bar{Z}\right). \tag{115}$$

Using the notations defined in the earlier section, we get

$$\bar{y}_{s_3}^c - \bar{Y} = \bar{Y}\left[(\xi_3 - 1)\bar{Y} + \bar{Y}\xi_3\varepsilon_0 + \theta_3\bar{X}\varepsilon_1 + \delta_3\bar{Z}\varepsilon_2\right]. \tag{116}$$

Squaring and taking expectation both sides of (116), we will get the MSE of the estimator up to first order of approximation as follows

$$\text{MSE}\left(\bar{y}_{s_3}^c\right) = \bar{Y}^2\Big[(\xi_3 - 1)^2\bar{Y}^2 + \xi_3^2\bar{Y}^2V_{200} + \theta_3^2\bar{X}^2V_{020} + \delta_3^2\bar{Z}^2V_{002} + 2\xi_3\theta_3\bar{X}\bar{Y}V_{110} + 2\xi_3\delta_3\bar{Z}\bar{Y}V_{101} + 2\theta_3\delta_3\bar{X}\bar{Z}V_{011}\Big]. \tag{117}$$

The optimum values of ξ_3, θ_3, and δ_3 can be obtained by minimizing (117) w.r.t. ξ_3, θ_3, and δ_3 as follows:

$$\xi_{3(\text{opt})} = \frac{1}{\left[1 + V_{200} - \left(\left(V_{020}V_{101}^2 + V_{002}V_{110}^2 - 2V_{110}V_{101}V_{011}\right)/\left(V_{020}V_{002} - V_{011}^2\right)\right)\right]} = \frac{B_3}{A_3}(say), \quad (118)$$

$$\theta_{3(\text{opt})} = \xi_{3(\text{opt})}\left(\frac{\bar{Y}}{\bar{X}}\right)\frac{\left(V_{101}V_{011} - V_{002}V_{110}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}, \quad (119)$$

$$\delta_{3(\text{opt})} = \xi_{3(\text{opt})}\left(\frac{\bar{Y}}{\bar{Z}}\right)\frac{\left(V_{110}V_{011} - V_{020}V_{101}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}, \quad (120)$$

where $A_3 = [1 + V_{200} - ((V_{020}V_{101}^2 + V_{002}V_{110}^2 - 2V_{110}V_{101}V_{011})/(V_{020}V_{002} - V_{011}^2))]$ and $B_3 = 1$.

Putting $\xi_{3(\text{opt})}$, $\theta_{3(\text{opt})}$, and $\delta_{3(\text{opt})}$ in (117), we get the minimum MSE as follows:

$$\text{minMSE}\left(\bar{y}_{s_3}^c\right) = \bar{Y}^2\left(1 - \xi_{3(\text{opt})}\right) = \bar{Y}^2\left(1 - \frac{B_3^2}{A_3}\right). \quad (121)$$

In a similar way, we can calculate the MSEs of other estimators $\bar{y}_{s_i}^c$, $i = 1, 2, 4, 5$, as follows

$$\text{MSE}\left(\bar{y}_{s_i}^c\right) = \bar{Y}^2\left[1 + \xi_i^2 A_i - 2\xi_i B_i\right], \quad (122)$$

where

$$A_1 = 1 + V_{200} + 2\theta_1^2 V_{020} + 2\delta_1^2 V_{002} - 2\theta_1 V_{020} - 2\delta_1 V_{002} + 4\theta_1 V_{110} + 4\delta_1 V_{101} + 4\theta_1\delta_1 V_{011},$$

$$B_1 = 1 + \frac{\theta_1^2}{2}V_{020} + \frac{\delta_1^2}{2}V_{002} - \theta_1 V_{020} - \delta_1 V_{002} + \theta_1 V_{110} + \delta_1 V_{101} + \theta_1\delta_1 V_{011},$$

$$A_2 = 1 + V_{200} + \theta_2^2 V_{020} + \delta_2^2 V_{002} - \theta_2 V_{020} - \delta_2 V_{002} + 4\theta_2 V_{110} + 4\delta_2 V_{101} + 4\theta_2\delta_2 V_{011},$$

$$B_2 = 1 - \frac{\theta_2}{2}V_{020} - \frac{\delta_2}{2}V_{002} + \theta_2 V_{110} + \delta_2 V_{101} + \theta_2\delta_2 V_{011},$$

$$A_4 = 1 + V_{200} + \theta_4 V_{020} + \delta_4 V_{002} + 2\theta_4^2 V_{020} + 2\delta_4^2 V_{002} - 4\theta_4 V_{110} - 4\delta_4 V_{101} + 4\theta_4\delta_4 V_{011},$$

$$B_4 = 1 + \frac{\theta_4(\theta_4 + 1)}{2}V_{020} + \frac{\delta_4(\delta_4 + 1)}{2}V_{002} - \theta_4 V_{110} - \delta_4 V_{101} + \theta_4\delta_4 V_{011},$$

$$A_5 = 1 + V_{200} + 3\theta_5^2 V_{020} + 3\delta_5^2 V_{002} - 4\theta_5 V_{110} - 4\delta_5 V_{101} + 4\theta_5\delta_5 V_{011},$$

$$B_5 = 1 + \theta_5^2 V_{020} - \theta_5 V_{110} + \delta_5^2 V_{002} - \delta_5 V_{101} + \theta_5\delta_5 V_{011},$$

$$v_1 = \frac{a\bar{X}}{a\bar{X} + b},$$

$$v_2 = \frac{c\bar{Z}}{c\bar{Z} + d}. \quad (123)$$

The optimum values of the scalars involved are given hereunder:

$$\xi_{i(\text{opt})} = \frac{B_i}{A_i}, \quad (124)$$

$$\theta_{1(\text{opt})} = \frac{\left(V_{101}V_{011} - V_{002}V_{110}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)} = \theta_{2(\text{opt})}, \quad (125)$$

$$\delta_{1(\text{opt})} = \frac{\left(V_{110}V_{011} - V_{020}V_{101}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)} = \delta_{2(\text{opt})}, \quad (126)$$

$$\theta_{4(\text{opt})} = \frac{\left(V_{002}V_{110} - V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)} = \theta_{5(\text{opt})}, \quad (127)$$

$$\delta_{4(\text{opt})} = \frac{\left(V_{020}V_{101} - V_{110}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)} = \delta_{5(\text{opt})}. \quad (128)$$

D. MSEs of Proposed Separate Estimators

This section addresses the outline of the proof of Theorem 3 and Corollary 4 of Section 3.2.

Consider the estimator

$$\bar{y}_{s_3}^s = \sum_{h=1}^{L} W_h\left[\xi_{3_h}\bar{y}_{h[\text{RSS}]} + \theta_{3_h}\left(\bar{x}_{h(\text{RSS})} - \bar{X}_h\right) + \delta_{3_h}\left(\bar{z}_{h[\text{RSS}]} - \bar{Z}_h\right)\right]. \quad (129)$$

Using the notations defined in the earlier section, we get

$$\bar{y}_{s_3}^{s} - \bar{Y}_h = \sum_{h=1}^{L} W_h \bar{Y}_h \left[\left(\xi_{3_h} - 1\right)\bar{Y} + \bar{Y}_h \xi_{3_h}\varepsilon_{0_h} + \theta_{3_h}\bar{X}_h\varepsilon_{1_h} + \delta_{3_h}\bar{Z}_h\varepsilon_{2_h} \right]. \quad (130)$$

Squaring and taking expectation both sides of (130), we will get the MSE of the estimator up to the first order of approximation as follows:

$$\begin{aligned}\mathrm{MSE}\left(\bar{y}_{s_3}^{s}\right) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Big[&\left(\xi_{3_h} - 1\right)^2 \bar{Y}_h^2 + \xi_{3_h}^2 \bar{Y}_h^2 U_{200} \\ &+ \theta_{3_h}^2 \bar{X}_h^2 U_{020} + \delta_{3_h}^2 \bar{Z}_h^2 U_{002} + 2\xi_{3_h}\theta_{3_h}\bar{X}_h\bar{Y}_h U_{110} \\ &+ 2\xi_{3_h}\delta_{3_h}\bar{Z}_h\bar{Y}_h U_{101} + 2\theta_{3_h}\delta_{3_h}\bar{X}_h\bar{Z}_h U_{011}\Big]. \end{aligned} \quad (131)$$

The optimum values of ξ_{3_h}, θ_{3_h}, and δ_{3_h} can be obtained by minimizing (131) w.r.t. ξ_{3_h}, θ_{3_h}, and δ_{3_h} as follows

$$\xi_{3_h(\mathrm{opt})} = \frac{1}{\left[1 + U_{200} - \left(\left(U_{020}U_{101}^2 + U_{002}U_{110}^2 - 2U_{110}U_{101}U_{011}\right)/\left(U_{020}U_{002} - U_{011}^2\right)\right)\right]} = \frac{B_{3_h}}{A_{3_h}}\,(say), \quad (132)$$

$$\theta_{3_h(\mathrm{opt})} = \xi_{3_h(\mathrm{opt})}\left(\frac{\bar{Y}_h}{\bar{X}_h}\right)\frac{\left(U_{101}U_{011} - U_{002}U_{110}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)}, \quad (133)$$

$$\delta_{3(\mathrm{opt})} = \xi_{3_h(\mathrm{opt})}\left(\frac{\bar{Y}_h}{\bar{Z}_h}\right)\frac{\left(U_{110}U_{011} - U_{020}U_{101}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)}, \quad (134)$$

where $A_{3_h} = [1 + U_{200} - ((U_{020}U_{101}^2 + U_{002}U_{110}^2 - 2U_{110}U_{101}U_{011})/(U_{020}U_{002} - U_{011}^2))]$ and $B_{3_h} = 1$.

Putting $\xi_{3_h(\mathrm{opt})}$, $\theta_{3_h(\mathrm{opt})}$ and $\delta_{3_h(\mathrm{opt})}$ in (131), we get the minimum MSE as

$$\mathrm{minMSE}\left(\bar{y}_{s_3}^{s}\right) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left(1 - \xi_{3_h(\mathrm{opt})}\right) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left(1 - \frac{B_{3_h}^2}{A_{3_h}}\right). \quad (135)$$

Similarly, we can calculate the MSE of other estimators $\bar{y}_{s_i}^{s}$, $i = 1, 2, 4, 5$, as follows

$$\mathrm{MSE}\left(\bar{y}_{s_i}^{s}\right) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[1 + \xi_{i_h}^2 A_{i_h} - 2\xi_{i_h} B_{i_h}\right], \quad (136)$$

where

$$\begin{aligned} A_{1_h} &= \Big\{1 + U_{200} + 2\theta_{1_h}^2 U_{020} + 2\delta_{1_h}^2 U_{002} - 2\delta_{1_h} U_{020} \\ &\quad - 2\delta_{1_h} U_{002} + 4\theta_{1_h} U_{110} + 4\delta_{1_h} U_{101} + 4\theta_{1_h}\delta_{1_h} U_{011}\Big\}, \\ B_{1_h} &= 1 + \frac{\theta_{1_h}^2}{2} U_{020} + \frac{\delta_{1_h}^2}{2} U_{002} - \theta_1 U_{020} - \delta_{1_h} U_{002} + \theta_{1_h} U_{110} \\ &\quad + \delta_{1_h} U_{101} + \theta_{1_h}\delta_{1_h} U_{011}, \\ A_{2_h} &= \Big\{1 + U_{200} + \theta_{2_h}^2 U_{020} + \delta_{2_h}^2 U_{002} - \theta_{2_h} U_{020} - \delta_{2_h} U_{002} \\ &\quad + 4\theta_{2_h} U_{110} + 4\delta_{2_h} U_{101} + 4\theta_{2_h}\delta_{2_h} U_{011}\Big\}, \\ B_{2_h} &= 1 - \frac{\theta_{2_h}}{2} U_{020} - \frac{\delta_{2_h}}{2} U_{002} + \theta_{2_h} U_{110} + \delta_{2_h} U_{101} + \theta_{2_h}\delta_{2_h} U_{011}, \\ A_{4_h} &= \Big\{1 + U_{200} + \theta_{4_h} U_{020} + \delta_{4_h} U_{002} + 2\theta_{4_h}^2 U_{020} + 2\delta_4^2 U_{002} \\ &\quad - 4\theta_{4_h} U_{110} - 4\delta_{4_h} U_{101} + 4\theta_{4_h}\delta_{4_h} U_{011}\Big\}, \\ B_{4_h} &= 1 + \frac{\theta_{4_h}\left(\theta_{4_h} + 1\right)}{2} U_{020} + \frac{\delta_{4_h}\left(\delta_{4_h} + 1\right)}{2} U_{002} - \theta_{4_h} U_{110} \\ &\quad - \delta_{4_h} U_{101} + \theta_{4_h}\delta_{4_h} U_{011}, \\ A_{5_h} &= 1 + U_{200} + 3\theta_{5_h}^2 U_{020} + 3\delta_{5_h}^2 U_{002} - 4\theta_{5_h} U_{110} - 4\delta_{5_h} U_{101} \\ &\quad + 4\theta_{5_h}\delta_{5_h} U_{011}. \end{aligned} \quad (137)$$

The optimum values of the scalars involved are given hereunder as follows:

$$\xi_{i_h(\mathrm{opt})} = \frac{B_{i_h}}{A_{i_h}}, \quad (138)$$

$$\theta_{1_h(\mathrm{opt})} = \frac{\left(U_{101}U_{011} - U_{002}U_{110}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)} = \theta_{2_h(\mathrm{opt})}, \quad (139)$$

$$\delta_{1_h(\mathrm{opt})} = \frac{\left(U_{110}U_{011} - U_{020}U_{101}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)} = \delta_{2_h(\mathrm{opt})}, \quad (140)$$

$$\theta_{4_h(\mathrm{opt})} = \frac{\left(U_{002}U_{110} - U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)} = \theta_{5_h(\mathrm{opt})}, \quad (141)$$

$$\delta_{4_h(\mathrm{opt})} = \frac{\left(U_{020}U_{101} - U_{110}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)} = \delta_{5_h(\mathrm{opt})}. \quad (142)$$

Data Availability

The descriptive statistics of the artificially generated populations are given in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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References

[1] N. Koyuncu and C. Kadilar, "Family of estimators of population mean using two auxiliary variables in stratified random sampling," *Communications in Statistics-Theory and Methods*, vol. 38, no. 14, pp. 2398–2417, 2009.

[2] R. Tailor, S. Chouhan, R. Tailor, and N. Garg, "A ratio-cum-product estimator of population mean in stratified random sampling using two auxiliary variables," *Statistica*, vol. 72, no. 3, pp. 287–297, 2012.

[3] R. Tailor and S. Chouhan, "Ratio-cum-product type exponential estimator of finite population mean in stratified random sampling," *Communications in Statistics-Theory and Methods*, vol. 43, no. 2, pp. 343–354, 2014.

[4] H. A. Lone, R. Tailor, and H. P. Singh, "Generalized ratio-cum-product type exponential estimator in stratified random sampling," *Communications in Statistics-Theory and Methods*, vol. 45, no. 11, pp. 3302–3309, 2016.

[5] H. A. Lone, R. Tailor, and M. R. Verma, "Efficient separate class of estimators of population mean in stratified random sampling," *Communications in Statistics-Theory and Methods*, vol. 46, no. 2, pp. 554–573, 2016.

[6] S. Muneer, J. Shabbir, and A. Khalil, "Estimation of finite population mean in simple random sampling and stratified random sampling using two auxiliary variables," *Communications in Statistics-Theory and Methods*, vol. 46, no. 5, pp. 2181–2192, 2017.

[7] S. Muneer, A. Khalil, and J. Shabbir, "A parent-generalized family of chain ratio exponential estimators in stratified random sampling using supplementary variables," *Communications in Statistics: Simulation and Computation*, vol. 51, no. 8, pp. 4727–4748, 2020.

[8] G. Mclntyre, "A method for unbiased selective sampling using ranked sets," *Australian Journal of Agricultural Research*, vol. 3, no. 4, pp. 385–390, 1952.

[9] K. Takahasi and K. Wakimoto, "On unbiased estimates of the population mean based on the sample stratified by means of ordering," *Annals of the Institute of Statistical Mathematics*, vol. 20, no. 1, pp. 1–31, 1968.

[10] T. R. Dell and J. L. Clutter, "Ranked set sampling theory with order statistics background," *Biometrics*, vol. 28, no. 2, pp. 545–555, 1972.

[11] H. A. Muttlak, "Parameters estimation in a simple linear regression using rank set sampling," *Biometrical Journal*, vol. 37, no. 7, pp. 799–810, 1995.

[12] H. M. Samawi and H. A. Muttlak, "Estimation of ratio using rank set sampling," *Biometrical Journal*, vol. 38, no. 6, pp. 753–764, 1996.

[13] S. Bhushan and A. Kumar, "Log type estimators of population mean under ranked set sampling," in *Predictive Analytics Using Statistics and Big Data: Concepts and Modeling*, vol. 28, pp. 47–74, Bentham Science Publisher, 2020.

[14] S. Bhushan and A. Kumar, "Predictive estimation approach using difference and ratio type estimators in ranked set sampling," *Journal of Computational and Applied Mathematics*, vol. 410, article 114214, 2022.

[15] S. Bhushan and A. Kumar, "On optimal classes of estimators under ranked set sampling," *Communications in Statistics - Theory and Methods*, vol. 51, no. 8, pp. 2610–2639, 2022.

[16] S. Bhushan and A. Kumar, "An efficient class of estimators based on ranked set sampling," *Life Cycle Reliability and Safety Engineering*, vol. 11, no. 1, pp. 39–48, 2022.

[17] S. Bhushan, A. Kumar, and S. Singh, "Some efficient classes of estimators under stratified sampling," *Communications in Statistics - Theory and Methods*, pp. 1–30, 2021.

[18] S. Bhushan, A. Kumar, and S. A. Lone, "On some novel classes of estimators using ranked set sampling," *Alexandria Engineering Journal*, vol. 61, no. 7, pp. 5465–5474, 2022.

[19] W. A. Abu-Dayyeh, M. S. Ahmed, R. A. Ahmed, and H. A. Muttlak, "Some estimators for the population mean using auxiliary information under ranked set sampling," *Journal of Modern Applied Statistical Methods*, vol. 8, no. 1, pp. 253–265, 2009.

[20] I. Olkin, "Multivariate ratio estimation for finite Populations," *Biometrika*, vol. 45, no. 1-2, pp. 154–165, 1958.

[21] N. Mehta and V. L. Mandowara, "Improved ratio estimators using two auxiliary variables in raked set sampling," *Journal of International Academic Research for Multidisciplinary*, vol. 2, no. 7, pp. 2320–5083, 2014.

[22] L. Khan and J. Shabbir, "Generalized exponential-type ratio-cum ratio estimators of population mean in ranked set and stratified ranked set sampling," *Journal of Statistics & Management Systems*, vol. 20, no. 1, pp. 133–151, 2017.

[23] F. Smarandache, "Neutrosophic set-a generalization of the intuitionistic fuzzy set," *International Journal of Pure and Applied Mathematics*, vol. 24, no. 3, p. 287, 2005.

[24] F. Smarandache, *Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability*, Infinite Study, 2013.

[25] R. Alhabib, M. M. Ranna, H. Farah, and A. A. Salama, "Some neutrosophic probability distributions," *Neutrosophic Sets and Systems*, vol. 22, pp. 30–38, 2018.

[26] M. Aslam, O. H. Arif, and R. A. K. Sherwani, "New diagnosis test under the neutrosophic statistics: an application to diabetic patients," *BioMed Research International*, vol. 2020, Article ID 2086185, 7 pages, 2020.

[27] M. Aslam, R. A. K. Sherwani, and M. Saleem, "Vague data analysis using neutrosophic Jarque-Bera test," *PLoS One*, vol. 16, no. 12, article e0260689, 2021.

[28] Z. Tahir, H. Khan, M. Aslam, J. Shabbir, Y. Mahmood, and F. Smarandache, "Neutrosophic ratio-type estimators for estimating the population mean," *Complex & Intelligent Systems*, vol. 7, no. 6, pp. 2991–3001, 2021.

[29] G. K. Vishwakarma and A. Singh, "Generalized estimator for computation of population mean under neutrosophic ranked set technique: an application to solar energy data," *Computational and Applied Mathematics*, vol. 41, no. 4, pp. 1–29, 2022.

[30] M. Khoshnevisan, R. Singh, P. Chauhan, N. Sawan, and F. Smarandache, "A general family of estimators for estimating population means using known value of some population parameter(s)," *Far East Journal of Statistics*, vol. 22, no. 2, pp. 181–191, 2007.

[31] D. T. Searls, "The utilization of a known coefficient of variation in the estimation procedure," *Journal of the American Statistical Association*, vol. 59, no. 308, pp. 1225-1226, 1964.

[32] S. Bhushan, R. Gupta, S. Singh, and A. Kumar, "Some Improved classes of estimators using auxiliary information," *International Journal for Research in Applied Science and Engineering Technology*, vol. 8, no. 6, pp. 1088–1098, 2020.

[33] S. Bhushan, R. Gupta, S. Singh, and A. Kumar, "Some new improved classes of estimators using multiple auxiliary information," *Global Journal of Pure and Applied Mathematics*, vol. 16, no. 3, pp. 515–528, 2020.

[34] W. G. Cochran, *Sampling Techniques*, John Wiley and Sons, New York, 1977.

[35] C. E. Sarndal, B. Swensson, and J. Wretman, *Model Assisted Survey Sampling*, Springer-Verlag, 1992.

[36] National Horticulture Baard, 2010, http://nhb.gov.in/statistics/area-productionstatistics.html.